Singularities in the Weak Turbulence Regime for the Quintic Schrödinger Equation

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Abstract. In this paper, we discuss the problem of derivation of kinetic equations from the theory of weak turbulence for the quintic Schrödinger equation. We study the quintic Schrödinger equation on $L\mathbb{T}$, with $L \gg 1$ and with a non-linearity of size $\varepsilon \ll 1$. We consider the correlations $f(T)$ of the Fourier coefficients of the solution at times $t = T\varepsilon^{-2}$ when $\varepsilon \to 0$ and $L \to \infty$. Our results can be summed up in the following way: there exists a regime for ε and L such that for T dyadic, $f(T)$ has the form expected from the Physics literature for kinetic regimes, but such that f has an infinite number of discontinuity points. This discontinuity appears in the context of finite-box effects.

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Contents

1 Introduction

In this paper, we discuss the problem of derivation of kinetic equations from the theory of weak turbulence for the quintic Schrödinger equation.
Wave turbulence has been introduced by Peierls in the late 1920s, in [29] for cristals. It describes the statistical dynamics of random non-linear waves. It was followed by works by Brout and Prigogine, [3] and early developments are described in the book by Prigogine [30]. In the 1960s, wave turbulence was developed for plasma physics [32, 36], and in the context of water waves and primitive equations, for example the works by Benney and co-authors [2], by Hasselman [18, 19] and Zakharov and Filonenko [34, 35]. These works started to describe the statistical dynamics of waves that were far from equilibrium, as opposed to a perturbative study. Zakharov introduced in particular the now-called Kolmogorov-Zakharov spectra, [33], that modelises a specific momentum transport.
An extensive literature has been developed since in Physics, as it is reviewed in the book by Nazarenko, [28].
The derivation of kinetic equations consists in describing the dynamics of the statistic of random waves, that is to derive an equation satisfies by the moments or the law of a solution to a generic Hamiltonian equation. It is called kinetic in analogy with the kinetic equations for large systems of particles (eg the Vlasov equation) but instead of having particles that interacts to form a non-linearity,

one considers the repartition of waves, and these waves interacts to form a non-linearity.
We first describe the expected result before describing our framework and results.
Consider the quintic Schrödinger Cauchy problem :

$$\left\{ \begin{array}{c} i\partial_t u_{\varepsilon,L} = -\bigtriangleup u_{\varepsilon,L} + \varepsilon |u_{\varepsilon,L}|^4 u_{\varepsilon,L} \\ u_{\varepsilon,L}(t=0) = a_L \end{array} \right. \tag{1}$$

on the torus $L\mathbb{T}$, $L \gg 1$, with $\varepsilon \ll 1$ and with initial datum

$$a_L = \sum_{k\in\mathbb{Z}} a(k/L) \frac{e^{ikx/L}}{\sqrt{2\pi L}} g_k$$

where $(g_k)_k$ is a sequence of independent centered and normalized Gaussian variables, and a is a smooth, compactly supported map.
The map $u_{\varepsilon,L}$ is a map from $\mathbb{R} \times (L\mathbb{T}) \times \Omega$ where Ω is a probability space supporting the $(g_k)_k$, and $\bigtriangleup$ is the Laplace-Beltrami operator on $L\mathbb{T}$.
The issue at stake is the description of the dynamics of

$$\mathbb{E}(|\hat{u}_{\varepsilon,L}(t,k)|^2)$$

as ε goes to 0 and L goes to ∞, for any $k \in \frac{1}{L}\mathbb{Z}$, where

$$\hat{u}_{\varepsilon,L}(t,k) := \frac{1}{\sqrt{2\pi L}} \int_{L\mathbb{T}} e^{-ikx} u_{\varepsilon,L}(t,x) dx$$

is the Fourier transform of $u_{\varepsilon,L}$.
Note that because the law of the initial datum a_L and the equation are invariant under the action of space translations, we have

$$\mathbb{E}(\overline{\hat{u}_{\varepsilon,L}(t,k')} \hat{u}_{\varepsilon,L}(t,k)) = 0$$

at all times $t \in \mathbb{R}$, if $k \neq k'$.
The ersatz is the following. Approaching $u_{\varepsilon,L}$ by its 0-th, first and second Picard iterates, we get up to second order in ε,

$$u_{\varepsilon,L}(t) \sim e^{it\bigtriangleup} a_L + \varepsilon b_L(t) + \varepsilon^2 c_L(t),$$

where b_L and c_L are solutions to the equations

$$\left\{ \begin{array}{c} i\partial_t b_L = -\bigtriangleup b_L + |e^{it\bigtriangleup} a_L|^4 (e^{it\bigtriangleup} a_L) \\ i\partial_t c_L = -\bigtriangleup c_L + 3|e^{it\bigtriangleup} a_L|^4 b_L + 2\overline{b_L} |e^{it\bigtriangleup} a_L|^2 (e^{it\bigtriangleup} a_L)^2 \end{array} \right.$$

with initial datum $b_L(t=0) = c_L(t=0) = 0$.
Up to second order, we thus get

$$\begin{aligned} &\partial_t \mathbb{E}(|\hat{u}_{\varepsilon,L}(t,k)|^2) \\ &\sim 2\varepsilon \mathrm{Re}\Big(\overline{\widehat{e^{it\bigtriangleup} a_L}(t,k)} \partial_t \hat{b}_L(t,k)\Big) + 2\varepsilon^2 \mathrm{Re}\Big(\overline{\widehat{e^{it\bigtriangleup} a_L}(t,k)} \partial_t \hat{c}_L(t,k) + \overline{\hat{b}_L(t,k)} \partial_t \hat{b}_L(t,k)\Big). \end{aligned}$$

Because of probabilistic cancellations due to the law of the initial datum a_L, the term

$$\mathrm{Re}\Big(\overline{\widehat{e^{it\triangle}a_L}(t,k)}\partial_t\hat{b}_L(t,k)\Big)$$

involves only first order trivial resonances, which can be removed from the solution by multiplicating y a phase, see [10, 26] for a discussion in different contexts.
This suggests that the right scale of time is ε^{-2}, and thus we consider the quantity

$$U_{\varepsilon,L}(t,k) = \mathbb{E}(|\hat{u}_{\varepsilon,L}(t\varepsilon^{-2},k)|^2)$$

with

$$\partial_t U_{\varepsilon,L}(t,k) \sim 2\mathrm{Re}\Big(\overline{\widehat{e^{it\varepsilon^{-2}\triangle}a_L}(k)}\partial_t\hat{c}_L(t\varepsilon^{-2},k) + \overline{\hat{b}_L(t\varepsilon^{-2},k)}\partial_t\hat{b}_L(t\varepsilon^{-2},k)\Big).$$

This is valid only if the right hand side is not null.
By taking $L\to\infty$ and $\varepsilon\to 0$, we expect to get

$$U_{\varepsilon,L}(t,k) = \frac{3}{2\pi^3}\int_{\mathbb{R}^5}\delta(k+\sum_{j=1}^{5}(-1)^j k_j)\delta(\Delta(k,\vec{k}))\frac{1}{k-k_1+k_2-k_3}|a(k)|^2\prod_{j=1}^{5}|a(k_j)|^2 F_a(k,\vec{k})d\vec{k}, \tag{2}$$

where $\vec{k}=(k_1,\dots,k_5)$, where

$$F_a(k,\vec{k}) = \Big(\frac{1}{|a(k)|^2} - \frac{1}{|a(k_1)|^2} + \frac{1}{|a(k_2)|^2} - \frac{1}{|a(k_3)|^2} + \frac{1}{|a(k_4)|^2} - \frac{1}{|a(k_5)|^2}\Big),$$

where $\Delta(k,\vec{k}) = k^2-k_1^2+k_2^2-k_3^2+k_4^2-k_5^2$ and where the δs are Dirac deltas. The integral converges.
The aim of this paper is to illustrate the fact that this limit is far from being obvious in generic regimes. The reason is that the Picard expansion does not uniformly converge in L for times of order ε^{-2}, the L^2 Lebesgue norm of the initial datum a_L growing with L. The series of paper [4, 5, 6, 10] have reached larger and larger times by developing fine analytic estimates for the cubic Schrödinger equation due to the algebraic structure of the non-linearity. In particular, they estimate so-called Dyson series. We also mention [1, 11].
However, full derivation of the kinetic equation for a slightly modified laplacian and for the cubic Schrödinger equation for the regime $\varepsilon = L^{-1}$ was reached in [9] (in dimension $d\geq 3$). In [31], the authors reach full derivation of the kinetic equation for KdV type equations in dimension $d\geq 14$, by taking $L\to\infty$ and then $\varepsilon\to 0$, as it is done in the physics literature.
In the series of papers [15, 13, 14], Dymov and Kuksin reach a satisfying result at times of order ε^{-1}, by using quasi-solutions for the cubic Schrödinger equation, a notion coming from the Physics literature. We take here an approach

closer to their point of view. We also mention the work [26] on cubic equations on nets, [7, 8] on quadratic equations coming fluid mechanics, and [17]. For a general bibliographical review on connected subjects, we refer to [6] and references therein.

Apart from the model (quintic Vs cubic or quadratic equations), chosen to get a nontrivial manifold for the first order resonances, even when the dimension is 1, the main differences with the above literature are threefold. One important aspect of this work is that, to justify the use of quasi-solutions, we use the stochastic tool of Wick products. A side advantage of this is that we avoid certain possible correlations, that actually cancel each other out but that taken individually would diverge at times of order ε^{-2}. Another main difference is that, in order to reach times of order ε^{-2}, we prepare the approximation of the equation in $L\mathbb{T}$. We justify this approximation in the next subsections. It allows to reach very large times.

Finally, we exhibit a regime for ε and L such that depending on the time $t \in \mathbb{R}$, we can get different behaviors for the sequence $U_{\varepsilon,L}(t,k)$. This regime falls within the framework of *finite-box effects*, that is well-known in the Physics literature - it is for instance described in Chapter 10 of [28] - because it is something that appears when doing experimental measurements. These finite box effects appears when the nonlinear frequency broadening is not big compared to the frequency spacing in the finite box. In our case, this translates as when ε^2 is not very big compared to $\frac{1}{L^2}$. In other words, we expect to see the kinetic regime when

$$\varepsilon \gg L^{-1}.$$

The regime we describe here is such that $\varepsilon = o(L^{-1})$. In this case, one expects to be in the regime of discrete wave turbulence. This means that the set of 5-waves interactions that contribute to the final limit is expected to be depleted to the exact resonances. This is the context of the Diophantine problem formulated by Kartashova in [22, 23]. This is what can be seen when one takes first ε to 0 and then L to ∞. However, in the context of a regime in ε and L, one must take into account the 2π-periodicity of sin – we mean the sine that appears in (9). Here, we take indeed ε such that

$$\varepsilon \ll L^{-1} \quad \Leftrightarrow \quad L^{-2}\varepsilon^{-2} \gg 1$$

but we have that

$$L^{-2}\varepsilon^{-2}\text{mod}[2\pi] \ll 1$$

which help us get close to a kinetic regime at specific given times in Theorem 1.1. However, we still observe a depletion of contributing 5-waves interactions at other times, which is what happens in Theorem 1.2.

The regime of discrete wave turbulence is discussed in [6]. This is when – using the notation of this paper – $T_{kin} \gg 1$. In this paper, they also treat the case when $T_{kin} \ll 1$, and in a previous paper [5], they dealt with $T_{kin} \sim 1$. We mention that the work by Dymov and Kuksin [15] deals with the kinetic

regime case, but that [16] is about the derivation of effective equation in the discrete turbulent regime.
Note that a mesoscopic behavior has been exhibited, combining both the kinetic behavior of larger boxes and the discrete behavior of small boxes, as in [27].
We now describe our framework and results.

1.1 Framework

Let $(\xi_k)_{k\in\mathbb{Z}}$ be a sequence of real, independent, centered and normalized Gaussian variables. We write $(\Omega,\mathcal{F},\mathbb{P})$ their underlying probability space and $\mathcal{A}$ the σ-algebra generated by the $(\xi_k)_k$. Given a well-chosen sequence $(q_k)_{k\in\mathbb{Z}}$ of positive real numbers, we define $\mathcal{S}_{-1}(H^s(L\mathbb{T}))$, $s>\frac{1}{2}$ the space of Kondratiev's distributions of $(\Omega,\mathcal{A},\mathbb{P})$ on $H^s(L\mathbb{T})$. For the exact definition of Kondratiev's distributions, we refer to Subsection 2.1. Here, we use the terminology of [21, 25], and for the general definition of Kondratiev's distributions, we refer to the original work [24]. The Wick product (see again Subsection 2.1) is well-defined on $\mathcal{S}_{-1}(H^s(L\mathbb{T}))$, we denote it $\diamond$.
Let F_L be the map defined on Fourier mode by, for all $k\in\frac{1}{L}\mathbb{Z}$,

$$\widehat{F_L}(\alpha,\beta,\gamma,\delta,\eta)(k)=\frac{1}{(2\pi L)}\sum_{C_L(k)}\hat{\alpha}(k_1)\diamond\overline{\hat{\beta}(k_2)}\diamond\gamma(k_3)\diamond\overline{\delta}(k_4)\diamond\eta(k_5),$$

where

$$C_L(k)=\Big\{(k_1,k_2,k_3,k_4,k_5)\in\Big(\frac{1}{L}\mathbb{Z}\Big)^5\ \Big|\ k_1-k_2+k_3-k_4+k_5=k,$$
$$|k-k_1+k_2-k_3|\geq\mu^{-1}(L),\quad |k^2-k_1^2+k_2^2-k_3^2+k_4^2-k_5^2|\geq\nu^{-1}(L)\Big\}\tag{3}$$

and where we used the abuse of notation

$$\hat{\alpha}(k_1)\diamond\hat{\beta}(k_2)=\Big((\hat{\alpha}(k_1)e^{ik_1x})\diamond(\hat{\beta}(k_2)e^{ik_2x})\Big)e^{-i(k_1+k_2)x}.$$

Here, μ and ν are sequences indexed by $L\in\mathbb{N}^*$ that go to ∞ as L goes to ∞. The sequence ν helps with avoiding the first order resonances, and μ compensates ν when taking the final limits and obtaining the Dirac deltas.
Note that taking (u_L) a sequence of maps such that $u_L\in\mathcal{S}^{-1}(H^s(L\mathbb{T}))$ and such that for any $\alpha\in\mathbb{N}_f^{\mathbb{Z}}$, the sequence $\|(u_L)_\alpha\|_{H^s(L\mathbb{T})}$ is uniformly bounded in L (see Remark 3.1 for the relevance of this property), then we have that for all smooth and compactly supported map f, and all $\alpha\in\mathbb{N}_f^{\mathbb{Z}}$,

$$\langle f,F_L(u,u,u,u,u)_\alpha-(u\diamond\bar{u}\diamond u\diamond\bar{u}\diamond u)_\alpha\rangle\underset{L\to\infty}{\to}0,$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}$, as soon as ν goes to ∞ with L.
We consider the Cauchy problem

$$\begin{cases} i\partial_t u_{\varepsilon,L}=-\triangle u_{\varepsilon,L}+\varepsilon F_L(u,u,u,u,u)\\ u_{\varepsilon,L}(t=0)=a_L\end{cases}\tag{4}$$

where a_L is given by

$$a_L = \sum_{k\in\mathbb{Z}} a(k/L)\frac{e^{ikx/L}}{\sqrt{2\pi L}} g_k,$$

where $g_k = \frac{1}{\sqrt{2}}(\xi_{\varphi(0,k)} + i\xi_{\varphi(1,k)})$ with φ a bijection from $\{0\}\times\mathbb{Z}\sqcup\{1\}\times\mathbb{Z}$ to $\mathbb{Z}$. The sequence $(g_k)_k$ is a sequence of independent, centered, normalized complex Gaussian variables. This is the classical way of extending real Gaussian Hilbert spaces to complex ones.

We call P_N the projection onto the Wiener chaos of degree at most N.

We are now ready to state the results.

1.2 Results

Theorem 1.1. *For any $L \in \mathbb{N}^*$, there exists a Banach algebra $X \subseteq \mathcal{S}_{-1}(H^s(L\mathbb{T}))$ into which the Cauchy problem (4) is globally well-posed and such that $a_L \in X$. Set $N \in \mathbb{N}^*$, $M \geq N$, and set f and g two smooth compactly supported maps of $\mathbb{R}$.*

Assume that ε writes

$$\varepsilon^{-2} = 2\pi L^2 2^L + \rho(L),$$

that ρ, ν and μ satisfy the following relationships:

$$\begin{cases} \exists\alpha > 0,\ \nu(L)^{1+\alpha} = o(L^{1/2}), \\ \rho(L) = o(\mu(L)), \\ \rho(L)\mu(L) = o(\nu(L)), \\ \ln^2(\mu(L)) = o(\rho^{1/4}), \\ \mu(L) \to \infty, \\ \rho(L) \to \infty \end{cases} \tag{5}$$

and finally assume that t is dyadic and not null.

Then, we have

$$\lim_{L\to\infty} \partial_t \mathbb{E}(\langle P_N U_L, f\rangle\langle g, P_M U_L\rangle)(t) =$$

$$\frac{3}{4\pi^4}\int_{\mathbb{R}^6} \delta(k + \sum_{j=1}^{5}(-1)^j k_j)\delta(\Delta(\vec{k}))\frac{1}{k - k_1 + k_2 - k_3}\prod_{j=1}^{5}|a(k_j)|^2 \hat{f}(k)\overline{\hat{g}(k)}dk\prod_{j=1}^{5}dk_j$$

and besides the integral converges.

Above, we have $U_L = u_{\varepsilon,L}(t\varepsilon^{-2})$, $\Delta(\vec{k}) = k^2 - k_1^2 + k_2^2 - k_3^2 + k_4^2 - k_5^2$ and the δs are Dirac deltas.

Remark 1.1. *The regime $\nu = L^\alpha$, $\mu = L^\beta$, $\rho = L^\gamma$ with $0 < \gamma < \beta < \alpha < \frac{1}{2}$ and $\beta + \gamma < \alpha$ satisfies the assumptions (5).*

Remark 1.2. *On the one hand, this theorem applies to a class of times that are dense in $\mathbb{R}$. If the dyadic numbers are not satisfying, they can be changed to rational numbers by choosing the regime*

$$\varepsilon^{-2} = 2\pi L^2 L! + \rho(L).$$

However, this will serve us to prove that

$$t \mapsto \lim_{L\to\infty} \partial_t \mathbb{E}(\langle P_N U_L, f\rangle \langle g, P_M U_L\rangle)(t)$$

has a chaotic behavior.

Theorem 1.2. *With the same notations as in Theorem 1.1, assuming that ε writes*

$$\varepsilon^{-2} = 2\pi L^2 2^L + \rho(L),$$

that ρ, ν and μ satisfy (5) and that $t \in \frac{1}{3} + \mathbb{D}$, $\mathbb{D}$ being the set of dyadic numbers, we have

$$\lim_{L\to\infty} \partial_t \mathbb{E}(\langle P_N U_L, f\rangle \langle g, P_M U_L\rangle)(t) =$$
$$\frac{1}{12\pi^4}\int_{\mathbb{R}^6} \delta(k + \sum_{j=1}^5 (-1)^j k_j)\delta(\Delta(\vec{k}))\frac{1}{k-k_1+k_2-k_3}\prod_{j=1}^5 |a(k_j)|^2 \hat{f}(k)\overline{\hat{g}(k)}dk \prod_{j=1}^5 dk_j$$

and besides the integral converges.

Remark 1.3. *The difference is in the constant in front of the integral.*

Remark 1.4. *The behavior on the sequence $\partial_t \mathbb{E}(\langle P_N U_L, f\rangle \langle g, P_M U_L\rangle)$ depends mainly on the behavior of the sequence*

$$2^L t - \lfloor 2^L t \rfloor.$$

In the case when t is rational, because the sequence $2^L t - \lfloor 2^L t\rfloor$ is pre-periodic, we believe that

$$\lim_{L\to\infty} \partial_t \mathbb{E}(\langle P_N U_L, f\rangle \langle g, P_M U_L\rangle)(t) =$$
$$C(t)\int_{\mathbb{R}^6} \delta(k + \sum_{j=1}^5 (-1)^j k_j)\delta(\Delta(\vec{k}))\frac{1}{k-k_1+k_2-k_3}\prod_{j=1}^5 |a(k_j)|^2 \hat{f}(k)\overline{\hat{g}(k)}dk \prod_{j=1}^5 dk_j$$

or at least that the sequence admits a finite number of adherence values of this form.
The behavior of $2^L t - \lfloor 2^L t\rfloor$ when t is irrational is not so obvious. We recall that the closure of

$$\{2^L t - \lfloor 2^L t\rfloor \mid L \in \mathbb{N}\}$$

is either the torus $\mathbb{R}/\mathbb{Z}$ or a subset of null Haar measure, but that it is almost surely the torus. When $(2^L t - \lfloor 2^L t\rfloor)_L$ is dense in the torus, we believe that the sequence

$$(\partial_t \mathbb{E}(\langle P_N U_L, f\rangle \langle g, P_M U_L\rangle)(t))_L$$

has at least an infinite number of adherence values.
For a complete description of the behavior of the sequence $2^L t - \lfloor 2^L t\rfloor$, we refer to [12].

REMARK 1.5. *Given a fixed t, a similar argument to ours will yield that there exists a regime $\varepsilon(t, L)$ such that*

$$\lim_{L\to\infty} \partial_t \mathbb{E}(\langle P_N U_L, f\rangle\langle g, P_M U_L\rangle)(t) =$$

$$\frac{3}{4\pi^4}\int_{\mathbb{R}^6} \delta(k + \sum_{j=1}^{5}(-1)^j k_j)\delta(\Delta(\vec{k}))\frac{1}{k-k_1+k_2-k_3}\prod_{j=1}^{5}|a(k_j)|^2 \hat{f}(k)\overline{\hat{g}(k)}dk\prod_{j=1}^{5}dk_j$$

or written differently that

$$\liminf_{\varepsilon\to 0, L\to\infty}\Big|\partial_t \mathbb{E}(\langle P_N U_{\varepsilon,L}, f\rangle\langle g, P_M U_{\varepsilon,L}\rangle)(t)$$
$$-\frac{3}{4\pi^4}\int_{\mathbb{R}^6}\delta(k+\sum_{j=1}^{5}(-1)^j k_j)\frac{\delta(\Delta(\vec{k}))}{k-k_1+k_2-k_3}\prod_{j=1}^{5}|a(k_j)|^2\hat{f}(k)\overline{\hat{g}(k)}dk\prod_{j=1}^{5}dk_j\Big| = 0.$$

This may be compared with the result of [26].

REMARK 1.6. *We restricted ourselves to the dimension 1 for seek of clarity but the techniques used do not depend on dimension, except when the dimension plays a role in the structure of first order resonances. The results could be carefully extended to higher dimension with some adaptation.*

The proof relies on the following strategy. We first describe solutions $u_{\varepsilon,L}$ as a series (converging in Kondratiev's distributions) where each term corresponds to a certain degree in terms of Wiener chaos decomposition. Then, we describe each term of the series thanks to trees, or so-called Feynman diagrams. Finally, we analyze each of these trees and decide which of them are contributing to the considered limit.

The relevant trees are the ones with only one node, or of Wiener chaos of degree 1. The others are irrelevant for mainly two reasons: either they give a contribution of size $\varepsilon^n t^{2n} f(L)$ with $f(L) \to 0$, which gives, taking time t of order ε^{-2} something that goes to 0 when L goes to ∞; or they give a contribution of size $\varepsilon^{2n} t^m g(L)$ with $m < n$, in which case, taking as time scale ε^{-2}, we get a contribution of size $\varepsilon^{2(n-m)} g(L)$ and we use that $\varepsilon^{2(n-m)}$ can compensate the behavior of $g(L)$, as long as it has at most polynomial growth.

The first case arises when, in the history of interactions between the different wavelengths, special resonances occurs. This translates as constraint equations on the wavelengths and is explained in Subsection 4.2.

The second case arises in a more general context, which is explained in Subsection 4.1.

The reason we ask in $C_L(k)$ (and thus in the non-linearity) that $|\Delta(\vec{k})| \geq \nu^{-1}(L)$ and then, in Assumptions (5), $\nu^{1+\alpha} = O(L^{1/2})$ is to deal with the trees presenting constraint estimates. It is probably not optimal, as explained in Remark 4.1 but the proof suggests that the optimal assumption is $\nu^{1+\alpha} = o(L)$. The issue is that simply assuming $\Delta(\vec{k}) = 0$ ensures only that $|\Delta(\vec{k})| \geq L^{-2}$. Hence, we need the condition on ν to be far enough from first order resonances.

We now explain the special regime for ε. If

$$\varepsilon^{-2}t \in \rho_t(L) + 2\pi L^2\mathbb{Z}$$

we need $\rho_t(L) = o(L)$ while $\rho_t(L)$ goes to ∞ to get the result. But if

$$\varepsilon^{-2}t = \rho_t(L) = o(L)$$

then $\varepsilon(L)$ cannot be small enough to close the argument. So, we decided to use the degree of freedom in $2\pi L^2\mathbb{Z}$. To include the dyadic numbers, we chose

$$\varepsilon^{-2} = 2\pi L^2 2^L + \rho(L)$$

and thus ε was small enough to be able to be far from optimality regarding the trees contributing as $\varepsilon^{2n}t^m g(L)$.
Finally, the reason we ask $|k-k_1+k_2-k_3| \geq \mu^{-1}(L)$ is to be able to manipulate integrals when passing from sum to integral or when getting Dirac deltas.

1.3 Organization of the paper

In Section 2, we review the definitions of Kondratiev's distributions and Wick's product. We prove global well-posedness of Equation (4).
In Section 3, we define quintic trees and ordered quintic trees, that we use to describe the solution $u_{\varepsilon,L}$ as a sum indexed by these trees.
In Section 4, we estimates the different contributions of the trees.
In Section 5, we take the final limits that yield to our result.
Finally, for the rest of the paper, we write $\mathbb{Z}/L = \frac{1}{L}\mathbb{Z}$ to lighten notations.

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2 Wick's product, well-posedness of the equation

2.1 Wick's product, Kondratiev's distributions

Let $\Omega, \mathcal{A}, \mathbb{P}$ be a probability onto which one can define $(\xi_k)_{k\in\mathbb{Z}}$ a sequence of independent Gaussian variables centered and normalized. For any $\alpha \in \mathbb{N}^{\mathbb{Z}}$ with finite support, we define

$$\xi_\alpha := \prod_{k\in\mathbb{Z}} \frac{H_{\alpha_k}(\xi_k)}{\sqrt{\alpha_k!}},$$

where H_{α_k} is the α_k-th Hermite polynomial. It is a well-known fact that $\prod_k \sqrt{\alpha_k!}\xi_\alpha$ is the orthogonal (in $L^2((\Omega,\mathcal{A},\mathbb{P}))$) projection of

$$\prod_{k\in\mathbb{Z}} \xi_k^{\alpha_k}$$

on the orthogonal of the polynomials of degree at most $|\alpha| - 1 = \sum_k \alpha_k - 1$. Let $\mathcal{F}$ be the sigma-algebra generated by the sequence $(\xi_k)_k$. We recall that for any $\phi \in L^2((\Omega, \mathcal{F}, \mathbb{P}), H^s(L\mathbb{T}))$, we have the decomposition (called Wiener chaos decomposition)

$$\phi = \sum_{\alpha \in \mathbb{N}_f^{\mathbb{Z}}} \phi_\alpha \xi_\alpha$$

where $\phi_\alpha \in H^s(L\mathbb{T})$ and $\mathbb{N}_f^{\mathbb{Z}}$ is the set of sequences in $\mathbb{N}^{\mathbb{Z}}$ with finite support. What is more,

$$\|\phi\|^2_{L^2(\Omega, H^s(L\mathbb{T}))} = \sum_\alpha \|\phi_\alpha\|^2_{H^s(L\mathbb{T})}.$$

For more information on Wiener chaos, we refer to [20]. For the rest of this section, we omit the dependence in $L\mathbb{T}$ of the Sobolev spaces.
Let $q = (q_k)_{k\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$, $\alpha, \beta \in \mathbb{N}_f^{\mathbb{Z}}$, we introduce the notations

$$q^\alpha = \prod_k q_k^{\alpha_k}, \alpha! = \prod_k \alpha_k!, |\alpha| = \sum_k \alpha_k, C^\alpha_{\alpha+\beta} = \frac{(\alpha+\beta)!}{\alpha!\beta!}.$$

Moreover, we write (0) the sequence in $\mathbb{N}^{\mathbb{Z}}$ identically equal to 0.
We define the Wick's product of ξ_α and ξ_β as the orthogonal projection in $L^2((\Omega, \mathcal{F}, \mathbb{P}))$ of

$$\xi_\alpha \xi_\beta$$

on the orthogonal of the polynomials of degree at most $|\alpha| + |\beta| - 1$, that is

$$\xi_\alpha \diamond \xi_\beta = \sqrt{C^\alpha_{\alpha+\beta}} \xi_{\alpha+\beta}.$$

Let $(q_k)_k$ be a sequence of increasing, positive numbers such that for all $D > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_k \frac{1}{q_k^N} < \frac{1}{4D}.$$

We recall that the space of Kondratiev distributions $S_{-1}(H^s)$ is defined as the inductive limit of spaces $(S_{-1,-l}(H^s))_{l\in\mathbb{N}}$ where $S_{-1,-l}(H^s)$ is the closure of $L^2(\Omega, H^s)$ with regards to the norm

$$\|\phi\|^2_{-1,-l} = \sum_\alpha \frac{1}{\alpha!} q^{-l\alpha} \|\phi_\alpha\|^2_{H^s}.$$

The space of Kondratiev's distributions is the dual of the space

$$S_1(H^{-s}) = \bigcap_{l\in\mathbb{N}} S_{1,l}(H^{-s})$$

where $S_{1,l}(H^{-s})$ is induced by the norm

$$\|\phi\|^2_{S_{1,l}} = \sum_\alpha \alpha! q^{l\alpha} \|\phi_\alpha\|^2_{H^{-s}}.$$

For any $\phi \in S_{-1}(H^s)$, we have the chaos expansion

$$\phi = \sum_\alpha \phi_\alpha \xi_\alpha$$

therefore, we can define the Wick's product of two elements of $S_{-1}(H^s)$, ϕ and ψ by

$$(\phi \diamond \psi)_\alpha = \sum_{\alpha_1+\alpha_2=\alpha} \sqrt{C_\alpha^{\alpha_1}} \phi_{\alpha_1} \psi_{\alpha_2}.$$

Note that for any $s > \frac{1}{2}$, if $\phi \in S_{-1,-l_1}(H^s)$ and $\psi \in S_{-1,-l_2}(H^s)$, taking $l = \max(l_1 + A, l_2)$, where

$$\sum_k \frac{1}{q_k^A} < 1,$$

we have since H^s is an algebra,

$$\|\phi \diamond \psi\|^2_{-1,-l} = \sum_\alpha \frac{1}{\alpha!} q^{-l\alpha} \|(\phi \diamond \psi)_\alpha\|^2_{H^s} =$$
$$\sum_\alpha \sum_{\alpha_1,\alpha_2} \Big(\frac{q^{-l(\alpha-\alpha_1)/2}}{\sqrt{(\alpha-\alpha_1)!}} \|\psi_{\alpha-\alpha_1}\|_{H^s} \frac{q^{-l(\alpha-\alpha_2)/2}}{\sqrt{(\alpha-\alpha_2)!}} \|\psi_{\alpha-\alpha_2}\|_{H^s}$$
$$\frac{q^{-l\alpha_1/2}}{\sqrt{\alpha_1!}} \|\phi_{\alpha_1}\|_{H^s} \frac{q^{-l\alpha_2/2}}{\sqrt{\alpha_2!}} \|\phi_{\alpha_1}\|_{H^s} \Big).$$

By Cauchy-Schwarz, we have

$$\|\phi \diamond \psi\|^2_{-1,-l} \leq \Big(\sum_\alpha \frac{q^{-l\alpha/2}}{\sqrt{\alpha!}} \|\phi_\alpha\|_{H^s} \Big)^2 \|\psi\|^2_{-1,-l}$$

where indeed, $\|\psi\|_{-1,-l} < \infty$. Again by Cauchy-Schwarz, we have

$$\|\phi \diamond \psi\|^2_{-1,-l} \leq \sum_\alpha q^{-(l-l_1)\alpha} \|\phi\|^2_{-1,-l_1} \|\psi\|^2_{-1,-l}.$$

We recognize

$$\sum_\alpha q^{-(l-l_1)\alpha} = \sum_{N\in\mathbb{N}} \Big(\sum_k \frac{1}{q_k^{l-l_1}} \Big)^N$$

which is finite since

$$\sum_k \frac{1}{q_k^{l-l_1}}$$

is strictly less than 1.

Therefore the Wick product of 2 elements of $S_{-1}(H^s)$ is well-defined.

The space $S_{-1}(H^s)$ is called the space of Kondratiev distributions of H^s.

2.2 Embeddings

Definition 2.1. Let C_α be defined as

$$C_0 = 0, C_\alpha = 1, \text{ for all } |\alpha| = 1$$

and

$$C_\alpha = \sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1} C_{\alpha_2}.$$

Lemma 2.2. *We have for all* $\alpha \in \mathbb{N}_f^{\mathbb{Z}}$,

$$C_\alpha \leq 4^{|\alpha|}.$$

Proof. Let $(z_k)_k \in \mathbb{R}^{\mathbb{Z}}$ be sequence with finite support such that

$$\Big|\sum z_k\Big| < \frac{1}{4}.$$

Set $M \in \mathbb{N}$ and

$$F_M(z) = \sum_{|\alpha| \leq M} z^\alpha C_\alpha.$$

This is a finite sum, therefore, it converges.
We have

$$F_M(z) = \sum_{|\alpha|=1} z^\alpha + \sum_{1<|\alpha|\leq M} \sum_{\alpha_1+\alpha_2=\alpha} z^\alpha C_{\alpha_1} C_{\alpha_2}.$$

therefore

$$F_M(z) = \sum_k z_k + F_M(z)^2 - \sum_{\mathcal{A}_M} C_{\alpha_1} C_{\alpha_2} z^\alpha$$

where $\mathcal{A}_M = \{(\alpha_1, \alpha_2) |\, |\alpha_1|, |\alpha_2| \leq M, |\alpha_1 + \alpha_2| > M\}$. We work on the union of nonempty submanifolds of $\mathbb{R}^d$, $d \in \mathbb{N}^*$,

$$\{z \in \mathbb{R}_f^{\mathbb{Z}} \;\Big|\; |\sum z_k| < \frac{1}{4} \text{ and } |\sum_k z_k - \sum_{\mathcal{A}_M} C_{\alpha_1} C_{\alpha_2} z^\alpha| < \frac{1}{4}\}.$$

Writing

$$G_M(z) = \sum_{\mathcal{A}_M} C_{\alpha_1} C_{\alpha_2} z^\alpha$$

we have

$$F_M(z) = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4\sum_k z_k + 4G_M}.$$

Because $F_M(0) = 0$, we have

$$F_M(z) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\sum_k z_k + 4G_M}.$$

Expanding the square root, we get

$$F_M(z) = \sum_k z_k - G_M + \sum_{n>1} \frac{(2n-3)!}{n!(n-1)!}\Big(\sum_k z_k - G_M\Big)^N.$$

Keeping in mind that F_M is a polynomial of degree at most M in z, and that G_M is a polynomial of degree at least $M+1$, we have

$$F_M(z) = \sum_k z_k + \sum_{1<n\le M} \frac{(2n-3)!}{n!(n-1)!}\Big(\sum_k z_k\Big)^N.$$

Therefore

$$F_M(z) = \sum_k z_k + \sum_{1<n\le M}\sum_{|\alpha|=n} \frac{(2n-3)!}{n!(n-1)!} z^\alpha.$$

Thus, for all $1 < |\alpha| \le M$, we have

$$C_\alpha = \frac{(2|\alpha|-3)!}{|\alpha|!(|\alpha|-1)!} \le 4^{|\alpha|}.$$

Besides, for all $|\alpha| = 1$, $C_\alpha = 1 \le 4$, which concludes the proof. □

Definition 2.3. Let $D > 0$, and $X(D)$ be the space of Kondratiev's distributions ϕ such that $\phi_0 = 0$ and

$$\sup_\alpha \frac{\|\phi_\alpha\|_{H^s}}{\sqrt{\alpha!}C_\alpha D^{|\alpha|}} < \infty$$

endowed with the norm

$$\|\phi\|_{X(D)} = \sup_\alpha \frac{\|\phi_\alpha\|_{H^s}}{\sqrt{\alpha!}C_\alpha D^{|\alpha|}}.$$

Proposition 2.4. *The space $X(D)$ is a Banach algebra (for the Wick product).*

Partial proof. We prove that $X(D)$ is complete. Let ϕ^n be a Cauchy sequence in $X(D)$. We have that by definition of $X(D)$, at α fixed, the sequence ϕ^n_α is Cauchy in H^s and thus converges towards some ϕ_α in H^s. By the usual arguments, we have

$$\sup_\alpha \frac{\|\phi_\alpha\|_{H^s}}{\sqrt{\alpha!}C_\alpha D^{|\alpha|}} < \infty$$

and

$$\sup_\alpha \frac{\|\phi_\alpha - \phi^n_\alpha\|_{H^s}}{\sqrt{\alpha!}C_\alpha D^{|\alpha|}} \to 0$$

when n goes to ∞. The issue at stake is to prove that $(\phi_\alpha)_\alpha$ is indeed a Kondratiev distribution.

We have for l big enough

$$\sum_\alpha \|\phi_\alpha\|_{H^s}^2 \frac{q^{-l\alpha}}{\alpha!} \leq \sum_\alpha C_\alpha^2 D^{2|\alpha|} q^{-l\alpha} \|\phi\|_{X(D)}^2.$$

Because of the bound on C_α, we have

$$\sum_\alpha \|\phi_\alpha\|_{H^s}^2 \frac{q^{-l\alpha}}{\alpha!} \leq \sum_\alpha (4D)^{2|\alpha|} q^{-l\alpha} \|\phi\|_{X(D)}^2.$$

We recognize

$$\sum_\alpha (4D)^{2|\alpha|} q^{-l\alpha} = \sum_n (4D)^{2n} \Big(\sum_k \frac{1}{q_k^l} \Big).$$

For l big enough

$$\sum_k \frac{1}{q_k^l} < \frac{1}{(4D)^2}$$

hence $(\phi_\alpha)_\alpha$ defines an element of $S_{-1,-l}(H^s)$ and thus a Kondratiev distribution.

We omit the proof that $X(D)$ is an algebra as the fact is irrelevant for the sequel and the proof is fairly straightforward. □

2.3 Global well-posedness of the equation

Let $X_t(D)$ be the space of $\phi \in \mathcal{C}(\mathbb{R}, \mathcal{S}_{-1}(H^s))$ such that for all $t \in \mathbb{R}$, $\phi(t)_{(0)} = 0$ and induced by the norm :

$$\|\phi\|_{X_t(D)} = \sup_{t\in[-T,T]} \langle t\rangle^{1/4} \|\phi\|_{X(D\langle t\rangle^{1/4})}.$$

Proposition 2.5. *Let $D_0 > 0$. There exists $D > D_0$ such that for all ϕ_0 in the unit ball of $X(D_0)$ with $(\phi_0)_{(0)} = 0$, the Cauchy problem*

$$\begin{cases} i\partial_t \phi = - \triangle \phi + \varepsilon F_L(\phi) \\ \phi(t=0) = \phi_0 \end{cases} \tag{6}$$

admits a unique global solution in $X_t(D)$ and the flow thus defined is continuous in the initial datum.

Proof. We solve the fix point problem

$$\phi(t) = A(\phi) := S(t)\phi_0 - i \int_0^T S(t-\tau) F_L(\phi(\tau)) d\tau$$

in the ball of $X_t(D)$ of radius η for D big enough. Since the linear flow preserves the H^s norm, and since for all $t \in \mathbb{R}$ and Φ such that $\Phi_{(0)} = 0$,

$$\langle t\rangle^{1/4} \|\Phi\|_{X(D\langle t\rangle^{1/4})} = \sup_{|\alpha|>1} \frac{\|\Phi_\alpha\|_{H^s}}{\sqrt{\alpha!} C_\alpha D^{|\alpha|} \langle t\rangle^{(|\alpha|-1)/4}} \leq \|\Phi\|_{X(D)},$$

we have

$$\langle t\rangle^{1/4}\|A(\phi)(t)\|_{X(D\langle t\rangle^{1/4})} \leq \|\phi_0\|_{X(D)} + |t|\langle t\rangle^{1/4}\varepsilon \sup_{\tau\in[0,t]} \|F_L(\phi)(\tau)\|_{X(D\langle t\rangle^{1/4})}.$$

We have by definition

$$\hat{F}_L(\phi)(k) = \frac{1}{(2\pi L)^2}\sum_{C_L(k)} \hat{\phi}(k_1)\diamond\overline{\hat{\phi}(k_2)}\diamond\hat{\phi}(k_3)\diamond\overline{\hat{\phi}(k_4)}\diamond\hat{\phi}(k_5).$$

where

$$C_L(k) = \{(k_1,\dots,k_5)\in\mathbb{Z}/L \mid |k_1-k_2+k_3-k_4+k_5 = k,$$
$$|\Delta(\vec{k})| \geq \nu^{-1}, \quad |k_1-k_2+k_3| \geq \mu^{-1}\}.$$

Therefore, for $\alpha\in\mathbb{N}_f^{\mathbb{Z}}$ such that $|\alpha|\geq 5$ (for $|\alpha|<5$, $\hat{F}_L(\phi)_\alpha = 0$),

$$\hat{F}_L(\phi)_\alpha(k) = \frac{1}{(2\pi L)^2}\sum_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5=\alpha}\frac{\sqrt{\alpha!}}{\sqrt{\alpha_1!\alpha_2!\alpha_3!\alpha_4!\alpha_5!}}$$
$$\sum_{C_L(k)}\hat{\phi}_{\alpha_1}(k_1)\overline{\hat{\phi}_{\alpha_2}(k_2)}\hat{\phi}_{\alpha_3}(k_3)\overline{\hat{\phi}_{\alpha_4}(k_4)}\hat{\phi}_{\alpha_5}(k_5).$$

We get

$$|\hat{F}_L(\phi)_\alpha(k)| \leq \frac{1}{(2\pi L)^2}\sum_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5=\alpha}\frac{\sqrt{\alpha!}}{\sqrt{\alpha_1!\alpha_2!\alpha_3!\alpha_4!\alpha_5!}}$$
$$\sum_{k_1-k_2+k_3-k_4+k_5=k}\Big|\hat{\phi}_{\alpha_1}(k_1)\overline{\hat{\phi}_{\alpha_2}(k_2)}\hat{\phi}_{\alpha_3}(k_3)\overline{\hat{\phi}_{\alpha_4}(k_4)}\hat{\phi}_{\alpha_5}(k_5)\Big|.$$

By convexity of $\langle x\rangle^{2s}$ and Cauchy-Schwarz (and using that $s>1/2$), we get

$$\|F_L(\phi)_\alpha\|_{H^s} \lesssim$$
$$\sum_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5=\alpha}\frac{\sqrt{\alpha!}}{\sqrt{\alpha_1!\alpha_2!\alpha_3!\alpha_4!\alpha_5!}}\|\phi_{\alpha_1}\|_{H^s}\|\phi_{\alpha_2}\|_{H^s}\|\phi_{\alpha_3}\|_{H^s}\|\phi_{\alpha_4}\|_{H^s}\|\phi_{\alpha_5}\|_{H^s}.$$

We have using the norm of ϕ, for some constant C_s depending on the Sobolev regularity,

$$\|F_L(\phi)_\alpha(\tau)\|_{H^s} \leq C_s\sqrt{\alpha!}(\langle\tau\rangle^{1/4}D)^{|\alpha|}C_\alpha\|\phi\|^5_{X(\langle\tau\rangle^{1/4}D)},$$

that is

$$\|F_L(\phi)_\alpha(\tau)\|_{H^s} \leq C_s\sqrt{\alpha!}(\langle\tau\rangle^{1/4}D)^{|\alpha|}C_\alpha\eta^5\langle\tau\rangle^{-5/4}.$$

Note that this C_s does not depend on (or is uniformly bounded in) $L\in\mathbb{N}^*$ as its dependence in L is characterized by the value of

$$\frac{1}{L}\sum_{k\in\mathbb{Z}/L}\langle k\rangle^{-2s} \leq \frac{1}{L} + 2\int dx\langle x\rangle^{-2s}.$$

Because $|\alpha| \geq 5$, and $\langle \tau \rangle \leq \langle t \rangle$, we have

$$\|F_L(\phi)_\alpha(\tau)\|_{H^s} \leq C_s(\langle t \rangle^{1/4} D)^{|\alpha|} C_\alpha \eta^5 \langle t \rangle^{-5/4}.$$

We deduce

$$\|F_L(\phi)(\tau)\|_{X(D\langle t \rangle)} \leq C_s \langle t \rangle^{-5/4} \eta^5.$$

Taking η such that $C_s \eta^4 \varepsilon \leq \frac{1}{2}$, we get

$$\|A(\phi)\|_{X_t(D)} \leq \frac{D_0}{D} \|\phi_0\|_{X(D_0)} + \frac{1}{2}\eta.$$

Taking D such that

$$\frac{D_0}{D} \leq \frac{1}{2}\eta$$

we get that the ball of $X_t(D)$ of radius η is stable under A. A similar argument yields that A is contracting for η such that

$$\tilde{C}_s \varepsilon \eta^4 \leq \frac{1}{2}$$

where $\tilde{C}_s$ is a constant independent from L but depending on the Sobolev regularity s. □

3 Trees, description of the solution

3.1 Picard expansion

Let φ be a bijection from $\{0,1\} \times \mathbb{Z}$ to $\mathbb{Z}$. We call, by abuse of notation,

$$\xi_{\iota,k} = \xi_{\varphi(k)}.$$

We set

$$g_k = \xi_{0,k} + i\xi_{1,k}.$$

Let, for all $x \in (L\mathbb{T})^2$,

$$a_L(x) = \sum_{k \in \mathbb{Z}} g_k \frac{e^{ikx/L}}{\sqrt{2\pi}} a(\frac{k}{L})$$

where a is a smooth function with compact support. We call

$$u_{\varepsilon,L}$$

the solution to

$$i\partial_t u_{\varepsilon,L} = - \triangle u_{\varepsilon,L} + \varepsilon F_L(u_{\varepsilon,L})$$

with initial datum a_L in $X_t(D)$ for some D big enough.

First, note that, setting Supp $\alpha = \{k \in \mathbb{Z} \mid \alpha_k \neq 0\}$, we have

$$(a_L)_\alpha = \begin{cases} \frac{e^{ikx/L}}{\sqrt{L}} a(k/L) & \text{if } |\alpha| = 1 \text{ and } \mathrm{Supp}(\alpha) = \{\varphi(0,k)\} \\ i\frac{e^{ikx/L}}{\sqrt{L}} a(k/L) & \text{if } |\alpha| = 1 \text{ and } \mathrm{Supp}(\alpha) = \{\varphi(1,k)\} \\ 0 & \text{otherwise.} \end{cases}.$$

Hence

$$\|(a_L)_\alpha\|_{H^s} \leq \sup_{x\in\mathbb{R}} \langle x\rangle^s |a(x)|$$

and therefore a_L is in the unit ball of $X(D_0)$ for any $D_0 \geq \|\langle x\rangle^s a\|_{L^\infty}$. Hence $u_{\varepsilon,L}$ is well-defined.

Remark 3.1. *We have that a_L belongs to some $X(D_0)$ with D_0 uniformly bounded in L. According to the proof of Proposition 2.5, we have that the solution*

$$u_{\varepsilon,L}$$

belongs to $X_t(D)$ with

$$D = c_s D_0 \varepsilon^{1/4}$$

where c_s is a constant depending only on the Sobolev regularity s and thus D is uniformly bounded in L.
Finally, we have

$$\|(u_{\varepsilon,L}(t))_\alpha\|_{H^s(L\mathbb{T})} \leq \sqrt{\alpha!} C_\alpha D^{|\alpha|} \langle t\rangle^{\frac14(|\alpha|-1)}$$

and thus for all $\alpha \in \mathbb{N}_f^{\mathbb{Z}}$ and all $t \in \mathbb{R}$, we have that

$$\|(u_{\varepsilon,L}(t))_\alpha\|_{H^s(L\mathbb{T})}$$

is uniformly bounded in L.

Proposition 3.1. *Set $u_{0,L} = S(t)a_L := e^{it\Delta} a_L$ and define the sequence $u_{n,L}$ by induction on n: for $n \in \mathbb{N}$, $u_{n+1,L}$ is the solution to*

$$i\partial_t u_{n+1,L} = -\triangle u_{n+1,L} + \sum_{n_1+n_2+n_3+n_4+n_5=n} F_L(u_{n_1,L}, u_{n_2,L}, u_{n_3,L}, u_{n_4,L}, u_{n_5,L})$$

with initial datum 0.
Then, the series

$$\sum_{n\in\mathbb{N}} \varepsilon^n u_{n,L}$$

converges in $X_t(D)$ for D big enough.
The solution $u_{\varepsilon,L}$ to (4) satisfies

$$u_{\varepsilon,L} = \sum_{n=0}^{\infty} \varepsilon^n u_{n,L}.$$

What is more, setting P_N the projection over chaos of degree at most N, we have

$$P_N u_{\varepsilon,L} = \sum_{n=0}^{M} \varepsilon^n u_{n,L}$$

with $M = \lfloor \frac{N-1}{4} \rfloor$.

Proof. We prove that the series

$$\sum_{n=0}^{\infty} \varepsilon^n u_{n,L}$$

converges in $X_t(D)$ for D big enough.
We prove that $u_{n,L}$ is either 0 or a Wiener chaos of exact degree $4n+1$ (by which we mean a sum of monomials of exact degree $4n+1$) and that there exists D big enough and η small enough such that $\|u_{n,L}\|_{X_t(D)} \leq \eta \leq 1$. This is true for $u_{0,L}$, we have for any $\eta > 0$,

$$\|u_{0,L}\|_{X_t(\frac{\|\langle x\rangle^s a\|_{L^\infty}}{\eta})} \leq \eta.$$

We prove that this is true for any n by induction.
We have

$$u_{n+1,L}(t) = -i\int_0^t S(t-\tau) \sum_{n_1+n_2+n_3+n_4+n_5=n} F_L(u_{n_1}, u_{n_2}, u_{n_3}, u_{n_4}, u_{n_5})$$

where we recall that $S(t)$ is the linear flow $e^{it\Delta}$. Since u_{n_j} is of exact degree $4n_j+1$, we get that

$$F_L(u_{n_1}, u_{n_2}, u_{n_3}, u_{n_4}, u_{n_5})$$

is of degree $4(n_1+n_2+n_3+n_4+n_5)+5 = 4n+5 = 4(n+1)+1$ or null. Besides, for any α, we have

$$\|u_{n+1,L}(t)_\alpha\|_{H^s} \leq C(\alpha!)^{1/2}|t| \sum_{n_1+n_2+n_3+n_4+n_5=n} \sum_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5=\alpha} \sup_{\tau\leq t} \prod_j \|u_{n_j}(\tau)_{\alpha_j}\|_{H^s}(\alpha_j!)^{-1/2}.$$

Using that

$$\|u_{n_j}(\tau)_{\alpha_j}\|_{H^s}(\alpha_j!)^{-1/2}$$

is 0 if $|\alpha_j| \neq 4n_j+1$ and is less that

$$D^{4n_j+1}\langle\tau\rangle^{n_j} C_{\alpha_j}\sqrt{\alpha_j!}\nu$$

we get, if $|\alpha| = 4n+5$,

$$\|u_{n+1,L}(t)_\alpha\|_{H^s} \leq C\nu^5(\alpha!)^{1/2}|t| \sum_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5=\alpha} \prod_j C_{\alpha_j} D^{4(n+1)+1}\langle t\rangle^n = C\nu^5 C_\alpha D^{4(n+1)+1}\langle t\rangle^{n+1}$$

and is 0 if $|\alpha| \neq 4n+5$.
Hence

$$\|u_{n+1,L}\|_{X_t(D)} \leq C\nu^5.$$

For ν small enough such that $C\nu^4 \leq 1$, we have indeed $\|u_{n+1,L}\|_{X_t(D)} \leq \nu$. Therefore, since $\nu \leq 1$ and $\varepsilon < 1$, the series

$$\sum_n \varepsilon^n u_{n,L}$$

converges in $X_t(D)$ for $D = \frac{\|\langle x\rangle^s a\|_{L^\infty}}{\nu}$.
Since this series satisfies equation

$$i\partial_t u = - \triangle u + \varepsilon F_L(u)$$

with initial datum $u_{0,L}$, we have

$$u_{\varepsilon,L} = \sum_n \varepsilon^n u_{n,L}.$$

What is more,

$$P_N u_{\varepsilon,L} = \sum_n \varepsilon^n P_N u_{n,L}$$

and because $u_{n,L}$ is of degree $4n+1$, we get $P_N u_{n,L} = u_{n,L}$ if $4n+1 \leq N$ and is 0 otherwise. □

We set

$$\hat{u}_{n,L}(k,t) = \frac{1}{2\pi L}\int_{\mathbb{T}L} u_{n,L}(x,t)e^{-ikx}dx.$$

We get that $\hat{u}_{n,L}(k)$ satisfies for $n \geq 1$,

$$i\partial_t \hat{u}_{n,L} = k^2 \hat{u}_{n,L}(k) + \frac{1}{(2\pi L)^2} \sum_{n_1+n_2+n_3+n_4+n_5=n-1} \sum_{C_L(k)} \hat{u}_{n_1,L}(k_1) \diamond \overline{\hat{u}_{n_2,L}(k_2)} \diamond \hat{u}_{n_3,L}(k_3) \diamond \overline{\hat{u}_{n_4,L}(k_4)} \diamond \hat{u}_{n_5,L}(k_5).$$

We set $v_{n,L}(k,t) = e^{ik^2 t}\hat{u}_{n,L}(k,t)$ and get the equation

$$v_{0,L}(k,t) = a(k)$$

and

$$i\partial_t v_{n,L}(k) = \frac{1}{(2\pi L)^2} \sum_{n_1+n_2+n_3+n_4+n_5=n-1} \sum_{C_L(k)} e^{i\Delta(\vec{k})t} v_{n_1,L}(k_1) \diamond \overline{v_{n_2,L}(k_2)} \diamond v_{n_3,L}(k_3) \diamond \overline{v_{n_4,L}(k_4)} \diamond v_{n_5,L}(k_5). \quad (7)$$

3.2 QUINTIC TREES

All the definitions in this subsection and the next one have examples in Appendix A.

DEFINITION 3.2. Let $k \in \mathbb{Z}/L$. We define by induction the set $\mathcal{T}_n[k]$ of quintic trees with n nodes by

$$\mathcal{T}_0[k] = \{(k)\}$$

and

$$\mathcal{T}_{n+1}[k] = \{(T_1, T_2, T_3, T_4, T_5, k) \mid \forall j = 1, \dots, 5, T_j \in \mathcal{T}_{n_j}[k_j] \text{ with } \sum n_j = n, (k_1, \dots, k_5) \in C_L(k)\}.$$

We set

$$l\mathcal{T}_n = \bigcup_{k \in \mathbb{Z}/L} \mathcal{T}_n[k].$$

DEFINITION 3.3. For any tree we define the following functions or random variables. First, we set

$$F_{(k)}(t) = 1, \quad g_{(k)} = g_{Lk}, \quad A_{(k)} = a(k),$$

and if $T = (T_1, T_2, T_3, T_4, T_5, k)$ with $T_j \in \mathcal{T}_{n_j}[k_j]$, we set

$$F_T(t) = -i \int_0^t e^{i\Delta\tau} F_{T_1}(\tau) \overline{F_{T_2}(\tau)} F_{T_3}(\tau) \overline{F_{T_4}(\tau)} F_{T_5}(\tau) d\tau$$

and

$$g_T = g_{T_1} \diamond \overline{g_{T_2}} \diamond g_{T_3} \diamond \overline{g_{T_4}} \diamond g_{T_5}, \quad A_T = A_{T_1} \overline{A_{T_2}} A_{T_3} \overline{A_{T_4}} A_{T_5}.$$

DEFINITION 3.4. Finally, we define the labels $\vec{T}$ of the leaves of a tree T in the following way

$$(\vec{k}) = k \in \mathbb{Z}/L$$

and for $T = (T_1, T_2, T_3, T_4, T_5, k)$ writing $(k_{j,1}, \dots, k_{j,4n_j+1}) = \vec{T_j}$ with $T_j \in \mathcal{T}_{n_j}[l_j]$, we set

$$\vec{T} = (k_{1,1}, \dots, k_{1,4n_1+1}, k_{2,1}, \dots, k_{2,4n_2+1}, k_{3,1}, \dots, k_{3,4n_3+1}, k_{4,1}, \dots, k_{4,4n_4+1}, k_{5,1}, \dots, k_{5,4n_5+1}).$$

REMARK 3.2. *The definition is consistent as $\vec{T} \in \mathbb{R}^{4n+1}$ for any $T \in l\mathcal{T}_n$.*

PROPOSITION 3.5. *Let $T \in \mathcal{T}_n[k]$ and $\vec{T} = (k_1, \dots, k_{4n+1})$, we have*

$$k = \sum_{j=1}^{4n+1} (-1)^{j+1} k_j, \quad A_T = \prod_{j=1}^{4n+1} a(k_j, (-1)^{j+1}), \quad g_T = \mathop{\diamondsuit}_{j=1}^{4n+1} g_{k_j,(-1)^{j+1}}.$$

We used the notation

$$a(k_j, 1) = a(k_j) \quad \textit{and} \quad a(k_j, -1) = \overline{a(k_j)}$$
$$g_{k_j,1} = g_{k_j L} \quad \textit{and} \quad g_{k_j,-1} = \overline{g_{k_j L}}$$

Proof. By induction starting from Equation (7). ☐

Proposition 3.6. *We have for all* $(n,k,t) \in \mathbb{N} \times \mathbb{Z}/L \times \mathbb{R}$*,*

$$v_{n,L}(k,t) = \Big[\frac{1}{(2\pi L)^2}\Big]^n \sum_{T \in \mathcal{T}_n[k]} F_T(t) A_T g_T.$$

Proof. By induction. ☐

We introduce unlabelled quintic trees.

Definition 3.7. Let $\mathcal{T}_n$ defined by induction by

$$\mathcal{T}_0 = \{\perp\}$$

and

$$\mathcal{T}_{n+1} = \{(T_1,T_2,T_3,T_4,T_5) | \forall j = 1,\dots,5, T_j \in \mathcal{T}_{n_j} \sum_{j=1}^5 n_j = n\}.$$

Given $\vec{k} = (k_1,\dots,k_{4n+1}) \in (\mathbb{Z}/L)^{4n+1}$, we write

$$\perp(\vec{k}) = (k_1)$$

and

$$(T_1,T_2,T_3,T_4,T_5)(\vec{k}) = (T_1(\vec{k}_1), T_2(\vec{k}_2), T_3(\vec{k}_3), T_4(\vec{k}_4), T_5(\vec{k}_5), k)$$

with $T_j \in \mathcal{T}_{n_j}$ and $\vec{k}_j = (k_{\tilde{n}_j+1},\dots,k_{\tilde{n}_j+4n_j+1})$ with

$$\tilde{n}_j = \sum_{l=1}^{j-1}(4n_l+1) \quad \text{and} \quad k = -\sum_{j=1}^{4n+1}(-1)^j k_n.$$

We now give a definition of labels for the nodes and the leaves that helps seeing the "history" of the tree, as the paternity of nodes.

Definition 3.8. For any tree $T \in \bigcup_n \mathcal{T}_n$ we define by induction

$$N(\perp) = \emptyset, \tilde{N}(\perp) = \{0\}$$

and if $T = (T_1,T_2,T_3,T_4,T_5)$, we write

$$\begin{aligned} N(T) = \{0\} &\sqcup (\{1\} \times N(T_1)) \\ &\sqcup (\{2\} \times N(T_2)) \sqcup (\{3\} \times N(T_3)) \sqcup (\{4\} \times N(T_4)) \sqcup (\{5\} \times N(T_5)) \end{aligned}$$

and

$$\begin{aligned} \tilde{N}(T) = \{0\} &\sqcup (\{1\} \times \tilde{N}(T_1)) \\ &\sqcup (\{2\} \times \tilde{N}(T_2)) \sqcup (\{3\} \times \tilde{N}(T_3)) \sqcup (\{4\} \times \tilde{N}(T_4)) \sqcup (\{5\} \times \tilde{N}(T_5)). \end{aligned}$$

Definition 3.9. Let $\vec{k} = (k_1, \dots, k_{4n+1}) \in (\mathbb{Z}/L)^{4n+1}$. We define

$$k_{T,\vec{k}} : \tilde{N}(T) \to \mathbb{Z}/L$$

such that $k_{\perp,\vec{k}}(0) = k_1$ and if $T = (T_1, T_2, T_3, T_4, T_5)$ with $T_j \in \mathcal{T}_{n_j}$ and $\vec{k}_j = (k_{\tilde{n}_j+1}, \dots, k_{\tilde{n}_j+4n_j+1})$ with

$$\tilde{n}_j = \sum_{l=1}^{j-1}(4n_l + 1)$$

we define

$$k_{T,\vec{k}}(0) = \sum_{j=1}^{4n+1}(-1)^{j+1}k_j \quad \text{and} \quad k_{T,\vec{k}}((j,m)) = k_{T_j,\vec{k}_j}(m).$$

We define also

$$\Omega_{T,\vec{k}} : N(T) \to \mathbb{R}$$

such that

$$\Omega_{T,\vec{k}}(0) = k^2_{T,\vec{k}}(0) - k^2_{T,\vec{k}}((1,0)) + k^2_{T,\vec{k}}((2,0)) - k^2_{T,\vec{k}}((3,0)) + k^2_{T,\vec{k}}((4,0)) - k^2_{T,\vec{k}}((5,0))$$

and

$$\Omega_T((j,l)) = (-1)^{j+1}\Omega_{T_j}(l).$$

We use the notation Ω instead of Δ because of the discussion of the sign. Finally, we define a condition on $\vec{k}$ such that $T(\vec{k}) \in \mathcal{T}_n[k]$.

Definition 3.10. We define by induction $C_T(k)$ by

$$C_\perp(k) = \{k\}$$

and $\vec{k} \in C_T(k)$ iff

$$k_{T,\vec{k}}(0) = k, \quad |k_{T,\vec{k}}(1,0) - k_{T,\vec{k}}(2,0) + k_{T,\vec{k}}(3,0) - k_{T,\vec{k}}(0)| \geq \mu^{-1},$$

and

$$|\Omega_{T,\vec{k}}(0)| \geq \nu^{-1}, \quad \forall j = 1, \dots, 5, \vec{k}_j \in C_{T_j}(k_{T,\vec{k}}(j,0))$$

with $T = (T_1, \dots, T_5)$, $T_j \in \mathcal{T}_{n_j}$ and

$$\vec{k}_j = (k_{\tilde{n}_j+1,\dots,k_{\tilde{n}_j+4n_j+1}})$$

with $\tilde{n}_j = \sum_{l<j}(4n_l + 1)$.

Proposition 3.11. *We have*

$$\mathcal{T}_n[k] = \{T(\vec{k}) | T \in \mathcal{T}_n \textit{ and } \vec{k} \in C_T(k)\}.$$

Proof. By induction using Definition 3.7. □

Proposition 3.12. *We have for all $(n,k,t) \in \mathbb{N} \times \mathbb{Z}/L \times \mathbb{R}$,*

$$v_{n,L}(k,t) = \Big[\frac{1}{(2\pi L)^2}\Big]^n \sum_{T\in\mathcal{T}_n} \sum_{\vec{k}\in C_T(k)} F_{T(\vec{k})}(t) A_{\vec{k}} g_{\vec{k}}$$

where

$$A_{\vec{k}} = \prod_{j=1}^{4n+1} a(k_j,(-1)^{j+1}) \quad \textit{and} \quad g_{\vec{k}} = \mathop{\Diamond}_{j=1}^{4n+1} g_{k_j,(-1)^{j+1}}.$$

Proof. Direct consequence of Propositions 3.6, 3.11, and 3.5. □

3.3 Ordered trees

Definition 3.13. We define on $N(T)$ the partial order relation R_T such that for all $l \in N(T)$,

$$j\ R_T\ 0$$

for all $l_1 = (j_1, m_1) \in \{j_1\} \times N(T_{j_1})$ and $l_2 = (j_2, m_2) \in \{j_2\} \times N(T_{j_2})$, we have

$$l_1 R_T l_2 \Leftrightarrow j_1 = j_2 \text{ and } m_1 R_{T_{j_1}} m_2.$$

Remark 3.3. *The partial order R_T represents parenthood in the tree.*

Proposition 3.14. *The cardinal of $N(T)$ is n. We have*

$$F_{T(\vec{k})}(t) = (-i)^{N_T} \int_{I_T(t)} \prod_{l\in N(T)} e^{it_l \Omega_{T,\vec{k}}(l)} dt_l$$

where $I_T(t) = \{(t_l)_{l\in N(T)} \in [0,t]^n \mid l_1 R l_2 \Rightarrow t_{l_1} \le t_{l_2}\}$ and where N_T is defined by induction on the number of nodes of T by $N_\perp = 0$ and for $T = (T_1, T_2, T_3, T_4, T_5)$,

$$N_T = 1 - \sum_{j=1}^{5} (-1)^j N_{T_j}.$$

Lemma 3.15. *We have*

$$I_T(t) = \{(t_l)_{l\in N(T)} \in [0,t]^n \mid \forall j = 1\dots 5, \quad (t_{(j,l_j)})_{l_j\in N(T_j)} \in I_{T_j}(t_0)\}.$$

Proof. We recall that 0 is bigger than any other element of $N(T)$ and that if j_1 is different from j_2, (j_1, m_1) and (j_2, m_2) are not comparable while $(j, m_1)R(j, m_2)$ iff $m_1 R_{T_j} m_2$. □

Proof of Proposition 3.14. From Lemma 3.15, the induction follows almost directly, it remains to see that

$$\overline{e^{i\Delta t}} = e^{-i\Delta t}$$

which justifies in the definition of the function $\Omega_T(j,m)$ the sign -1 when j is even, as in

$$\Omega_T(j,m) = -\Omega_{T_j}(m).$$

□

Definition 3.16. Let $T \in \mathcal{T}_n$, we define $\mathfrak{S}_T$ be the set of bijection from $[1,n] \cap \mathbb{N}$ to $N(T)$such that $\varphi \in \mathfrak{S}_T$ if and only if

$$\forall l_1, l_2 \in N(T), \quad l_1 R_T l_2 \Rightarrow \varphi^{-1}(l_1) \le \varphi^{-1}(l_2).$$

We set

$$F^{\varphi}_{T,\vec{k}}(t) = \int_{0\le t_1 \le \ldots \le t_n \le t} \prod_{j=1}^{n} e^{i\Omega_{T,\vec{k}}(\varphi(j))t_j} dt_j.$$

Proposition 3.17. *We have*

$$F_{T(\vec{k})}(t) = (-i)^{N_T} \sum_{\varphi \in \mathfrak{S}_T} F^{\varphi}_{T,\vec{k}}(t).$$

Proof. Description of $I_T(t)$. □

3.4 Description of the solution

Lemma 3.18. *Let $\vec{k}_1 \in (\mathbb{Z}/L)^{4n_1+1}$ and $\vec{k}_2 \in (\mathbb{Z}/L)^{4n_2+1}$. We have*

$$\mathbb{E}(\overline{g_{\vec{k}_1}} g_{\vec{k}_2}) = 0$$

unless $n_1 = n_2 =: n$ and unless there exists a bijection $\sigma \in \mathfrak{S}_{4n+1}$ such that for all $j = 1, \ldots, 4n+1$,

$$k^1_{\sigma(j)} = k^2_j.$$

Besides, if for all $j_1 \neq j_2$, $k^1_{j_1} \neq k^1_{j_2}$ then σ is uniquely defined and conserves the parity.

Proof. We have

$$g_{\vec{k}_l} = \mathop{\Diamond}_{j=1}^{4n_l+1} g_{k^l_j,(-1)^{j+1}}$$

that is by definition of $g_{k^l_j,(-1)^{j+1}}$,

$$g_{\vec{k}_l} = \mathop{\Diamond}_{j=1}^{4n_l+1} \Big(\frac{\xi_{0,Lk^l_j} + i(-1)^{j+1}\xi_{1,Lk^l_j}}{\sqrt{2}} \Big).$$

Expanding the product, we get

$$(\sqrt{2})^{4n_1+1} g_{\vec{k}_l} = \sum_{\iota:[1,4n_l+1]\cap\mathbb{N}\to\{0,1\}} \mathop{\Diamond}_{j=1}^{4n_l+1} \xi_{\iota(j),Lk^l_j} i^{\iota(j)} (-1)^{(j+1)\iota(j)}.$$

Therefore,

$$2^{2(n_1+n_2)+1}\mathbb{E}(\overline{g_{\vec{k}_1}}g_{\vec{k}_2}) = \sum_{\iota_1,\iota_2}\mathbb{E}\Big[\mathop{\Diamond}_{j_1=1}^{4n_1+1}\xi_{\iota_1(j_1),Lk^1_{j_1}}i^{-\iota_1(j_1)}(-1)^{(j_1+1)\iota_1(j_1)} \mathop{\Diamond}_{j_2=1}^{4n_2+1}\xi_{\iota_2(j_2),Lk^2_{j_2}}i^{-\iota_2(j_2)}(-1)^{(j_2+1)\iota_2(j_2)}\Big].$$

First, for

$$E\Big[\mathop{\Diamond}_{j_1=1}^{4n_1+1}\xi_{\iota_1(j_1),Lk^1_{j_1}}i^{-\iota_1(j_1)}(-1)^{(j_1+1)\iota_1(j_1)}\mathop{\Diamond}_{j_2=1}^{4n_2+1}\xi_{\iota_2(j_2),Lk^2_{j_2}}i^{-\iota_2(j_2)}(-1)^{(j_2+1)\iota_2(j_2)}\Big]$$

not to be zero, one needs the existence of a bijection from $[1, 4n_2 + 1] \cap \mathbb{N}$ to $[1, 4n_1 + 1] \cap \mathbb{N}$ such that

$$k^1_{\sigma(j_2)} = k^2_{j_2}.$$

This implies that $n_1 = n_2$ but σ is not necessarily uniquely defined unless the k^1_j are all different. In this case, we get

$$2^{4n_1+1}\mathbb{E}(\overline{g_{\vec{k}_1}}g_{\vec{k}_2}) = \sum_{\iota_1,\iota_2}\mathbb{E}\Big[\mathop{\Diamond}_{j_1=1}^{4n_1+1}\xi_{\iota_1(j_1),Lk^1_{j_1}}i^{-\iota_1(j_1)}(-1)^{(j_1+1)\iota_1(j_1)} \mathop{\Diamond}_{j_2=1}^{4n_2+1}\xi_{\iota_2\circ\sigma(j_2),Lk^1_{j_2}}i^{-\iota_2\circ\sigma(j_2)}(-1)^{(\sigma(j_2)+1)\iota_2\circ\sigma(j_2)}\Big].$$

For

$$\mathbb{E}\Big[\mathop{\Diamond}_{j_1=1}^{4n_1+1}\xi_{\iota_1(j_1),Lk^1_{j_1}}i^{-\iota_1(j_1)}(-1)^{(j_1+1)\iota_1(j_1)} \mathop{\Diamond}_{j_2=1}^{4n_2+1}\xi_{\iota_2\circ\sigma(j_2),Lk^1_{j_2}}i^{-\iota_2\circ\sigma(j_2)}(-1)^{(\sigma(j_2)+1)\iota_2\circ\sigma(j_2)}\Big]$$

not to be 0, we need $\iota_2 \circ \sigma = \iota_1$, therefore, setting $k_j = k^1_j$ and $n = n_1 = n_2$, we get

$$2^{4n+1}\mathbb{E}(\overline{g_{\vec{k}_1}}g_{\vec{k}_2}) = \sum_{\iota}\mathbb{E}\Big[\Big|\mathop{\Diamond}_{j=1}^{4n+1}\xi_{\iota(j),Lk_j}\Big|^2\Big]\prod(-1)^{(j+\sigma(j))\iota(j)}.$$

If σ conserves parity, we have

$$2^{4n+1}\mathbb{E}(\overline{g_{\vec{k}_1}}g_{\vec{k}_2}) = \sum_{\iota}\mathbb{E}\Big[\Big|\mathop{\Diamond}_{j=1}^{4n+1}\xi_{\iota(j),Lk_j}\Big|^2\Big] > 0.$$

Otherwise, let j_0 such that $j_0 + \sigma(j_0)$ is odd. For any ι, we write ι' such that $\iota'(j_0) = -\iota(j_0)$ and $\iota'(j) = \iota(j)$ for all $j \neq j_0$. Since $\iota \mapsto \iota'$ is a bijection of

$$[1, 4n + 1] \cap \mathbb{N} \to \{0, 1\},$$

we have

$$\mathbb{E}(\overline{g_{\vec{k}_1}} g_{\vec{k}_2}) = \sum_{\iota} \mathbb{E}\Big[\Big| \mathop{\Diamond}_{j=1}^{4n+1} \xi_{\iota'(j),Lk_j} \Big|^2\Big] \prod (-1)^{(j+\sigma(j))\iota'(j)}.$$

Since the k_j are all different replacing $\xi_{\iota(j_0),Lk_{j_0}}$ by $\xi_{\iota'(j_0),Lk_{j_0}}$ consists in exchanging the roles of $\xi_{0,Lk_{j_0}}$ and $\xi_{1,Lk_{j_0}}$. In other words, since

$$(\xi_{\iota(j),k_j})_j \text{ and } (\xi_{\iota'(j),k_j})_j$$

have the same law

$$\mathbb{E}\Big[\Big| \mathop{\Diamond}_{j=1}^{4n+1} \xi_{\iota'(j),k_j} \Big|^2\Big] = \mathbb{E}\Big[\Big| \mathop{\Diamond}_{j=1}^{4n+1} \xi_{\iota(j),Lk_j} \Big|^2\Big].$$

Since

$$\prod_j (-1)^{(j+\sigma(j))\iota'(j)} = -\prod_j (-1)^{(j+\sigma(j))\iota'(j)}$$

we get

$$\mathbb{E}(\overline{g_{T_1}} g_{T_2}) = -\mathbb{E}(\overline{g_{T_1}} g_{T_2}) = 0.$$

□

Proposition 3.19. *We have*

$$\mathbb{E}(\overline{v_{n_1,L}(k,t)} v_{n_2,L}(k,t)) = 0$$

unless $n_1 = n_2 = n$ *and in this case*

$$\mathbb{E}(\overline{v_{n,L}(k,t)} v_{n,L}(k,t)) = \frac{1}{(2\pi L)^{4n}} \sum_{(T_1,T_2)\in\mathcal{T}_n^2} (-i)^{N_{T_2}-N_{T_1}}$$
$$\sum_{\sigma\in\mathfrak{S}_{4n+1}} \sum_{\varphi_j\in\mathfrak{S}_{T_j}} \sum_{\vec{k}\in C(T_1,T_2,\sigma,k)} G^{\varphi_1,\varphi_2}_{T_1,T_2,\vec{k},\vec{k}_\sigma}(t) \overline{A_{\vec{k}}} A_{\vec{k}_\sigma} \frac{\mathbb{E}(\overline{g_{\vec{k}}} g_{\vec{k}_\sigma})}{c(\sigma,\vec{k})}$$

where setting $\vec{k} = (k_1, \dots, k_{4n+1})$, *we used the notations*

$$k_\sigma = (k_{\sigma(1)}, \dots, k_{\sigma(4n+1)}),$$

$$C(T_1,T_2,\sigma,k) = \{\vec{k} \in C_{T_1}(k) \mid \vec{k}_\sigma \in C_{T_2}(k)\},$$

$$G^{\varphi_1,\varphi_2}_{T_1,T_2,\vec{k},\vec{k}_\sigma} = \overline{F^{\varphi_1}_{T_1,\vec{k}}} F^{\varphi_2}_{T_2,\vec{k}_\sigma}$$

and

$$c(\sigma,\vec{k}) = \#\{\sigma' \in \mathfrak{S}_{4n+1} \mid \vec{k}_\sigma = \vec{k}_{\sigma'}\}.$$

Proof. We use Proposition 3.12 to get that

$$v_{n,L}(k,t) = \frac{1}{(2\pi L)^{2n}} \sum_{T\in\mathcal{T}_n} \sum_{\vec{k}\in C_T(k)} F_{T(\vec{k})}(t) A_{\vec{k}} g_{\vec{k}}.$$

Therefore if $g_{\vec{k}}$ is involved in this sum, then $\vec{k} \in C_T(k)$ for some $T \in \mathcal{T}_n$ and thus $\vec{k} \in (\mathbb{Z}/L)^{4n+1}$.

Let $n_1, n_2 \in \mathbb{N}$ and assume that $g_{\vec{k}_1}$ appears in the sum describing $v_{n_1,L}(k,t)$ and $g_{\vec{k}_2}$ appears in the sum describing $v_{n_2,L}(k,t)$ then according to Lemma 3.18, we have that

$$\mathbb{E}(\bar{g}_{\vec{k}_1} g_{\vec{k}_2}) \neq 0$$

implies that $n_1 = n_2$ and that there exists $\sigma \in \mathfrak{S}_{4n_1+1}$ such that for all $j = 1, \ldots, 4n_1+1$, $k_j^2 = k_j^1$, in other words $\vec{k}_2 = (\vec{k}_1)_\sigma$.

Therefore, if $n_1 \neq n_2$, we have indeed:

$$\mathbb{E}(\overline{v_{n_1,L}(k,t)} v_{n_2,L}(k,t)) = 0.$$

If $n_1 = n_2 = n$, we have

$$E(|v_{n,L}(k,t)|^2) = \frac{1}{(2\pi L)^{4n}} \sum_{(T_1,T_2)\in\mathcal{T}_n^2} \sum_{\vec{k}_j\in C_{T_j}(k)} \overline{F_{T_1(\vec{k}_1)}(t)} F_{T_2(\vec{k}_2)}(t) \bar{A}_{\vec{k}_1} A_{\vec{k}_2} \mathbb{E}(\bar{g}_{\vec{k}_1} g_{\vec{k}_2}).$$

We remark that if $\vec{k}_1 = \vec{k}$ and $\vec{k}_2 = \vec{k}_\sigma$ then $\vec{k}_\sigma \in C_{T_2}(k)$. Now, if $\vec{k}$ is such that there exists $j_1 \neq j_2$ such that $k_{j_1} = k_{j_2}$ there exists $\sigma \neq \sigma'$ such that $\vec{k}_\sigma = \vec{k}_{\sigma'}$. We should not count these twice and thus

$$E(|v_{n,L}(k,t)|^2) = \frac{1}{(2\pi L)^{4n}} \sum_{(T_1,T_2)\in\mathcal{T}_n^2} \sum_{\sigma\in\mathfrak{S}_{4n+1}} \sum_{\vec{k}\in C(T_1,T_2,\sigma,k)} \overline{F_{T_1(\vec{k})}(t)} F_{T_2(\vec{k}_\sigma)}(t) \bar{A}_{\vec{k}} A_{\vec{k}_\sigma} \frac{\mathbb{E}(\bar{g}_{\vec{k}} g_{\vec{k}_\sigma})}{c(\sigma,k)}.$$

We conclude thanks to Proposition 3.17. We have

$$F_{T_1(\vec{k})}(t) = (-i)^{N_{T_1}} \sum_{\varphi_1\in\mathfrak{S}_{T_1}} F^{\varphi}_{T_1,\vec{k}}(t)$$

and thus

$$\begin{aligned}\overline{F_{T_1(\vec{k})}(t)} F_{T_2(\vec{k}_\sigma)}(t) &= (-i)^{N_{T_2}-N_{T_1}} \sum_{\varphi_j\in\mathfrak{S}_{T_j}} \overline{F^{\varphi_1}_{T_1,\vec{k}}(t)} F^{\varphi_2}_{T_2,\vec{k}_\sigma}(t) \\ &= (-i)^{N_{T_2}-N_{T_1}} \sum_{\varphi_j\in\mathfrak{S}_{T_j}} G^{\varphi_1,\varphi_2}_{T_1,T_2,\vec{k},\vec{k}_\sigma}(t).\end{aligned}$$

□

Corollary 3.20. *We have*

$$\partial_t \mathbb{E}(|v_{n,L}(k,t)|^2) = \frac{1}{(2\pi L)^{4n}} \sum_{(T_1,T_2)\in\mathcal{T}_n^2} (-i)^{N_{T_2}-N_{T_1}}$$

$$\sum_{\sigma\in\mathfrak{S}_{4n+1}} \sum_{\varphi_j\in\mathfrak{S}_{T_j}} \sum_{\vec{k}\in C(T_1,T_2,\sigma,k)} \partial_t G^{\varphi_1,\varphi_2}_{T_1,T_2,\vec{k},\vec{k}_\sigma}(t) \overline{A_{\vec{k}}} A_{\vec{k}_\sigma} \frac{\mathbb{E}(\overline{g_{\vec{k}}} g_{\vec{k}_\sigma})}{c(\sigma,\vec{k})}$$

Proof. We derive under the (finite) sum, using that the only terms that depend on time are the functions

$$G^{\varphi_1,\varphi_2}_{T_1,T_2,\vec{k},\vec{k}_\sigma}.$$

□

Remark 3.4. *Note that since a is of compact support, all the sums are finite and thus converge.*

4 Analysis

4.1 Time estimates

In Subsections 4.1 and 4.2, we define the behavior of certain quantities at large time (and large L). In these subsection we take $|t| \geq 1$.

Proposition 4.1. *Let $\Omega_1, \dots, \Omega_M \in \mathbb{Z}^*/L^2$, we define*

$$G_M(t) = \int_{0\leq t_1\leq\ldots\leq t_M\leq t} \prod_{j=1}^{M} e^{i\Omega_j t_j} dt_j.$$

We have

$$G_M(t) = O_M(L^{2M}|t|^{\lfloor\frac{M-1}{2}\rfloor})$$

unless M is even and for all $j = 1$ to $M/2$, we have $\Omega_{2j-1} + \Omega_{2j} = 0$, in which case

$$G_M(t) = \prod_{j=1}^{M/2} \frac{1}{\Omega_{2j}} \frac{t^{M/2}}{(M/2)!} + O(L^{2M}|t|^{\lfloor\frac{M-1}{2}\rfloor}).$$

Proof. We prove it by induction. If $M = 0$, we have

$$G_M(t) = 1 = \prod_{j=1}^{0} \frac{1}{\Omega_{2j}} \frac{t^0}{0!} + O(L^0|t|^{\lfloor\frac{M-1}{2}\rfloor}).$$

We assume that the proposition is true for all $m \leq M$, we prove it for $M+1$. We have if $M = 0$,

$$G_{M+1}(t) = \frac{e^{i\Omega_1 t} - 1}{i\Omega_1} = O(L^2).$$

Otherwise, since

$$G_{M+1}(t) = \int_0^t e^{i\Omega_{M+1} t_{M+1}} G_M(t_{M+1}) dt_{M+1},$$

by integration by parts, we get

$$G_{M+1}(t) = \frac{e^{i\Omega_{M+1} t}}{i\Omega_{M+1}} G_M(t) - \int_0^t \frac{e^{i(\Omega_{M+1}+\Omega_M)\tau}}{i\Omega_{M+1}} G_{M-1}(\tau) d\tau.$$

First case, $M+1$ is odd. We have since M is even

$$|G_M(t)| = O_M(t^{M/2} L^M) + O_M(t^{(M-2)/2} L^{2M}) = O_M(L^{2M} t^{M/2}),$$

therefore

$$\frac{e^{i\Omega_{M+1} t}}{i\Omega_{M+1}} G_M(t) = O_M(t^{M/2} L^{2(M+1)}).$$

Since $M-1$ is odd, we have

$$\frac{e^{i(\Omega_{M+1}+\Omega_M)\tau}}{i\Omega_{M+1}} G_{M-1}(\tau) = O_M(\tau^{(M-2)/2} L^{2(M-1)})$$

thus

$$\int_0^t d\tau \frac{e^{i(\Omega_{M+1}+\Omega_M)\tau}}{i\Omega_{M+1}} G_{M-1}(\tau) = O_M(t^{M/2} L^{2M})$$

which concludes the induction when $M+1$ is odd.
Second case, $M+1$ is even. We have

$$G_M(t) = O_M(L^{2M} t^{(M-1)/2})$$

hence

$$\frac{e^{i\Omega_{M+1} t}}{i\Omega_{M+1}} G_M(t) = O_M(L^{2(M+1)} t^{(M-1)/2})$$

Then, if $\Omega_{M+1} + \Omega_M \neq 0$, we have either $M = 1$ and then

$$\int_0^t d\tau \frac{e^{i(\Omega_{M+1}+\Omega_M)\tau}}{i\Omega_{M+1}} G_{M-1}(\tau) = \frac{e^{i(\Omega_{M+1}+\Omega_M)t} - 1}{-\Omega_{M+1}(\Omega_{M+1}+\Omega_M)} = O(L^4)$$

or $M > 1$ and by integration by parts

$$\int_0^t \frac{e^{i(\Omega_{M+1}+\Omega_M)\tau}}{i\Omega_{M+1}} G_{M-1}(\tau) d\tau =$$
$$\frac{e^{i(\Omega_{M+1}+\Omega_M)t}}{-(\Omega_{M+1}+\Omega_M)\Omega_{M+1}} G_{M-1}(t) + \int_0^t d\tau \frac{e^{i(\Omega_{M+1}+\Omega_M)t}}{-(\Omega_{M+1}+\Omega_M)\Omega_{M+1}} e^{i\Omega_{M-1}\tau} G_{M-2}(\tau).$$

We have

$$\frac{e^{i(\Omega_{M+1}+\Omega_M)t}}{-(\Omega_{M+1}+\Omega_M)\Omega_{M+1}} G_{M-1}(t) = O_M(L^4 L^{2(M-1)} t^{(M-1)/2})$$

and

$$\frac{e^{i(\Omega_{M+1}+\Omega_M)t}}{-(\Omega_{M+1}+\Omega_M)\Omega_{M+1}} e^{i\Omega_{M-1}\tau} G_{M-2}(\tau) = O_M(L^4 L^{2(M-2)} t^{(M-3)/2})$$

hence

$$G_{M+1}(t) = O_M(L^{2(M+1)} t^{(M-1)/2}).$$

If $\Omega_{M+1} + \Omega_M = 0$, either at least $\Omega_{2j-1} + \Omega_{2j} \neq 0$ for one $j = 1$ to $(M-1)/2$, in which case

$$\int_0^t d\tau \frac{e^{i(\Omega_{M+1}+\Omega_M)\tau}}{i\Omega_{M+1}} G_{M-1}(\tau)$$
$$= \int_0^t O_M(L^2 L^{2(M-1)} \tau^{(M-3)/2}) = O_M(L^{2(M+1)} t^{(M-1)/2})$$

or $\Omega_{2j-1} + \Omega_{2j} = 0$ for all $j = 1$ to $(M-1)/2$ (which includes $M = 1$), in which case

$$\int_0^t d\tau \frac{e^{i(\Omega_{M+1}+\Omega_M)\tau}}{i\Omega_{M+1}} G_{M-1}(\tau) =$$
$$\int_0^t \frac{1}{i\Omega_{M+1}} \Big(\prod_{j=1}^{(M-1)/2} \frac{1}{i\Omega_{2j}} \frac{\tau^{(M-1)/2}}{(M-1)/2!} + O_M(L^{2(M-1)} \tau^{(M-3)/2}) \Big) =$$
$$\prod_{j=1}^{(M+1)/2} \frac{1}{i\Omega_{2j}} \frac{t^{(M+1)/2}}{(M+1)/2!} + O_M(L^{2(M+1)} t^{(M-1)/2}).$$

□

PROPOSITION 4.2. *Let M be odd. We are in one of the following cases:*

1. *$M = 1$ then*
$$G_M(t) = \frac{e^{i\Omega_M t} - 1}{i\Omega_M};$$

2. *we are not in case 1, $\forall j = 1, \ldots, \frac{M-1}{2}$, $\Omega_{2j} = -\Omega_{2j-1} = -\Omega_M$ then*
$$G_M(t) = (i\Omega_M)^{-(M+1)/2} (-1)^{(M-1)/2} \frac{t^{(M-1)/2}}{(M-1)/2!} (e^{i\Omega_M t} - 1) + O(L^{2M} t^{(M-3)/2});$$

3. *we are not in cases 1 or 2, $\forall j = 1, \ldots, \frac{M-1}{2}$, $\Omega_{2j} + \Omega_{2j-1} = 0$ then*
$$G_M(t) = (-1)^{(M-1)/2} \prod_{j=0}^{(M-1)/2} \frac{1}{i\Omega_{2j+1}} e^{i\Omega_M t} \frac{t^{(M-1)/2}}{(M-1)/2!} + O(L^{2M} t^{(M-3)/2});$$

4. *we are not in cases 1, 2, or 3,* $\forall j = 1, \dots, \frac{M-1}{2}$, $\Omega_{2j} + \Omega_{2j+1} = 0$ *then*

$$G_M(t) = (-1)^{(M+1)/2} \prod_{j=0}^{(M-1)/2} \frac{1}{i\Omega_{2j+1}} \frac{t^{(M-1)/2}}{(M-1)/2!} + O(L^{2M} t^{(M-3)/2});$$

5. *we are not in cases 1, 2, 3 or 4, there exists* $j_0 \in [1, \frac{M-1}{2}] \cap \mathbb{N}$ *such that* $\Omega_{2j_0+1} + \Omega_{2j_0} + \Omega_{2j_0-1} = 0$ *and* $\forall j > j_0$, $\Omega_{2j+1} + \Omega_{2j} = 0$ *and* $\forall j < j_0$, $\Omega_{2j} + \Omega_{2j-1} = 0$ *then*

$$G_M(t) = (-1)^{(M-1)/2} \prod_{j=0}^{(M-1)/2} \frac{1}{i\Omega_{2j+1}} \frac{t^{(M-1)/2}}{(M-1)/2!} + O(L^{2M} t^{(M-3)/2});$$

6. *we are not in cases 1, 2, 3, 4, or 5, then*

$$G_M(t) = O(L^{2M} t^{(M-3)/2}).$$

Before proving the proposition, we prove the following lemma.

Lemma 4.3. *Let* $n \in \mathbb{N}$, *and* $\alpha \neq 0$, *we have*

$$\int_0^t e^{i\alpha\tau} \tau^n d\tau = \frac{e^{i\alpha t}}{i\alpha} \sum_{k=0}^{n} \Big(-\frac{1}{i\alpha}\Big)^k \frac{n!}{(n-k)!} t^{n-k} + \Big(-\frac{1}{i\alpha}\Big)^{n+1} n!. \tag{8}$$

Proof. Writing

$$I_n = \int_0^t e^{i\alpha\tau} \tau^n d\tau$$

the proof follows from

$$I_0 = \frac{e^{i\alpha t - 1}}{i\alpha}$$

and the induction relation

$$I_{n+1} = \frac{e^{i\alpha t}}{i\alpha} t^n - \frac{n}{i\alpha} I_{n-1}.$$

□

Proof of Proposition 4.2. We omit in the proof the dependance in M of the constant. We proceed by induction over M.
If $M = 1$, we are in case 1.
We assume that the proposition is true up to $M - 2$, and we prove it for M.
Case 2 We have by definition

$$G_M(t) = \int_0^t dt_M e^{i\Omega_M t_M} \int_{0 \leq t_1 \leq \dots \leq t_{M-1} \leq t_M} \prod_{n=1}^{M-1} e^{i\Omega_n t_n} dt_n.$$

Because of the hypothesis on the Ω_n, we have

$$G_M(t) = \int_0^t dt_M e^{i\Omega_M t_M} \int_{0\leq t_1\leq \ldots \leq t_{M-1}\leq t_M} \prod_{n=1}^{(M-1)/2} e^{i\Omega_M(t_{2n-1}-t_{2n})} dt_{2n-1}dt_{2n}.$$

By integration by parts, we get

$$G_M(t) = \frac{e^{i\Omega_M t}}{i\Omega_M} G_{M-1}(t) - \int_0^t \frac{1}{i\Omega_M} G_{M-2(\tau)} d\tau.$$

By Proposition 4.1, we have

$$\frac{e^{i\Omega_M t}}{i\Omega_M} G_{M-1}(t) = (-1)^{(M-1)/2} \frac{1}{(i\Omega_M)^{(M+1)/2}} \frac{t^{(M-1)/2}}{(M-1)/2!} + O(t^{(M-3)/2} L^{2M}).$$

We have that G_{M-2} is either in case 1 or 2, hence

$$G_{M-2}(\tau) = (-1)^{(M-3)/2} (e^{i\Omega_M \tau} - 1) \frac{1}{(i\Omega_M)^{(M-1)/2}} \frac{\tau^{(M-3)/2}}{(M-3)/2!} + O(\tau^{(M-5)/2} L^{2(M-2)}).$$

We use (8) to get

$$G_M(t) = \frac{t^{(M-1)/2}}{(M-1)/2!} (-1)^{(M-1)/2} (e^{i\Omega_M t} - 1)(i\Omega_M)^{-(M+1)/2} + O(t^{(M-3)/2} L^{2M}),$$

which is the desired result.
Case 3 By integration by parts, we have

$$G_M(t) = \frac{e^{i\Omega_M t}}{i\Omega_M} G_{M-1}(t) - \int_0^t \frac{e^{i(\Omega_M+\Omega_{M-1})\tau}}{i\Omega_M} G_{M-2}(\tau) d\tau.$$

By Proposition 4.1, we have

$$G_{M-1}(t) = (-1)^{(M-1)/2} \prod_{j=0}^{(M-3)/2} \frac{1}{i\Omega_{2j+1}} \frac{t^{(M-1)/2}}{(M-1)/2!} + O(t^{(M-3)/2} L^{2(M-1)})$$

therefore

$$(-1)^{(M-1)/2} G_M(t) = e^{i\Omega_M t} \prod_{j=0}^{(M-1)/2} \frac{1}{i\Omega_{2j+1}} \frac{t^{(M-1)/2}}{(M-1)/2!} - \int_0^t \frac{e^{i(\Omega_M+\Omega_{M-1})\tau}}{i\Omega_M} G_{M-2}(\tau) d\tau + O(t^{(M-3)/2} L^{2M}).$$

We have that G_{M-2} is either in case 1, 2 or 3. If it is in case 1 or 2, then $\Omega_M + \Omega_{M-1} \neq 0$ otherwise G_M is in case 3 and $\Omega_M + \Omega_{M-1} + \Omega_{M-2} = \Omega_M \neq 0$, therefore,

$$G_M(t) = e^{i\Omega_M t} \prod_{j=0}^{(M-1)/2} \frac{1}{i\Omega_{2j+1}} \frac{t^{(M-1)/2}}{(M-1)/2!} - \int_0^t \frac{e^{i(\Omega_M + \Omega_{M-1})\tau}}{i\Omega_M} \frac{e^{i\Omega_{M-2}\tau} - 1}{i\Omega_{M-2}} d\tau + O(t^{(M-3)/2} L^{2M})$$

if G_{M-2} is in case 1 and

$$G_M(t) = e^{i\Omega_M t} \prod_{j=0}^{(M-1)/2} \frac{1}{i\Omega_{2j+1}} \frac{t^{(M-1)/2}}{(M-1)/2!}$$
$$- \int_0^t \frac{e^{i(\Omega_M + \Omega_{M-1})\tau}}{i\Omega_M} (-1)^{(M-3)/2} \frac{e^{i\Omega_{M-2}\tau} - 1}{(i\Omega_{M-2})^{(M-1)/2}} \frac{\tau^{(M-3)/2}}{(M-3)/2!} d\tau + O(t^{(M-3)/2} L^{2M})$$

if G_{M-2} is in case 2. In both cases, by (8), we get

$$G_M(t) = e^{i\Omega_M t} \prod_{j=0}^{(M-1)/2} \frac{1}{i\Omega_{2j+1}} \frac{t^{(M-1)/2}}{(M-1)/2!} + O(t^{(M-3)/2} L^{2M})$$

which is the desired result.
If G_{M-2} is in case 3, then

$$G_{M-2}(t) = (-1)^{(M-3)/2} e^{i\Omega_{M-2} t} \prod_{j=0}^{(M-1)/2} \frac{1}{i\Omega_{2j+1}} \frac{t^{(M-3)/2}}{(M-3)/2!} + O(t^{(M-5)/2} L^{2M}).$$

Since $\Omega_M + \Omega_{M-1} + \Omega_{M-2} = \Omega_M \neq 0$, we have by (8)

$$G_M(t) = e^{i\Omega_M t} \prod_{j=0}^{(M-1)/2} \frac{1}{i\Omega_{2j+1}} \frac{t^{(M-1)/2}}{(M-1)/2!} + O(t^{(M-3)/2} L^{2M})$$

Case 4 The integration by parts yields

$$G_M(t) = \frac{e^{i\Omega_M t}}{i\Omega_M} G_{M-1}(t) - \int_0^t \frac{1}{i\Omega_M} G_{M-2}(\tau) d\tau.$$

Because we are not in case 2 or 3, by Proposition 4.1, we have

$$G_{M-1}(t) = O(t^{(M-3)/2} L^{2(M-1)})$$

thus

$$G_M(t) = -\int_0^t \frac{1}{i\Omega_M} G_{M-2}(\tau) d\tau + O(t^{(M-3)/2} L^{2M}).$$

We have that G_{M-2} is either in case 1, 2, or 4. If G_{M-2} is in case 1 then

$$-\int_0^t \frac{1}{i\Omega_M} G_{M-2}(\tau)d\tau = -\int_0^t \frac{e^{i\Omega_{M-2}\tau}-1}{i\Omega_M i\Omega_{M-2}} = \frac{t}{i\Omega_M i\Omega_{M-2}} + O(L^6)$$

which yields the desired result.
If G_{M-2} is in case 2, then

$$G_M(t) = (-1)^{(M-1)/2}\int_0^t \frac{1}{i\Omega_M}\frac{e^{i\Omega_{M-2}\tau}-1}{(i\Omega_{M-2})^{(M-3)/2}}\frac{\tau^{(M-3)/2}}{(M-3)/2!}d\tau + O(t^{(M-3)/2}L^{2M}).$$

By (8), we get

$$G_M(t) = (-1)^{(M+1)/2}\frac{1}{i\Omega_M}\frac{1}{(i\Omega_{M-2})^{(M-3)/2}}\frac{t^{(M-1)/2}}{(M-1)/2!} + O(t^{(M-3)/2}L^{2M}).$$

If G_{M-2} is in case 4, then

$$G_M(t) = -(-1)^{(M-1)/2}\int_0^t \frac{1}{i\Omega_M}\prod_{j=0}^{(M-3)/2}\frac{1}{i\Omega_{2j+1}}\frac{\tau^{(M-3)/2}}{(M-3)/2!}d\tau + O(t^{(M-3)/2}L^{2M})$$

therefore

$$G_M(t) = (-1)^{(M+1)/2}\prod_{j=0}^{(M-1)/2}\frac{1}{i\Omega_{2j+1}}\frac{t^{(M-1)/2}}{(M-1)/2!} + O(t^{(M-3)/2}L^{2M}).$$

CASE 5 Let j_0 such that $\Omega_{2j_0+1} + \Omega_{2j_0} + \Omega_{2j_0-1} = 0$. We cannot have $\Omega_{2j_0} + \Omega_{2j_0-1} = 0$ hence

$$G_{M-1}(t) = O(t^{M-3)/3}L^{2(M-1)})$$

by Proposition 4.1.
CASE 5.1 : $j_0 = \frac{M-1}{2}$. We recall that by integration by parts

$$G_M(t) = \frac{e^{i\Omega_M t}}{i\Omega_M}G_{M-1}(t) - \int_0^t \frac{e^{i(\Omega_M+\Omega_{M-1})\tau}}{i\Omega_M}G_{M-2}(\tau)d\tau.$$

We have that G_{M-2} is either in case 1, 2, or 3. If G_{M-2} is either in case 1 or 2, we have

$$G_{M-2}(\tau) = (-1)^{(M-3)/2}\frac{\tau^{(M-3)/2}}{(M-3)/2!}\frac{e^{i\Omega_{M-2}}-1}{(i\Omega_{M-2})^{(M-1)/2}} + O(\tau^{(M-5)/2}L^{2(M-2)}).$$

Therefore by (8), since $\Omega_M + \Omega_{M-1} = -\Omega_{M-2} \neq 0$ and $\Omega_M + \Omega_{M-1} + \Omega_{M-2} = 0$, we have

$$G_M(t) = (-1)^{(M-1)/2}\frac{1}{i\Omega_M(i\Omega_{M-2})^{(M-1)/2}}\frac{\tau^{(M-1)/2}}{(M-1)/2!} + O(t^{(M-3)/2L^{2M}}).$$

If G_{M-2} is in case 3, we have

$$G_M(t) = -\int_0^t \frac{e^{i(\Omega_M+\Omega_{M-1}+\Omega_{M-2})\tau}}{i\Omega_M}(-1)^{(M-3)/2}\prod_{j=0}^{(M-3)/2}\frac{1}{i\Omega_{2j+1}}\frac{\tau^{(M-3)/2}}{(M-3)/2!} + O(t^{(M-3)/2L^{2M}}).$$

And since $\Omega_M + \Omega_{M-1} + \Omega_{M-2} = 0$, we get

$$G_M(t) = (-1)^{(M-1)/2}\prod_{j=0}^{(M-1)/2}\frac{1}{i\Omega_{2j+1}}\frac{t^{(M-1)/2}}{(M-1)/2!} + O(t^{(M-3)/2L^{2M}}).$$

This is the desired result in case 5.1.
Case 5.2 $j_0 \neq \frac{M-1}{2}$. In this case, $\Omega_M + \Omega_{M-1} = 0$ and G_{M-2} is in case 5. Therefore

$$G_M(t) = -\int_0^t \frac{d\tau}{i\Omega_M}(-1)^{(M-3)/2}\prod_{j=0}^{(M-3)/2}\frac{1}{i\Omega_{2j+1}}\frac{\tau^{(M-3)/2}}{(M-3)/2!}+O(t^{(M-3)/2}L^{2M})$$

which yields by exact computation

$$G_M(t) = (-1)^{(M-1)/2}\prod_{j=0}^{(M-1)/2}\frac{1}{i\Omega_{2j+1}}\frac{t^{(M-1)/2}}{(M-1)/2!} + O(t^{(M-3)/2}L^{2M}).$$

Case 6 Because we are not in cases 1, 2, or 3, we have

$$G_{M-1}(t) = O(t^{(M-3)/2}L^{2(M-1)}).$$

If G_{M-2} is in case 1 or 2, we have $\Omega_M + \Omega_{M-1} + \Omega_{M-2} \neq 0$ otherwise G_M is in case 5 (and even 5.1) and $\Omega_M + \Omega_{M-1} \neq 0$ otherwise G_M is in case 2 or 4. Therefore by integration by parts, induction hypothesis on G_{M-2} and (8), we get

$$G_M(t) = O(t^{(M-3)/2}L^{2M}).$$

If G_{M-2} is in case 3, then $\Omega_M + \Omega_{M-1} + \Omega_{M-2} \neq 0$ otherwise G_M is in case 5, and we conclude by the above strategy, as in Case 3, with G_{M-2} in case 3.
If G_{M-2} is in case 4, then $\Omega_M + \Omega_{M-1} \neq 0$ otherwise G_M is in case 4, and we conclude by the above strategy, as in Case 3, when G_{M-2} is in case 1 or 2.
If G_{M-2} is in case 5, then $\Omega_M + \Omega_{M-1} \neq 0$ otherwise G_M is in case 5 and we conclude by the above strategy, as in Case 3, when G_{M-2} is in case 1 or 2.
If G_{M-2} is in case 6, then

$$G_{M-2}(\tau) = O(\tau^{(M-5)/2}L^{2(M-2)})$$

and we can conclude. □

4.2 Constraint estimates

Given the description of

$$\partial_t E(|v_n|^2)$$

in Corollary 3.20, we separate the sum in three parts, in the case $n > 1$, either Case A $\sigma, \varphi, T_1, T_2$ are such that for all $\vec{k} \in C(T_1, T_2, \sigma, k)$, we have that

$$\partial_t G^{\varphi}_{T_1,T_2,\vec{k},\vec{k}_\sigma} \neq O(t^{n-2}L^{4n})$$

or Case B σ, φ are not such but however

$$\partial_t G^{\varphi}_{T_1,T_2,\vec{k},\vec{k}_\sigma} \neq O(t^{n-2}L^{4n})$$

or Case C

$$\partial_t G^{\varphi}_{T_1,T_2,\vec{k},\vec{k}_\sigma} = O(t^{n-2}L^{4n}).$$

We recall that in general, we have

$$\partial_t G^{\varphi}_{T_1,T_2,\vec{k},\vec{k}_\sigma} = O(t^n L^{4n})$$

but that if the term of higher order is null then

$$\partial_t G^{\varphi}_{T_1,T_2,\vec{k},\vec{k}_\sigma} = O(t^{n-2}L^{4n}).$$

We first explain why Case A never happens.

Proposition 4.4. *There does not exist any* (T_1, T_2, σ, k) *such that for all* $\vec{k} \in C(T_1, T_2, \sigma, k)$*, we have that*

$$\partial_t G^{\varphi}_{T_1,T_2,\vec{k},\vec{k}_\sigma} \neq O(t^{n-2}L^{4n})$$

Proof. Let $T_1, T_2 \in \mathcal{T}_n$, $\sigma \in \mathfrak{S}_{4n+1}$, $\varphi_1 \in \mathfrak{S}_{T_1}$, $\varphi_2 \in \mathfrak{S}_{T_2}$ and $k \in \mathbb{Z}/L$. Assume that for all $\vec{k} \in C(T_1, T_2, \sigma, k)$, we have

$$\partial_t G^{\varphi}_{T_1,T_2,\vec{k},\vec{k}_\sigma} \neq O(t^{n-2}L^{4n})$$

and let us prove that it yields to a contradiction.
We recall that

$$\partial_t G^{\varphi}_{T_1,T_2,\vec{k},\vec{k}_\sigma} = \overline{\partial_t F^{\varphi_1}_{T_1,\vec{k}}}\, F^{\varphi_2}_{T_2,\vec{k}_\sigma} + \overline{F^{\varphi_1}_{T_1,\vec{k}}}\, \partial_t F^{\varphi_2}_{T_2,\vec{k}_\sigma}$$

and that F and $\partial_t F$ take the form

$$F^{\varphi_1}_{T_1,\vec{k}}(t) = \int_{0\leq t_1\leq \dots \leq t_n\leq t} \prod_{j=1}^{n} e^{i\Omega_{T_1,\vec{k}}(\varphi_1(j))t_j} dt_j,$$

and

$$\partial_t F^{\varphi_1}_{T_1,\vec{k}}(t) = e^{i\Omega_{T_1,\vec{k}}(\varphi_1(n))t} \int_{0\le t_1\le \ldots \le t_{n-1}\le t} \prod_{j=1}^{n-1} e^{i\Omega_{T_1,\vec{k}}(\varphi_1(j))t_j} dt_j.$$

Case 1: the integer n is even.
The maximal order for t in $F^{\varphi_2}_{T_2,\vec{k}_\sigma}$ is $\frac{n}{2}$ and for t in $\overline{\partial_t F^{\varphi_1}_{T_1,\vec{k}}}$ is $\frac{n-2}{2}$ for a total order of

$$n-1.$$

The situation for the other term in $\partial_t G$ is symmetric.
Lower orders are

$$\mathcal{O}(L^{4n-2}t^{n-2}).$$

For $F^{\varphi_2}_{T_2,\vec{k}_\sigma}$ to be of order $\frac{n}{2}$ in t, one needs to be in the situation that for $j=1$ to $\frac{n}{2}$,

$$\Omega_{T_2,\vec{k}_\sigma}(\varphi_2(2j-1)) + \Omega_{T_2,\vec{k}_\sigma}(\varphi_2(2j)) = 0.$$

But $\Omega_{T_2,\vec{k}_\sigma}(\varphi(2j-1))$ and $\Omega_{T_2,\vec{k}_\sigma}(\varphi_2(2j))$ are two different second order polynomials in $\vec{k}$ and therefore their sum cannot be identically 0 for all $\vec{k} \in C(T_1,T_2,\sigma,k)$, which yields to a contradiction.
Case 2: the integer n is odd.
The maximal order for t in $F^{\varphi_2}_{T_2,\vec{k}_\sigma}$ is $\frac{n-1}{2}$ and for t in $\overline{\partial_t F^{\varphi_1}_{T_1,\vec{k}}}$ is $\frac{n-1}{2}$ for a total order of

$$n-1.$$

The situation for the other term in $\partial_t G$ is symmetric.
Lower orders are

$$\mathcal{O}(L^{4n-2}t^{n-2}).$$

For $\overline{\partial_t F^{\varphi_1}_{T_1,\vec{k}}}$ to be of order $\frac{n-1}{2}$ in t, one needs to be in the situation that for $j=1$ to $\frac{n-1}{2}$,

$$\Omega_{T_1,\vec{k}}(\varphi_1(2j-1)) + \Omega_{T_1,\vec{k}}(\varphi_1(2j)) = 0.$$

But $\Omega_{T_1,\vec{k}}(\varphi_1(2j-1))$ and $\Omega_{T_1,\vec{k}}(\varphi_1(2j))$ are two different second order polynomials in $\vec{k}$ and therefore their sum cannot be identically 0 for all $\vec{k} \in C(T_1,T_2,\sigma,k)$, which yields to a contradiction. □

We now prove that Case B happens only for a few $\vec{k} \in C(T_1,T_2,\sigma,k)$ such that the total measure of

$$\{\vec{k} \in C(T_1,T_2,\sigma,k) \mid \text{we are in Case B}\}$$

is very small compared to L^{4n}.

PROPOSITION 4.5. *Let $T_1, T_2 \in \mathcal{T}_n$, $\sigma \in \mathfrak{S}_{4n+1}$, $\varphi_1 \in \mathfrak{S}_{T_1}$, $\varphi_2 \in \mathfrak{S}_{T_2}$ and $k \in \mathbb{Z}/L$. Set*

$$V(T_1,T_2,\sigma,k,\varphi_1,\varphi_2) = \frac{1}{(4\pi L)^n} \sum_{\vec{k}\in C(T_1,T_2,\sigma,k)} \partial_t G^{\varphi_1,\varphi_2}_{T_1,T_2,\vec{k},\vec{k}_\sigma}(t)\overline{A_{\vec{k}}}A_{\vec{k}_\sigma}\mathbb{E}(\overline{g_{\vec{k}}}g_{\vec{k}_\sigma}).$$

We have for all $\alpha > 0$,

$$V(T_1,T_2,\sigma,k,\varphi_1,\varphi_2) = O_{a,T_1,T_2,\varphi_1,\varphi_2,\sigma,\alpha}\Big[t^{n-1}\nu^{(1+\alpha)n/2}L^{-(n-1)/2}\Big] + O_{a,n}(L^{4n-2}t^{n-2}).$$

Proof. In all the proof, we omit the dependence in $T_1, T_2, \sigma, \varphi_1, \varphi_2$.
We write

$$\underline{C} = \{\vec{k}\in C \mid \overline{\partial_t F^{\varphi_1}_{T_1,\vec{k}}} F^{\varphi_2}_{T_2,\vec{k}_\sigma} \neq O_n(L^{4n-2}t^{n-2})\}$$

and

$$\tilde{C} = \{\vec{k}\in C \mid \overline{F^{\varphi_1}_{T_1,\vec{k}}}\partial_t F^{\varphi_2}_{T_2,\vec{k}_\sigma} \neq O_n(L^{4n-2}t^{n-2})\}.$$

We set

$$W = \frac{1}{(4\pi L)^n}\sum_{\vec{k}\in\underline{C}} \overline{\partial_t F^{\varphi_1}_{T_1,\vec{k}}} F^{\varphi_2}_{T_2,\vec{k}_\sigma}\overline{A_{\vec{k}}}A_{\vec{k}_\sigma}\mathbb{E}(\overline{g_{\vec{k}}}g_{\vec{k}_\sigma})$$

and

$$W' = \frac{1}{(4\pi L)^n}\sum_{\vec{k}\in\tilde{C}} \overline{F^{\varphi_1}_{T_1,\vec{k}}}\partial_t F^{\varphi_2}_{T_2,\vec{k}_\sigma}\overline{A_{\vec{k}}}A_{\vec{k}_\sigma}\mathbb{E}(\overline{g_{\vec{k}}}g_{\vec{k}_\sigma}).$$

Since for all $\vec{k}\in C$,

$$\Big|\overline{A_{\vec{k}}}A_{\vec{k}_\sigma}\mathbb{E}(\overline{g_{\vec{k}}}g_{\vec{k}_\sigma})\Big| \leq (4n+1)!\prod_{j=1}^{4n+1}|a(k_j)|^2$$

and since $\prod_{j=1}^{4n+1}|a(k_j)|^2$ is integrable on

$$\{\vec{k} \mid \sum_{j=1}^{4n+1} k_j(-1)^{j+1} = k\}$$

we get that

$$V = W + W' + \mathcal{O}_{n,a}(L^{4n-2}t^{n-2}).$$

The estimate on W' being symmetric, we only estimate W.
CASE 1: the integer n is even.
Let $\vec{k}\in\underline{C}$, we have

$$F^{\varphi_2}_{T_2,\vec{k}_\sigma} = \frac{t^{n/2}}{(n/2)!}\prod_{j=1}^{n/2}\frac{1}{i\Omega_{T_2,\vec{k}_\sigma}(\varphi_2(2j))} + O_n(L^{2n}t^{n/2-1})$$

and

$$\left|\partial_t F^{\varphi_1}_{T_1,\vec{k}}\right| \leq 2\frac{t^{(n-2)/2}}{((n-2)/2)!}\prod_{j=0}^{(n-2)/2}\frac{1}{|\Omega_{T_1,\vec{k}}(\varphi_1(2j+1))|} + O_n(L^{2n-2}t^{n/2-2}).$$

What is more, $\vec{k}$ belongs to $\underline{C}^2 \cap \underline{C}^1$ with

$$\underline{C}^2 = \{\vec{k} \in C \mid \forall j = 1, \dots, \frac{n}{2}, \quad \Omega_{T_2,\vec{k}_\sigma}(\varphi_2(2j-1)) + \Omega_{T_2,\vec{k}_\sigma}(\varphi_2(2j)) = 0\}$$

and

$$\underline{C}^1 = \underline{C_1}^1 \cup \underline{C_2}^1 \cup \underline{C_3}^1$$

where

$$\underline{C_1}^1 = \{\vec{k} \in C \mid \forall j = 1, \dots, \frac{n-2}{2}, \quad \Omega_{T_1,\vec{k}}(\varphi_1(2j-1)) + \Omega_{T_1,\vec{k}}(\varphi_1(2j)) = 0\},$$

$$\underline{C_2}^1 = \{\vec{k} \in C \mid \forall j = 1, \dots, \frac{n-2}{2}, \quad \Omega_{T_1,\vec{k}}(\varphi_1(2j+1)) + \Omega_{T_1,\vec{k}}(\varphi_1(2j)) = 0\},$$

and finally

$$\underline{C_3}^1 = \Bigg\{\vec{k} \in C \;\Bigg|\; \exists j_0, \begin{array}{c} \forall j = j_0+1, \dots, \frac{n-2}{2}, \quad \Omega_{T_1,\vec{k}}(\varphi_1(2j+1)) + \Omega_{T_1,\vec{k}}(\varphi_1(2j)) = 0 \\ \Omega_{T_1,\vec{k}}(\varphi_1(2j_0+1)) + \Omega_{T_1,\vec{k}}(\varphi_1(2j_0)) + \Omega_{T_1,\vec{k}}(\varphi_1(2j_0-1)) = 0 \\ \forall j = 1, \dots, j_0 - 1, \quad \Omega_{T_1,\vec{k}}(\varphi_1(2j-1)) + \Omega_{T_1,\vec{k}}(\varphi_1(2j)) = 0 \end{array}\Bigg\}.$$

The set $\underline{C}^2$ is built such that $F^{\varphi_2}_{T_2,\vec{k}_\sigma}(t)$ is of maximal order, and the set $\underline{C}^1$ is built such that $\partial_t F^{\varphi_1}_{T_1,\vec{k}}(t)$ is of maximal order. The sets $\underline{C_1}^1$, $\underline{C_2}^1$ and $\underline{C_3}^1$ corresponds to the different cases in Proposition 4.2.
Because of the estimate on $\overline{\partial_t F^{\varphi_1}_{T_1,\vec{k}}} F^{\varphi_2}_{T_2,\vec{k}_\sigma}$, we get

$$W \leq \frac{2(4n+1)!}{((n-2)/2)!(n/2)!}\frac{t^{n-1}}{(4\pi L)^n}\sum_{\vec{k}\in\underline{C}}\prod_{j=1}^{n/2}\frac{1}{|\Omega_{T_1,\vec{k}}(\varphi_1(2j-1))|\,|\Omega_{T_2,\vec{k}_\sigma}(\varphi_2(2j))|}\prod_{j=1}^{4n+1}|a(k_j)|^2 + \mathcal{O}_{n,a}(L^{4n-2}t^{n-2}).$$

By Cauchy-Schwarz inequality, we get

$$W \leq C_n t^{n-1} W_1 W_2 + \mathcal{O}_{n,a}(L^{4n-2}t^{n-2})$$

where C_n is a constant depending only on n,

$$W_1^2 = \frac{1}{(4\pi L)^n}\sum_{\vec{k}\in\underline{C}^1}\prod_{j=0}^{(n-2)/2}\frac{1}{|\Omega_{T_1,\vec{k}}(\varphi_1(2j+1))|^2}\prod_{j=1}^{4n+1}|a(k_j)|^2$$

and

$$W_2^2 = \frac{1}{(4\pi L)^n} \sum_{\vec{k}\in \underline{C}^2} \prod_{j=1}^{n/2} \frac{1}{|\Omega_{T_2,\vec{k}_\sigma}(\varphi_2(2j))|^2} \prod_{j=1}^{4n+1} |a(k_j)|^2.$$

We estimate W_2. For each $j=1$ to n, and for each $m=1$ to 5, we write

$$l_m(j) = k_{T_2,\vec{k}_\sigma}(s_m(\varphi_2(j))$$

where $s_m : N(T_2) \to \tilde{N}(T_2)$ is defined by induction by $s_m(0) = (m,0)$ and $s_m(m_1,l) = (m_1, s_m(l))$. In other words, $s_m(j)$ is the label of the mth subnode or leaf of the node indexed by j. For example, if $T = (\perp,\perp,\perp,\perp,\perp)$ and $\vec{k} = (k_1,k_2,k_3,k_4,k_5)$, we have $s_m(0) = m$ and $l_m(0) = k_m$.
We also write

$$l(j) = l_1(j)-l_2(j)+l_3(j)-l_4(j)+l_5(j) \quad \text{and} \quad \bar{l}(j) = l(j)-l_1(j)+l_2(j)-l_3(j).$$

We have that $S_{T_2,\sigma,\varphi_2} : \vec{k} \mapsto \vec{l} = ((l_1(j),l_2(j),l_3(j),l_4(j))_{1\leq j\leq n}$ is linear and injective. Indeed, if $((l_1(j),l_2(j),l_3(j),l_4(j))_{1\leq j\leq n}$ is fixed, then, since $l(n) = k_{T_2,\vec{k}_\sigma}(0) = k$, we get that $l_5(n)$ is fixed. Now, since $l(n-1)$ is one of the $l_m(n)$ for $m=1$ to 5, we get that $l_5(n-1)$ is fixed. By going down the tree, we get to know the full $(l_m(j))_{m,j}$ for $m=1$ to 5 and $j=1$ to n. In particular we know the labels of the leaves for all the nodes at the bottom of the tree. We know the labels of the leaves, in other words, $\vec{k}_\sigma$ and knowing σ, we know $\vec{k}$.
We set $\Omega_j(\vec{l}) = \Omega_{T_2,\vec{k}_\sigma}(\varphi_2(j))$.
The image of $\underline{C}^2$ by S_{T_2,σ,φ_2} is included in

$$\begin{aligned}\underline{S}^2 = \{\vec{l} \in (\mathbb{Z}/L)^{4n} \mid \forall j=1,\dots,n, \quad |\bar{l}(j)| \geq \mu^{-1}, \\ |\Omega_j(\vec{l})| \geq \nu^{-1}, \quad \forall j = 1,\dots,\frac{n}{2}, \quad \Omega_{2j-1}(\vec{l}) + \Omega_{2j}(\vec{l}) = 0\}.\end{aligned}$$

We get, with the new notations

$$W_2^2 \leq \frac{\nu^{(1+\alpha)n/2)}}{(2\pi L)^{4n}} \sum_{\vec{l}\in \underline{S}^2} \prod_{j=1}^{n/2} \frac{1}{|\Omega_{2j}(\vec{l})|^{1-\alpha}} \prod_{j=1}^{4n+1} |a(S_{T_2,\sigma,\varphi_2}(\vec{l})_j|^2$$

with the convention $S_{T_2,\sigma,\varphi_2}(\vec{l})_j = k_j$. Because a has compact support, so has $a\circ S_{T_2,\sigma}$ and therefore, there exists $K = K(a,\sigma,T_2,\varphi_2)$ such that

$$W_2^2 \leq \max |a|^{2(4n+1)} \frac{\nu^{(1+\alpha)n/2)}}{(2\pi L)^{4n}} \sum_{\vec{l}\in \underline{S}^2\cap[-K,K]^{4n}} \prod_{j=1}^{n/2} \frac{1}{|\Omega_{2j}(\vec{l})|^{1-\alpha}}.$$

We count the degrees of freedom in $\underline{S}^2$. We start by fixing $l_1(n), l_2(n), l_3(n), l_4(n)$. As we remarked earlier, this fixes automatically

$l_5(n)$ and therefore, $l(n-1)$, and $\Omega_n(\vec{l})$. Because we are in $\underline{S}^2$, this fixes too $\Omega_{n-1}(\vec{l})$.
We then fix arbitrarily $l_1(n-1), l_2(n-1)$ and $l_3(n-1)$, which fixes automatically $\bar{l}(n-1)$. We recall that $\Omega_{n-1}(\vec{l})$ is fixed and that

$$\Omega_{n-1}(\vec{l}) = l^2(n-1)-l_1^2(n-1)+l_2^2(n-1)-l_3^2(n-1)-\bar{l}(n-1)-2\bar{l}(n-1)l_4(n-1).$$

Therefore, this fixes $l_4(n-1)$ and in turn $l_5(n-1)$.
Going down the tree, we get that fixing $(l_1(2j), l_2(2j), l_3(2j), l_4(2j), l_1(2j-1), l_2(2j-1), l_3(2j-1))$ for $j = n/2$ to 1 is sufficient to recover the whole $\vec{l}$. Therefore, we have

$$W_2^2 \leq C_a \frac{\nu^{(1+\alpha)n/2)}}{(2\pi L)^{4n}} \sum_{\vec{l}\in(\mathbb{Z}/L\cap[-K,K])^{7n/2}} \prod_{j=1}^{n/2} \frac{1}{|\Omega_{2j}(\vec{l})|^{1-\alpha}},$$

where $[-K,K]^{4n}$ has been replaced by $[-K,K]^{7n/2}$.
Because

$$\Omega_{2j}(\vec{l}) = -2\bar{l}(2j)\Big(l_4(2j) - \frac{1}{2\bar{l}(2j)}(l^2(2j) - l_1^2(2j) + l_2^2(2j) - l_3^2(2j) - \bar{l}^2(2j)\Big),$$

By integrating in the following order $l_3(1)$, $l_2(1)$, $l_1(1)$, $l_4(2)$, $l_3(2)$, $l_2(2)$, $l_1(2), \dots, l_3(n-1)$, $l_2(n-1)$,$l_1(n-1)$, $l_4(n)$, $l_3(n)$, $l_2(n)$, $l_1(n)$, we get that

$$\prod_{j=1}^{n/2} \frac{1}{|\Omega_{2j}(\vec{l})|^{1-\alpha}}$$

is integrable on compacts and therefore

$$W_2^2 \lesssim_{a,T_2,\sigma,\varphi_2} \frac{\nu^{(1+\alpha)n/2)}}{L^{n/2}}.$$

For W_1, there are in each subset forming $\underline{C}^1$, $\frac{n-2}{2}$ constraint estimates on the Ωs, while we integrate $\frac{n}{2}$ different Ω^{-2}s. Therefore, by applying the same strategy as for W_2, we get

$$W_1^2 \lesssim_{a,T_1,\varphi_1} \frac{\nu^{(1+\alpha)n/2}}{L^{(n-2)/2}}.$$

We deduce

$$W = \mathcal{O}_{a,T_1,T_2,\sigma,\varphi_1,\varphi_2,n}\Big(\frac{\nu^{(1+\alpha)n/2)}}{L^{(n-1)/2}}t^{n-1}\Big) + \mathcal{O}_{n,a}(L^{4n-2}t^{n-2})$$

which yields the result when n is even.
Case 2 : n is odd.

Let $\vec{k} \in \underline{C}$. We have

$$|F^{\varphi_2}_{T_2,\vec{k}_\sigma}| \leq 2\frac{t^{(n-1)/2}}{((n-1)/2)!}\prod_{j=0}^{(n-1)/2}\frac{1}{|\Omega_{T_2,\vec{k}_\sigma}(\varphi_2(2j+1))|} + \mathcal{O}_n(L^{2n}t^{(n-1)/2-1})$$

and

$$|\partial_t F^{\varphi_1}_{T_1,\vec{k}}| \leq \frac{t^{(n-1)/2}}{((n-1)/2)!}\prod_{j=1}^{(n-1)/2}\frac{1}{|\Omega_{T_1,\vec{k}}(\varphi_1(2j))|} + \mathcal{O}_n(L^{2n}t^{(n-1)/2-1}).$$

What is more, $\vec{k}$ belongs to $\underline{C}^1 \cap \underline{C}^2$ with

$$\underline{C}^1 = \{\vec{k} \in C \mid \forall j = 1, \dots, \frac{n-1}{2}, \quad \Omega_{T_1,\vec{k}}(\varphi_1(2j-1)) + \Omega_{T_1,\vec{k}}(\varphi_1(2j)) = 0\}$$

and

$$\underline{C}^2 = \underline{C_1}^2 \cup \underline{C_2}^2 \cup \underline{C_3}^2$$

where

$$\underline{C_1}^2 = \{\vec{k} \in C \mid \forall j = 1, \dots, \frac{n-1}{2}, \quad \Omega_{T_2,\vec{k}_\sigma}(\varphi_2(2j-1)) + \Omega_{T_2,\vec{k}_\sigma}(\varphi_2(2j)) = 0\},$$

$$\underline{C_2}^2 = \{\vec{k} \in C \mid \forall j = 1, \dots, \frac{n-1}{2}, \quad \Omega_{T_2,\vec{k}_\sigma}(\varphi_2(2j+1)) + \Omega_{T_2,\vec{k}_\sigma}(\varphi_2(2j)) = 0\},$$

and finally

$$\underline{C_3}^2 = \Big\{\vec{k} \in C \;\Big|\; \exists j_0, \left.\begin{array}{c} \forall j = j_0+1, \dots, \frac{n-1}{2}, \quad \Omega_{T_2,\vec{k}_\sigma}(\varphi_2(2j+1)) + \Omega_{T_2,\vec{k}_\sigma}(\varphi_2(2j)) = 0, \\ \Omega_{T_2,\vec{k}_\sigma}(\varphi_2(2j_0-1)) + \Omega_{T_2,\vec{k}_\sigma}(\varphi_2(2j_0)) + \Omega_{T_2,\vec{k}_\sigma}(\varphi_2(2j_0+1)) = 0, \\ \forall j = 1, \dots, j_0-1, \quad \Omega_{T_2,\vec{k}_\sigma}(\varphi_2(2j-1)) + \Omega_{T_2,\vec{k}_\sigma}(\varphi_2(2j)) = 0 \end{array}\right\}.$$

We get as previously by Cauchy-Schwarz inequality,

$$W \leq C_n W_1 W_2 t^{n-1} + \mathcal{O}_{n,a}(L^{4n-2}t^{n-2})$$

with

$$W_1^2 = \frac{1}{(2\pi L)^{4n}}\sum_{\vec{k}\in\underline{C}^1}\prod_{j=1}^{(n-1)/2}\frac{1}{|\Omega_{T_1,\vec{k}}(\varphi_1(2j))|}\prod_{j=1}^{4n+1}|a(k_j)|^2$$

and

$$W_2^2 = \frac{1}{(2\pi L)^{4n}}\sum_{\vec{k}\in\underline{C}^1}\prod_{j=0}^{(n-1)/2}\frac{1}{|\Omega_{T_2,\vec{k}_\sigma}(\varphi_2(2j+1))|}\prod_{j=1}^{4n+1}|a(k_j)|^2.$$

We repeat the same strategy as in the case n even.

For W_1, there are $\frac{n-1}{2}$ quantities $|\Omega|^{-2}$ to integrate and $\frac{n-1}{2}$ constraints estimates. Hence,

$$W_1^2 \lesssim_{a,T_1,\varphi_1,\alpha} \frac{\nu^{(1+\alpha)(n-1)/2}}{L^{(n-1)/2}}.$$

For W_2, there are $\frac{n+1}{2}$ quantities $|\Omega|^{-2}$ to integrate and $\frac{n-1}{2}$ constraint equations, hence

$$W_1^2 \lesssim_{a,T_2,\varphi_2,\alpha,\sigma} \frac{\nu^{(1+\alpha)(n+1)/2}}{L^{(n-1)/2}}.$$

Therefore, we have

$$W = \mathcal{O}_{a,T_1,T_2,\varphi_1,\varphi_2,\sigma,\alpha}\Big(\frac{\nu^{(1+\alpha)n/2}}{L^{(n-1)/2}}\Big) + \mathcal{O}_{a,n}(L^{4n-2}t^{n-2})$$

which concludes the proof when n is odd. □

Remark 4.1. *The application of the Cauchy-Schwarz inequality prevents us from being optimal. The worst case scenario we can think of (n odd, $T_1 = T_2$, $\varphi_1 = \varphi_2$, $\sigma = Id$) yielding a bound of the form*

$$\frac{\nu^{(1+\alpha)(n-1)/2}}{L^{(n-1)/2}}\nu^{\alpha}.$$

4.3 Case $n = 1$

We now deal with the case $n = 1$.
The set $\mathcal{T}_1$ is reduced to 1 element $T = (\perp,\perp,\perp,\perp,\perp)$. We write $C(k) = C_T(k)$ and for all $\sigma \in \mathfrak{S}_5$, we write $C_\sigma(k) = C(T,T,\sigma,k)$. We also write $\mathfrak{A}$ the set of $\sigma \in \mathfrak{S}_5$ that conserves parity.

Proposition 4.6. *We have*

$$\partial_t \mathbb{E}(|v_1(t,k)|^2) = \sum_{\sigma\in\mathfrak{A}} \sum_{\vec{k}\in C_\sigma(k)} \frac{2}{(2\pi L)^4} \frac{\sin(\Delta(\vec{k})t)}{\Delta(\vec{k})} \prod_{j=1}^{5} |a(k_j)|^2 + O_a(L^{-1}\nu).$$

Proof. By definition, we have

$$\partial_t \mathbb{E}(|v_1(t,k)|^2) = 2\mathrm{Re}\Big[\sum_{\sigma\in\mathfrak{S}_5} \sum_{\vec{k}\in C_\sigma(k)} \frac{1}{(2\pi L)^4} \frac{e^{i\Delta(\vec{k})t}-1}{i\Delta(\vec{k})} e^{-i\Delta(\vec{k}_\sigma)t} A_{\vec{k}} \overline{A_{\vec{k}_\sigma}} \frac{\mathbb{E}(g_{\vec{k}}\overline{g_{\vec{k}_\sigma}})}{c(\sigma,k)}\Big].$$

We write

$$\partial_t \mathbb{E}(|v_1(t,k)|^2) = A + B$$

with

$$A = 2\mathrm{Re}\Big[\sum_{\sigma\in\mathfrak{A}} \sum_{\vec{k}\in C_\sigma(k)} \frac{1}{(2\pi L)^4} \frac{e^{i\Delta(\vec{k})t}-1}{i\Delta(\vec{k})} e^{-i\Delta(\vec{k}_\sigma)t} A_{\vec{k}} \overline{A_{\vec{k}_\sigma}} \frac{\mathbb{E}(g_{\vec{k}}\overline{g_{\vec{k}_\sigma}})}{c(\sigma,k)}\Big]$$

and

$$B = 2\mathrm{Re}\Big[\sum_{\sigma\in\mathfrak{B}}\sum_{\vec{k}\in C_\sigma(k)}\frac{1}{(2\pi L)^4}\frac{e^{i\Delta(\vec{k})t}-1}{i\Delta(\vec{k})}e^{-i\Delta(\vec{k}_\sigma)t}A_{\vec{k}}\overline{A_{\vec{k}_\sigma}}\frac{\mathbb{E}(g_{\vec{k}}\overline{g_{\vec{k}_\sigma}})}{c(\sigma,k)}\Big]$$

where $\mathfrak{B}$ is the complementary of $\mathfrak{A}$ in $\mathfrak{S}_5$.
If $\sigma\in\mathfrak{A}$, then $\Delta(\vec{k})=\Delta(\vec{k}_\sigma)$, $A_{\vec{k}}\overline{A_{\vec{k}_\sigma}}=\prod_{j=1}^5|a(k_j)|^2$ and $\mathbb{E}(g_{\vec{k}}\overline{g_{\vec{k}_\sigma}})=\mathbb{E}(|g_{\vec{k}}|^2)$. Therefore

$$A=\sum_{\sigma\in\mathfrak{A}}\sum_{\vec{k}\in C_\sigma(k)}\frac{2}{(2\pi L)^4}\frac{\sin(\Delta(\vec{k})t)}{\Delta(\vec{k})}\prod_{j=1}^5|a(k_j)|^2\frac{\mathbb{E}(|g_{\vec{k}}|^2)}{c(\sigma,k)}.$$

Write

$$C_{\sigma,\neq}(k)=\{\vec{k}\in C_\sigma(k)\mid\forall j_1\neq j_2,k_{j_1}\neq k_{j_2}\}$$

and $C_{\sigma,=}(k)$ its complementary in $C_\sigma(k)$. For all $\vec{k}\in C_{\sigma,\neq}(k)$ we have, according to Lemma 3.18 $\mathbb{E}(|g_k|^2)=1$ and $c(\sigma,k)=1$ and thus

$$A=A_{\neq}+A_=$$

with

$$A_{\neq}=\sum_{\sigma\in\mathfrak{A}}\sum_{\vec{k}\in C_\sigma(k)}\frac{2}{(2\pi L)^4}\frac{\sin(\Delta(\vec{k})t)}{\Delta(\vec{k})}\prod_{j=1}^5|a(k_j)|^2$$

and

$$A_==\sum_{\sigma\in\mathfrak{A}}\sum_{\vec{k}\in C_{\sigma,=}(k)}\frac{2}{(2\pi L)^4}\frac{\sin(\Delta(\vec{k})t)}{\Delta(\vec{k})}\prod_{j=1}^5|a(k_j)|^2\Big(\frac{\mathbb{E}(|g_{\vec{k}}|^2)}{c(\sigma,k)}-1\Big).$$

Since $|\frac{\mathbb{E}(|g_{\vec{k}}|^2)}{c(\sigma,k)}-1|\leq 5!$ and $|\Delta(\vec{k})|\geq\nu^{-1}$, we get

$$|A_=|\leq 5!\nu\sum_{\sigma\in\mathfrak{A}}\sum_{\vec{k}\in C_{\sigma,=}(k)}\frac{2}{(2\pi L)^4}\prod_{j=1}^5|a(k_j)|^2.$$

Because satisfying "$\exists j\neq l$ such that $k_j=k_l$" is a constraint independent from $\sum_{j=1}^5(-1)^jk_j+k=0$, we get that

$$A_==O_a(L^{-1}\nu).$$

We estimate B. We recall that if σ does not conserve parity and $\vec{k}\in C_\sigma(k)$ then $\mathbb{E}(g_{\vec{k}}\overline{g_{\vec{k}_\sigma}})\neq 0$ implies $\vec{k}\in C_{\sigma,=}(k)$. Therefore

$$B=2\mathrm{Re}\Big[\sum_{\sigma\in\mathfrak{B}}\sum_{\vec{k}\in C_{\sigma,=}(k)}\frac{1}{(2\pi L)^4}\frac{e^{i\Delta(\vec{k})t}-1}{i\Delta(\vec{k})}e^{-i\Delta(\vec{k}_\sigma)t}A_{\vec{k}}\overline{A_{\vec{k}_\sigma}}\frac{\mathbb{E}(g_{\vec{k}}\overline{g_{\vec{k}_\sigma}})}{c(\sigma,k)}\Big].$$

Since $|\frac{\mathbb{E}(g_{\vec{k}}\overline{g_{\vec{k}_\sigma}})}{c(\sigma,k)}| \leq 5!$ and since

$$\Big|\frac{e^{i\Delta(\vec{k})t}-1}{i\Delta(\vec{k})}e^{-i\Delta(\vec{k}_\sigma)t}\Big| \leq 2\nu$$

we get

$$|B| \leq 2\cdot 5!\nu \sum_{\sigma\in\mathfrak{B}}\sum_{\vec{k}\in C_{\sigma,=}(k)} \frac{1}{(2\pi L)^4}\prod_{j=1}^5 |a(k_j)|^2.$$

Because belonging to $C_{\sigma,=}(k)$ implies two independent linear constraint on $\vec{k}$, we get

$$B = O_a(L^{-1}\nu).$$

□

5 Final limits and proof of the result

We sum up what we have done so far. Since for all $n>1$, $T_1, T_2 \in \mathcal{T}_n$,$\sigma \in \mathfrak{S}_{4n+1}$, $k\in\mathbb{Z}/L$ and $t\in\mathbb{R}$, we have thanks to Proposition 4.5

$$V(T_1,T_2,\sigma,\varphi_1,\varphi_2,k,t) = O_{T_1,T_2,\sigma,\varphi_1,\varphi_2,a,\alpha}(t^{n-1}\nu^{(1+\alpha)n/2}L^{-(n-1)/2}) + O_{n,a}(t^{n-2}L^{4n-2}),$$

we get

$$\partial_t\mathbb{E}(|v_{n,L}(k,t)|^2) = O_{n,\alpha,a}(t^{n-1}\nu^{(1+\alpha)n/2}L^{-(n-1)/2}) + O_{n,a}(t^{n-2}L^{4n-2}).$$

We deduce that for all $t\in\mathbb{R}$, we have that

$$\varepsilon^{2(n-1)}\partial_t\mathbb{E}(|v_{n,L}(k,t\varepsilon^{-2})|^2) = O_{n,a,t,\alpha}(\nu^{(1+\alpha)n/2}L^{-(n-1)/2}) + O_{n,a,t}(\varepsilon^2 L^{4n-2}).$$

Indeed, when $t=0$,

$$\varepsilon^{2(n-1)}\partial_t\mathbb{E}(|v_{n,L}(k,t\varepsilon^{-2})|^2) = 0$$

and when $t\neq 0$ then for ε small enough (or L big enough) $|t|\varepsilon^{-2}\geq 1$. Therefore, we have

$$\sum_{n=2}^N \varepsilon^{2(n-1)}\partial_t\mathbb{E}(|v_{n,L}(k,t\varepsilon^{-2})|^2) = O_{N,a,t}\Big(\frac{\nu^{1+\alpha}}{L^{1/2}}\Big) + O_{N,a,t}(\varepsilon^2 L^{4N-2}).$$

We recall that thanks to Proposition 4.6, we have

$$\partial_t\mathbb{E}(|v_1(t,k)|^2) = \sum_{\sigma\in\mathfrak{A}}\sum_{\vec{k}\in C_\sigma(k)} \frac{2}{(2\pi L)^4}\frac{\sin(\Delta(\vec{k})t)}{\Delta(\vec{k})}\prod_{j=1}^5 |a(k_j)|^2 + O_a(L^{-1}\nu).$$

We set

$$I_{\varepsilon,L}(k,t) = \sum_{\sigma\in\mathfrak{A}} \sum_{\vec{k}\in C_\sigma(k)} \frac{2}{(2\pi L)^4} \frac{\sin(\Delta(\vec{k})t\varepsilon^{-2})}{\Delta(\vec{k})} \prod_{j=1}^{5} |a(k_j)|^2. \tag{9}$$

We introduce test functions.

Proposition 5.1. *Let f, g be two smooth, compactly supported functions on $\mathbb{R}$. Let $M \geq N \in \mathbb{N}^*$. We have*

$$\partial_t \mathbb{E}\Big(\langle P_N U_L, f\rangle\langle g, P_M U_L\rangle\Big) = \frac{1}{2\pi L}\sum_{k\in Z/L} \hat{f}(k)\overline{\hat{g}(k)} I_{\varepsilon,L}(k,t) \\ + O_{N,a,k,t,f,g}\Big(\frac{\nu^{1+\alpha}}{\sqrt{L}} + \varepsilon^2 L^{N-3}\Big).$$

Proof. Set $n_0 = \lfloor \frac{N-1}{4} \rfloor$. We have

$$P_N U_L(t) = \sum_{n=0}^{n_0} \varepsilon^n u_{n,L}(\varepsilon^{-2}t),$$

hence

$$\langle P_N U_L(t), f\rangle = \sum_{n=0}^{n_0} \varepsilon^n \langle u_{n,L}(\varepsilon^{-2}t), f\rangle.$$

What is more,

$$u_{n,L}(t\varepsilon^{-2}) = \sum_{k\in\mathbb{Z}/L} \hat{u}_{n,L}(t\varepsilon^{-2},k)\frac{e^{ikx}}{\sqrt{2\pi L}}$$

and we recall that because the support of a is compact, the sum over k is finite. Hence

$$\langle P_N U_L(t), f\rangle = \frac{1}{\sqrt{2\pi L}} \sum_{k\in\mathbb{Z}/L}\sum_{n=0}^{n_0} \varepsilon^n \overline{\hat{u}_{n,L}(t\varepsilon^{-2},k)}\hat{f}(k).$$

For the same reasons

$$\langle g, P_M U_L(t)\rangle = \frac{1}{\sqrt{2\pi L}} \sum_{k'\in\mathbb{Z}/L}\sum_{m=0}^{m_0} \varepsilon^m \hat{u}_{m,L}(t\varepsilon^{-2},k')\overline{\hat{g}(k')}$$

where $m_0 = \lfloor \frac{M-1}{4} \rfloor$.
Therefore,

$$\mathbb{E}\Big(\langle P_N U_L, f\rangle\langle g, P_M U_L\rangle\Big) = \\ \frac{1}{2\pi L}\sum_{k,k'} \hat{f}(k)\overline{\hat{g}(k')}\sum_{n,m}\varepsilon^{n+m}\mathbb{E}(\overline{\hat{u}_{n,L}(t\varepsilon^{-2},k)}\hat{u}_{m,L}(t\varepsilon^{-2},k')).$$

We recall that

$$\mathbb{E}(\overline{\hat{u}_{n,L}(t\varepsilon^{-2},k)}\hat{u}_{m,L}(t\varepsilon^{-2},k'))$$

is null unless $n=m$ and $k=k'$. Hence,

$$\mathbb{E}\Big(\langle P_N U_L, f\rangle\langle g, P_M U_L\rangle\Big) = \frac{1}{2\pi L}\sum_{k\in\mathbb{Z}/L}\hat{f}(k)\overline{\hat{g}(k)}\sum_{n=0}^{n_0}\varepsilon^{2n}\mathbb{E}(|\hat{u}_{n,L}(t\varepsilon^{-2},k)|^2).$$

We have

$$\mathbb{E}(|\hat{u}_{n,L}(t\varepsilon^{-2},k)|^2) = \mathbb{E}(|v_{n,L}(t\varepsilon^{-2},k)|^2)$$

hence

$$\partial_t\mathbb{E}\Big(\langle P_N U_L, f\rangle\langle g, P_M U_L\rangle\Big) \\ = \frac{1}{2\pi L}\sum_{k\in\mathbb{Z}/L}\hat{f}(k)\overline{\hat{g}(k)}\sum_{n=1}^{n_0}\varepsilon^{2(n-1)}\partial_t\mathbb{E}(|v_{n,L}(k)|^2)(t\varepsilon^{-2}).$$

We recall

$$\sum_{n=2}^{n_0}\varepsilon^{2(n-1)}\partial_t\mathbb{E}(|v_{n,L}(k)|^2)(t\varepsilon^{-2}) = O_{n_0,a,t}\Big(\frac{\nu}{L^{1/2}} + \varepsilon^2 L^{4n_0-2}\Big)$$

hence

$$\frac{1}{2\pi L}\sum_{k\in\mathbb{Z}/L}\hat{f}(k)\overline{\hat{g}(k)}\sum_{n=2}^{n_0}\varepsilon^{2(n-1)}\partial_t\mathbb{E}(|v_{n,L}(k)|^2)(t\varepsilon^{-2}) \\ = O_{n_0,a,t,f,g}\Big(\frac{\nu}{L^{1/2}} + \varepsilon^2 L^{4n_0-2}\Big).$$

Finally,

$$\partial_t\mathbb{E}(|v_{1,L}(k)|^2)(t\varepsilon^{-2}) = I_{\varepsilon,L}(k,t) + O_a(L^{-1}\nu)$$

hence

$$\frac{1}{2\pi L}\sum_{k\in\mathbb{Z}/L}\hat{f}(k)\overline{\hat{g}(k)}\partial_t\mathbb{E}(|v_{1,L}(k)|^2)(t\varepsilon^{-2}) = \frac{1}{2\pi L}\sum_{k\in\mathbb{Z}/L}\hat{f}(k)\overline{\hat{g}(k)}I_{\varepsilon,L}(k,t) \\ + O_{a,f,g}(L^{-1}\nu)$$

which concludes the proof. □

We set

$$I_L(t) = \frac{1}{2\pi L}\sum_{k\in\mathbb{Z}/L}\hat{f}(k)\overline{\hat{g}(k)}I_{\varepsilon,L}(k,t).$$

5.1 Case t is dyadic

Proposition 5.2. *Assume that* $\varepsilon^{-2} = 2\pi 2^L L^2 + \rho(L)$ *and that* t *is dyadic. We have*

$$I_L(t) = \frac{3}{4\pi^5}\int_{\vec{k}\in B} \frac{\sin(\Delta(\vec{k})t\rho}{\Delta(\vec{k})}\prod_{j=1}^{5}|a(k_j)|^2\delta(k+\sum_{j=1}^{5}(-1)^j k_j)\hat{f}(k)\overline{\hat{g}(k)}d\vec{k} + O_a(\frac{\rho^2}{L}+\frac{\rho\mu}{\nu}+\frac{\rho}{\mu})$$

where $B = \{(k,k_1,k_2,k_3,k_4,k_5)\in\mathbb{R}^6 \mid |k-k_1+k_2-k_3| \geq \frac{1}{\mu}\}$*, where*

$$\Delta(\vec{k}) = k^2 + \sum_{j=1}^{5}(-1)^j k_j^2$$

and $d\vec{k} = dk\prod_{j=1}^{5} dk_j$*.*

Proof. Let $\vec{k}\in C_\sigma := \{(k,\vec{k}')|\vec{k}'\in C_\sigma(k)\}$, we have

$$\Delta(\vec{k})t\varepsilon^{-2} = \Delta(\vec{k})L^2 t 2\pi 2^L + \Delta(\vec{k})t\rho.$$

Therefore, since t is dyadic, for L big enough

$$\Delta(\vec{k})t\varepsilon^{-2} \in \Delta(\vec{k})t\rho + 2\pi\mathbb{Z}.$$

Hence

$$I_{\varepsilon,L}(k,t) = \sum_{\sigma\in\mathfrak{A}}\sum_{\vec{k}\in C_\sigma(k)}\frac{2}{(2\pi L)^4}\frac{\sin(\Delta(\vec{k})t\rho)}{\Delta(\vec{k})}\prod_{j=1}^{5}|a(k_j)|^2.$$

Since

$$\Big|\frac{\sin(\Delta(\vec{k})t\rho)}{\Delta(\vec{k})}\Big| \leq t\rho$$

and

$$\Big|\nabla\Big(\frac{\sin(\Delta(\vec{k})t\rho}{\Delta(\vec{k})}\Big)\Big| \leq |t\rho|^2\,|\nabla\Delta|\Big\|\frac{x\cos x - \sin x}{x^2}\Big\|_{L^\infty}$$

we get the convergence towards the Riemann integral since $\rho^2 = o(\nu^2) = o(L)$,

$$I_L(t) = \sum_{\sigma\in\mathfrak{A}}\int_{B_\sigma}\frac{2}{(2\pi)^5}\frac{\sin(\Delta(\vec{k})t\rho}{\Delta(\vec{k})}\prod_{j=1}^{5}|a(k_j)|^2\delta(k+\sum_{j=1}^{5}(-1)^j k_j)\hat{f}(k)\overline{\hat{g}(k)}d\vec{k} + O_{a,f,g,t}(\rho^2 L^{-1})$$

where

$$B_\sigma = \{(k,k_1,k_2,k_3,k_4,k_5) \mid |k-k_1+k_2-k_3| \geq \mu^{-1}, |k-k_{\sigma(1)}+k_{\sigma(2)}-k_{\sigma(3)}| \geq \mu^{-1},\ \Delta(\vec{k}) \geq \nu^{-1}\}.$$

We have

$$\int_{|\Delta(\vec{k})|<\frac{1}{\nu},|k-k_1+k_2-k_3|\geq\frac{1}{\mu}} \prod_{j=1}^{5}|a(k_j)|^2\delta(k+\sum_{j=1}^{5}(-1)^j k_j)|\hat{f}(k)\hat{g}(k)|d\vec{k} = \int_{|k-k_1+k_2-k_3|\geq\frac{1}{\mu}} dkdk_1dk_2dk_3|a(k_1)|^2|a(k_2)|^2|a(k_3)|^2|\hat{f}(k)\hat{g}(k)| \int_{|\Delta(\vec{k})|<\frac{1}{\nu}} dk_4|a(k_4)|^2|a(k+\sum_{j=1}^{4}(-1)^j k_j)|^2.$$

Since

$$\Delta(\vec{k}) = k^2 - k_1^2 + k_2^2 - k_3^4 - (k+k+\sum_{j=1}^{3}(-1)^j k_j)^2 - 2k_4(k+\sum_{j=1}^{3}(-1)^j k_j)$$

we get that $|\Delta(\vec{k})| \leq \frac{1}{\nu}$ implies

$$|k_4 - \frac{k^2 - k_1^2 + k_2^2 - k_3^4 - (k+k+\sum_{j=1}^{3}(-1)^j k_j)^2}{2(k+\sum_{j=1}^{3}(-1)^j k_j)}| \leq \frac{\mu}{\nu}$$

hence

$$\int_{|\Delta(\vec{k})|<\frac{1}{\nu},|k-k_1+k_2-k_3|\geq\frac{1}{\mu}} \prod_{j=1}^{5}|a(k_j)|^2\delta(k+\sum_{j=1}^{5}(-1)^j k_j)|\hat{f}(k)\hat{g}(k)|d\vec{k} \leq \sup|a|^4\frac{\mu}{\nu}\int dkdk_1dk_2dk_3|a(k_1)|^2|a(k_2)|^2|a(k_3)|^3|\hat{f}(k)\hat{g}(k)|.$$

Thus, we get

$$I_L(t) = \sum_{\sigma\in\mathfrak{A}}\int_{B'_\sigma}\frac{2}{(2\pi)^5}\frac{\sin(\Delta(\vec{k})t\rho}{\Delta(\vec{k})}\prod_{j=1}^{5}|a(k_j)|^2\hat{f}(k)\overline{\hat{g}(k)}\delta(k+\sum_{j=1}^{5}(-1)^j k_j)\prod_{j=1}^{5}dk_j + O_{a,t,f,g}(\rho^2L^{-1}) + O_{a,t,f,g}(\frac{\rho\mu}{\nu})$$

where

$$B'_\sigma = \{(k,k_1,k_2,k_3,k_4,k_5) \mid |k-k_1+k_2-k_3| \geq \frac{1}{\mu}\ |k-k_{\sigma(1)}+k_{\sigma(2)}-k_{\sigma(3)}| \geq \frac{1}{\mu}\}.$$

We have

$$\int_{|k+\sum_{j=1}^{3}(-1)^j k_{\sigma(j)}|<\frac{1}{\mu}} |\hat{f}(k)\hat{g}(k)|\prod_{j=1}^{5}|a(k_j)|^2\delta(k+\sum_{j=1}^{5}(-1)^j k_j)d\vec{k} = \int dkdk_1dk_2dk_4|a(k_1)|^2|a(k_2)|^2|a(k_4)|^2|\hat{f}(k)\hat{g}(k)| \int_{|k+\sum_{j=1}^{3}(-1)^j k_{\sigma(j)}|<\frac{1}{\mu}} dk_3|a(k_3)|^2|a(k+\sum_{j=1}^{4}(-1)^j k_j)|^2.$$

We get

$$\int_{|k+\sum_{j=1}^{3}(-1)^j k_{\sigma(j)}|<\frac{1}{\mu}} \prod_{j=1}^{5}|a(k_j)|^2\delta(k+\sum_{j=1}^{5}(-1)^j k_j)|\hat{f}(k)\hat{g}(k)|d\vec{k}$$
$$\leq \int dk dk_1 dk_2 dk_4 |a(k_1)|^2|a(k_2)|^2|a(k_4)|^2|\hat{f}(k)\hat{g}(k)|\sup|a|^4\frac{1}{\mu}.$$

Thus, we get

$$I_L(t) = \sum_{\sigma\in\mathfrak{A}}\int_B \frac{2}{(2\pi L)^5}\frac{\sin(\Delta(\vec{k})t\rho}{\Delta(\vec{k})}\prod_{j=1}^{5}|a(k_j)|^2\hat{f}(k)\overline{\hat{g}(k)}\delta(k+\sum_{j=1}^{5}(-1)^j k_j)d\vec{k}$$
$$+ O_{a,t,f,g}(\rho^2 L^{-1} + \frac{\rho\mu}{\nu} + \frac{\rho}{\mu})$$

which concludes the proof since the cardinal of $\mathfrak{A}$ is $3!2! = 12$. □

Let

$$J_L(t) = \frac{3}{4\pi^5}\int_{\vec{k}\in B}\frac{\sin(\Delta(\vec{k})t\rho}{\Delta(\vec{k})}\prod_{j=1}^{5}|a(k_j)|^2\hat{f}(k)\overline{\hat{g}(k)}\delta(k+\sum_{j=1}^{5}(-1)^j k_j)d\vec{k}.$$

Proposition 5.3. *We have that*

$$\lim_{L\to\infty} J_L(t) =$$
$$\frac{3}{4\pi^4}\int_{\mathbb{R}^6}\delta(k+\sum_{j=1}^{5}(-1)^j k_j)\delta(\Delta(\vec{k}))\frac{1}{k-k_1+k_2-k_3}\prod_{j=1}^{5}|a(k_j)|^2\hat{f}(k)\overline{\hat{g}(k)}d\vec{k}$$

and besides the integral converges.

Proof. We have

$$J_L(t) = \frac{3}{4\pi^5}\int_{\vec{k}\in C}\frac{\sin(D(\vec{k})t\rho)}{D(\vec{k})}\prod_{j=1}^{4}|a(k_j)|^2|a(k+\sum_{j=1}^{4}(-1)^j k_j)|^2\hat{f}(k)\overline{\hat{g}(k)}d\vec{k}$$

where

$$C = \{(k,k_1,\dots,k_4)|\ |k-k_1+k_2-k_3|\geq\frac{1}{\mu}\}$$

and

$$D(k,k_1,\dots,k_4) = \Delta(k,k_1,\dots,k_4,k-\sum_{j=1}^{4}(-1)^j k_j).$$

To lighten the notations, we write equally

$$C = \{(k_1,k_2,k_3)|\ |k-k_1+k_2-k_3|\geq\frac{1}{\mu}\}.$$

We get

$$J_L(t) = \frac{3}{4\pi^5}\int_C dk dk_1 dk_2 dk_3 \hat{f}(k)\overline{\hat{g}(k)}|a(k_1)|^2|a(k_2)|^2|a(k_3)|^2 J_L^4(t,k,k_1,k_2,k_3)$$

where

$$J_L^4(t,k,k_1,k_2,k_3) = \int dk_4 \frac{\sin(D(\vec{k})t\rho)}{D(\vec{k})}|a(k_4)|^2|a(k+\sum_{j=1}^4(-1)^j k_j)|^2.$$

Given a fixed $k,k_1,k_2,k_3 \in C$, the derivative of $k_4 \mapsto D(\vec{k})$ being

$$-2(k-k_1+k_2-k_3)$$

and denoting $\bar{k} = k-k_1+k_2-k_3 \neq 0$, we get by integration by parts

$$J_L^4(t,k,k_1,k_2,k_3) = J_{L,1}^4(t,k,k_1,k_2,k_3) + J_{L,2}^4(t,k,k_1,k_2,k_3)$$

with

$$J_{L,1}^4(t,k,k_1,k_2,k_3) = \int dk_4 \frac{1-\cos(D(\vec{k})t\rho)}{D^2(\vec{k})t\rho}|a(k_4)|^2|a(\bar{k}+k_4)|^2$$

and

$$J_{L,2}^4(t,k,k_1,k_2,k_3) = \int dk_4 \frac{1-\cos(D(\vec{k})t\rho)}{\bar{k}D(\vec{k})t\rho}\mathrm{Re}\left(a'(k_4)\bar{a}(k_4)|a(\bar{k}+k_4)|^2 + |a(k_4)|^2 a'(\bar{k}+k_4)\bar{a}(\bar{k}+k_4)\right).$$

For $j=1,2$, write

$$J_{L,j}(t) = \frac{3}{4\pi^5}\int_C dk dk_1 dk_2 dk_3 \hat{f}(k)\overline{\hat{g}(k)}|a(k_1)|^2|a(k_2)|^2|a(k_3)|^2 J_{L,j}^4(t,k_1,k_2,k_3).$$

We estimate $J_{L,2}(t)$. First, we have

$$\frac{1-\cos(D(\vec{k})t\rho)}{D(\vec{k})t\rho} \lesssim \frac{1}{(D(\vec{k})t\rho)^{1/2}}$$

and thus

$$J_{L,2}^4(t,k,k_1,k_2,k_3) \le \|a'\|_{L^\infty}\|a\|_{L^\infty}^2 \frac{1}{|\bar{k}|\sqrt{t\rho}}\int dk_4 \frac{|a(k_4)|}{\sqrt{D(\vec{k})}}$$

which implies in turn

$$J_{L,2}(t) \lesssim_a \frac{1}{\sqrt{t\rho}}\int_C dk dk_1 dk_2 dk_3 dk_4 |\hat{f}(k)\hat{g}(k)| \frac{|a(k_4)|\prod_{j=1}^3 |a(k_j)|^2}{|\bar{k}|\sqrt{D(\vec{k})}}.$$

Then, we see that

$$D(\vec{k}) = \bar{D}(k_1, k_2, k_3) - 2\bar{k}k_4$$

with

$$\bar{D}(k_1, k_2, k_3) = k^2 - k_1^2 + k_2^2 - k_3^2 - \bar{k}^2 = \tilde{D}(k_1, k_2) + 2\bar{k}k_3$$

with

$$\tilde{D}(k_1, k_2) = k^2 - k_1^2 + k_2^2 - (k - k_1 + k_2)^2 = -2(k - k_1)(k_2 - k_1).$$

We divide the domain of integration of $J_{L,2}$ in three parts as

$$J_{L,2}(t) = J_{L,3}(t) + J_{L,4}(t) + J_{L,5}(t)$$

with

$$\frac{4\pi^5}{3} J_{L,3}(t) = \int_{C \cap \{\bar{D} \leq \tilde{D}/2\}} dk dk_1 dk_2 dk_3 |\hat{f}(k)\hat{g}(k)||a(k_1)|^2|a(k_2)|^2|a(k_3)|^2 J_{L,2}^4(t, k_1, k_2, k_3),$$

$$J_{L,4}(t,k) = \frac{3}{4\pi^5} \int_{C \cap \{\bar{D} > \tilde{D}/2, D \leq \bar{D}/2\}} dk dk_1 dk_2 dk_3 \, |\hat{f}(k)\hat{g}(k)||a(k_1)|^2|a(k_2)|^2|a(k_3)|^2 J_{L,2}^4(t, k_1, k_2, k_3)$$

and

$$J_{L,5}(t,k) = \frac{3}{4\pi^5} \int_{C \cap \{\bar{D} > \tilde{D}/2, D > \bar{D}/2\}} dk dk_1 dk_2 dk_3 \, |\hat{f}(k)\hat{g}(k)||a(k_1)|^2|a(k_2)|^2|a(k_3)|^2 J_{L,2}^4(t, k_1, k_2, k_3).$$

For $J_{L,5}$, we use that $D > \tilde{D}/4$ to get

$$|J_{L,5}| \lesssim_a \frac{1}{\sqrt{t\rho}} \int_C dk dk_1 dk_2 dk_3 dk_4 |\hat{f}(k)\hat{g}(k)| \frac{|a(k_4)| \prod_{j=1}^3 |a(k_j)|^2}{|\bar{k}|\sqrt{\tilde{D}(k_1, k_2)}}.$$

We use that we integrate over C to get

$$|J_{L,5}| \lesssim_a \frac{\ln^2 \mu}{\sqrt{t\rho}} \int dk dk_1 dk_2 dk_3 dk_4 |\hat{f}(k)\hat{g}(k)| \frac{|a(k_4)| \prod_{j=1}^3 |a(k_j)|^2}{|\bar{k}| \ln^2(|\bar{k}|)\sqrt{(k - k_1)(k_2 - k_1)}}.$$

We recall that $\bar{k} = k - k_1 + k_2 - k_3$, therefore, by integrating first in k_4 then in k_3 then in k_2 then in k_1 and finally in k, as in

$$\frac{\sqrt{t\rho}}{\ln^2 \mu} |J_{L,5}| \lesssim_a \int dk |\hat{f}(k)\hat{g}(k)| \int dk_1 \frac{|a(k_1)|^1}{\sqrt{|k - k_1|}} \int dk_2 \frac{|a(k_2)|^2}{\sqrt{|k_2 - k_1|}} \int dk_3 \frac{|a(k_3)|^3}{|\bar{k}| \ln^2(|\bar{k}|)} \int dk_4 |a(k_4)|$$

we get

$$|J_{L,5}| \lesssim_{a,f,g} \frac{\ln^2\mu}{\sqrt{t\rho}}$$

which goes to 0 as L goes to ∞ as $\ln^2\mu = o(\sqrt{\rho})$.
For $J_{L,4}$ we use that since $D \leq \bar{D}/2$, we have that

$$|k_4| = \frac{|\bar{D}-D|}{2|\bar{k}|} \geq \frac{|\bar{D}|}{4|\bar{k}|}$$

and therefore since $|\bar{D}| > |\tilde{D}|/2$, we get

$$|k_4| \geq \frac{|\tilde{D}|}{8|\bar{k}|} \text{ and } |a(k_4)| \leq \frac{1}{\sqrt{|k_4|}\langle k_4\rangle^2}\|\sqrt{|x|}\langle x\rangle^2 a\|_{L^\infty} \lesssim_a \frac{\sqrt{|\bar{k}|}}{\sqrt{\tilde{D}}\langle k_4\rangle^2}$$

from which we get

$$\frac{\sqrt{t\rho}}{\ln^2\mu}|J_{L,4}(t,k)| \lesssim_a$$
$$\int dk dk_1 dk_2 dk_3 dk_4 |\hat{f}(k)\hat{g}(k)| \frac{|a(k_1)|^2}{\sqrt{|k-k_1|}} \frac{|a(k_2)|^2}{\sqrt{|k_2-k_1|}} \frac{|a(k_3)|^3}{|\bar{k}|\ln^2(|\bar{k}|)} \frac{\sqrt{|\bar{k}|}}{\langle k_4\rangle^2\sqrt{D(\vec{k})}}.$$

Since $\frac{D(\vec{k})}{|\bar{k}|} = -2k_4 + \alpha(k_1,k_2,k_3)$ where α is a map depending only on k_1, k_2, k_3 we get

$$|J_{L,4}(t,k)| \lesssim_{a,f,g} \frac{\ln^2\mu}{\sqrt{t\rho}}.$$

For $J_{L,3}$ we use that since $|\bar{D}| \leq |\tilde{D}|/2$ we have that

$$|k_3| = \frac{|\tilde{D}-\bar{D}|}{2|\bar{k}|} \geq \frac{|\tilde{D}|}{4|\bar{k}|}$$

and therefore,

$$|a(k_3)| \lesssim_a \sqrt{\frac{|\bar{k}|}{\tilde{D}}}.$$

We get as previously

$$\frac{\sqrt{t\rho}}{\ln^2\mu}|J_{L,3}(t,k)| \lesssim_a$$
$$\int dk dk_1 dk_2 dk_3 dk_4 |\hat{f}(k)\hat{g}(k)| \frac{|a(k_1)|^2}{\sqrt{|k-k_1|}} \frac{|a(k_2)|^2}{\sqrt{|k_2-k_1|}} \frac{|a(k_3)|}{|\bar{k}|\ln^2(|\bar{k}|)} |a(k_4)| \frac{\sqrt{|\bar{k}|}}{\sqrt{D(\vec{k})}}$$

and we conclude as for $J_{L,4}$.

We now compute the limit of $J_{L,1}$. By the change of variable $\xi = D(\vec{k})t\rho$, we have

$$J^4_{L,1}(t,k,k_1,k_2,k_3) = \int \frac{d\xi}{2|\bar{k}|}\frac{1-\cos(\xi)}{\xi^2}|a(\frac{\bar{D}-\xi/(t\rho)}{2\bar{k}})|^2|a(\bar{k}+\frac{\bar{D}-\xi/(t\rho)}{2\bar{k}})|^2.$$

We divide the integral into two parts as

$$J^4_{L,1}(t,k,k_1,k_2,k_3) = K^4_{L,1}(t,k,k_1,k_2,k_3) + K^4_{L,2}(t,k,k_1,k_2,k_3)$$

with

$$K^4_{L,1}(t,k,k_1,k_2,k_3) = \int_{|\xi|\leq|\bar{D}|t\rho/2} \frac{d\xi}{2|\bar{k}|}\frac{1-\cos(\xi)}{\xi^2}|a(\frac{\bar{D}-\xi/(t\rho)}{2\bar{k}})|^2|a(\bar{k}+\frac{\bar{D}-\xi/(t\rho)}{2\bar{k}})|^2$$

and

$$K^4_{L,2}(t,k,k_1,k_2,k_3) = \int_{|\xi|>|\bar{D}|t\rho/2} \frac{d\xi}{2|\bar{k}|}\frac{1-\cos(\xi)}{\xi^2}|a(\frac{\bar{D}-\xi/(t\rho)}{2\bar{k}})|^2|a(\bar{k}+\frac{\bar{D}-\xi/(t\rho)}{2\bar{k}})|^2.$$

We denote for $j = 1,2$,

$$K_{L,j}(t,k) = \frac{3}{4\pi^5}\int_C dkdk_1dk_2dk_3\hat{f}(k)\overline{\hat{g}(k)}|a(k_1)|^2|a(k_2)|^2|a(k_3)|^2K^4_{L,j}(t,k,k_1,k_2,k_3).$$

For $K_{L,2}$, using the bound on $|\xi|$ we get

$$|K^4_{L,2}(t,k,k_1,k_2,k_3)| \lesssim_a \frac{1}{|\bar{k}|(t\rho)^{1/4}|\bar{D}|^{1/4}}\int d\xi\frac{1-\cos(\xi)}{|\xi|^{7/4}} \lesssim_a \frac{1}{|\bar{k}|(t\rho)^{1/4}|\bar{D}|^{1/4}}.$$

We divide the integral on k_1,k_2,k_3 in two as previously as

$$|K_{L,2}(t)| \lesssim_a K_{L,3}(t) + K_{L,4}(t)$$

with

$$K_{L,3}(t) = \int_{\bar{D}\leq\tilde{D}/2\cap C} \frac{1}{|\bar{k}|(t\rho)^{1/4}|\bar{D}|^{1/4}}|\hat{f}(k)\hat{g}(k)||a(k_1)|^2|a(k_2)|^2|a(k_3)|^2dk_1dk_2dk_3$$

and

$$K_{L,4}(t) = \int_{\bar{D}>\tilde{D}/2\cap C} \frac{1}{|\bar{k}|(t\rho)^{1/4}|\bar{D}|^{1/4}}|\hat{f}(k)\hat{g}(k)||a(k_1)|^2|a(k_2)|^2|a(k_3)|^2dkdk_1dk_2dk_3.$$

For $K_{L,4}$ we use the inequalities on $\bar{k}$ and $\bar{D}$ to get

$$\frac{(t\rho)^{1/4}}{\ln^2(\mu)}K_{L,4}(t) \lesssim \int \frac{dkdk_1dk_2dk_3}{|\bar{k}|\ln^2(|\bar{k}|)|k-k_1|^{1/4}|k_1-k_2|^{1/4}}|\hat{f}(k)\hat{g}(k)||a(k_1)|^2|a(k_2)|^2|a(k_3)|^2$$

which is integrable as previously, hence

$$K_{L,4}(t,k) \lesssim_{a,f,g} \frac{\ln^2(\mu)}{(t\rho)^{1/4}}.$$

For $K_{L,3}$ we recall that the inequality on $\bar{D}$ implies that

$$|k_3| \geq \frac{\tilde{D}}{|\bar{k}|}.$$

We use that

$$|a(k_3)| \lesssim_a \frac{|\bar{k}|^{3/4}}{|\tilde{D}|^{3/4}}.$$

We deduce

$$K_{L,3} \lesssim_a \frac{1}{(t\rho)^{1/4}}\int \frac{1}{|\bar{D}|^{1/4}|\tilde{D}|^{3/4}|\bar{k}|^{1/4}}|\hat{f}(k)\hat{g}(k)||a(k_1)|^2|a(k_2)|^2|a(k_3)|dkdk_1dk_2dk_3.$$

We have that

$$|\bar{D}| \geq 2|k_3-\beta(k_1,k_2)|\;|k_3-\gamma(k_1,k_2)|$$

where β and γ are two maps depending only on k_1,k_2. By using Hölder's inequality on k_3, we get

$$K_{L,3} \lesssim_a \frac{1}{(t\rho)^{1/4}}\int dk|\hat{f}(k)\hat{g}(k)|\int dk_1\frac{|a(k_1)|^2}{|k-k_1|^{3/4}} \int dk_2\frac{|a(k_2)|^2}{|k_2-k_1|^{3/4}}\|\frac{|a|^{1/3}}{|k_3-\beta|^{1/4}}\|_{L^3(k_3)}\|\frac{|a|^{1/3}}{|k_3-\gamma|^{1/4}}\|_{L^3(k_3)}\|\frac{|a|^{1/3}}{|\bar{k}|^{1/4}}\|_{L^3(k_3)}.$$

Since $\frac{3}{4}<1$, we get

$$K_{L,3} \lesssim_{f,g} \frac{1}{(t\rho)^{1/4}}.$$

We now turn to $K_{L,1}$. We set

$$f_L(k,k_1,k_2,k_3,\xi) = \mathbf{1}_{|\bar{k}|\geq\frac{1}{\mu}}\mathbf{1}_{|\xi|\leq|\bar{D}|t\rho/2}\frac{1}{2|\bar{k}|}\frac{1-\cos(\xi)}{\xi^2} |a(\frac{\bar{D}-\xi/(t\rho)}{2\bar{k}})|^2|a(\bar{k}+\frac{\bar{D}-\xi/(t\rho)}{2\bar{k}})|^2\hat{f}(k)\overline{\hat{g}(k)}|a(k_1)|^2|a(k_2)|^2|a(k_3)|^2.$$

When L goes to ∞ we have that $\rho \to \infty$ and thus f_L converges almost surely to

$$f_\infty(k, k_1, k_2, k_3, \xi) = \frac{1}{2|\bar{k}|}\frac{1-\cos(\xi)}{\xi^2}|a(\frac{\bar{D}}{2\bar{k}})|^2|a(\bar{k}+\frac{\bar{D}}{2\bar{k}})|^2\hat{f}(k)\overline{\hat{g}(k)}|a(k_1)|^2|a(k_2)|^2|a(k_3)|^2.$$

What is more, because $|\xi| \le |\bar{D}|t\rho/2$, we have that

$$\Big|\frac{\bar{D}-\xi/(t\rho)}{2\bar{k}}\Big| \ge \Big|\frac{\bar{D}}{4\bar{k}}\Big|$$

from which we deduce that

$$|a(\frac{\bar{D}-\xi/(t\rho)}{2\bar{k}})|^2 \lesssim_a \frac{|\bar{k}|^{1/2}}{|\bar{D}|^{1/2}}.$$

Therefore, if $|\bar{D}| \ge |\tilde{D}|/2$, we get

$$|f_L(k, k_1, k_2, k_3, \xi)| \lesssim_a \frac{1}{|\bar{k}|^{1/2}|\tilde{D}|^{1/2}}\frac{1-\cos(\xi)}{\xi^2}|\hat{f}(k)\hat{g}(k)||a(k_1)|^2|a(k_2)|^2|a(k_3)|^2.$$

On the other hand, if $|\bar{D}| < |\tilde{D}|/2$ then

$$|k_3| > \frac{|\tilde{D}|}{4|\bar{k}|}$$

and we get

$$|f_L(k, k_1, k_2, k_3, \xi)| \lesssim_a \frac{1}{|\bar{k}|^{1/2}|\tilde{D}|^{1/2}}\frac{1-\cos(\xi)}{\xi^2}|\hat{f}(k)\hat{g}(k)||a(k_1)|^2|a(k_2)|^2|a(k_3)|.$$

Hence, for k, k_1, k_2, k_3, k_4 we have

$$|f_L(k_1, k_2, k_3, \xi)| \lesssim_a \frac{1}{|\bar{k}|^{1/2}|\tilde{D}|^{1/2}}\frac{1-\cos(\xi)}{\xi^2}|\hat{f}(k)\hat{g}(k)||a(k_1)|^2|a(k_2)|^2|a(k_3)|,$$

the map on the right hand side being integrable we can apply DCT and get that

$$\lim_{L\to\infty} K_{L,1}(t) = \int f_\infty(k, k_1, k_2, k_3, \xi)$$
$$= \frac{3}{4\pi^4}\int dk_1 dk_2 dk_3 \frac{1}{|\bar{k}|}|a(\frac{\bar{D}}{2\bar{k}})|^2|a(\bar{k}+\frac{\bar{D}}{2\bar{k}})|^2|a(k_1)|^2|a(k_2)|^2|a(k_3)|^2$$

and the map below the integral is integrable. □

5.2 Case $t = \frac{1}{3}$

Proposition 5.4. *Assume that* $\varepsilon^{-2} = 2\pi 2^L L^2 + \rho(L)$ *and that* $t = \frac{1}{3}$. *We have*

$$I_L(t) = \frac{1}{12\pi^5}\int_{\vec{k}\in B} \frac{\sin(\Delta(\vec{k})t\rho}{\Delta(\vec{k})} \prod_{j=1}^5 |a(k_j)|^2 \delta(k+\sum_{j=1}^5 (-1)^j k_j)\hat{f}(k)\overline{\hat{g}(k)}d\vec{k} + O_a\Big(\frac{\mu\nu}{L}+\frac{\rho^2}{L}+\frac{\rho\mu}{\nu}+\frac{\rho}{\mu}\Big)$$

where $B = \{(k,k_1,k_2,k_3,k_4,k_5)\in\mathbb{R}^6 \mid |k-k_1+k_2-k_3|\geq \frac{1}{\mu}\}$, *where*

$$\Delta(\vec{k}) = k^2 + \sum_{j=1}^5 (-1)^j k_j^2$$

and $d\vec{k} = dk\prod_{j=1}^5 dk_j$.

Proof. First, we see that

$$t = \frac{1}{3} = \sum_{n\geq 1}\frac{1}{2^{2n}}$$

and therefore

$$2^L t \in \frac{x_L}{3} + \mathbb{N}$$

where $x_L = 1$ if L is even and $x_L = 2$ if L is odd. We deduce that

$$\Delta(\vec{k})\varepsilon^{-2}t \in \Delta(\vec{k})t\rho + \frac{2\pi x_L}{3}\Big[(kL)^2 + \sum_{j=1}^5 (-1)^j (Lk_j)^2\Big] + 2\pi\mathbb{Z}.$$

And therefore, we get

$$\sin(\Delta(\vec{k})\varepsilon^{-2}t) = \sin(\Delta(\vec{k})\rho t)\cos\Big(\frac{2\pi x_L}{3}\Big[(kL)^2 + \sum_{j=1}^5 (-1)^j (Lk_j)^2\Big]\Big) + \cos(\Delta(\vec{k})\rho t)\sin\Big(\frac{2\pi x_L}{3}\Big[(kL)^2 + \sum_{j=1}^5 (-1)^j (Lk_j)^2\Big]\Big).$$

To know the values of

$$\cos\Big(\frac{2\pi x_L}{3}\Big[(kL)^2 + \sum_{j=1}^5 (-1)^j (Lk_j)^2\Big]\Big)$$

and

$$\sin\Big(\frac{2\pi x_L}{3}\Big[(kL)^2 + \sum_{j=1}^5 (-1)^j (Lk_j)^2\Big]\Big)$$

it is sufficient to know the congruence of Lk, Lk_j modulo 3. Therefore, we write

$$I_L(t) = \frac{2}{(2\pi)^5} \sum_\kappa I_{\kappa,L}(t)$$

with κ a map from $[0,4] \cap \mathbb{N}$ to $\{-1, 0, 1\}$ and

$$I_{\kappa,L}(t) = \sum_{\sigma \in \mathfrak{A}} \frac{1}{L^5} \sum_{\vec{k} \in C_{\sigma,\kappa}} \frac{\sin(\Delta(\vec{k})t\rho) b_{\kappa,L} + \cos(\Delta(\vec{k})t\rho) c_{\kappa,L}}{\Delta(\vec{k})} \prod_{j=1}^5 |a(k_j)|^2 \hat{f}(k)\overline{\hat{g}(k)}$$

where

$$C_{\sigma,\kappa} = \{(k, k_1, k_2, k_3, k_4, k_5) \in C_\sigma \mid Lk \in \kappa(0) + 3\mathbb{Z}, \quad \forall j, \quad Lk_j \in \kappa(j) + 3\mathbb{Z}\}$$

and

$$b_{\kappa,L} = \cos\Big(\frac{2\pi x_L}{3} \sum_{j=0}^5 (-1)^j \kappa(j)^2\Big)$$

and

$$c_{\kappa,L} = \sin\Big(\frac{2\pi x_L}{3} \sum_{j=0}^5 (-1)^j \kappa(j)^2\Big).$$

We divide $I_{\kappa,L}$ in two as

$$I_{\kappa,L} = J_{\kappa,L} + K_{\kappa,L}$$

with

$$J_{\kappa,L} = b_{\kappa,L} \sum_{\sigma \in \mathfrak{A}} \frac{1}{L^5} \sum_{\vec{k} \in C_{\sigma,\kappa}} \frac{\sin(\Delta(\vec{k})t\rho)}{\Delta(\vec{k})} \prod_{j=1}^5 |a(k_j)|^2 \hat{f}(k)\overline{\hat{g}(k)}$$

and

$$K_{\kappa,L} = c_{\kappa,L} \sum_{\sigma \in \mathfrak{A}} \frac{1}{L^5} \sum_{\vec{k} \in C_{\sigma,\kappa}} \frac{\cos(\Delta(\vec{k})t\rho)}{\Delta(\vec{k})} \prod_{j=1}^5 |a(k_j)|^2 \hat{f}(k)\overline{\hat{g}(k)}.$$

With the same strategy as in the proof of Proposition 5.2, we have

$$J_{\kappa,L} = b_{\kappa,L} \frac{12}{3^5} \int_{\vec{k} \in B} \frac{\sin(\Delta(\vec{k})t\rho)}{\Delta(\vec{k})} \prod_{j=1}^5 |a(k_j)|^2 \hat{f}(k)\overline{\hat{g}(k)} d\vec{k} + O_{a,f,g}\Big(\frac{\rho^2}{L} + \frac{\rho\mu}{\nu} + \frac{\rho}{\mu}\Big).$$

According to the program set in Appendix B, $b_{\kappa,L} = 1$ in 99 cases and is equal to

$$\cos(\frac{2\pi}{3}) = \cos(-\frac{2\pi}{3}) = -\frac{1}{2}$$

in 144 cases. Therefore,

$$\sum_\kappa J_{\kappa,L} =$$

$$12\frac{99-72}{3^5}\int_{\vec k\in B}\frac{\sin(\Delta(\vec k)t\rho)}{\Delta(\vec k)}\prod_{j=1}^5 |a(k_j)|^2\hat f(k)\overline{\hat g(k)}d\vec k + O_{a,f,g}\Big(\frac{\rho^2}{L}+\frac{\rho\mu}{\nu}+\frac{\rho}{\mu}\Big).$$

Still according to the program in Appendix B, $c_{\kappa,L}=0$ in 99 cases, is equal to $\frac{\sqrt3}{2}$ in 72 cases and is equal to $-\frac{\sqrt3}{2}$ in 72 cases. We fix an involution $\kappa\mapsto\bar\kappa$ such that $c_{\bar\kappa,L}=-c_{\kappa,L}$. We get

$$\begin{aligned}K_{\kappa,L}+K_{\bar\kappa,L} = c_{\kappa,L}\sum_{\sigma\in\mathfrak A}\frac{1}{L^5}\sum_{\vec k\in C_{\sigma,\kappa}}\frac{\cos(\Delta(\vec k)t\rho)-1}{\Delta(\vec k)}\prod_{j=1}^5|a(k_j)|^2\hat f(k)\overline{\hat g(k)}\\ - c_{\kappa,L}\sum_{\sigma\in\mathfrak A}\frac{1}{L^5}\sum_{\vec k\in C_{\sigma,\bar\kappa}}\frac{\cos(\Delta(\vec k)t\rho)-1}{\Delta(\vec k)}\prod_{j=1}^5|a(k_j)|^2\hat f(k)\overline{\hat g(k)}\\ + c_{\kappa,L}\sum_{\sigma\in\mathfrak A}\frac{1}{L^5}\sum_{\vec k\in C_{\sigma,\kappa}}\frac{1}{\Delta(\vec k)}\prod_{j=1}^5|a(k_j)|^2\hat f(k)\overline{\hat g(k)}\\ - c_{\kappa,L}\sum_{\sigma\in\mathfrak A}\frac{1}{L^5}\sum_{\vec k\in C_{\sigma,\bar\kappa}}\frac{1}{\Delta(\vec k)}\prod_{j=1}^5|a(k_j)|^2\hat f(k)\overline{\hat g(k)}.\end{aligned}\tag{10}$$

For the same reasons as in the proof of Proposition 5.2, we get that

$$\sum_{\sigma\in\mathfrak A}\frac{1}{L^5}\sum_{\vec k\in C_{\sigma,\kappa}}\frac{\cos(\Delta(\vec k)t\rho)-1}{\Delta(\vec k)}\prod_{j=1}^5|a(k_j)|^2\hat f(k)\overline{\hat g(k)} =$$

$$\frac{12}{3^5}\int_{\vec k\in B}\frac{\cos(\Delta(\vec k)t\rho)-1}{\Delta(\vec k)}\prod_{j=1}^5|a(k_j)|^2\hat f(k)\overline{\hat g(k)} + O_{a,f,g}\Big(\frac{\rho^2}{L}+\frac{\rho\mu}{\nu}+\frac{\rho}{\mu}\Big)$$

the key points being that $\frac{\cos(\Delta(\vec k)t\rho)-1}{\Delta(\vec k)}\leq t\rho$ and that the derivative of $\frac{\cos x-1}{x}$ is continuous and bounded.
This erases the first two lines in (10) as in

$$\begin{aligned}K_{\kappa,L}+K_{\bar\kappa,L} = c_{\kappa,L}\sum_{\sigma\in\mathfrak A}\frac{1}{L^5}\sum_{\vec k\in C_{\sigma,\kappa}}\frac{1}{\Delta(\vec k)}\prod_{j=1}^5|a(k_j)|^2\hat f(k)\overline{\hat g(k)}\\ - c_{\kappa,L}\sum_{\sigma\in\mathfrak A}\frac{1}{L^5}\sum_{\vec k\in C_{\sigma,\bar\kappa}}\frac{1}{\Delta(\vec k)}\prod_{j=1}^5|a(k_j)|^2\hat f(k)\overline{\hat g(k)}\\ + O_{a,f,g}\Big(\frac{\rho^2}{L}+\frac{\rho\mu}{\nu}+\frac{\rho}{\mu}\Big).\end{aligned}\tag{11}$$

We deal with the remainder, we set

$$K_{\kappa,L,2} =$$
$$\frac{1}{L^5}\sum_{\vec k\in C_{\sigma,\kappa}}\frac{1}{\Delta(\vec k)}\prod_{j=1}^5 |a(k_j)|^2\hat f(k)\overline{\hat g(k)} - \frac{1}{L^5}\sum_{\vec k\in C_{\sigma,\bar\kappa}}\frac{1}{\Delta(\vec k)}\prod_{j=1}^5 |a(k_j)|^2\hat f(k)\overline{\hat g(k)}.$$

Writing $C_\kappa = \{\vec k | Lk\in \kappa(0)+3\mathbb{Z},\ \forall j,\ Lk_j\in\kappa(j)+3\mathbb{Z}\}$ and $\vec j = \vec k + \frac{\bar\kappa-\kappa}{L}$, we get

$$K_{\kappa,L,2} = \frac{1}{L^5}\sum_{\vec k\in C_\kappa}\Big[\frac{\mathbf{1}_{\vec k\in C_\sigma}-\mathbf{1}_{\vec j\in C_\sigma}}{\Delta(\vec k)} + \mathbf{1}_{\vec j\in C_\sigma}\Big(\frac{1}{\Delta(\vec k)}-\frac{1}{\Delta(\vec j)}\Big)\Big]\prod_{j=1}^5 |a(k_j)|^2\hat f(k)\overline{\hat g(k)}$$
$$+\frac{1}{L^5}\sum_{\vec k\in C_\kappa}\frac{\mathbf{1}_{\vec j\in C_\sigma}}{\Delta(\vec j)}\Big(\prod_{j=1}^5 |a(k_j)|^2\hat f(k)\overline{\hat g(k)} - \prod_{j=1}^5 |a(j_j)|^2\hat f(j)\overline{\hat g(j)}\Big).$$

We divide again $K_{\kappa,L,2}$ into three parts as

$$K_{\kappa,L,2} = K_{\kappa,L,3}+K_{\kappa,L,4}+K_{\kappa,L,5}$$

with

$$K_{\kappa,L,3} = \frac{1}{L^5}\sum_{\vec k\in C_\kappa}\frac{\mathbf{1}_{\vec k\in C_\sigma}-\mathbf{1}_{\vec j\in C_\sigma}}{\Delta(\vec k)}\prod_{j=1}^5 |a(k_j)|^2\hat f(k)\overline{\hat g(k)},$$

$$K_{\kappa,L,4} = \frac{1}{L^5}\sum_{\vec k\in C_\kappa}\mathbf{1}_{\vec j\in C_\sigma}\Big(\frac{1}{\Delta(\vec k)}-\frac{1}{\Delta(\vec j)}\Big)\prod_{j=1}^5 |a(k_j)|^2\hat f(k)\overline{\hat g(k)}$$

and

$$K_{\kappa,L,5} = \frac{1}{L^5}\sum_{\vec k\in C_\kappa}\frac{\mathbf{1}_{\vec j\in C_\sigma}}{\Delta(\vec j)}\Big(\prod_{j=1}^5 |a(k_j)|^2\hat f(k)\overline{\hat g(k)} - \prod_{j=1}^5 |a(j_j)|^2\hat f(j)\overline{\hat g(j)}\Big).$$

We first deal with $K_{\kappa,L,5}$. We have $|\vec k-\vec j|\leq \frac{2}{L}$. Hence,

$$\Big|\frac{\mathbf{1}_{\vec j\in C_\sigma}}{\Delta(\vec j)}\Big(\prod_{j=1}^5 |a(k_j)|^2\hat f(k)\overline{\hat g(k)} - \prod_{j=1}^5 |a(j_j)|^2\hat f(j)\overline{\hat g(j)}\Big)\Big| \lesssim_{a,f,g}\frac{\mathbf{1}_{\vec j\in C_\sigma}\nu}{L}.$$

Besides a has compact support hence there exists $M>0$ such that

$$|K_{\kappa,L,5}|\lesssim_{a,f,g}\frac{1}{L^5}\sum_{\vec k\in C_\kappa\cap[-M,M]^6}\frac{\mathbf{1}_{\vec j\in C_\sigma}\nu}{L}\lesssim_{a,f,g}\frac{\nu}{L}.$$

We turn to $K_{\kappa,L,4}$. Given again the fact that a has compact support, we get on the support of $\prod_{j=1}^5|a(k_j)|^2$ remembering that $k=-\sum_{j=1}^5(-1)^jk_j$,

$$\Big|\mathbf{1}_{\vec j\in C_\sigma}\Big(\frac{1}{\Delta(\vec k)}-\frac{1}{\Delta(\vec j)}\Big)\Big|\lesssim_a\frac{\mathbf{1}_{\vec j\in C_\sigma}\nu}{|\Delta(\vec k)|L}.$$

Since $|\Delta(\vec{k})| \geq |\Delta(\vec{j})| - |\Delta(\vec{j}) - \Delta(\vec{k})|$ and since $|\Delta(\vec{j}) - \Delta(\vec{k})| \lesssim_a L^{-1}$ we get that for L big enough, since $\nu = o(\sqrt{L})$,

$$\Big|\mathbf{1}_{\vec{j}\in C_\sigma}\Big(\frac{1}{\Delta(\vec{k})} - \frac{1}{\Delta(\vec{j})}\Big)\Big| \lesssim_a \frac{\mathbf{1}_{\vec{j}\in C_\sigma}\nu^2}{L}.$$

And therefore, we get

$$|K_{\kappa,L,4}| \lesssim_{a,f,g} \frac{\nu^2}{L}.$$

We now turn to $K_{\kappa,L,3}$ and estimate the numbers of $\vec{k}$ such that

$$\mathbf{1}_{\vec{k}\in C_\sigma} - \mathbf{1}_{\vec{j}\in C_\sigma}$$

is not null. We assume without loss of generality that $\vec{k} \in C_\sigma$ but that $\vec{j} \notin C_\sigma$.
First Case: $|\Delta(\vec{j})| < \nu^{-1}$. We have

$$\nu^{-1} \leq |\Delta(\vec{k})| \leq |\Delta(\vec{j})| + |\Delta(\vec{k}) - \Delta(\vec{j})|.$$

Since on the support of $\prod_{j=1}^5 |a(k_j)|^2$, we have

$$|\Delta(\vec{k}) - \Delta(\vec{j})| \lesssim_a L^{-1},$$

we get

$$\nu^{-1} \leq |\Delta(\vec{k})| \leq |\Delta(\vec{j})| + C_a L^{-1}.$$

We recall that $\Delta(\vec{k}) = \bar{D}(k,k_1,k_2,k_3) - 2\bar{k}k_4$ and thus

$$\frac{1}{\nu|\bar{k}|} \leq |k_4 - \frac{\bar{D}}{2\bar{k}}| \leq \frac{1}{\nu|\bar{k}|} + C_a\frac{\mu}{\nu}.$$

Hence k_4 belongs to the reunion of two intervals of size $C_a\frac{\mu}{L}$.
Second case: $|\bar{j}| < \mu^{-1}$. We have

$$\mu^{-1} \leq |\bar{k}| \leq |\bar{j}| + |\bar{j} - \bar{k}| \leq \mu^{-1} + \frac{2}{L}.$$

Therefore, k_3 belongs to the reunion of two intervals of size $\frac{2}{L}$.
Third case: $|\bar{j}_\sigma| < \mu^{-1}$. Similar to second case.
Therefore,

$$|K_{\kappa,L,3}| \lesssim_{a,f,g} \frac{\mu\nu}{L} + \frac{\nu}{L}$$

and we can conclude. □

From Proposition 5.3, we therefore get

$$\lim_{L\to\infty} I_L(t) =$$
$$\frac{1}{12\pi^4}\int_{\mathbb{R}^6} \delta(k + \sum_{j=1}^5 (-1)^j k_j)\delta(\Delta(\vec{k}))\frac{1}{k - k_1 + k_2 - k_3}\prod_{j=1}^5 |a(k_j)|^2 \hat{f}(k)\overline{\hat{g}(k)}d\vec{k}$$

and get Theorem 1.2.

A Tree glossary

A.1 Labeled trees

We draw some trees corresponding to Definition 3.2 in Figures 1 and 2. The squares represent leaves and the circles represent nodes. We have $T_0 \in \mathcal{T}_0[k]$, $T_1 \in \mathcal{T}_1[k]$ and $T_2 \in \mathcal{T}_2[k]$. We fix $\Delta = \Delta_1 = k^2 - k_1^2 + k_2^2 - k_3^2 + k_4^2 - k_5^2$, and $\Delta_2 = k_1^2 - j_1^2 + j_2^2 - j_3^2 + j_4^2 - j_5^2$. Note that, in mathematical writing, we have

$$T_0 = (k), \quad T_1 = ((k_1), (k_2), (k_3), (k_4), (k_5), k),$$

and

$$T_2 = (((j_1), (j_2), (j_3), (j_4), (j_5), k_1), (k_2), (k_3), (k_4), (k_5), k).$$

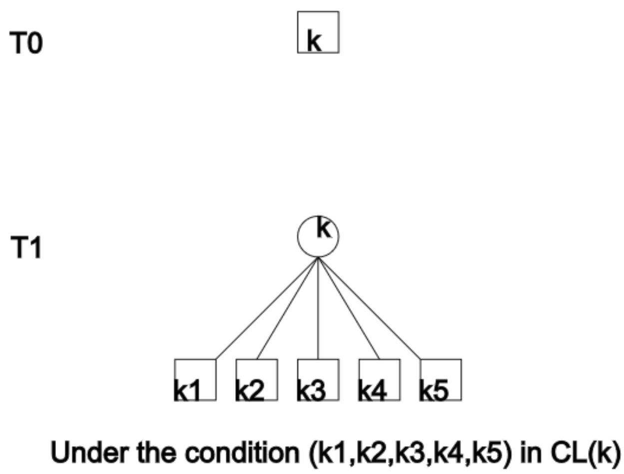

Figure 1: Labelled trees with 0 and 1 nodes

Keeping in mind the previous examples, we have corresponding to Definition 3.3:

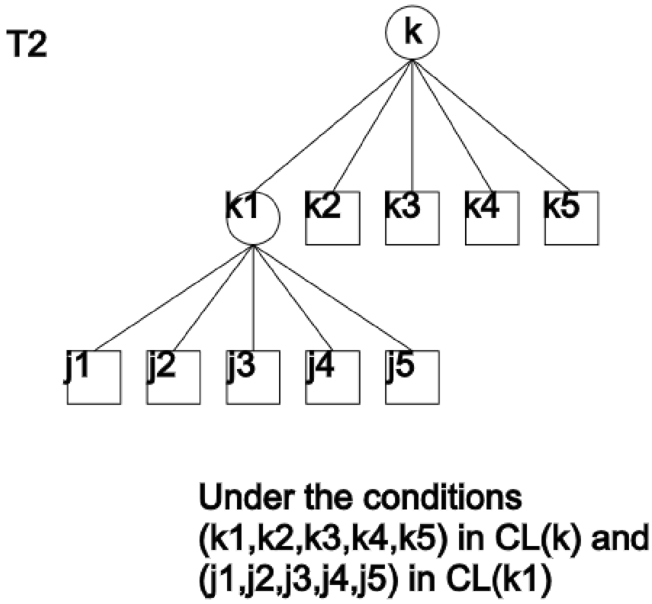

Figure 2: Labelled tree with 2 nodes

$$\begin{aligned}
F_{T_0}(t) &= 1,\\
F_{T_1}(t) &= -i\int_0^t e^{i\Delta\tau}d\tau,\\
F_{T_2}(t) &= -\int_0^t e^{i\Delta_1\tau}\int_0^\tau e^{i\Delta_2 s}dsd\tau,\\
g_{T_0} &= g_{Lk},\\
g_{T_1} &= g_{Lk_1}\diamond \bar g_{Lk_2}\diamond g_{Lk_3}\diamond \bar g_{Lk_4}\diamond g_{Lk_5},\\
g_{T_2} &= g_{Lj_1}\diamond \bar g_{Lj_2}\diamond g_{Lj_3}\diamond \bar g_{Lj_4}\diamond g_{Lj_5}\diamond \bar g_{Lk_2}\diamond g_{Lk_3}\diamond \bar g_{Lk_4}\diamond g_{Lk_5},
\end{aligned}$$

and

$$\begin{aligned}
A_{T_0} &= a(k),\\
A_{T_1} &= a(k_1)\bar a(k_2)a(k_3)\bar a(k_4)a(k_5),\\
A_{T_1} &= a(j_1)\bar a(j_2)a(j_3)\bar a(j_4)a(j_5)\bar a(k_2)a(k_3)\bar a(k_4)a(k_5).
\end{aligned}$$

We have, corresponding to Definition 3.4,

$$\vec{k}_0 := \vec{T}_0 = k, \quad \vec{k}_1 := \vec{T}_1 = (k_1, k_2, k_3, k_4, k_5),$$
$$\vec{k}_2 := \vec{T}_2 = (j_1, j_2, j_3, j_4, j_5, k_2, k_3, k_4, k_5).$$

A.2 Unlabeled trees

We draw a picture corresponding to Definition 3.7 in Figure 3. In mathematical writing, we have $\mathtt{t}_0 = \perp$, $\mathtt{t}_1 = (\perp, \perp, \perp, \perp, \perp)$ and $\mathtt{t}_2 = ((\perp, \perp, \perp, \perp, \perp), \perp, \perp, \perp, \perp)$. We also have $T_0 = \mathtt{t}_0(k)$, $T_1 = \mathtt{t}_1((k_1, k_2, k_3, k_4, k_5))$ and

$$T_2 = \mathtt{t}_2((j_1, j_2, j_3, j_4, j_5, k_2, k_3, k_4, k_5)).$$

We draw a picture corresponding to Definition 3.8 in Figure 4. We forgot some parenthesis as they were redundant.
Corresponding to Definition 3.9, we have

$$k_{\mathtt{t}_0, \vec{k}_0} : 0 \mapsto k, \quad k_{\mathtt{t}_1, \vec{k}_1} : \begin{array}{c} 0 \mapsto k \\ (l, 0) \mapsto k_l \end{array}$$

with $k = \sum_{l=1}^{5} (-1)^{l+1} k_l$ and

$$k_{\mathtt{t}_2, \vec{k}_2} : \begin{array}{c} 0 \mapsto k \\ (l, 0) \mapsto k_l \\ (1, l, 0) \mapsto j_l \end{array}$$

with $k_1 = \sum_{l=1}^{5} (-1)^{l+1} j_l$ and $k = \sum_{l=1}^{5} (-1)^{l+1} k_l$.
We also have

$$\Omega_{\mathtt{t}_1, \vec{k}_1} : 0 \mapsto \Delta, \quad \Omega_{\mathtt{t}_2, \vec{k}_2} : \begin{array}{c} 0 \mapsto \Delta_1 \\ (1, 0) \mapsto \Delta_2 \end{array} .$$

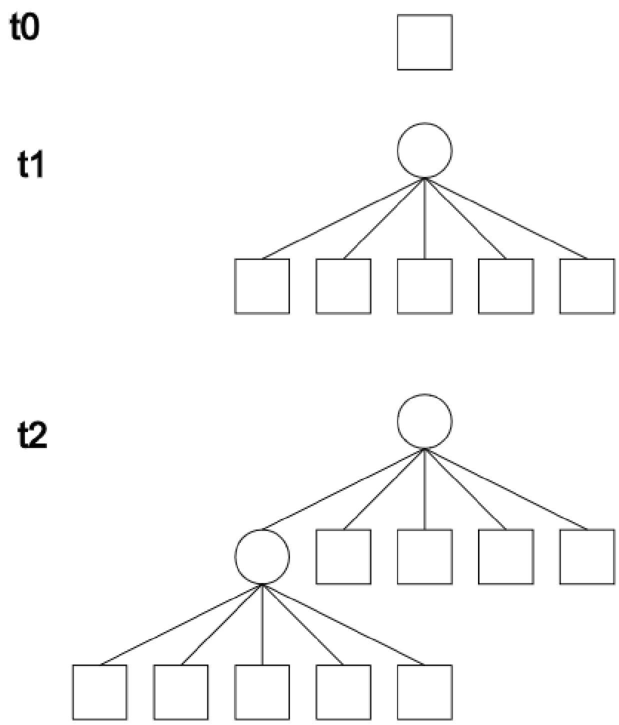

Figure 3: Unlabelled trees with 0, 1 and 2 nodes

A.3 Node ordering

Finally, corresponding to Definition 3.13, we write explicitly the partial order $R_{\mathtt{t}_4}$ on the tree with four nodes of Figure 5. We have

$$(1,4,0)R_{\mathtt{t}_4}(1,0)R_{\mathtt{t}_4}0 \quad \text{and} \quad (5,0)R_{\mathtt{t}_4}0$$

and the other nodes are not comparable. In other words, $(1,0)$ is not comparable to $(5,0)$ but also $(1,4,0)$ is not comparable to $(5,0)$.

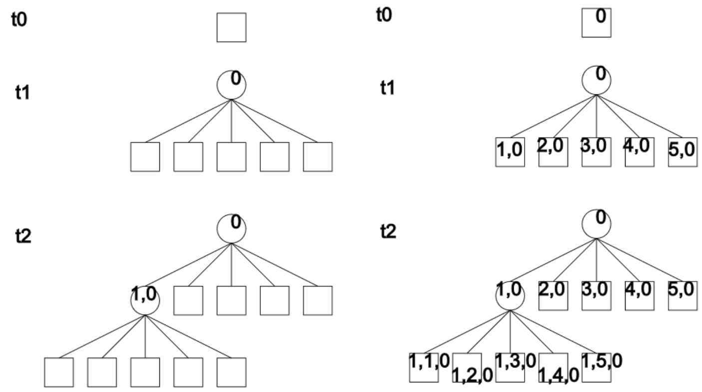

Figure 4: Ordering labels of nodes and leaves

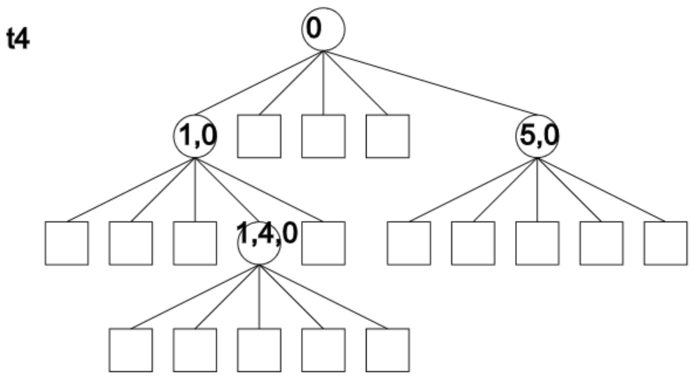

Figure 5: A tree with four nodes

B A program in Python

Here, we present a program on Python designed to give the number of maps $\kappa : [0,4] \to \{-1,0,1\}$ such that the number

$$\kappa(0)^2 + \sum_{j=1}^{5} (-1)^j \kappa(j)^2$$

is equal to 0, 1 or -1 in $\mathbb{F}_3$.

```
 def hs(k) :
a,b,c = 0,0,0
for x in range(3):
for x1 in range(3):
for x2 in range(3):
for x3 in range(3):
for x4 in range(3):
if (x**2-x1**2+x2**2-x3**2+x4**2-(x-x1+x2-x3+x4)**2)%3 == 0:
a=a+1
else:
if (x**2-x1**2+x2**2-x3**2+x4**2-(x-x1+x2-x3+x4)**2)%3 == 1:
b=b+1
else:  c=c+1
return a,b,c
```

The program returns: $(99, 72, 72)$.

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