

THE CLOSURE OF DOUBLE RAMIFICATION LOCI
VIA STRATA OF EXACT DIFFERENTIALS

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ABSTRACT. Double ramification loci, also known as strata of 0-differentials, are algebraic subvarieties of the moduli space of smooth curves parametrizing Riemann surfaces such that there exists a rational function with prescribed ramification over 0 and ∞ . We describe the closure of double ramification loci inside the Deligne-Mumford compactification in geometric terms. To a rational function we associate its exact differential, which allows us to realize double ramification loci as linear subvarieties of strata of meromorphic differentials. We then obtain a geometric description of the closure using our recent results on the boundary of linear subvarieties. Our approach yields a new way of relating the geometry of loci of rational functions and Teichmüller dynamics. We also compare our results to a different approach using admissible covers.

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1 INTRODUCTION

Inside the moduli space $\mathcal{M}_{g,n}$ of pointed smooth curves (X, \mathbf{x}) of genus g with n marked points $\mathbf{x} = (x_1, \dots, x_n)$ for any integer $k \geq 0$ there exist natural closed subvarieties

$$\mathcal{H}_g^k(\mu) := \left\{ (X, \mathbf{x}) : \mathcal{O}_X \left(\sum_{i=1}^n \mu_i x_i \right) \simeq \omega_X^{\otimes k} \right\} \subseteq \mathcal{M}_{g,n},$$

where $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ is an integer partition of $k(2g - 2)$.

For $k \geq 1$, the above condition is equivalent to the existence of a meromorphic k -differential on X with prescribed vanishing order at the marked points. In this case $\mathcal{H}^k(\mu)$ is called a *stratum of meromorphic k -differentials of type μ* and they have been studied extensively, especially from the viewpoint of Teichmüller dynamics. See for example the surveys [Che17, Wri15, Zor06]. Strata of k -differentials are not compact, and finding geometrically meaningful compactifications is an active area of research. One possible way of compactifying the stratum $\mathcal{H}_g^k(\mu) \subseteq \mathcal{M}_{g,n}$ is by simply taking the closure inside $\overline{\mathcal{M}}_{g,n}$. The closure of $\mathcal{H}_g^k(\mu)$ in $\overline{\mathcal{M}}_{g,n}$ has been determined for $k = 1$ in [BCGGM18] in terms of *twisted differentials*, and for $k > 1$ in [BCGGM19a], by using a covering construction to reduce it to the $k = 1$ case. In [BCGGM19b] the authors use the explicit description of the closure of $\mathcal{H}_g^k(\mu)$ to construct the *moduli space of multi-scale differentials* $\Xi \overline{\mathcal{M}}_{g,n}(\mu)$: a smooth, modular compactification of the stratum $\mathcal{H}_g^1(\mu)$, which has been extended to $k > 1$ in [CMZ20a].

In the case $k = 0$, the isomorphism $\mathcal{O}_X \left(\sum_{i=1}^n \mu_i x_i \right) \simeq \mathcal{O}_X$ is equivalent to the existence of a rational function $f : X \rightarrow \mathbb{P}^1$ with prescribed ramification over 0 and ∞ with the preimages of 0 and ∞ being all marked points x_k such that $\mu_k > 0$ and $\mu_k < 0$, respectively. The subvariety $\text{DR}_g(\mu) := \mathcal{H}_g^0(\mu)$ of $\mathcal{M}_{g,n}$ is called a *double ramification locus* or a *stratum of 0-differentials*. Equivalently, double ramification loci can be defined as the pullback of the zero-section from the universal Jacobian under the Abel-Jacobi map. As a warning to the reader, the double ramification locus is a subvariety of \mathcal{M}_g and should not be confused with double ramification cycles, which are classes in the Chow ring of $\overline{\mathcal{M}}_{g,n}$ and will not play a part in the sequel.

In 2001 Eliashberg posed the problem of extending double ramification cycles to $\overline{\mathcal{M}}_{g,n}$ for the development of symplectic field theory [EGH00]. Since then various different ways of extending the cycle class of $\text{DR}_g(\mu)$ to $\overline{\mathcal{M}}_{g,n}$ have been studied in the literature. One possible extension is via relative Gromov-Witten theory using the space of rubber maps to \mathbb{P}^1 , see [Li01, Li02, GV05]. A different approach to extending double ramification loci is to extend the Abel-Jacobi map. The Abel-Jacobi naturally extends to curves of compact types.

The corresponding cycle class was computed by Hain in [Hai13] and the computation was later simplified in [GZ14]. However, in general the Abel-Jacobi map does not extend to all of $\overline{\mathcal{M}}_{g,n}$; nonetheless different extensions of double ramification loci to $\overline{\mathcal{M}}_{g,n}$ using the Abel-Jacobi map have been proposed, see for example [HKP18, Hol19, MW17]. Another extension has been proposed in [FP18]. The natural cycle classes on $\overline{\mathcal{M}}_{g,n}$ that arises in these approaches are called double ramification cycles and a conjectural formula for the class in the tautological ring of $\overline{\mathcal{M}}_{g,n}$ was proposed by Pixton and later proved in [JRPZ17]. We stress that the proposed extensions do not coincide with the cycle class of the closure of $\text{DR}_g(\mu)$ in $\overline{\mathcal{M}}_{g,n}$ but rather contain additional contributions from the boundary.

One way of finding a geometric description of the closure of $\text{DR}_g(\mu)$ is using the theory of admissible covers, first introduced by [HMS2]. The space of admissible covers $\text{Adm}_{g,n}(\sigma)$ is a proper Deligne-Mumford stack compactifying the space of maps $X \rightarrow \mathbb{P}^1$ with a fixed ramification profile σ . Contributions to the intersection theory of admissible cover have been made in [SvZ20, Lia20].

We will recall admissible covers in more detail in Section 5. Using the stack of admissible covers $\text{Adm}_{g,n}(\sigma)$ a description of the boundary of double ramification loci is as follows (we state it more precisely in Proposition 5.2): a stable curve $(X, \mathbf{x}) \in \overline{\mathcal{M}}_{g,n}$ is contained in the closure $\overline{\text{DR}}_g(\mu)$ of the double ramification locus if and only if there exists a suitable admissible cover on a semistable model X' of X . While certainly geometric, the existence of an admissible cover on a semistable model is hard to verify in practice due to the combinatorial complexity. In particular it would be useful to have a description solely in terms of the given stable curve X , which is the goal of this paper.

To obtain a more explicit geometric criterion in this paper we take a different approach, similar in spirit to the approach for $k \geq 1$ in [BCGGM18, BCGGM19a] by relating double ramification loci to strata of meromorphic differentials. By associating to a rational function $f : X \rightarrow \mathbb{P}^1$ the exact differential df we can use the moduli space of multi-scale differentials $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$, constructed in [BCGGM19b], to understand degenerations of df . This gives a new way of relating double ramification loci to strata of differentials. Since exact differentials are described by the vanishing of all absolute periods, this requires an analysis of periods near the boundary of $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$. The results obtained in [Ben20] and further refined in [BDG20] describe the boundary of subvarieties of strata of meromorphic differentials given by linear equations on periods in $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$, and can thus be applied to the case of exact differentials. The idea of studying Hurwitz spaces via exact differentials has also appeared in [Sav17, Sec. 4] and [Mul20].

We now define the key notions, which will allow us to state the main result.

TWISTABLE RATIONAL FUNCTIONS

Let Γ be the dual graph of a marked stable curve. We consider Γ as a triple (V, E, H) where V are the vertices, E are the edges and H are the *legs* cor-

responding to the marked points. We denote h_k the leg corresponding to the marked point x_k .

Recall from [BCGGM18] that a *level graph* $\bar{\Gamma}$ is a dual graph Γ of a stable curve together with a level function $\ell : V(\Gamma) \rightarrow \mathbb{Z}_{\leq 0}$. A level function induces a partial order on the set of vertices by setting $v \succcurlyeq v'$ if $\ell(v) \geq \ell(v')$. Here we understand partial ordering in a weak sense and allow $\ell(v) = \ell(v')$ even if $v \neq v'$.

DEFINITION 1.1. Let μ be a partition of zero as above, and $\bar{\Gamma}$ a level graph. A *twistable rational function* (X, \mathbf{x}, f) of type μ compatible with $\bar{\Gamma}$ on a stable pointed curve (X, \mathbf{x}) is a collection $f = (f_v)_{v \in V(\Gamma)}$ such that $f_v : \tilde{X}_v \rightarrow \mathbb{P}^1$ is a rational function on the normalization \tilde{X}_v of each irreducible component X_v of X , satisfying

1. (PRESCRIBED ORDER OF VANISHING) Each rational function f_v is holomorphic away from the marked points and nodes. Furthermore,

$$\text{ord}_{x_k} f = \mu_k \text{ for all } k.$$

If a component X_v contains any marked zero x_k of f_v , then all zeros of f_v are at marked points or nodes.

2. (MATCHING ORDER AT NODES) If a node of X identifies $q_1 \in \tilde{X}_{v_1}$ with $q_2 \in \tilde{X}_{v_2}$. Then

$$\text{ord}_{q_1}(df_{v_1}) + \text{ord}_{q_2}(df_{v_2}) \geq -2. \quad (1.1)$$

3. (COMPATIBILITY WITH THE LEVEL GRAPH) If furthermore q_2 is a pole of f_{v_2} , i.e. $\text{ord}_{q_2} f < 0$, then $\ell(v_1) > \ell(v_2)$.

Some remarks about this definition are in order. We note that condition (2) is vacuous if $\text{ord}_{q_1} f \geq 0$ and $\text{ord}_{q_2} f \geq 0$. If $\text{ord}_{q_2} f < 0$ then it can be rephrased as

$$\text{mult}_{q_1} f_{v_1} \geq \text{mult}_{q_2} f_{v_2},$$

where $\text{mult}_q f_v - 1$ is the ramification index of the map $f_v : X_v \rightarrow \mathbb{P}^1$ at q .

Remark 1.2. We stress that while all poles of f are either at a marked point \mathbf{x} or at a node, there can be zeros of f which are not marked and not at a node. There are thus two kinds of irreducible components depending on whether or not they contain marked zeros of f_v . On an irreducible component containing marked zeros, the pointed curve $(\tilde{X}_v, (\mathbf{x}_v, \mathbf{q}_v))$, where \mathbf{x}_v and \mathbf{q}_v are all the marked points and nodes contained in \tilde{X}_v , is contained in a double ramification locus $\text{DR}_{g(X_v)}(\mu_v)$ for a partition μ_v depending only on μ and the level graph $\bar{\Gamma}$ and the orders of vanishing of f at the nodes. On the remaining components, which do not contain any marked zero of f_v , we instead have a rational function with conditions on the ramification indices at the nodes.

1.1 THE EVALUATION MORPHISM OF A TWISTABLE RATIONAL FUNCTION

The definition of twistable rational functions so far is similar to the definition of a twisted differential in the sense of [BCGGM18], except that in (2) we only have inequality. We now introduce the crucial new ingredient that will allow us to determine whether a twistable rational function can be smoothed to a rational function contained in a double ramification locus.

Let (X, \mathbf{x}, f) be a twistable rational function of type μ compatible with a level graph $\bar{\Gamma}$. We partition $\mathbf{x} = (\mathbf{z}, \mathbf{p})$ such that \mathbf{z} consists of zeros of f and \mathbf{p} of poles. From now on we usually assume that $\mu_i \neq 0$ at all marked points. All our results can be extended to the case where $\mu_i = 0$ at some marked point by including all marked points with $\mu_i = 0$ in \mathbf{z} .

We consider the dual graph $\bar{\Gamma}$ with legs corresponding to the marked points \mathbf{x} as a 1-dimensional cell complex. By abuse of notation we call x_k the endpoint of the leg $h(x_k)$ and we can thus consider the marked points \mathbf{x} as a subset of $\bar{\Gamma}$. For a given level $i \in \mathbb{Z}_{\leq 0}$ we denote $\bar{\Gamma}_{(\leq i)}$ the subgraph consisting of all vertices v such that $\ell(v) \leq i$, together with all edges between them.

We define $\bar{\Gamma}_{(i)}$ similarly, except that for every edge e connecting a vertex v of level i and a vertex of level lower than i , we attach an additional leg $h(q_e^+)$ to v . Pictorially, we cut each edge connecting a vertex of level i to a vertex of lower levels in the middle. If we denote $\bar{\Gamma}$ some level graph as shown on the left in Figure 1 then $\bar{\Gamma}_{(0)}$ is the graph on the right, consisting of a single vertex and two newly attached half-legs.

We say a relative homology class $[\gamma] \in H_1(\bar{\Gamma}, \mathbf{z}; \mathbb{Z})$ has *top level* at most i if it can be represented by a collection of paths γ contained in $\bar{\Gamma}_{(\leq i)}$. We then let $L_{\leq i}(\bar{\Gamma}) \subseteq H_1(\bar{\Gamma}, \mathbf{z}; \mathbb{Z})$ be the subspace of all homology classes of level at most i . We can now describe the *evaluation morphism*

$$\text{ev}_f^{(i)} : L_{\leq i}(\bar{\Gamma}) \rightarrow \mathbb{C}$$

of level i as follows. Suppose that $[\gamma] \in L_{\leq i}(\bar{\Gamma})$ is represented by a collection of paths γ contained in $\bar{\Gamma}_{(\leq i)}$. We then restrict γ to $\bar{\Gamma}_{(i)}$ and evaluate f , with signs, at the endpoints of the restriction. For example, if the restriction of γ connects q_1^+ and q_2^+ , as depicted in Figure 1, then we have

$$\text{ev}_f^{(0)}([\gamma]) = f(q_2^+) - f(q_1^+),$$

and if the restriction of γ is a closed path then we have $\text{ev}_f^{(i)}([\gamma]) = 0$. We will revisit the evaluation map in more detail in Section 2.1 and in particular check that it is well-defined in Proposition 2.3. We can now state our main result.

MAIN THEOREM. *Let (X, \mathbf{x}) be a stable curve in $\overline{\mathcal{M}}_{g,n}$ with dual graph Γ . Then (X, \mathbf{x}) is contained in the closure of $\text{DR}_g(\mu)$ if and only if the following conditions are satisfied:*

1. *There exists a level graph structure $\bar{\Gamma}$ on Γ and a twistable rational function (X, \mathbf{x}, f) of type μ compatible with $\bar{\Gamma}$.*

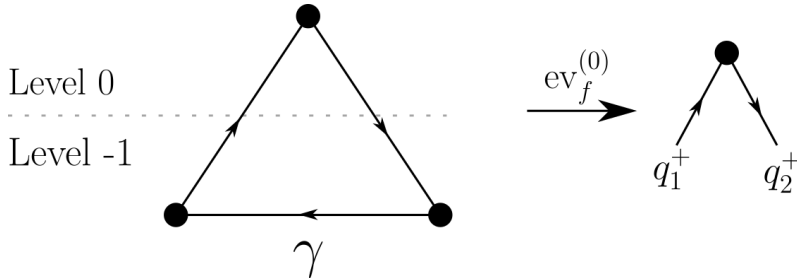


Figure 1: The evaluation map

2. The evaluation morphism $\text{ev}_f^{(i)}$ vanishes identically for all levels i .

For example for some level graph as in Figure 1 the only closed path we need to consider is γ and

$$\text{ev}_f^{(0)}([\gamma]) = f(q_2^+) - f(q_1^+).$$

Condition (2) says exactly that the values of the twistable rational function f at q_1^+ and q_2^+ are the same,

$$f(q_1^+) = f(q_2^+).$$

In general one needs to find a basis for the absolute homology of the level graph and then one obtains a relation for every basis element.

Being a twistable rational function with vanishing evaluation morphism is a condition that can be verified explicitly. On each irreducible component one has a rational function satisfying a condition similar to double ramification loci, and the vanishing of the evaluation morphism is a combinatorial condition relating the values of the rational functions on different irreducible components. In Section 6 we discuss several examples of twistable rational functions highlighting the features of the evaluation morphism, showing the explicit nature of twistable rational functions. The Abel-Jacobi map extends to curves of compact type and thus degenerations of the double ramification locus in this case are well understood. Furthermore, in [GZ14] the authors extend the Abel-Jacobi map to curves with one non-separating node and use it to describe the closure of the double ramification locus. Our techniques allow us to describe arbitrary degenerations. As a demonstration, going beyond what has been studied in the literature, in Example 6.5 we study twistable rational functions on dollar curves, i.e. stable curves with two irreducible components meeting transversely at three nodes. We discuss the possible level graphs and the different types of conditions that the vanishing of the evaluation morphism imposes. At first glance the notion of a twistable rational function is very similar to the notion of twisted differentials in [FP18] and [BCGGM18]. A twisted differential is a collection of meromorphic differential forms on each irreducible component. In particular, it determines a unique partial order on the dual

graph by comparing the vanishing orders at the nodes. In [BCGGM18] the authors introduce the global residue condition for twisted differentials, which is the crucial ingredient that allows to smooth out a twisted differential. The global residue condition does depend not only the partial ordering but on the actual level graph, as can be seen in [BCGGM18, Ex 3.4]. The situation is somewhat different for twistable rational functions. In Example 6.3 we will see that a twistable rational function does not determine a unique partial ordering, i.e. there exist two different partial orders on the dual graph of the same underlying twistable rational function both satisfying Condition (3) in Definition 1.1. The level graph only comes into play when considering the evaluation morphism. There are several cases where a fixed partial ordering on the dual graph of a twistable rational function uniquely determines the level graph (up to automorphism) and thus also determines the conditions imposed by the vanishing of the evaluation morphism, for example if every irreducible component contains a marked zero of f . On the other hand in Example 6.4 we construct an example of two different level graphs, inducing the same partial ordering, for which the vanishing of the evaluation morphism imposes different conditions.

Remark 1.3. (Prestable and stable curves) For technical reasons, we will have to additionally mark the critical points of the rational functions during the proof of the Main theorem. By forgetting the marked critical points we are then naturally forced to consider prestable models of a given stable curves. A combinatorial observation (Lemma 3.3) allows us to state our results only in terms of the stable curve without alluding to a prestable model. On the other hand, using admissible covers one can show that a stable curve X lies in the boundary of $\overline{\mathrm{DR}}_g(\mu)$ if there exists an admissible cover on a semistable model of X and it seems hard in general to provide a criterion for the existence of such admissible cover only involving the stable curve X .

POTENTIAL APPLICATIONS

The determination of the closure of strata inside the Hodge bundle via twisted differentials in [BCGGM18] allowed to study the geometry of strata of differentials with algebro-geometric methods. For example, in [Gen15] the author obtains first information about the Kodaira dimension of strata using twisted differentials. In [MUW17] the authors solve the realization problem for tropical canonical divisors using the smoothability of twisted differentials. Using the smoothability of twistable rational functions one can hope to study similar questions for the double ramification locus using differentials.

In forthcoming work we use the view point of exact differentials to study admissible cover cycles $[\overline{\mathrm{DR}}_g(\mu)] \in \mathrm{CH}^*(\overline{\mathcal{M}}_{g,n})$. The cycle class $[\overline{\mathrm{DR}}_g(\mu)]$ is obtained as pushforward of the locus of exact differentials inside the moduli space of multi-scale differentials and thus computing the class of the locus of exact differentials allows a new approach to the computation of $[\overline{\mathrm{DR}}_g(\mu)]$. The locus of exact differentials is described by the vanishing of all absolute periods

and can be realized as the zero locus of a vector bundle on the moduli space of multi-scale differentials $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$, which allows class computations via intersection theory.

By taking the closure of a double ramification locus or more generally of a Hurwitz space of covers of \mathbb{P}^1 inside the moduli space of multi scale differentials one can produce smooth compactifications as we have seen in [BDG20]. It follows from our discussion in Section 5.1 that this yields birational models different from stacks of admissible covers and it seems worthwhile to investigate the relationship between the two.

OUTLINE OF THE PROOF

The key idea to prove the **Main theorem** is to consider exact differentials instead of rational functions. On the double ramification locus $\text{DR}_g(\mu)$ only the zeros and poles of f are marked, but not the remaining zeros of df , which correspond to the critical points of f . In Section 2, we construct a rigidification of $\text{DR}_g(\mu)$, which we denote $\text{DR}_g^c(\tilde{\mu})$, by additionally marking the critical points. The advantage of using $\text{DR}_g^c(\tilde{\mu})$ is that the associated exact differential is contained in a fixed stratum. Furthermore, being an exact differential is equivalent to the vanishing of all absolute periods, a condition that is linear in period coordinates of the stratum. We can then describe the boundary of $\text{DR}_g^c(\tilde{\mu})$ in terms of *twisted rational functions* using the main result from [Ben20] and some further consequences from [BDG20]. A twisted rational function is a twistable rational function where we additionally mark all the critical points of the rational function, and such that the associated exact differential is a twisted differential in the sense of [BCGGM18].

In Proposition 2.4 we then show that the boundary of $\text{DR}_g^c(\tilde{\mu})$ can be described in terms of twisted rational functions with vanishing evaluation morphism.

So far we have artificially marked the critical points of f and now we need to forget them again. It then only remains to study the relationship between twisted rational functions and twistable rational functions. In Section 3 we show that twistable rational functions arise exactly by forgetting the critical points of a twisted rational function and then stabilizing the resulting prestable curve, which is then enough to prove the main theorem.

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2 EXACT DIFFERENTIALS AND TWISTED RATIONAL FUNCTIONS

LEVEL GRAPHS

We recall some additional notation for level graphs from [BCGGM18]. An edge of $\bar{\Gamma}$ is called *horizontal* if it joins vertices of the same level, and *vertical* otherwise. If an edge e joins the vertices v and v' such that $\ell(v) \geq \ell(v')$ then we let $\ell(e+)$ be the level of v and similarly $\ell(e-)$ the level of v' . Furthermore we set $v(e+) := v$ and $v(e-) := v'$. Similarly, we denote q_e^+ and q_e^- the preimages of the node e on $v(e+)$ and $v(e-)$ respectively. At horizontal nodes we make a random choice. Similarly, for a leg h we let $v(h)$ be the vertex connected to h and $\ell(h)$ be the level of $v(h)$.

MODULI SPACES OF EXACT DIFFERENTIALS

Let X be a stable curve with dual graph Γ and $f = (f_v)_{v \in V(\Gamma)}$ a collection of rational functions on the irreducible components of X . To f we can associate the collection of meromorphic differentials $df = (d(f_v))_{v \in V(\Gamma)}$. We have

$$\text{ord}_x(df) = \begin{cases} \text{mult}_x f - 1 & \text{if } x \notin f^{-1}(\infty) \\ -\text{mult}_x f - 1 & \text{if } x \in f^{-1}(\infty). \end{cases} \tag{2.1}$$

where $\text{mult}_x f$ is the ramification index of the map $f_v : X_v \rightarrow \mathbb{P}^1$ at x . To study degenerations of f we want to use the compactification of strata of meromorphic differentials given by multi-scale differentials. In order to ensure that df is contained in a fixed stratum we need to control the zero and pole order of f as well as the ramification.

We define the projectivized Hodge bundle

$$\mathbb{P}(E) := \mathbb{P} \left(\pi_* \omega_{\mathcal{C}_{g,n}/\mathcal{M}_{g,n}} \left(\sum_{\mu_k < 0} (-\mu_k x_k) \right) \right),$$

where $\pi : \mathcal{C}_{g,n} \rightarrow \mathcal{M}_{g,n}$ denotes the universal curve. Since exact differentials are exactly the meromorphic differentials with zero absolute periods, we can identify $\text{DR}_g(\mu)$ with a locally closed subset of $\mathbb{P}(E)$ as follows. We define a new partition μ' by setting $\mu'_k := \mu_k - 1$ and call μ' the partition *associated* to μ . Then the locally closed subset

$$\widetilde{\text{DR}}_g(\mu) := \left\{ (X, \mathbf{x}, \omega) : \int_{\gamma} \omega = 0 \forall \gamma \in H_1(X \setminus \mathbf{p}, \mathbf{z}; \mathbb{Z}), \text{ord}_{x_k} \omega = \mu'_k \text{ for all } k \right\}$$

of $\mathbb{P}(E)$ is isomorphic to $\text{DR}_g(\mu)$.

MARKING THE CRITICAL POINTS

We will need analogs of $\widetilde{\text{DR}}_g(\mu)$ where all critical points are marked and the zero order of df is prescribed at all zeros, nodes and critical points of f . Let

(X, \mathbf{x}, f) be a twistable rational function. We denote \mathbf{c} the remaining critical points of f that are neither zeros, poles or nodes, and set

$$\tilde{\mathbf{x}} = (\mathbf{z}, \mathbf{p}, \mathbf{c}).$$

We stress that a marked point is always a smooth point of \tilde{X} , thus \mathbf{x} consists of zeros and poles of f that are not nodes, and \mathbf{c} consists of all critical points of \tilde{f} that are not zeros or poles of f or nodes.

We fix a partition μ of length n . We say a partition $\tilde{\mu}$ of $2g - 2$ extends μ if it can be written as $\tilde{\mu} = (\mu', \mu'')$ such that μ' is the partition associated to μ and all entries of μ'' are positive. We denote the length of $\tilde{\mu}$ by n' . Let

$$\widetilde{\text{DR}}_g^c(\tilde{\mu}) := \left\{ (X, \tilde{\mathbf{x}}, \omega) : \int_{\gamma} \omega = 0 \forall \gamma \in H_1(X \setminus \mathbf{p}, \mathbf{z}; \mathbb{Z}), \text{ord}_{\tilde{x}_k} \omega = \tilde{\mu}_k \text{ for all } k \right\}$$

We similarly let $\text{DR}_g^c(\tilde{\mu})$ be the image of $\widetilde{\text{DR}}_g^c(\tilde{\mu})$ under the forgetful map $\mathbb{P}\mathcal{H}(\tilde{\mu}) \rightarrow \mathcal{M}_{g,n'}$. We summarize the relationship of all the different moduli spaces in the following diagram. Adding a tilde means that we mark a differential form, while adding the superscript c indicates that we mark *all* critical points of the rational function or equivalently all zeros and poles of the exact differential form.

$$\begin{array}{ccc} \widetilde{\text{DR}}_g(\mu) & \longleftarrow & \widetilde{\text{DR}}_g^c(\tilde{\mu}) \\ \downarrow & & \downarrow \\ \text{DR}_g(\mu) & \longleftarrow & \text{DR}_g^c(\tilde{\mu}) \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,n} & \longleftarrow & \overline{\mathcal{M}}_{g,n'} \end{array}$$

We now define a notion analogous to twistable rational function, taking into account also the total ramification of f .

DEFINITION 2.1. Let $\tilde{\mu}$ be a partition extending μ . A pointed stable curve $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ with a collection $d\tilde{f} = (df_v)_{v \in V}$ of rational functions is called a *twisted rational function* if $d\tilde{f}$ is a twisted differential of type $\tilde{\mu}$ compatible with $\bar{\Gamma}$ in the sense of [BCGGM18].

Let us unravel this definition. First, note that $d\tilde{f}$ cannot have simple poles and thus $\bar{\Gamma}$ cannot have any horizontal edges. Therefore the matching residue condition is vacuous. Furthermore, since $d\tilde{f}$ is exact, all of its periods over absolute homology classes are zero and thus it has no residues at higher order poles either. In particular the global residue condition is also vacuous. Thus equivalently we can define a twisted rational function as follows.

DEFINITION 2.2. Let $\tilde{\mu}$ be a partition extending μ . We call a pointed stable curve $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ together with a collection of rational functions and a decomposition $\tilde{\mathbf{x}} = (\mathbf{x}, \mathbf{c})$ a *twisted rational function* of type $\tilde{\mu}$ compatible with $\bar{\Gamma}$ if the following conditions are satisfied.

1. (PRESCRIBED ORDER OF VANISHING) Each rational function \tilde{f}_v is holomorphic away from the marked points and nodes. If some marked section x_k is a zero for \tilde{f}_v , then \tilde{f}_v is non-zero away from the marked points and nodes. Furthermore, $\text{ord}_{\tilde{x}_k}(d\tilde{f}) = \tilde{\mu}_k$ for all k .
2. (MATCHING ORDER AT NODES) Suppose a node of X identifies $q_1 \in \tilde{X}_{v_1}$ with $q_2 \in \tilde{X}_{v_2}$, then

$$\text{ord}_{q_1}(d\tilde{f}_{v_1}) + \text{ord}_{q_2}(d\tilde{f}_{v_2}) = -2$$

3. (COMPATIBILITY WITH THE LEVEL GRAPH) Furthermore, $\ell(v_1) > \ell(v_2)$ if and only if $\text{ord}_{q_2}(d\tilde{f}_{v_2}) < 0$.

We call \mathbf{x} the *marked zeros and poles* of \tilde{f} and \mathbf{c} the *remaining critical points*. Sometimes we want to decompose further and write $\mathbf{x} = (\mathbf{z}, \mathbf{p})$ where \mathbf{z} and \mathbf{p} are zeros and poles of \tilde{f} , respectively. We stress that, as in the definition of twistable rational function, not all zeros of \tilde{f} have to be marked points in \mathbf{x} or nodes, since there can be irreducible components not containing any marked point of \mathbf{x} .

The reason for using twisted rational functions instead of twistable rational functions is that we can use the smoothing results from [Ben20, BDG20].

2.1 THE EVALUATION MORPHISM

Let $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ be a twisted rational function with marked points $\tilde{\mathbf{x}} = (\mathbf{z}, \mathbf{p}, \mathbf{c})$. The embedding $(\bar{\Gamma}_{(\leq i)}, \mathbf{z}) \rightarrow (\bar{\Gamma}, \mathbf{z})$ induces a natural map

$$H_1(\bar{\Gamma}_{(\leq i)}, \mathbf{z}; \mathbb{Z}) \rightarrow H_1(\bar{\Gamma}, \mathbf{z}; \mathbb{Z}).$$

We define the *level filtration* $L_\bullet(\bar{\Gamma})$ of $H_1(\bar{\Gamma}, \mathbf{z}; \mathbb{Z})$ by

$$L_{\leq i}(\bar{\Gamma}) := \text{Im} (H_1(\bar{\Gamma}_{(\leq i)}, \mathbf{z}; \mathbb{Z}) \rightarrow H_1(\bar{\Gamma}, \mathbf{z}; \mathbb{Z})) \tag{2.2}$$

and we say that a cycle $[\gamma] \in L_{\leq i}(\bar{\Gamma}) \setminus L_{\leq i-1}(\bar{\Gamma})$ has *top level i* .

We are now going to define the evaluation morphism

$$\text{ev}_{\tilde{f}}^{(i)} : L_{\leq i}(\bar{\Gamma}) \rightarrow \mathbb{C}.$$

We recall that $\bar{\Gamma}_{(i)}$ has additional legs $h(q_e^+)$ corresponding to nodes with $\ell(e+) = i, \ell(e-) < i$. By embedding the leg $h(q_e^+)$ into the edge e we can consider $\bar{\Gamma}_{(i)}$ as a subspace of $\bar{\Gamma}_{(\leq i)}$. We stress that this is a continuous map of cell complexes and not a graph morphism.

The idea to construct $\text{ev}_{\tilde{f}}^{(i)}$ is to take a path in $\bar{\Gamma}_{(\leq i)}$ and restrict it to $\bar{\Gamma}_{(i)}$.

Let $[\gamma] \in L_{\leq i}(\bar{\Gamma})$ be represented by a collection of smooth paths γ which are contained in $\bar{\Gamma}_{(\leq i)}$. The restriction of γ to $\bar{\Gamma}_{(i)}$ decomposes into a sum of disjoint paths $\alpha_1, \dots, \alpha_m$ which are either closed or have endpoints at the marked zeros

or preimages of nodes $\mathbf{z} \cup \mathbf{q}_{(i)}^+$. Here $\mathbf{q}_{(i)}^+$ are the half-legs corresponding to nodes with $\ell(e+) = i, \ell(e-) < i$. We then define

$$\text{ev}_{\tilde{f}}^{(i)} := \sum_{k=1}^m \tilde{f}(\alpha_k(1)) - \tilde{f}(\alpha_k(0)).$$

If all α_k are closed paths this is automatically zero. Note that it follows from condition (3) in the definition of twisted rational function that $\tilde{f}(q_e^+)$ is always finite.

For a twistable rational function (X, \mathbf{x}, f) we define $\text{ev}_f^{(i)}$ in the same way, where we decompose $\mathbf{x} = (\mathbf{z}, \mathbf{p})$.

PROPOSITION 2.3. *Suppose \tilde{f} is a twistable rational function or a twisted rational function, then the map $\text{ev}_{\tilde{f}}^{(i)} : L_{\leq i}(\bar{\Gamma}) \rightarrow \mathbb{C}$ is well-defined.*

Proof. The proof is similar to [Ben20, Prop. 4.2] for 2-dimensional cell complexes instead of graphs. From the long exact sequence of the triple

$$\mathbf{z} \subseteq \bar{\Gamma}_{(\leq i-1)} \cup \mathbf{z} \subseteq \bar{\Gamma}_{(\leq i)} \cup \mathbf{z}$$

we obtain the exact sequence

$$\begin{array}{ccccccc} H_1(\bar{\Gamma}_{(\leq i-1)}, \mathbf{z}; \mathbb{Z}) & \longrightarrow & H_1(\bar{\Gamma}_{(\leq i)}, \mathbf{z}; \mathbb{Z}) & \xrightarrow{\nu_i} & H_1(\bar{\Gamma}_{(\leq i)}, \bar{\Gamma}_{(\leq i-1)} \cup \mathbf{z}; \mathbb{Z}) & & \\ & & & & \downarrow \simeq & & \\ & & & & H_1(\bar{\Gamma}_{(i)}, \mathbf{q}_{(i)}^+ \cup \mathbf{z}; \mathbb{Z}) & & \end{array}$$

where the vertical isomorphism is induced by excising $\bar{\Gamma}_{(\leq i-1)}$. We denote by

$$g_i : H_1(\bar{\Gamma}_{(\leq i)}, \mathbf{z}; \mathbb{Z}) \rightarrow H_1(\bar{\Gamma}_{(i)}, \mathbf{q}_{(i)}^+ \cup \mathbf{z}; \mathbb{Z})$$

the composition of ν_i with the excision map. For a path γ in $\bar{\Gamma}_{(\leq i)}$ the map g_i is given by $g_i(\gamma) = \sum_k \alpha_k$ where $\sum_k \alpha_k$ is the restriction of γ to $\bar{\Gamma}_{(i)}$, considered as paths with endpoints in $\mathbf{z} \cup \mathbf{q}_{(i)}^+$. Furthermore, there exists a boundary map

$$\delta_i : H_1(\bar{\Gamma}_{(i)}, \mathbf{z} \cup \mathbf{q}_{(i)}^+; \mathbb{Z}) \rightarrow H_0(\mathbf{z} \cup \mathbf{q}_{(i)}^+; \mathbb{Z})$$

from the long exact sequence of the pair $(\bar{\Gamma}_{(i)}, \mathbf{z} \cup \mathbf{q}_{(i)}^+)$ which sends a path α to $[\alpha(1)] - [\alpha(0)]$. We define

$$\begin{aligned} \text{ev} : H_0(\mathbf{z} \cup \mathbf{q}^+; \mathbb{Z}) &\rightarrow \mathbb{C}, \\ \sum_k c_k [z_k] + \sum_e d_e [q_e^+] &\mapsto \sum_k c_k \tilde{f}(z_k) + \sum_e d_e \tilde{f}(q_e^+). \end{aligned}$$

Then we have

$$\text{ev}_{\tilde{f}}^{(i)} = \text{ev} \circ \delta_i \circ g_i$$

and thus $\text{ev}_{\tilde{f}}^{(i)}$ is well-defined. □

We are now able to characterize the closure of $\text{DR}_g^c(\tilde{\mu}) \subseteq \overline{\mathcal{M}}_{g,n'}$ using the results of [Ben20].

PROPOSITION 2.4. *A marked stable curve $(\tilde{X}, \tilde{\mathbf{x}})$ is contained in the closure of $\text{DR}_g^c(\tilde{\mu})$ in $\overline{\mathcal{M}}_{g,n'}$ if and only if there exists a level graph $\bar{\Gamma}$ for \tilde{X} and a collection of rational functions \tilde{f} on \tilde{X} such that $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ is a twisted rational function compatible with $\bar{\Gamma}$ and $\text{ev}_{\tilde{f}}(i) = 0$ for all i .*

Proof. Let $(\tilde{X}, \tilde{\mathbf{x}})$ be in the closure of $\text{DR}_g^c(\tilde{\mu})$. Then there exists a level graph $\bar{\Gamma}$ on \tilde{X} and a twisted differential η on \tilde{X} compatible with $\bar{\Gamma}$ such that $(\tilde{X}, \tilde{\mathbf{x}}, \eta)$ is in the closure of $\widetilde{\text{DR}}_g^c(\tilde{\mu})$ inside $\Xi\mathcal{M}_{g,n'}(\tilde{\mu})$. Note that η cannot have residues, since every residue is the limit of an absolute period $\int_{\gamma} \omega$ for $\gamma \in H_1(X \setminus \mathbf{p}; \mathbb{Z})$ along a family of differentials (X, ω) degenerating to (\tilde{X}, η) . In particular, $\bar{\Gamma}$ has no horizontal nodes.

The subvariety $\widetilde{\text{DR}}_g^c(\tilde{\mu}) \subseteq \mathbb{P}\mathcal{H}(\tilde{\mu})$ is a linear subvariety cut out by the equations

$$\int_{\gamma} \omega = 0 \quad \text{for all } \gamma \in H_1(X_0 \setminus \mathbf{p}_0, \mathbf{z}_0; \mathbb{Z}),$$

near a point $(X_0, \tilde{\mathbf{x}}_0, \omega) \in \widetilde{\text{DR}}_g^c(\tilde{\mu})$. We have

$$T_{(X_0, \tilde{\mathbf{x}}_0, \omega)} \widetilde{\text{DR}}_g^c(\tilde{\mu}) = \text{Ann}(H_1(X_0 \setminus \mathbf{p}_0, \mathbf{z}_0; \mathbb{Z})).$$

Thus by [Ben20, Prop. 10.1] the intersection of $\widetilde{\text{DR}}_g^c(\tilde{\mu})$ with the open boundary stratum corresponding to $\bar{\Gamma}$ is a linear subvariety given by $\text{Ann}\left(\bigoplus_{i=-L(\bar{\Gamma})}^0 \text{Im } \tilde{g}_i\right)$. Here the map $\tilde{g}_i : H_1(X_{(\leq i)} \setminus \mathbf{p}, \mathbf{z}; \mathbb{Z}) \rightarrow H_1(X_{(i)} \setminus \mathbf{p}, \mathbf{z} \cup \mathbf{q}_{(i)}^+; \mathbb{Z})$ can be described as follows. For a simple closed curve $\gamma \in X_{(\leq i)}$, we have $\tilde{g}_i(\gamma) = \sum_k \alpha_k$ where $\sum_k \alpha_k$ is the restriction of γ to $X_{(i)}$, considered as paths with endpoints in $\mathbf{z} \cup \mathbf{q}_{(i)}^+$. In particular we have $\int_{\gamma} \eta = 0$ for any cycle contained in $X_v \setminus \mathbf{q}$ and thus η is a collection of exact differentials. Thus there exists a collection of rational functions \tilde{f} on \tilde{X} such that $d\tilde{f} = \eta$. Furthermore for any $\gamma \in H_1(X \setminus \mathbf{p}, \mathbf{z})$ we have

$$0 = \int_{\tilde{g}_i(\gamma)} \eta = \int_{\tilde{g}_i(\gamma)} d\tilde{f} = \sum_{k=1}^m \int_{\alpha_k} d\tilde{f} = \sum_{k=1}^m \tilde{f}(\alpha_k(1)) - \tilde{f}(\alpha_k(0)) = \tilde{\text{ev}}_{\tilde{f}}^{(i)}(\gamma).$$

It remains to show that if an irreducible component X_v contains a marked zero z_k , then \tilde{f}_v is non-zero away from the marked points \mathbf{x} and nodes. We fix a marked zero z_0 of \tilde{f}_v and let z' be a zero on the same component that is not a node and is not marked. Note that z' is necessarily a simple zero and we can thus construct a holomorphic section marking $z'(t)$ along a family $(X(t), \omega(t))$ degenerating to (\tilde{X}, η) . Suppose that $(X(t), \omega(t))$ is a holomorphic family of exact differentials degenerating to η . Then

$$\int_{z_0(t)}^{z'(t)} \omega(t) = t \cdot h(t)$$

where $h(t)$ is analytic in a neighborhood of the origin and thus the corresponding rational function $f(t)$ on $X(t)$ has a zero in a small neighborhood of $z'(t)$, disjoint from all marked zeros of $f(t)$. But since on $X(t)$ all zeros of \tilde{f} are marked, this is a contradiction. This shows that $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ is a twisted rational function compatible with $\bar{\Gamma}'$.

It remains to show that we can smooth out a twisted rational function, while staying inside a double ramification locus. Let $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ be a twisted rational function with vanishing evaluation morphism and $\eta = d\tilde{f}$ the associated twisted differential. We denote $D \subseteq \Xi\overline{\mathcal{M}}_{g,n}(\mu)$ the boundary of the moduli space of multi-scale differentials. Let U be a small open neighborhood of $(\tilde{X}, \tilde{\mathbf{x}}, \eta)$ in $\Xi\overline{\mathcal{M}}_{g,n}(\mu)$. On $U \setminus D$ we can define a local analytic subvariety by

$$Z := \left\{ (X, \omega) : \int_{\gamma} \omega = 0 \quad \forall \gamma \in H_1(X \setminus \mathbf{p}, \mathbf{z}; \mathbb{Z}) \right\} \subseteq U \setminus D.$$

By [BDG20, Cor 4.3], Z is smooth and transverse to the boundary and we can thus smooth out $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$. Note that even though Z is only defined locally, the proof of (loc. cit.) applies verbatim in this situation. Alternatively, the smoothness of Z also follows from the proof of [Ben20, Prop 8.9]. \square

3 TWISTING A TWISTABLE RATIONAL FUNCTION

So far we have obtained a smoothing result for twisted rational function and now we need to transfer it to a smoothing result for twistable rational functions. In this section we thus collect various results relating twistable rational functions and twisted rational functions.

DEFINITION 3.1. Let $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ be a twisted rational function of type $\tilde{\mu}$ compatible with $\bar{\Gamma}'$, where $\tilde{\mu}$ extends μ . We recall that $\tilde{\mathbf{x}} = (\mathbf{x}, \mathbf{c})$ consists of some marked zeros and poles of \tilde{f} as well as the remaining critical points, that are neither zeros, poles of \tilde{f} or nodes. In particular we can consider \mathbf{x} as a subset of $\tilde{\mathbf{x}}$. We consider the prestable curve (\tilde{X}, \mathbf{x}) where we forget the remaining critical points and let (X, \mathbf{x}) be its stabilization. Here we abused notation and denote the images of \mathbf{x} under the stabilization map $\tilde{X} \rightarrow X$ again by \mathbf{x} . The stabilization induces a graph morphism $\phi : \Gamma' \rightarrow \Gamma$ contracting edges that correspond to unstable chains of rational curves. The dual graph Γ inherits a level graph structure $\bar{\Gamma}$ from $\bar{\Gamma}'$ using the stabilization morphism ϕ . More concretely, for a vertex of $\bar{\Gamma}'$ corresponding to a stable component we define $\ell(\phi(v)) := \ell'(v)$, where ℓ and ℓ' are the level functions of $\bar{\Gamma}$ and $\bar{\Gamma}'$, respectively. Furthermore, X inherits a collection of rational functions $(f_v)_{v \in V(\Gamma)}$. We call (X, \mathbf{x}, f) the stabilization of $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ and similarly say that $\bar{\Gamma}$ is the stabilization of $\bar{\Gamma}'$. We call any component of \tilde{X} that is contracted under stabilization *unstable*. Similarly starting with a twistable rational function (X, \mathbf{x}, f) , if there exists a twisted rational function $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ such that (X, \mathbf{x}, f) is the stabilization of $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$, then we call $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ a *twist* of (X, \mathbf{x}, f) .

From now on $\bar{\Gamma}'$ will always denote the level graph of a twisted rational function and $\bar{\Gamma}$ the level graph of its stabilization.

The goal of this section is to show that we have the following correspondence: The stabilization of a twisted rational function is a twistable rational function and every twistable rational function admits a twist. We are now going to make this precise.

LEMMA 3.2. *Let $\tilde{\mu}$ be a partition extending μ . Furthermore, let $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ be a twisted rational function of type $\tilde{\mu}$ compatible with a level graph $\bar{\Gamma}'$. Then its stabilization (X, \mathbf{x}, f) is a twistable rational function of type μ compatible with $\bar{\Gamma}$.*

Proof. Since $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ is a twisted rational function, we have

$$\text{ord}_{x_k} \tilde{f} = \tilde{\mu}_k \text{ for all } k, \text{ord}_{q_e^+} d\tilde{f} + \text{ord}_{q_e^-} d\tilde{f} = -2 \text{ for all nodes } e \in E.$$

The stabilization (X, \mathbf{x}, f) is obtained by contracting all unstable components. Chains of unstable components come in two flavors. A chain either connects two nodes, in which case we call it a *rational bridge*, or it forms a *rational tail*. We first study the contraction of a rational tail. If the end of the rational tail does not contain a marked zero or a marked pole of \tilde{f} , then we do not mark the preimage of the node where the rational tail connects after stabilization and thus there is nothing to check at this point. We now claim that the end of a rational tail never contains a marked zero or a marked pole. This is part of the following Lemma 3.3. Thus, if an irreducible component X_v contains a marked point x_k , then X_v is not an unstable component and $\text{ord}_{x_k} f = \text{ord}_{x_k} \tilde{f} = \mu_k$. It remains to analyze the case of a rational bridge. Suppose we have a chain C_1, \dots, C_n of rational curves connecting two nodes preimages of nodes q and q' . Let c_1, \dots, c_m be the marked critical points contained in C_1, \dots, C_n . Then we have

$$\text{ord}_q d\tilde{f} + \text{ord}_{q'} d\tilde{f} = -2 + \sum_{k=1}^m \text{ord}_{c_k} d\tilde{f} > -2,$$

which proves the matching order at nodes for a twistable rational function. We also need to show the compatibility with the level graph, i.e. if $\text{ord}_{q'} f < 0$ then $\ell(v(q)) > \ell(v(q'))$. By the following Lemma 3.3, no component C_k of the rational bridge is a local maximum for the level order and thus the level can only increase along the chain C_1, \dots, C_n of rational curves. Since $\bar{\Gamma}'$ has no horizontal nodes, the level is strictly increasing and thus $\ell(v(q)) > \ell(v(q'))$. \square

LEMMA 3.3. *Let $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ be a twisted rational function. An unstable component X_v does not contain a marked zero or pole of \tilde{f}_v . Additionally, an unstable component with two nodes is not a local maximum for the level order.*

Proof. We first assume there exists an unstable component containing a marked zero or pole x of \tilde{f}_v . Then there exists an unstable rational tail C_1, \dots, C_n with $x \in C_n$ and a stable component C_0 which is separated from C_1 by a node q .

We let q_0 be the preimage of q contained in C_1 . Note that C_n has a single node. If x is a zero of \tilde{f}_{C_n} , then \tilde{f}_{C_n} has a pole at the node. Furthermore, since all zeros of \tilde{f}_{C_n} are marked, \tilde{f}_{C_n} has exactly one zero and one pole and thus no other critical points. This yields a contradiction since every unstable component contains a marked critical point.

We can thus assume that x is a pole for \tilde{f}_{C_n} . Let then c_1, \dots, c_m be the collection of marked critical point on the rational tail C_1, \dots, C_n . Then

$$\text{ord}_{q_0} d\tilde{f} = \text{ord}_x d\tilde{f} + \sum_{k=1}^m \text{ord}_{c_k} d\tilde{f} > \text{ord}_x d\tilde{f}.$$

Here we used that the order of $d\tilde{f}$ is positive at all critical points in \mathbf{c} . Note that in particular $\text{ord}_q \tilde{f} > \text{ord}_x \tilde{f}$. We now remove the rational tail C_1, \dots, C_n and if q_0 is a pole for \tilde{f} we mark it on the resulting curve X' . Furthermore, X' inherits a level graph by contracting all the edges corresponding to nodes in the rational tail C_1, \dots, C_n . We claim that the evaluation morphism still vanishes for $(X', \mathbf{x}, d\tilde{f})$. The dual graph $\bar{\Gamma}$ of (X, \mathbf{z}) deformation retracts onto the dual graph $\bar{\Gamma}'$ of (X', \mathbf{z}) , where the deformation retract can be chosen to fix the zeros \mathbf{z} . Let $\alpha : (\bar{\Gamma}, \mathbf{z}) \rightarrow (\bar{\Gamma}', \mathbf{z})$ be the resulting retract. Then $\alpha(L_{\leq i}(\bar{\Gamma})) = L_{\leq i}(\bar{\Gamma}')$ and the diagram

$$\begin{array}{ccc} L_{\leq i}(\bar{\Gamma}) & & \\ \alpha \downarrow & \searrow \text{ev}_f^{(i)}(\bar{\Gamma}) & \\ L_{\leq i}(\bar{\Gamma}') & \xrightarrow{\text{ev}_f^{(i)}(\bar{\Gamma}')} & \mathbb{C} \end{array}$$

commutes, see also the proof of Lemma 3.6 for a similar argument. This finishes the proof of the claim.

Thus, by what was already proven in Proposition 2.4, $(X', \mathbf{x}, d\tilde{f})$ can be smoothed out to an exact differential where all marked points \mathbf{z} stay zeros of $d\tilde{f}$ of order $\text{ord}_{z_k} d\tilde{f}$. But this exact differential comes from a rational function with more zeros than poles, since $\text{ord}_{q_0} \tilde{f} > \text{ord}_x \tilde{f}$ as noted above, which is impossible.

It remains to prove the second part. Assume that some unstable component X_v contains two nodes and is a local maximum for the level order. Then X_v has no poles at the nodes and thus needs to have a marked pole somewhere. Thus X_v contains two nodes and at least one marked pole and is not unstable, which is a contradiction. \square

Remark 3.4. The above lemma is crucial for us. It is the main reason why we are able to state the **Main theorem** only in terms of a given twistable rational function instead of having to say that there exists a twisted rational function that stabilizes to the given twistable rational function.

So far we established that stabilizing a twisted rational function yields a twistable rational function. We now show that conversely every twistable rational function admits a twist.

PROPOSITION 3.5. *Let (X, \mathbf{x}, f) be a twistable rational function of type μ compatible with a level graph $\bar{\Gamma}$. Then there exists a partition $\tilde{\mu}$ extending μ , a level graph $\bar{\Gamma}'$ and a twisted rational function $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ of type $\tilde{\mu}$ compatible with $\bar{\Gamma}'$ such that $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ is a twist of (X, \mathbf{x}, f) .*

Proof. As a first step we mark all zeros of df , that are not already marked points in \mathbf{x} , and declare them as critical points in \mathbf{c} . To turn (X, \mathbf{x}, f) into a twisted rational function we are going to insert rational curves at all nodes with $\text{ord}_{q_e^+} df + \text{ord}_{q_e^-} df > -2$. At such a node we insert a rational curve $(\mathbb{P}^1, (0, 1, \infty))$ by identifying q_e^+ and q_e^- with 0 and ∞ , respectively. We equip \mathbb{P}^1 with a rational function f' such that df' has order $-\text{ord}_{q_e^+} df - 2$ at 0 and order $-\text{ord}_{q_e^-} df - 2$ at ∞ and no other poles. If both $\text{ord}_{q_e^+} df$ and $\text{ord}_{q_e^-} df$ are positive, f' has poles at 0 and ∞ and thus such f' exists. If $\text{ord}_{q_e^-} df < 0$, then $\text{ord}_{q_e^+} df > 0$ by the compatibility with the level graph of a twistable rational function. In particular f' has a pole at 0 and a zero at ∞ . Note that in this case such a rational function exists since

$$\text{mult}_0 f' = \text{ord}_{q_e^+} df + 1 > -\text{ord}_{q_e^-} df - 1 = \text{mult}_\infty f'.$$

We declare all remaining critical points of f' to be marked points in \mathbf{c} . Note that in both cases no zero of f' is marked as a point in \mathbf{x} .

We let $(\tilde{X}, \tilde{\mathbf{x}})$ be the resulting curve obtained by inserting the rational curves, where we set $\tilde{\mathbf{x}} := (\mathbf{x}, \mathbf{c})$. Furthermore by construction there exists a collection of rational functions \tilde{f} on \tilde{X} . Our next goal is to describe the level graph on \tilde{X} . We start with the level function for $\bar{\Gamma}$ and add additional levels in between two levels i and $i - 1$ and call those *intermediate levels*. For each level there are two intermediate levels, the level i^+ is in between $i + 1$ and i and the level i^- is in between i and $i - 1$. Note that we have $(i - 1)^+ = i^-$ by construction. We then assign the level of an inserted rational curve as follows.

If q_e^+ is a pole of f , then the level of the inserted \mathbb{P}^1 needs to be larger than $\ell(q_e^+)$ and otherwise the level of the inserted \mathbb{P}^1 needs to be lower than $\ell(q_e^-)$. The same is true for q_e^- . Note that q_e^+ and q_e^- cannot both be poles of f because of the compatibility with the level graph in Definition 1.1. Thus if q_e^+ is a pole, we assign the level to be $\ell(q_e^+)^+$ and if q_e^- is a pole we assign it to be $\ell(q_e^-)^+$. If both q_e^+ and q_e^- are zeros of f we declare the level to be $\min(\ell(q_e^+), \ell(q_e^-))^-$. We denote $\bar{\Gamma}'$ the resulting level graph on \tilde{X} and by construction $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ is compatible with $\bar{\Gamma}'$.

To summarize, we have constructed a twisted rational function $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ where $\tilde{\mathbf{x}} = (\mathbf{x}, \mathbf{c})$ and \mathbf{x} consists of zeros and poles of \tilde{f} and \mathbf{c} contains all remaining critical points of \tilde{f} which are not nodes of \tilde{X} , together with a level graph $\bar{\Gamma}'$ for \tilde{X} . We let $\tilde{\mu}$ be the partition consisting of the order of vanishing of df at

all marked points $\tilde{\mathbf{x}}$. Then $\tilde{\mu}$ extends μ and furthermore $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ is of type $\tilde{\mu}$ and compatible with $\tilde{\Gamma}'$. After forgetting all additional critical points \mathbf{c} and stabilizing we recover (X, \mathbf{x}, f) . \square

Let $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ be a twist of (X, \mathbf{x}, f) . As a final ingredient we need to compare the evaluation morphisms of f and \tilde{f} . The stabilization morphism $(\tilde{X}, \tilde{\mathbf{x}}) \rightarrow (X, \mathbf{x})$ induces a surjective map $p : H_1(\tilde{\Gamma}', \mathbf{z}) \rightarrow H_1(\tilde{\Gamma}, \mathbf{z})$. Recall here that by abuse of notation we call the legs of $\tilde{\Gamma}'$ and of $\tilde{\Gamma}$ corresponding to the marked zeros of \tilde{f} and f both \mathbf{z} .

LEMMA 3.6. *The inclusion $p(L_{\leq i}(\tilde{\Gamma}')) \subseteq L_{\leq i}(\tilde{\Gamma})$ holds and furthermore the diagram*

$$\begin{array}{ccc} L_{\leq i}(\tilde{\Gamma}') & & \\ \downarrow p & \searrow \text{ev}_{\tilde{f}}^{(i)} & \\ L_{\leq i}(\tilde{\Gamma}) & \xrightarrow{\text{ev}_f^{(i)}} & \mathbb{C} \end{array}$$

commutes. In particular, $\text{ev}_f^{(i)}$ vanishes identically if and only if $\text{ev}_{\tilde{f}}^{(i)}$ vanishes identically.

Proof. The stabilization morphism removes some irreducible components but does not change the level of any stable component. Thus the top level can only go down under p , i.e. $p(L_{\leq i}(\tilde{\Gamma}')) \subseteq L_{\leq i}(\tilde{\Gamma})$.

Let $\gamma \in L_{\leq i}(\tilde{\Gamma}')$. If the evaluation morphism $\text{ev}_{\tilde{f}}^{(i)}(\gamma)$ has a contribution from a rational function f_v , then necessarily $\ell(v) = i$. Our goal is to show that the restriction of γ to any unstable component X_v of level i is null-homotopic. Since the level graph $\tilde{\Gamma}'$ is purely vertical, v has to be a local maximum for the level order on $\tilde{\Gamma}'$. Furthermore, X_v cannot have two nodes by Lemma 3.3. This forces X_v to be the end a rational tail, which does not have a marked zero, again by Lemma 3.3. But then the restriction of γ to X_v is null-homotopic. Thus γ is homotopic to a path that does not cross any unstable top level component and thus $\text{ev}_f^{(i)}([\gamma]) = \text{ev}_{\tilde{f}}^{(i)}([\gamma])$. \square

4 THE PROOF OF THE MAIN THEOREM

Proof of the Main theorem. We first prove necessity of the conditions; that is we need to show that given a marked curve (X, \mathbf{x}) in the closure of $\text{DR}_g(\mu)$ there exists a twistable rational function compatible with some level graph structure on the dual graph of X with vanishing evaluation map.

Choose a holomorphic map $\phi : \Delta \rightarrow \overline{\mathcal{M}}_{g,n}$ from the unit disk Δ with $\phi(\Delta^*) \subseteq \text{DR}_g(\mu)$ and $\phi(0) = (X, \mathbf{x})$. We denote α the identification of $\text{DR}_g(\mu)$ and $\widetilde{\text{DR}}_g(\mu) \subseteq \mathbb{P}(E)$. After shrinking Δ , the image of $\alpha \circ \phi$ inside $\mathbb{P}(E)$ is contained in a stratum $\mathbb{P}\mathcal{H}(\tilde{\mu})$ for some partition $\tilde{\mu}$ extending μ . Here $\mathbb{P}\mathcal{H}(\tilde{\mu})$ denotes an

unmarked stratum, i.e. not all zeros of the differential form are marked points. We are now going to lift $\alpha \circ \phi$ to the moduli space of multi-scale differentials $\mathbb{P}\Xi\mathcal{M}_{g,n'}(\tilde{\mu})$.

Claim: After a base change $\Delta^* \rightarrow \Delta^*, z \mapsto z^k$ for a suitable k , we can construct a lift $\phi' : \Delta \rightarrow \mathbb{P}\Xi\mathcal{M}_{g,n'}(\tilde{\mu})$ of $\alpha \circ \phi$ such that $\phi'(\Delta^*) \subseteq \text{DR}_g^c(\tilde{\mu})$.

Proof of the claim. Since the number of critical points is constant, we can locally mark the critical points over Δ^* . Due to monodromy when encircling the origin in Δ , those local sections might not be well-defined and might permute the critical points non-trivially. We then choose k divisible enough so that the k -th power of the permutation is trivial. If we denote $\psi : \Delta^* \rightarrow \Delta^*, z \mapsto z^k$, then the family of curves corresponding to $\alpha \circ \phi \circ \psi$ admits global sections marking the critical points and thus there exists a map $\phi' : \Delta \rightarrow \mathbb{P}\Xi\mathcal{M}_{g,n'}(\tilde{\mu})$ such that $\phi'(\Delta^*) \subseteq \text{DR}_g^c(\tilde{\mu})$ and

$$\begin{array}{ccc} & & \mathbb{P}\Xi\mathcal{M}_{g,n'}(\tilde{\mu}) \\ & \nearrow \phi' & \downarrow \pi \\ \Delta & \xrightarrow{\alpha \circ \phi \circ \psi} & \overline{\mathcal{M}}_{g,n} \end{array}$$

commutes. Now the claim is proven.

We denote $(\tilde{X}, \tilde{\mathbf{x}}, \eta) := \phi'(0) \in \mathbb{P}\Xi\mathcal{M}_{g,n'}(\tilde{\mu})$. Since $\pi(\tilde{X}, \tilde{\mathbf{x}}, \eta) = \phi(0) = (X, \mathbf{x})$, we conclude that X is the stabilization of \tilde{X} . By Proposition 2.4, we can write $\eta = d\tilde{f}$ such that $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ is a twisted rational function of type $\tilde{\mu}$ compatible with a level graph $\tilde{\Gamma}'$ and $\text{ev}_{\tilde{f}}^{(i)} = 0$ for all i . By Lemma 3.2 the stabilization (X, \mathbf{x}, f) of $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ is a twistable rational function of type μ compatible with $\bar{\Gamma}$, where $\bar{\Gamma}$ is the level graph for X obtained by stabilizing $\tilde{\Gamma}'$. We furthermore conclude that $\text{ev}_f^{(i)} = 0$ for all i by Lemma 3.6. We have thus proven the necessity part.

Now we proceed with sufficiency. Here it is enough to prove that given a twistable rational function (X, \mathbf{x}, f) of type μ compatible with a level graph $\bar{\Gamma}$ such that the evaluation morphism vanishes identically, we can construct a one-parameter family of smooth curves in $\text{DR}_g(\mu)$ degenerating to (X, \mathbf{x}, f) . By Proposition 3.5 there exists a twist $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ of (X, \mathbf{x}, f) which is of type $\tilde{\mu}$ for some partition extending μ compatible with some level graph $\tilde{\Gamma}'$. Again, by Lemma 3.6 we have $\text{ev}_{\tilde{f}}^{(i)} = 0$ for all i and thus $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f}) \in \overline{\text{DR}}_g^c(\tilde{\mu}) \subseteq \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n'}(\tilde{\mu})$ by Proposition 2.4. Therefore, there exists a holomorphic map $\phi' : \Delta \rightarrow \overline{\text{DR}}_g^c(\tilde{\mu})$ which is generically contained in $\widetilde{\text{DR}}_g^c(\tilde{\mu})$ and $\phi'(0) = (\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$. Composing with $\pi : \mathbb{P}\Xi\overline{\mathcal{M}}_{g,n'}(\tilde{\mu}) \rightarrow \overline{\mathcal{M}}_{g,n}$ we obtain a map $\phi := \pi \circ \phi' : \Delta \rightarrow \overline{\text{DR}}_g(\mu)$ which is generically contained in $\widetilde{\text{DR}}_g(\mu)$ and $\phi(0) = (X, \mathbf{x})$. \square

5 ADMISSIBLE COVERS VERSUS TWISTABLE RATIONAL FUNCTIONS

In this section we discuss the relationship between using admissible covers and twistable rational functions to describe the closure of double ramification loci. As a consequence we see that the existence of an admissible cover can be encoded in the level graph of a twistable rational function, a result that to the best of our knowledge is new.

We start by setting up our notation for admissible covers. As before let μ be a partition of zero. The *Hurwitz scheme* $\text{Hur}_{g,n}(\sigma)$ parameterizes maps $f : X \rightarrow \mathbb{P}^1$ with ramification over 0 and ∞ prescribed by μ , and simple ramification otherwise. We mark on X all preimages of branch points of f , see for example [JKK05], where the same variant of admissible covers is used.

We now recall admissible covers, see for example the original reference [HM82]. Note that here we do not require the ramification to be simple.

DEFINITION 5.1 (Admissible cover). Let (D, \mathbf{x}) be a stable n -pointed nodal curve of genus 0. An *admissible cover* $f : C \rightarrow D$ of degree d is a finite degree d morphism such that

- C is nodal and every node of C maps to a node of D ,
- the restriction of f to D_{gen} is étale, where D_{gen} is the complement of the marked zeros and nodes,
- if a node of C identifies x and y , then

$$\text{mult}_x f = \text{mult}_y f,$$

where $\text{mult}_x f - 1$ is the ramification index of f at x .

We additionally require that the marked points in C are exactly the preimages of marked points in D . The *type* of f records ramification profile of f at all marked points which are not nodes.

We let $\text{Adm}_{g,n}(\sigma)$ be the stack of all admissible covers of type $\sigma = (\mu, 1, \dots, 1)$ where we additionally mark all preimages of marked points in D . Then $\text{Adm}_{g,n}(\sigma)$ is a proper Deligne-Mumford stack and $\text{Hur}_{g,n}(\sigma)$ is a dense open substack. Note that $\text{Adm}_{g,n}(\sigma)$ is in general not smooth, but its normalization is, see for example [AVC03], where the authors also give a modular interpretation of the normalization in terms of twisted S_d -covers. Our notation differs from (loc. cit.) in that we mark all preimages of the marked points of D . Since $\text{Hur}_{g,n}(\sigma)$ is a dense open substack of $\text{Adm}_{g,n}(\sigma)$, we have the following.

PROPOSITION 5.2. *A stable n -pointed curve (X, \mathbf{x}) is contained in the closure of $\text{DR}_g(\mu)$ if and only if there exists a marked stable curve (X', \mathbf{x}') such that*

- *there exists an admissible cover $X' \rightarrow D$ of type $\sigma = (\mu, 1, \dots, 1)$,*
- *the stabilization of the prestable curve $(X', f^{-1}(\{0, \infty\}))$ is isomorphic to (X, \mathbf{x}) .*

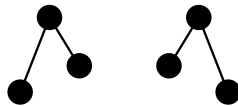


Figure 2: The cherry graph from Example 5.3 with two level graphs $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$

Proof. Generically a rational map in a double ramification locus has simple ramification and thus we can lift a family in $\text{DR}_g(\mu)$ to $\text{Hur}_{g,n}(\sigma)$ where $\sigma = (\mu, 1, \dots, 1)$. Since $\text{Hur}_{g,n}(\sigma)$ is dense in $\text{Adm}_{g,n}(\sigma)$, we get the desired result. \square

We have thus described the closure in two seemingly different ways, with admissible covers and with twistable rational functions. As a result we obtain a correspondence between the existence of a suitable admissible cover on a prestable model of X and the existence of a certain twistable rational function on X . In general this correspondence is far from being one-to-one.

Example 5.3 (Twistable rational functions and admissible covers do not determine each other). Even on a smooth curve X there can be multiple admissible covers. For example we can attach unstable components to X such that the resulting prestable curve X' curve maps to the union of two rational curves attached at a single node and X is the stabilization of X' .

On the other hand, given a fixed admissible cover there can be multiple level graphs compatible with it. For example we can construct an admissible cover on a “cherry level graph”, see Figure 2. The two level graphs $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ are obtained from each other by tilting the lower level components and are supported on the same dual graph Γ . Suppose now we have a twistable rational function (X, \mathbf{x}, f) such that all marked zeros are on the top level component. In this case the vanishing of the evaluation morphism of $\bar{\Gamma}_1$ for a twistable rational function on X is equivalent to the vanishing on $\bar{\Gamma}_2$. So given any admissible cover on X we can then construct two different twistable rational functions, one on $\bar{\Gamma}_1$ and one on $\bar{\Gamma}_2$. By choosing different genera for the irreducible components on the lower levels one can make sure that $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ are not isomorphic level graphs.

It appears very difficult to try to approach the correspondence between admissible covers and twistable rational functions directly, since the combinatorial difficulties are very significant. Given a twistable rational function one needs to insert unstable components in order to obtain a finite morphism but there are many different ways one can add unstable components. And in the other direction, given an admissible cover one needs to find a level graph on a given dual graph such that the evaluation morphism vanishes. The number of possible level graphs on a given dual grows exponentially, see for example [CMZ20b] for the problem of enumerating all possible level graphs.

The correspondence can be generalized to admissible covers with arbitrary fixed ramification multiplicities. The proof remains the same, at least as long

all poles of f are marked. In the case where the poles of f are marked it doesn't seem possible to state the theorem purely on the stable curve X but one can only phrase it in terms of a prestable model. In general the bookkeeping is tedious since one needs to remember which branch points lie in the same fiber and depending on that the evaluation morphisms changes. We leave the details to the reader.

5.1 BIRATIONAL MODELS OF MARKED DOUBLE RAMIFICATION LOCI

Consider the marked double ramification locus $\widetilde{\mathrm{DR}}_g^c(\tilde{\mu})$. By taking the closure of $\widetilde{\mathrm{DR}}_g^c(\tilde{\mu})$ inside the moduli space of multi-scale differentials $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n'}(\tilde{\mu})$, we obtain a smooth compactification $\widehat{\mathrm{DR}}_g(\tilde{\mu})$ of $\widetilde{\mathrm{DR}}_g^c(\tilde{\mu})$. A caveat of $\widehat{\mathrm{DR}}_g(\tilde{\mu})$ is that it does not naturally come with a target map. On each irreducible component one has a rational function but it is not clear whether the functions assemble together in a natural way. On the other hand the moduli stack of admissible covers $\mathrm{Adm}_{g,n'}(\sigma)$ with marked ramification yields a different compactification of $\widetilde{\mathrm{DR}}_g^c(\tilde{\mu})$ and comes with a generically finite map $\mathrm{Adm}_{g,n'}(\sigma) \rightarrow \overline{\mathcal{M}}_{0,k}$ by sending an admissible cover to the base curve. Comparing the two birational models of $\widetilde{\mathrm{DR}}_g^c(\tilde{\mu})$ seems an interesting problem. In this direction one can consider the fiber product

$$F = \mathrm{Adm}_{g,n'}(\sigma) \times_{\overline{\mathcal{M}}_{g,n'}} \widehat{\mathrm{DR}}_g(\tilde{\mu}).$$

A point in F corresponds to a tuple (X, p, f, μ) where X is a n -marked curve with admissible cover $f: \tilde{X} \rightarrow C$ from a semistable model \tilde{X} of X and on each irreducible component X_v of X an exact differential df_v . Since we marked all critical points of f on X and on C , the semistable model \tilde{X} is already stable and thus agrees with X . Furthermore on each component of C , the limits of the branch points of f are exactly the zeros and poles of the limit of dz on \mathbb{P}^1 . Hence on each component of $X = \tilde{X}$, the differential $\mu_v = df_v$ is the derivative of f_v . Additionally there exists a level graph $\bar{\Gamma}$ determining a total order on the vertices of X . The differential is now superfluous and one only needs to remember the data $(X, p, f, \bar{\Gamma})$. Thus F differs from $\mathrm{Adm}_{g,n}(\sigma)_{g,n'}(\sigma)$ by remembering the speed of degeneration of the rational function. In this description F is closely related to the moduli-space of multi-scale differentials and for example one can ask if there exists a blow-up description of F similar to how $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\mu)$ is obtained as a sequence of blow-ups as in [BCGGM19b].

6 EXAMPLES

We now illustrate the main theorem in a few examples and showcase some of the features of twistable rational functions. We will use the following conventions to depict level graphs. The legs denote the marked zeros and poles and critical points of the rational functions. Dashed legs correspond to marked critical

points which are not marked zeroes or poles in \mathbf{x} . The label associated to a half-leg is the order of vanishing of the exact differential df (and *not* the order of the rational function f) at the corresponding marked point. The label of a vertex is the genus of the normalization of the corresponding irreducible component. If a vertex is unlabeled, it corresponds to an irreducible component of genus zero.

The decorations on the edges are the numbers $\text{ord}_{q_+} df$ and $\text{ord}_{q_-} df$ respectively. The level order is implicit in the figures: the highest vertices correspond to the top level, lower levels are drawn below and vertices of the same height are of the same level. From now on (X, \mathbf{x}, f) always denotes a collection of rational functions on a stable curve such that (X, \mathbf{x}) is contained in the closure of some double ramification locus $\text{DR}_g(\mu)$ and (X, \mathbf{x}, f) is a twistable rational function for some partition $\tilde{\mu}$ extending μ compatible with some level graph $\bar{\Gamma}$.

Example 6.1 (Unmarked zeros). As a first example we consider a degeneration of $\text{DR}_2(1^3, -3)$, depicted in Figure 3a.

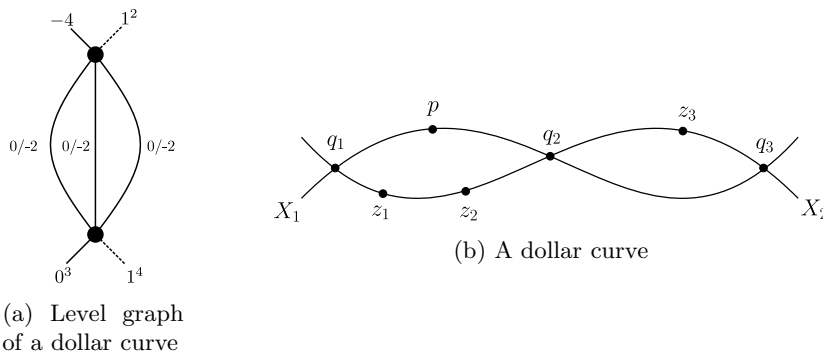


Figure 3: The dollar curve from Example 6.1

It is an example of a “dollar curve”, which we discuss more thoroughly in Example 6.5. We note that the top level component does not contain any marked zeros. In particular we see that both types of components mentioned in Remark 1.2 do appear.

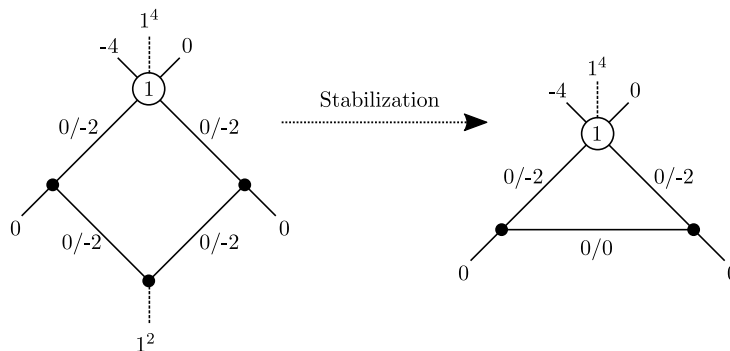
The evaluation morphism can be computed as follows. Let γ_1 be the path $(X_1, q_2, X_2, q_1, X_1)$ passing once through q_1 and q_2 and similarly let γ_2 be the path $(X_1, q_3, X_2, q_2, X_1)$ passing through q_2 and q_3 once. Then

$$\begin{aligned} \text{Im } \text{ev}_f^{(0)} &= \langle \text{ev}_f^{(0)}(\gamma_1), \text{ev}_f^{(0)}(\gamma_2) \rangle, \\ \text{ev}_f^{(0)}(\gamma_1) &= f_1(q_2^+) - f_1(q_1^+), \\ \text{ev}_f^{(0)}(\gamma_2) &= f_1(q_3^+) - f_1(q_2^+) \end{aligned}$$

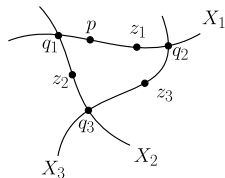
and the vanishing of the evaluation morphism is equivalent to $f_1(q_1^+) = f_1(q_2^+) = f_1(q_3^+)$, i.e. $\{q_1^+, q_2^+, q_3^+\}$ lie in the same fiber of f_1 . Note that the

evaluation morphism automatically vanishes at level -1 in this example since there are no non-trivial relative paths contained in level -1 . Note that such a rational function $f_1 : X \rightarrow \mathbb{P}^1$ always exists. Similarly on X_2 the rational function f_2 has three simple zeroes, three simple poles and four remaining simple branch points. Such a rational functions always exists as well.

Example 6.2 (Level graphs with horizontal nodes). We have seen that level graphs for twisted rational functions have no horizontal nodes. The next example shows that the level graph of a twistable rational function can have horizontal nodes. We consider again a degeneration of $\text{DR}_2(1^3, -3)$, see Figure 4.



(a) The level graph $\overline{\Gamma}'$ and its stabilization $\overline{\Gamma}$



(b) A stable curve X with level graph $\overline{\Gamma}$

Figure 4: The dual graphs from Example 6.2

The level graph $\overline{\Gamma}'$ is the level graph of a twisted rational function and $\overline{\Gamma}$ the level graph of its stabilization. Note that $\overline{\Gamma}'$ has three levels, under stabilization the bottom vertex is contracted and thus $\overline{\Gamma}$ has only two levels with a horizontal edge. Let (X, \mathbf{x}, f) be the twist of $(\tilde{X}, \tilde{\mathbf{x}}, \tilde{f})$ which is contained in the closure of $\text{DR}_2(1^3, -3)$. On the genus 1 component X_1 the rational function $f_1 : X_1 \rightarrow \mathbb{P}^1$ is a degree three map which has a pole of order three at p and simple zero at z_1 . For a fixed pointed elliptic curve $(X_1, z_1, p, q_1^+, q_2^+)$ there is a $h^0(\mathcal{O}_{X_1}(3p - z_1)) = 2$ -dimensional family of such maps.

We now compute the top level evaluations $\text{ev}_f^{(0)}, \text{ev}_f^{(-1)}$. Let $\gamma_{ij} \in H_1(\overline{\Gamma}, \mathbf{z})$ be the path (z_i, v_i, v_j, z_j) connecting the marked zeros z_i and z_j by only passing through a single node. Then the image of the evaluation map is generated by

the paths γ_{ij} , in particular

$$\begin{aligned} \text{Im ev}_f^{(0)} &= \langle \text{ev}_f^{(0)}(\gamma_{12}), \text{ev}_f^{(0)}(\gamma_{13}) \rangle_{\mathbb{Z}}, \\ \text{Im ev}_f^{(-1)} &= \langle \text{ev}_f^{(-1)}(\gamma_{23}) \rangle_{\mathbb{Z}}. \end{aligned}$$

We then compute

$$\begin{aligned} \text{ev}_f^{(0)}(\gamma_{12}) &= f_1(z_1) - f_1(q_1^+) = -f_1(q_1^+), \\ \text{ev}_f^{(0)}(\gamma_{13}) &= f_1(z_1) - f_1(q_2^+) = -f_1(q_2^+), \\ \text{ev}_f^{(-1)}(\gamma_{23}) &= f_2(q_3^+) - f_3(q_3^-). \end{aligned}$$

The vanishing of $\text{ev}_f^{(0)}(\gamma_{12})$ and $\text{ev}_f^{(0)}(\gamma_{13})$ then implies that

$$(f_1) = z_1 + q_1^+ + q_2^+ - 3p$$

and thus $(X_1, (z_1, q_1^+, q_2^+, p)) \in \text{DR}_1(1^3, -3)$.

Finally, the vanishing of $\text{ev}_f^{(-1)}(\gamma_{23})$ implies

$$f_2(q_3^+) - f_3(q_3^-) = 0.$$

Both f_2 and f_3 are isomorphisms to \mathbb{P}^1 sending z_2 and z_3 to zero, respectively. In particular we have $f_2(q_3^+) \neq 0, f_3(q_3^-) \neq 0$ and by rescaling f_3 we can always achieve that $f_2(q_3^+) - f_3(q_3^-) = 0$. Thus in this case the only condition for (X, \mathbf{x}, f) to be in the closure of $\text{DR}_2(1^3, -3)$ is given by $(X_1, (z_1, q_1^+, q_2^+, p)) \in \text{DR}_1(1^3, -3)$.

For the following examples we will omit the remaining critical points from the level graphs and only draw the marked zeros and poles of f .

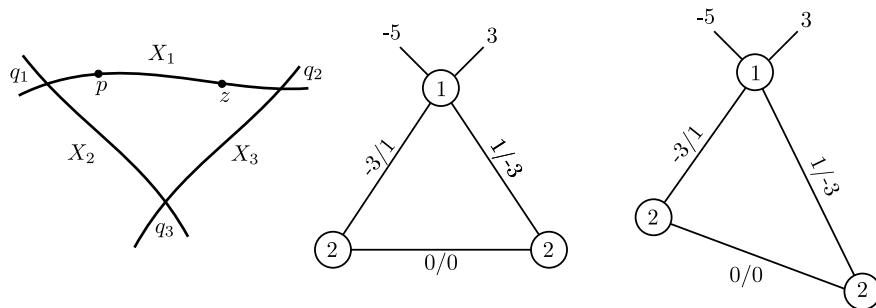


Figure 5: The different partial orders in Example 6.3

Example 6.3 (The partial ordering is not determined by the vanishing orders). In the case of twisted differentials (X, η) , a twisted differential induces a unique partial ordering on the dual graph. We will now see that the same is not true

for twistable rational functions. More precisely, we will construct a twistable rational function (X, \mathbf{x}, f) which lies in the boundary of $\text{DR}_5(4, -4)$ such that there are two different level graphs, inducing a different partial order, such that (X, \mathbf{x}, f) is compatible with both level graphs. By Condition (3) in Definition 1.1 this is only possible if there is a node such that the rational function is holomorphic at both preimages of the node.

We let $\bar{\Gamma}_1$ be the dual graph in the middle of Figure 5 and $\bar{\Gamma}_2$ the one on the right. Note that the two partial orderings induced by $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ are different since $\bar{\Gamma}_1$ has a horizontal node and $\bar{\Gamma}_2$ has not. To ensure that we can find a twistable rational function compatible with both level graphs we now unravel the vanishing of the evaluation morphism for $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$. Let γ_1 be the path (q_1, q_2, q_3) . Then

$$\text{Im } \text{ev}_f^{(0)}(f)(\bar{\Gamma}_k) = \langle \text{ev}_f^{(0)}(\bar{\Gamma}_k)(\gamma_1) \rangle \text{ for } k = 1, 2$$

and the evaluation morphism vanishes automatically at all other levels, since in both cases the level graph has no loops in lower levels and also no marked zeros on components of lower levels. We then compute

$$\text{ev}_f^{(0)}(\bar{\Gamma}_1)(\gamma_1) = \text{ev}_f^{(0)}(\bar{\Gamma}_2)(\gamma_1) = f_1(q_2^+) - f_1(q_1^+).$$

Thus for both $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ on X_1 the rational function f_1 is of degree 4, q_1^+, q_2^+ are in the same fiber of f_1 , and f_1 is simply ramified at q_1^+ and q_2^+ . Note that such a rational function always exists, for example by [PP06, Thm. 2.7]. Since the vanishing of the evaluation morphism is controlled by the top level, every twistable rational function that is compatible with $\bar{\Gamma}_1$ is automatically compatible with $\bar{\Gamma}_2$.

Example 6.4 (The evaluation morphism depends on the level graph). The notion of twistable rational function does not depend on the full structure of a level graph. The level graph only comes into play when considering the evaluation morphism. This example shows that there can be two different level graphs inducing the same partial ordering but the condition imposed by the evaluation morphism is different. In Figure 6 we see a degeneration of $\text{DR}_8(4, -4)$.

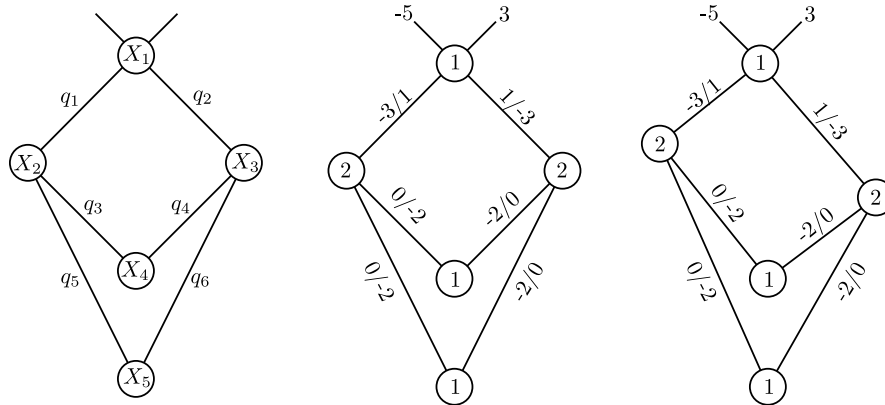


Figure 6: The different level graphs for Example 6.4

On the top level the vanishing of the evaluation morphism is independent of k and equivalent to $f_1(q_2^+) = f_1(q_1^+)$. Exactly as in Example 6.3 we see that there exists such a rational function. For level -1 we have

$$\begin{aligned} \text{ev}_f^{(-1)}(\overline{\Gamma}_1)(\gamma_2) &= f_2(q_3^+) - f_2(q_5^+) + f_3(q_6^+) - f_3(q_4^+), \\ \text{ev}_f^{(-1)}(\overline{\Gamma}_2)(\gamma_2) &= f_2(q_3^+) - f_2(q_5^+). \end{aligned}$$

Thus in the case of $\overline{\Gamma}_2$, for the evaluation morphism to vanish the preimages of the nodes q_3^+ and q_5^+ have to be conjugate under the hyperelliptic involution, while for $\overline{\Gamma}_1$ the condition $f_2(q_3^+) - f_2(q_5^+) = f_3(q_4^+) - f_3(q_6^+)$ can be satisfied without q_3^+ and q_5^+ being conjugate under the hyperelliptic involution.

In light of our discussion about admissible function the condition $f_2(q_3^+) - f_2(q_5^+) + f_3(q_6^+) - f_3(q_4^+)$ is, at first glance, surprising since it does not simply state that two points lie in the same fiber. But via an automorphism $z \mapsto z + c$ we can arrange that $f_2(q_3^+) = f_3(q_4^+)$. In which case $f_2(q_3^+) - f_2(q_5^+) + f_3(q_6^+) - f_3(q_4^+)$ simplifies to $f_2(q_3^+) - f_2(q_5^+) = f_3(q_6^+)$. Now those two conditions can be used to construct an admissible cover by mapping components of the same level to the same copy of \mathbb{P}^1 . Note that one has to attach additional unstable components for this.

Our next example is a thorough discussion of so-called *dollar curves*, i.e. two irreducible curves meeting transversally at three nodes. We study all possible level graphs and describe explicitly the conditions to be in the closure of double ramification locus. For curves of compact type, or for curves with at most one-separating node, one can extend the Abel-Jacobi map to describe the closure of double ramification locus, see [GZ14]. Thus dollar curves are a natural next step to illustrate our techniques.

Example 6.5 (“Dollar curves”). We let $X = X_1 \cup X_2$ be a stable curve in the boundary of $\text{DR}_g(\mu)$ with one component X_1 of genus g_1 and the other component X_2 of genus $g_2 := g - g_1 - 2$ meeting at three nodes q_1, q_2, q_3 , see

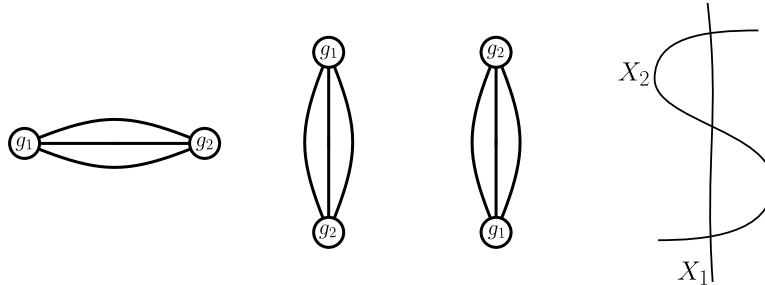


Figure 7: The possible level graphs $\bar{\Gamma}_1, \bar{\Gamma}_2, \bar{\Gamma}_3$ from Example 6.5 and the resulting stable curve X

Example 6.5. Without labeling the legs, there are only three possible level graphs, see Figure 7. The discussion for $\bar{\Gamma}_2$ and $\bar{\Gamma}_3$ is completely analogous and we thus only discuss $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$. We let \mathbf{x}_1 and \mathbf{x}_2 be all marked zeros or poles of f contained in X_1 and X_2 , respectively. Similarly, we let $\mathbf{q}^+ = (q_1^+, q_2^+, q_3^+)$ and $\mathbf{q}^- = (q_1^-, q_2^-, q_3^-)$ be the preimages of nodes contained in X_1 and X_2 respectively. The following conditions are necessary for a twistable rational function.

1. If $\text{ord}_{q_l^\pm} f < 0$ then $v(q_l^\pm)$ is of level -1 .
2. $\text{ord}_{q_l^+}(df_1) + \text{ord}_{q_l^-}(df_2) \geq -2$ for $l = 1, 2, 3$.

We now discuss the vanishing of the evaluation morphism for the different level graphs.

The dual graph $\bar{\Gamma}_1$: In this case both components X_1 and X_2 are of top level and thus $\text{ord}_{q_l^\pm} f \geq 0$ for $l = 1, 2, 3$. This is only possible if $\text{ord}_{q_l^\pm} f = 0$ for $l = 1, 2, 3$. Thus f_1 and f_2 have zeros and poles only at the marked points. In particular both components X_1 and X_2 contain marked zeros of f .

It remains to compute the evaluation morphism. Let γ_l be a path connecting a marked zero of X_1 and a marked zero of X_2 through the node q_l . Then the first homology $H_1(\bar{\Gamma}_1, \mathbf{z})$ of the dual graph relative to the marked zeros of f is generated by γ_1, γ_2 and γ_3 . We then have

$$\text{ev}_f^{(0)}(\gamma_l) = f_1(q_l^+) - f_2(q_l^-) \text{ for } l = 1, 2, 3,$$

and thus the vanishing of the evaluation morphism is equivalent to

$$f_1(q_l^+) = f_2(q_l^-) \text{ for } l = 1, 2, 3.$$

Since no preimage of a node q_l^\pm is a zero or pole of f , we can always rescale f_2 such that the condition $f_1(q_1^+) = f_2(q_1^-)$ is satisfied. But then the conditions for $l = 2, 3$ impose non-trivial restrictions.

The dual graph $\bar{\Gamma}_2$: We now consider the second case, where X_1 is of top level and X_2 of level -1 . The main difference is that now potentially one component contains no marked zeros of f . We thus distinguish two cases.

Case 1: Both components contain a marked zero

In this case $H_1(\bar{\Gamma}, \mathbf{x})$ is again generated by $\gamma_l, l = 1, 2, 3$. Since X_2 is of lower level the evaluation morphism now computes as

$$\text{ev}_f^{(0)}(\gamma_l) = f_1(q_l^+) \text{ for } l = 1, 2, 3$$

and thus the vanishing of the evaluation morphism is equivalent to $f_1(q_l^+) = 0$ for $l = 1, 2, 3$. In particular all preimages q_l^+ are zeros for f_1 . On X_2 there are no further restrictions.

Case 2: There exists a component without marked zeros

The evaluation morphism now computes differently, since there is no relative path crossing exactly one node. We let γ_{ab} to be the path $(X_1, q_a, X_2, q_b, X_1)$ be a path crossing only q_a and q_b once. Then γ_{12}, γ_{13} generate $H_1(\bar{\Gamma}, \mathbf{z})$ and

$$\begin{aligned} \text{ev}_f^{(0)}(\gamma_{12}) &= f_1(q_1^+) - f_1(q_2^+), \\ \text{ev}_f^{(0)}(\gamma_{13}) &= f_1(q_1^+) - f_1(q_3^+). \end{aligned}$$

Thus the vanishing of the evaluation morphism is equivalent to

$$f_1(q_1^+) = f_1(q_2^+) = f_1(q_3^+).$$

Combining our analysis we have shown that

PROPOSITION 6.6. *A marked curve (X, \mathbf{x}) with dual graph $\bar{\Gamma}$ lies in the closure of $\text{DR}_g(\mu)$ if and only if there exist rational functions f_k on X_k for $k = 1, 2$ satisfying the following conditions.*

1. $\text{ord}_{x_k} = \mu_k$ for all k .
2. $\text{ord}_{q_l^+}(df_1) + \text{ord}_{q_l^-}(df_2) \geq -2$ for $l = 1, 2, 3$.
3. *Either*

$$\begin{aligned} f_1(q_1^+) = f_1(q_2^+) = f_1(q_3^+), \text{ord}_{q_l^+} f_1 \geq 0 \text{ for } l = 1, 2, 3 \text{ or} \\ f_2(q_1^-) = f_2(q_2^-) = f_2(q_3^-), \text{ord}_{q_l^-} f_2 \geq 0 \text{ for } l = 1, 2, 3 \text{ or} \\ f_1(q_l^+) = f_2(q_l^-), \text{ord}_{q_l^\pm} f = 0 \text{ for } l = 1, 2, 3. \end{aligned}$$

We see that in this example we get a short list of conditions that characterize whether a curve lies in the closure of a double ramification locus. On each irreducible component one has a rational function contained in a double ramification locus, where potentially we have some additionally marked points that are not zeros or poles of f , together with some compatibility conditions between f_1 and f_2 . All the conditions either restrict the ramification profile on one irreducible components or state that some marked points on different components are mapped to the same point on \mathbb{P}^1 .

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