$\mathbb{A}^1$-Connected Components of Classifying Spaces and Purity for Torsors

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Abstract. In this paper, we study the Nisnevich sheafification $\mathcal{H}^1_{\text{ét}}(G)$ of the presheaf associating to a smooth scheme the set of isomorphism classes of G-torsors, for a reductive group G. We show that if G-torsors on affine lines are extended, then $\mathcal{H}^1_{\text{ét}}(G)$ is homotopy invariant and show that the sheaf is unramified if and only if Nisnevich-local purity holds for G-torsors. We also identify the sheaf $\mathcal{H}^1_{\text{ét}}(G)$ with the sheaf of $\mathbb{A}^1$-connected components of the classifying space $\mathrm{B}_{\text{ét}}G$. This establishes the homotopy invariance of the sheaves of components as conjectured by Morel. It moreover provides a computation of the sheaf of $\mathbb{A}^1$-connected components in terms of unramified G-torsors over function fields whenever Nisnevich-local purity holds for G-torsors.

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1 Introduction

1.1 The Asok–Fasel program

One of the most compelling recent successes in applying methods from homotopy theory to algebraic geometry is the Asok–Fasel program which attempts to classify vector bundles on smooth affine varieties by means of the Postnikov tower in Morel–Voevodsky's $\mathbb{A}^1$-homotopy theory [AF14a, AF14b, AF15]. Their methodology is based on the affine representability results of Morel [Mor12] and Asok, Hoyois and the third author [AHW17] which establish results to the effect that vector bundles on various smooth affine schemes are computed as maps in the $\mathbb{A}^1$-homotopy category, thus accessible via the Postnikov tower [MV99, AE17] and therefore understood in cohomological terms. A morphism

$$\mathcal{X} \to \mathcal{Y}$$

in the $\mathbb{A}^1$-homotopy category can be understood as the limit of a sequence of maps

$$\mathcal{X} \to \tau_{\leq j}\mathcal{Y} \qquad j \geq 0,$$

such that the obstruction to the extension problem:

$$\begin{array}{ccc} & & \tau_{\leq j+1}\mathcal{Y} \\ & \nearrow & \downarrow \\ \mathcal{X} & \longrightarrow & \tau_{\leq j}\mathcal{Y}, \end{array} \tag{1}$$

and their possible solutions lie in a (twisted) Nisnevich cohomology group.

These "affine representability" results, in turn, rely on the resolution of the Bass–Quillen conjecture in various cases, due to work of Lindel [Lin82] and Popescu [Pop89]. More specifically, let us write $\mathcal{V}_r(R)$ to be the set of isomorphism classes of projective modules of rank r on a ring R (equivalently, rank r vector bundles on $\operatorname{Spec} R$). It is proved in [AHW17, Theorem 5.2.1] that whenever A is a ring such that for any maximal ideal $\mathfrak{m}$, $A_{\mathfrak{m}}$ is ind-smooth over a Dedekind ring with perfect residue fields, the map

$$\mathcal{V}_r(A) \to \mathcal{V}_r(A[t_1, \cdots, t_j]),$$

is an isomorphism for any $j, r \geq 0$. Thus, in particular, $\mathbb{A}^1$-invariance holds for vector bundles on smooth affines schemes over perfect fields.

Since $\mathcal{V}_r(A)$ is isomorphic to the set of GL_r-torsors on $\operatorname{Spec} A$, it is natural to ask if the Asok–Fasel program can be used to study torsors under other reductive groups. For us, G-torsors are required to have local triviality with respect to the étale topology and hence isomorphism classes of G-torsors over a scheme X are in (pointed) bijection with the (pointed) étale cohomology set $\mathrm{H}^1_{\text{ét}}(X; G)$ (pointed at the isomorphism class of the constant torsor $G \times X \to X$). The analogous picture for $\mathrm{H}^1_{\mathrm{Nis}}(X; G)$ (where we require that torsors exhibit *Nisnevich-local* triviality) has been discussed in [AHW18] where affine representability was proved for isotropic reductive groups. In particular, whenever the Nisnevich and étale cohomological sets coincide (for example, when $G = \mathrm{SL}_n, \mathrm{GL}_n, \mathrm{Sp}_{2n}$), we can run the Asok–Fasel program to classify torsors under these groups. However, this coincidence between étale and Nisnevich torsors is not the "generic situation", and étale-locally trivial torsors are certainly more abundant in algebraic geometry — for example Nisnevich-locally trivial torsors over fields are all constant, a fact which is wildly false for étale-locally trivial torsors, e.g. for groups like $\mathrm{O}(n)$ (related to quadratic forms) or PGL_n (related to central simple algebras).

On first glance, it seems that the Asok–Fasel program will not have much to say about étale–locally trivial torsors (although frequently "stably trivial" torsors happen to be Nisnevich-locally trivial). Indeed, we have known for a long time that on a fixed affine scheme X, isomorphism classes of torsors under an arbitrary reductive group over X are not generally $\mathbb{A}^1$-invariant, i.e., the map of pointed sets

$$\mathrm{H}^1_{\text{ét}}(X, G) \to \mathrm{H}^1_{\text{ét}}(X \times \mathbb{A}^1, G), \tag{2}$$

need not be an isomorphism. Since pullback along the zero section provides a section of the map (2) what is at stake is the surjectivity of this map, i.e., whether or not a G-torsor on $X \times \mathbb{A}^1$ is *extended* from X. In other words, the analog of the Bass–Quillen conjecture fails and there is no hope to represent torsors under G as maps in the $\mathbb{A}^1$-homotopy category.

Our first result states that, in fact, the failure of $\mathbb{A}^1$-invariance *disappears* after Nisnevich sheafification. To formulate it, we fix a field k and let us write $\mathcal{H}^1_{\text{ét}}(G)$ for the Nisnevich sheafification of the presheaf on Sm_k given by

$$U \mapsto \mathrm{H}^1_{\text{ét}}(U; G).$$

Theorem 1.1 (Propositions 3.7, 3.9, 3.10). *Let k be a field and G be a k-group scheme which is either:*

1. *a connected reductive group if k is of characteristic zero, or*
2. $\mathrm{O}(n)$ *if k is characteristic not two, or*
3. PGL_n *if the characteristic of k is coprime to n.*

Then, $\mathcal{H}^1_{\text{ét}}(G)$ is $\mathbb{A}^1$-invariant.

In the meantime, the result has been extended by Balwe, Hogadi and Sawant to include all semisimple, simply connected groups over fields of arbitrary characteristic, cf. [BHS22].

To put our result in context, we note the following

Example 1.2 (Parimala). In the remarkable paper [Par78] constructed infinitely many, explicit rank 4 inner product spaces on $\mathbb{R}[x, y]$ which are not extended from $\mathbb{R}$. Translating this problem to torsors, she showed that the map

$$\mathrm{H}^1_{\text{ét}}(\mathbb{R}; \mathrm{O}(4)) \to \mathrm{H}^1_{\text{ét}}(\mathbb{R}[x, y]; \mathrm{O}(4))$$

is *not* an isomorphism. Furthermore, by the last paragraph of [Par78, page 923] and the exceptional isomorphism $\mathrm{PGL}_2 = \mathrm{SO}(3)$, we can further conclude that the map

$$\mathrm{H}^1_{\text{ét}}(\mathbb{R}; \mathrm{PGL}_2) \to \mathrm{H}^1_{\text{ét}}(\mathbb{R}[x, y]; \mathrm{PGL}_2)$$

is *not* an isomorphism. The reader is invited to consult [Lam06, Chapter VII] for further discussions.

In fact, Parimala proved more. She showed that her torsors are constant, and actually extended from $\mathbb{R}$, away from a principal open $\mathbb{A}^2_{\mathbb{R}}$, see [Lam06, Lemma VII.4.14]. Additionally, restricting her torsor to a local ring around each point of $\mathbb{A}^2_{\mathbb{R}}$ gives rise to a torsor which is extended from $\mathbb{R}$ [Par78, Theorem 2.1(ii)]. These addenda to her results encouraged us to look for a result of the form given in Theorem 1.1. □

Theorem 1.1 implies that, in particular, any counterexample to the surjectivity of (2) must vanish on a Nisnevich cover of X. Theorem 1.1 is proved by the following principle: the validity of the Grothendieck–Serre conjecture for G plus $\mathbb{A}^1$-invariance over fields implies Nisnevich-local $\mathbb{A}^1$-invariance; see Proposition 3.5 for a precise statement.

1.2 Towards motivic G-torsors and a conjecture of Morel's

To connect with the Asok–Fasel program, Theorem 1.1 serves as an input to the next result which belongs to $\mathbb{A}^1$-homotopy theory. To state it, let us recall some notation. Denote by $\mathrm{B}_{\text{ét}}G$ the classifying stack for the group G, so that evaluating this stack on a scheme X, the pointed set of connected component of the groupoid $\mathrm{B}_{\text{ét}}G(X)$ is canonically isomorphic to $\mathrm{H}^1_{\text{ét}}(X;G)$ (as pointed sets). Restricting $\mathrm{B}_{\text{ét}}G$ to the category of smooth schemes over a fixed field k we can take its motivic localization $L_{\text{mot}}\mathrm{B}_{\text{ét}}G$ so that if X is a smooth k-scheme, then the set of homotopy classes of maps from X into $L_{\text{mot}}\mathrm{B}_{\text{ét}}G$ computes maps in the $\mathbb{A}^1$-homotopy category:

$$\pi_0(L_{\text{mot}}\mathrm{B}_{\text{ét}}G(X)) \cong [X, \mathrm{B}_{\text{ét}}G]_{\mathbb{A}^1}.$$

We have a canonical map

$$\mathrm{H}^1_{\text{ét}}(X, G) \to [X, \mathrm{B}_{\text{ét}}G]_{\mathbb{A}^1} = \pi_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G)(X).$$

which induces a canonical map of Nisnevich sheaves

$$\mathcal{H}^1_{\text{ét}}(G) \to a_{\text{Nis}}\pi_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G). \tag{3}$$

In general, if $\mathcal{X}$ is a presheaf of spaces on Sm_k, the Nisnevich sheaf $a_{\text{Nis}}\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is the *sheaf of $\mathbb{A}^1$-connected components of $\mathcal{X}$*. A conjecture of Morel posits that $a_{\text{Nis}}\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is an $\mathbb{A}^1$-invariant sheaf. The next theorem furnishes new examples for the validity of this conjecture.

Theorem 1.3. *Let k be a field and G be a group satisfying the hypotheses of Theorem 1.1. Then:*

1. *the canonical map $\mathcal{H}^1_{\text{ét}}(G) \to a_{\text{Nis}}\pi_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G)$ is an isomorphism of Nisnevich sheaves of pointed sets;*
2. *the sheaf $a_{\text{Nis}}\pi_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G)$ is $\mathbb{A}^1$-invariant and is unramified in the sense of* [Mor12, Definition 2.1].

We make several remarks concerning Theorem 1.3.

Remark 1.4. A *motivic G-torsor* is a map $X \to \mathrm{B}_{\text{ét}}G$, i.e., an element of $[X, \mathrm{B}_{\text{ét}}G]_{\mathbb{A}^1}$. Theorem 1.3 provides the first step towards the classification of motivic G-torsors via Postnikov tower methods and gives an algebro-geometric meaning to such a classification. Indeed, in contrast to the situation for vector bundles where the sheaf of $\mathbb{A}^1$-connected components is trivial (which is basically a consequence of "Hilbert theorem 90"), we see that $\mathrm{B}_{\text{ét}}G$ is *not* $\mathbb{A}^1$-connected. Hence, the first stage of the lifting problem of (1) requires that we choose a map $X \to a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G)$ which, after Theorem 1.3 is a Nisnevich-local G-torsor. In many cases, we can say more explicitly what this means. For example, by Proposition 6.1, when $G = \mathrm{O}(n)$ this is the data of an unramified element of the Grothendieck-Witt group of X with a rank n-representative.

Remark 1.5. Theorem 1.3 also introduces the main challenge in attempting to run the Asok–Fasel program to classify motivic G-torsors. Since the classifying stack for GL_r-torsors is $\mathbb{A}^1$-connected, we can point the higher homotopy sheaves $a_{\mathrm{Nis}}\pi_j^{\mathbb{A}^1}(\mathrm{B}_{\mathrm{Zar}}\mathrm{GL}_r \simeq \mathrm{B}_{\text{ét}}\mathrm{GL}_r)$ at the trivial torsor/vector bundle. The lifting problems in (1) are then controlled by Nisnevich cohomology with coefficients in these sheaves. By Theorem 1.3, we have a multitude of choices for base points at which we form the higher homotopy sheaves of $\mathrm{B}_{\text{ét}}G$. This adds an additional complication that one has to deal with when trying to compute these homotopy sheaves. For example, we can point $a_{\mathrm{Nis}}\pi_j^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}\mathrm{PGL}_r)$ at various unramified Brauer classes (see Example 6.2) — at the trivial Brauer class the homotopy sheaves agree with the homotopy sheaves of $\mathrm{B}_{\mathrm{Nis}}\mathrm{SL}_r$ in a range but this will not be the case at other base points.

Remark 1.6. We give a brief survey of the state-of-the-art for Morel's conjecture.

1. Morel made his conjecture in [Mor12, Conjecture 1.12]. This should be contrasted with the main structural results from his book, [Mor12, Theorem 1.9], which states that the higher homotopy sheaves are strongly/strictly $\mathbb{A}^1$-invariant.

2. There are cases where Morel's conjecture holds rather easily — $\mathbb{A}^1$-rigid schemes as well as smooth curves over a field.

3. When G is a finite étale group scheme, the conjecture holds from [MV99, Page 137, Corollary 3.2]. Morel has also claimed [Mor12, Remark 1.14] that he has proved that (3) is a bijection on perfect fields when the group of irreducible components of G is order prime to k and indicated the possibility of a proof of his conjecture in this case, though no details were given.

4. Work of Choudhury [Cho14] has established Morel's conjecture for grouplike presheaves of H-spaces and for principal homogeneous spaces under them. Our arguments in Section 3 are inspired by some of his arguments.

5. In a different direction, work of Balwe–Hogadi–Sawant [BHS15] and Balwe–Sawant [BS20] has verified Morel's conjecture for smooth, projective surfaces in characteristic zero.

Remark 1.7. The key points in the proofs of Theorems 1.1 and 1.3 are the Grothendieck–Serre conjecture for G-torsors as established in [FP15, Pan17] and the $\mathbb{A}^1$-invariance for torsors over fields (established in the characteristic 0 case by [RR84]). The Grothendieck–Serre conjecture allows to reduce questions over irreducible smooth schemes to function fields, where $\mathbb{A}^1$-invariance can be used; this technique was used by Choudhury [Cho14] in his proof of Morel's conjecture for H-groups.

1.3 Unramified torsors after Colliot-Thélène and Sansuc

We now explain a consequence of Theorem 1.3 to the problem of understanding G-torsors on smooth schemes over a field. Let G be a connected reductive group over a field k and let X be an irreducible smooth k-scheme with function field K. We say that *purity* is satisfied for G-torsors over X if the map

$$\begin{aligned} &\operatorname{im}\left(\mathrm{H}^1_{\text{ét}}(X,G)\to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} K,G)\right)\\ \to &\bigcap_{x\in X^{(1)}} \operatorname{im}\left(\mathrm{H}^1_{\text{ét}}(\operatorname{Spec}\mathcal{O}_{X,x},G)\to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} K,G)\right)\end{aligned}$$

is surjective. An old question of Colliot-Thélène and Sansuc [CTS79, Question 6.4] asks when purity holds. The state-of-the-art is reviewed in Remark 4.3. On the one hand, purity is known for local X in a large number of cases. We observe in Section 4 that purity for henselian local X is equivalent to the unramifiedness of the sheaf $\mathcal{H}^1_{\text{ét}}(G)$. This establishes the unramifiedness of $a_{\text{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G$ in the situation of Theorem 1.3. On the other hand, the answer for general X is expected to be negative. More precisely, we have the following conjecture of Antieau and Williams [AW15]:

Conjecture 1.8 ([AW15, Conjecture 1.2]). *Let G be a non-special, semisimple k-group scheme. Then there exists a smooth, affine k-scheme X such that purity fails for G-torsors over X.*

Remark 1.9. The work of [AW15] proves Conjecture 1.8 in the setting of $k=\mathbb{C}$ and $G=\mathrm{PGL}_p$ by constructing examples of dimension $2p+2$. Since purity does hold in dimension ≤ 2, we do not know whether or not purity holds in dimension $d=3,4,5$.

Using our results on the sheaves of $\mathbb{A}^1$-connected components of classifying spaces, we can bring Conjecture 1.8 to the realm of $\mathbb{A}^1$-homotopy theory in a large range of cases, cf. Proposition 6.5:

Corollary 1.10. *Let G be a group over a field k satisfying the hypotheses of Theorem 1.1 and further assume that G satisfies local purity. Then* *if purity holds for G-torsors over X, the sheafification map $\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X) \to a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X)$ is surjective. In particular, if X is a smooth k-scheme for which the sheafification map $\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X) \to a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X)$ is* not *surjective, then we have a counterexample to purity in this setting.*

Proof. Since G satisfies local purity, by Lemma 4.7 $\mathcal{H}^1_{\text{ét}}(G)$ is unramified. Now it follows from Proposition 6.5 that the sheafification map $\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X) \to a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X)$ is surjective. □

It would be interesting to adapt some obstruction-theory methods to identify obstructions to surjectivity of $\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X) \to a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X)$ and thus to purity and better understand what properties of schemes X or groups G imply failure of purity. We defer this to a sequel.

1.4 Structure of the paper

We first recall a couple of preliminaries in Section 2. Homotopy invariance for Nisnevich sheafifications is discussed in Section 3 and unramifiedness for these sheafifications as well as the relation to purity in Section 4. Then we identify the Nisnevich sheaf of étale torsors with the $\mathbb{A}^1$-connected components of the classifying spaces in Section 5 and discuss a couple of examples and consequences in Section 6.

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2 Preliminaries and notation

In this paper, Sm_S denotes the category of smooth schemes over a base S. It is a full subcategory of the category EssSm_S of essentially smooth schemes. We will use (very moderately) the language of ∞-categories which is, by now, standard in the subject. The ∞-category of motivic spaces, as introduced by Morel–Voevodsky [MV99] is denoted by Spc(S) and we refer to [AE17, BH18] for the basics of this construction closer to the language of this paper; in particular, a motivic space is an $\mathbb{A}^1$-invariant Nisnevich sheaf of spaces, often denoted by $\mathcal{X}\colon \mathrm{Sm}_S^{\mathrm{op}} \to \mathcal{S}$. We will denote by $[\mathcal{X},\mathcal{Y}]_{\mathbb{A}^1} = \pi_0\mathrm{Maps}_{\mathrm{Spc(S)}}(\mathcal{X},\mathcal{Y})$ the set of homotopy classes of maps between two motivic spaces. The ∞-category Spc(S) is obtained from the ∞-category of presheaves of spaces on S (written as PreShv(S)) via a combination of two localization endofunctors

$$L_{\mathrm{Nis}}, L_{\mathbb{A}^1} : \mathrm{PreShv(S)} \to \mathrm{PreShv(S)}. \tag{4}$$

Here, L_{Nis} is the usual Nisnevich localization functor, while $L_{\mathbb{A}^1}$ can be modeled by the Sing construction of Suslin, as explained by Morel and Voevodsky in [MV99, Page 87]; see also [AE17, Definition 4.2]. In particular, if $\mathcal{X}$ is a presheaf of spaces, then

$$\pi_0(L_{\mathbb{A}^1}\mathcal{X}(T)) = \pi_0(\mathrm{Sing}\mathcal{X}(T)) = \mathrm{coeq}\left(\pi_0(\mathcal{X}(T \times \mathbb{A}^1)) \rightrightarrows \pi_0(\mathcal{X}(T)\right),$$

which is a formula we will need in this paper. In summary, Spc(S) is the essential image of the localization endofunctor

$$L_{\mathrm{mot}} : \mathrm{PreShv(S)} \to \mathrm{PreShv(S)},$$

which is computed as a transfinite composite of the two functors (4).
One consequence of the above discussion is the so-called unstable $\mathbb{A}^1$-connectivity theorem of Morel–Voevodsky ([MV99, Corollary 3.22], [AE17, Corollary 4.30]). To formulate it, we write $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ (resp. $\pi_0(\mathcal{X})$) for the presheaf $U \mapsto [U, \mathcal{X}]_{\mathbb{A}^1}$ (resp. $U \mapsto [U, \mathcal{X}]$) and we write $a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}(\mathcal{X})$ (resp. $a_{\mathrm{Nis}}\pi_0(\mathcal{X})$) for the Nisnevich sheafification (as presheaves of sets). Then, for any presheaf of spaces $\mathcal{X}$, the map

$$a_{\mathrm{Nis}}\pi_0(\mathcal{X}) \to a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}(\mathcal{X})$$

is an epimorphism of Nisnevich sheaf of sets.
Many arguments in this paper require passing to generic points or to stalks.

Notation 2.1. *Let $\mathcal{C}$ be an ∞-category with colimits. We adopt the following conventions: for a presheaf $\mathcal{F}\colon \mathrm{Sm}_S^{\mathrm{op}} \to \mathcal{C}$ we get a presheaf $\mathrm{EssSm}_S^{\mathrm{op}} \to \mathcal{C}$ by left Kan extension which we abusively also call $\mathcal{F}$. In particular any motivic space canonically extends to a functor out of EssSm_S.*

Remark 2.2. We have a fully faithful immersion $\mathrm{EssSm}_S \hookrightarrow \mathrm{Pro}(\mathrm{Sm}_S)$, where the EssSm_S is the subcategory of pro-objects with affine transition maps. Since any presheaf on Sm_S extends uniquely to one on $\mathrm{Pro}(\mathrm{Sm}_S)$ (via left Kan extension), it uniquely determines a presheaf on EssSm_S by restriction.
Note that the notion of essentially smooth schemes used above is more general than e.g. the one in Morel's book [Mor12], where the schemes are required to be noetherian and the transition maps are required to be étale. On the one hand, this means that the extension of a presheaf via left Kan extension above will in particular be defined for essentially smooth schemes in the stronger sense. On the other hand, the essentially smooth schemes we'll need for our purposes are mostly just (henselian) localizations of smooth schemes, so that they will indeed also be essentially smooth in the stronger sense.

2.1 Torsors

Let X be a scheme and G an X-group scheme. A *G-torsor* Y over a scheme X is an X-scheme Y equipped with a G-action $\rho\colon G \times_X Y \to Y$ such that the following two conditions are satisfied, cf. [stacks, Tag 049A]:

(T1) the map $(\rho, \mathrm{pr}_2)\colon G \times_X Y \to Y \times_X Y$ is an isomorphism of X-schemes, and

(T2) the map $Y \to X$ has sections locally in the fpqc topology; alternatively, the G-torsor Y is fpqc-locally over X isomorphic to the trivial G-torsor $G \times X$.

All of our group schemes in this paper are smooth (which we assume from now on) so that all G-torsors are locally trivial in the étale topology: indeed Y is a smooth X-scheme if G is smooth and thus admits sections locally in the étale topology on Y. A torsor is called *rationally trivial* if there is an open subset U of X such that $Y \times_X U$ is trivial. It follows from the Seshadri's result [Ses63] that all rationally trivial torsors over smooth schemes are locally trivial in Nisnevich topology.

2.2 Nisnevich sheaves of sets

We now recall some notions related to the Nisnevich topology. An étale cover $U \to X$ is Nisnevich if it is surjective on k-points for all fields k [Nis89]. The following pullback diagram

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{\ i\ } & X, \end{array}$$

in S-schemes in Sm_S is called a Nisnevich distinguished square, if i is an open immersion, p is étale and $p^{-1}(X/U) \to (X/U)$ is an isomorphism of schemes, where (X/U) has the reduced induced scheme structure. While the presheaves of interest in this paper come as presheaves of spaces, we will be mostly interested in their homotopy sheaves, in particular their π_0.

Nisnevich sheafification (mostly applied to presheaves of pointed sets) will be denoted by a_{Nis}. We note that the Nisnevich sheafification of a presheaf $\mathcal{F}\colon \mathrm{Sm}_S^{\mathrm{op}} \to \mathrm{Set}_*$ can be defined as follows, cf. [stacks, Tag 00W1]. The starting point is the zeroth Čech cohomology

$$\mathrm{H}^0(\mathcal{U}, \mathcal{F}) = \left\{ (s_i)_{i\in I} \in \prod_i \mathcal{F}(U_i) \middle| s_i|_{U_i \times_X U_j} = s_j|_{U_i \times_X U_j} \forall i, j \in I \right\}$$

for a covering $\mathcal{U} = \{U_i \to X\}_{i\in I}$, given by compatible families of sections over the elements U_i of the covering. One can then define a presheaf $\mathcal{F}^+$, whose sections $\mathcal{F}^+(X)$ over X are obtained by taking the colimit $\mathrm{colim}_{\mathcal{U}}\, \mathrm{H}^0(\mathcal{U}, \mathcal{F})$ over all coverings $\mathcal{U}$ of X. The sheafification is then be obtained as $\mathcal{F}^{++}$, iterating the $+$-construction twice. All we need from this discussion is that a section of the sheafification $a_{\mathrm{Nis}}\mathcal{F}$ is given by a collection of sections of $\mathcal{F}$ over the henselizations $\mathcal{O}^{\mathrm{h}}_{X,x}$ of the local rings of points $x \in X$, subject to compatibility conditions.

Let $\mathcal{F}(-) = \mathrm{H}^1_{\text{ét}}(-, G)$ denote the presheaf associating to a smooth scheme the set of isomorphism classes of G-torsors, pointed by the trivial torsor. The Nisnevich sheafification of this presheaf is denoted by $\mathcal{H}^1_{\text{ét}}(G)$.
The following is a list of properties of presheaves which we will use throughout the paper.

Definition 2.3. Let F be a field and $\mathcal{F}\colon \mathrm{EssSm}_F^{\mathrm{op}} \to \mathrm{Set}_*$ be a presheaf of pointed sets

1. The presheaf $\mathcal{F}$ is called *finitary* if it converts cofiltered limits to filtered colimits, i.e., it is left Kan extended from Sm_F.
2. The presheaf $\mathcal{F}$ is called *homotopy invariant* if for any $X \in \mathrm{EssSm}_F$ we have an induced isomorphism $\mathcal{F}(X) \xrightarrow{\cong} \mathcal{F}(X \times \mathbb{A}^1)$.
3. The presheaf $\mathcal{F}$ has the (strong) *Grothendieck–Serre property* if for any regular local F-algebra R with fraction field K, the map $\mathcal{F}(\operatorname{Spec} R) \to \mathcal{F}(\operatorname{Spec} K)$ has trivial kernel (is injective).
4. The presheaf $\mathcal{F}$ is *Nisnevich lexcisive* if for a Nisnevich distinguished square
$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ U & \longrightarrow & Y, \end{array}$$
the map
$$\mathcal{F}(Y) \to \mathcal{F}(X) \times_{\mathcal{F}(W)} \mathcal{F}(U)$$
is a surjective map of sets.
5. The presheaf $\mathcal{F}$ is *Nisnevich lexcisive* if it takes a Nisnevich distinguished square as above to a pullback square of pointed sets.

Remark 2.4. The Nisnevich lexcisive condition above may not be familiar. As the name suggests, it is a weaker form of excision which is satisfied for classifying spaces and allows just enough gluing of sections over curves sufficient for the homotopy invariance arguments in Section 3. It is also similar to a gluing condition appearing in other homotopy invariance proofs for torsors, cf. the formalism in [AHW20].

3 Homotopy invariance for Nisnevich sheafifications

In the following section we discuss the homotopy invariance of Nisnevich sheafifications, with a particular view toward $\mathcal{H}^1_{\text{ét}}(G)$. Essentially, any finitary presheaf which satisfies a strong version of the Grothendieck–Serre conjecture (i.e., that restricting sections from local rings to their function fields is injective) and $\mathbb{A}^1$-invariance over fields has a homotopy invariant Nisnevich sheafification. A similar method has already been used by Choudhury [Cho14] for proving that $a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is homotopy invariant for H-groups and principal homogeneous spaces under H-groups.

We first note that the strong Grothendieck–Serre property (which is about local rings) actually implies that sections over smooth schemes are detected on the function fields:

Proposition 3.1. *Let $\mathcal{F}$ be a finitary presheaf on* EssSm_F *which satisfies the strong Grothendieck–Serre property. Then for every irreducible essentially smooth F-scheme X with function field K the restriction map $a_{\mathrm{Nis}}\mathcal{F}(X) \to a_{\mathrm{Nis}}\mathcal{F}(\operatorname{Spec} K) \cong \mathcal{F}(\operatorname{Spec} K)$ to the generic point is injective.*

Proof. From the description of the sheafification, cf. Section 2, we have the following commutative square in which the left vertical map is injective:

$$\begin{array}{ccc} a_{\mathrm{Nis}}\mathcal{F}(X) & \longrightarrow & \mathcal{F}(\operatorname{Spec} K) \\ \big\downarrow & & \big\downarrow \\ \prod_{x\in X} \mathcal{F}(\operatorname{Spec} \mathcal{O}^{\mathrm{h}}_{X,x}) & \longrightarrow & \prod_{x\in X} \mathcal{F}(\operatorname{Spec} \operatorname{Frac}(\mathcal{O}^{\mathrm{h}}_{X,x})) \end{array}$$

It therefore suffices to check injectivity of the bottom map. But that follows since $\mathcal{F}$ has the strong Grothendieck–Serre property. ☐

Remark 3.2. Actually, the strong Grothendieck–Serre property implies the injectivity already for the Zariski sheafification. For the Nisnevich result, a strong Grothendieck–Serre property for henselian local rings would suffice.

In the following, we provide versions of [Cho14, Lemma 3.3, Theorem 3.1]. These say that sections of Nisnevich sheafifications of finitary presheaves satisfying Nisnevich descent are induced from presheaf sections defined at all codimension 1 points. These results will be needed for reducing the general homotopy invariance for the Nisnevich sheafification to the invariance for $\mathbb{A}^1$ over fields for the presheaf, as input in Proposition 3.5. They also play some role in the discussion of unramifiedness and purity later.

Proposition 3.3. *Let $\mathcal{F}$ be a finitary presheaf on* EssSm_F *satisfying Nisnevich lexcision and let R be an essentially smooth discrete valuation ring. Then the sheafification map is surjective*

$$\mathcal{F}(R) \twoheadrightarrow a_{\mathrm{Nis}}\mathcal{F}(R).$$

Proof. Denote by R^{h} the henselization of R at the maximal ideal, hence $a_{\mathrm{Nis}}\mathcal{F}(R^{\mathrm{h}}) \cong \mathcal{F}(R^{\mathrm{h}})$. The henselization R^{h} is a filtered colimit of étale neighbourhoods of the closed point of $\operatorname{Spec} R$. From the corresponding filtered limit of Nisnevich distinguished squares, we obtain

$$a_{\mathrm{Nis}}\mathcal{F}(R) = \mathcal{F}(R^{\mathrm{h}}) \times_{\mathcal{F}(\operatorname{Frac}(R^{\mathrm{h}}))} \mathcal{F}(\operatorname{Frac}(R)).$$

Now since $\mathcal{F}$ is lexcisive the map $\mathcal{F}(R) \to \mathcal{F}(R^{\mathrm{h}}) \times_{\mathcal{F}(\operatorname{Frac}(R^{\mathrm{h}}))} \mathcal{F}(\operatorname{Frac}(R))$ is surjective so the required surjectivity follows. ☐

Proposition 3.4. *Let $\mathcal{F}$ be a finitary presheaf on* EssSm_F *which satisfies Nisnevich lexcision. Then for any essentially smooth F-scheme X of dimension ≤ 1, the sheafification map $\mathcal{F}(X) \to a_{\mathrm{Nis}}\mathcal{F}(X)$ is surjective.*

Proof. The proof of [Cho14, Theorem 3.1] goes through almost verbatim. Consider an essentially smooth scheme $X \in \mathrm{EssSm}_F$ of dimension 1 with function field K. We can reduce to the case that X is connected. Since $\mathcal{F}$ is finitary, we can reduce to the case that X is noetherian.
Let $\sigma \in a_{\mathrm{Nis}}\mathcal{F}(X)$. By the description of Nisnevich sheafification, cf. Section 2, this means we are given $\sigma_x \in a_{\mathrm{Nis}}\mathcal{F}(\operatorname{Spec} \mathcal{O}^{\mathrm{h}}_{X,x})$ for any codimension 1 point $x \in$ and a generic section $\sigma_\eta \in a_{\mathrm{Nis}}\mathcal{F}(\operatorname{Spec} K) = \mathcal{F}(\operatorname{Spec} K)$ and the restrictions of σ_x to $\operatorname{Spec} K$ agree with σ_η. Take any cover of X, this cover can be refined by a distinguished Nisnevich square. Now since $\mathcal{F}$ is lexcisive and the local sections have to agree with σ_η we get the required section of $\mathcal{F}(X)$. ☐

The following results is in some way already contained in [Cho14]. Choudhury's proof of Morel's conjecture for H-groups is based on the fact that for sheaves of groups the weak and strong Grothendieck–Serre property agree and the weak version of the Grothendieck–Serre property follows since $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is a finitary homotopy-invariant presheaf satisfying Nisnevich excision.

Proposition 3.5. *Let $\mathcal{F}$ be a finitary and Nisnevich lexcisive presheaf on* EssSm_F *which satisfies the following:*

(a) $\mathcal{F}$ has the strong Grothendieck–Serre property.

(b) For any finitely generated field extension L/F the projection $\mathbb{A}^1_L \to \operatorname{Spec} L$ induces a bijection $a_{\mathrm{Nis}}\mathcal{F}(\operatorname{Spec} L) \xrightarrow{\cong} a_{\mathrm{Nis}}\mathcal{F}(\mathbb{A}^1_L)$.

Then $a_{\mathrm{Nis}}\mathcal{F}$ is homotopy invariant.

Proof. We want to show that for any smooth scheme X the projection $\mathrm{pr}_1 \colon X \times \mathbb{A}^1 \to X$ induces a bijection

$$a_{\mathrm{Nis}}\mathcal{F}(X) \xrightarrow{\cong} a_{\mathrm{Nis}}\mathcal{F}(X \times \mathbb{A}^1).$$

The map $X \xrightarrow{0} X \times \mathbb{A}^1 \xrightarrow{\mathrm{pr}_1} X$ is the identity. Hence showing the surjectivity of the above map is equivalent to injectivity of the map $a_{\mathrm{Nis}}\mathcal{F}(X \times \mathbb{A}^1) \to a_{\mathrm{Nis}}\mathcal{F}(X)$ induced by the zero section. Consider the following commutative diagram

$$\begin{array}{ccc} a_{\mathrm{Nis}}\mathcal{F}(X \times \mathbb{A}^1) & \longrightarrow & a_{\mathrm{Nis}}\mathcal{F}(\mathbb{A}^1_{K(X)}) \\ \downarrow & & \downarrow \\ a_{\mathrm{Nis}}\mathcal{F}(X) & \longrightarrow & \mathcal{F}(K(X)) \end{array}$$

where the vertical maps are restriction along zero sections and the horizontal maps are restrictions to open subschemes. The top and bottom maps are injective because $\mathcal{F}$ satisfies the strong Grothendieck–Serre property and Proposition 3.1. The right vertical map is a bijection because from assumption (b), the projection $\mathbb{A}^1_{K(X)} \to \operatorname{Spec} K(X)$ induces a bijection $a_{\mathrm{Nis}}\mathcal{F}(K(X)) \xrightarrow{\cong} a_{\mathrm{Nis}}\mathcal{F}(\mathbb{A}^1_{K(X)})$. Therefore, the left vertical map is also injective. □

Note that since we can glue torsors in the Nisnevich topology, the presheaf $\mathrm{H}^1_{\text{ét}}(-,G)$ satisfies Nisnevich lexcision. Now we now want to apply this result to prove homotopy invariance for the Nisnevich sheaf $\mathcal{H}^1_{\text{ét}}(G)$. To do that, we recall some results on $\mathbb{A}^1$-invariance of étale torsors over fields. The most general result we know of is the following, from [RR84].

Theorem 3.6. *Let F be a field with separable closure F^{sep} and let G be a connected reductive group. Then any G-torsor over $\mathbb{A}^1_F$ which becomes trivial over $\mathbb{A}^1_{F^{\mathrm{sep}}}$ is extended from F, i.e., pullback induces a bijection*

$$\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} F, G) \to \ker\left(\mathrm{H}^1_{\text{ét}}(\mathbb{A}^1_F, G) \to \mathrm{H}^1_{\text{ét}}(\mathbb{A}^1_{F^{\mathrm{sep}}}, G)\right).$$

In particular, for a field F of characteristic 0, étale torsors satisfy $\mathbb{A}^1$-invariance over all extension fields L/F in the sense that the pullback map $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} L, G) \to \mathrm{H}^1_{\text{ét}}(\mathbb{A}^1_L, G)$ is a bijection. In this case, we get homotopy invariance for $\mathcal{H}^1_{\text{ét}}(G)$ for any connected reductive group G:

Proposition 3.7. *Let F be a field of characteristic 0 and G be a connected reductive group over F. Then the sheaf $\mathcal{H}^1_{\text{ét}}(G)$ is homotopy invariant.*

Proof. We use Proposition 3.5 with the presheaf $U \mapsto \mathrm{H}^1_{\text{ét}}(U,G)$. The strong Grothendieck–Serre property follows from [FP15]. For the $\mathbb{A}^1$-invariance over extension fields L/F, we note that by Proposition 3.4 the map $\mathrm{H}^1_{\text{ét}}(\mathbb{A}^1_L, G) \to \mathcal{H}^1_{\text{ét}}(G)(\mathbb{A}^1_L)$ is surjective, i.e., any section of $\mathcal{H}^1_{\text{ét}}(G)$ is actually induced from a G-torsor over $\mathbb{A}^1_L$. After base change to the separable closure L^{sep}, the resulting G-torsor over $\mathbb{A}^1_{L^{\mathrm{sep}}}$ becomes trivial, by a theorem of Steinberg [Ste65, Theorem 1.9]. Thus we can apply Theorem 3.6 to see that the G-torsor over $\mathbb{A}^1_L$ must be constant, finishing the proof. □

Remark 3.8. This generalizes the observation that for Parimala's non-extended quadratic form over $\mathbb{A}^2_{\mathbb{R}}$, there is an open subset of $\mathbb{A}^2_{\mathbb{R}}$ where the torsor is constant and extended from the anisotropic form over $\mathbb{R}$, cf. [Lam06, Lemma VII.4.14 resp. Lemma VII.4.16].

We note two other cases of interest over fields of positive characteristic where we get homotopy invariance for $\mathcal{H}^1_{\text{ét}}(G)$.

Proposition 3.9. *Let F be a field of characteristic $\neq 2$. Then the sheaf $\mathcal{H}^1_{\text{ét}}(\mathrm{O}(n))$ is homotopy invariant.*

Proof. As in the proof of Proposition 3.7, we apply Proposition 3.5 to the presheaf $U \mapsto \mathrm{H}^1_{\text{ét}}(U,G)$ using the strong Grothendieck–Serre property from [FP15] (for infinite fields) and [Pan17] (for finite fields). The $\mathbb{A}^1$-invariance for $\mathrm{O}(n)$-torsors over fields of characteristic $\neq 2$ is Harder's theorem, cf. [Kne70, Theorem 13.4.3] resp. [Lam06, Theorem VII.3.13]. □

Proposition 3.10. *Let F be a field of characteristic p. If $p \nmid n$, then the sheaf $\mathcal{H}^1_{\text{ét}}(\mathrm{PGL}_n)$ is homotopy invariant.*

Proof. As before, we need only deal with the $\mathbb{A}^1$-invariance over fields. By Theorem 3.6, it suffices to show that for L a separably closed extension field of F all PGL_n-torsors over $\mathbb{A}^1_L$ are trivial. Consider the exact sequence (of pointed sets)

$$\mathrm{H}^1_{\text{ét}}(\mathbb{A}^1_L, \mathrm{GL}_n) \to \mathrm{H}^1_{\text{ét}}(\mathbb{A}^1_L, \mathrm{PGL}_n) \to \mathrm{H}^2_{\text{ét}}(\mathbb{A}^1_L, \mathbb{G}_{\mathrm{m}})$$

associated to the extension $\mathbb{G}_{\mathrm{m}} \to \mathrm{GL}_n \to \mathrm{PGL}_n$. Here $\mathrm{H}^2_{\text{ét}}(\mathbb{A}^1_L, \mathbb{G}_{\mathrm{m}})$ is the cohomological Brauer group which in turn is identified with the Brauer group $\mathrm{Br}(\mathbb{A}^1_L)$, cf. [Gro68, Corollary 2.2]. Exactness for the sequence of pointed sets then means that any PGL_n-torsor whose Brauer class is trivial comes from a vector bundle (and therefore has to be trivial). By the Auslander–Goldman theorem (together with our assumption that L is separably closed), $\mathrm{Br}(\mathbb{A}^1_L)$ is a p-torsion group. Since the boundary map factors as

$$\mathrm{H}^1_{\text{ét}}(\mathbb{A}^1_L, \mathrm{PGL}_n) \to \mathrm{H}^2_{\text{ét}}(\mathbb{A}^1_L, \mu_n) \to \mathrm{H}^2_{\text{ét}}(\mathbb{A}^1_L, \mathbb{G}_{\mathrm{m}})$$

and $\mathrm{H}^2_{\text{ét}}(\mathbb{A}^1_L, \mu_n)$ is an n-torsion group, our assumption $p \nmid n$ implies that PGL_n-torsors over $\mathbb{A}^1_L$ have trivial Brauer classes. The exact cohomology sequence now implies that every PGL_n-torsor over $\mathbb{A}^1_L$ is trivial, as required. □

Remark 3.11. In [KOS76], there are examples of PGL_n-torsors over the affine line $\mathbb{A}^1_L$ for separably closed but non-algebraically closed fields L which are not extended. These are related to non-trivial p-coverings of $\mathbb{A}^1_L$, where $p = \mathrm{char}(L)$ (in particular these torsors are not rationally trivial). This means that the divisibility condition in Proposition 3.10 is necessary and homotopy invariance generally fails for PGL_n-torsors if the characteristic of the base field divides n.

There are two results for exceptional groups, we observe the following homotopy invariance results which does not require our axiomatics.

Proposition 3.12.
1. *If F is a field of characteristic $\neq 2$, then the presheaf $\mathcal{H}^1_{\text{ét}}(\mathrm{G}_2)$ is homotopy invariant;*
2. *if F is a field of characteristic $\neq 2, 3$, then the presheaf $\mathcal{H}^1_{\text{ét}}(\mathrm{F}_4)$ is homotopy invariant.*

Proof. As in Proposition 3.10, we can reduce to the case of F separably closed (with the above assumptions on characteristics). For the groups we consider, a G-torsor $\mathcal{P}$ over $\mathbb{A}^1_L$ has trivial restriction to the function field $L(T)$ if and only if the relevant cohomological invariants are trivial, cf. [Ser95]. For $G = \mathrm{G}_2$, the classifying cohomological invariant is the class of the norm form in $\mathrm{H}^3_{\text{ét}}(-, \mu_2)$, and for $G = \mathrm{F}_4$, the cohomological invariants are Pfister forms $f_3 \in \mathrm{H}^3_{\text{ét}}(-, \mu_2)$ and $f_5 \in \mathrm{H}^5_{\text{ét}}(-, \mu_2)$ and the Rost invariant $g_3 \in \mathrm{H}^3_{\text{ét}}(-, \mu_3)$. Since L is separably closed, all these étale cohomology groups vanish for $L(T)$

which has étale cohomological dimension 1. Therefore, the restriction of the G-torsor $\mathcal{P}$ to the generic point of $\mathbb{A}^1_L$ is trivial because it has trivial cohomological invariants. Since rationally trivial torsors satisfy homotopy invariance, $\mathcal{P}$ has to be trivial. □

Remark 3.13. This argument via rationally trivial torsors could also be helpful to establish homotopy invariance of $\mathcal{H}^1_{\text{ét}}(G)$ for other cases of connected reductive groups G in positive characteristic. Extensions of these arguments were used in [BHS22, Theorem 3.8] to show $\mathbb{A}^1$-invariance of $\mathcal{H}^1_{\text{ét}}(G)$ for semisimple, simply connected groups over fields in arbitrary characteristic.
The most general result one could expect here is that $\mathbb{A}^1$-invariance (and thus homotopy invariance generally) holds for a reductive group G over a field F of characteristic p if both the group $\pi_0(G)$ of connected components of G and the Chevalley fundamental group $\Pi_1(G)$ of G have orders prime to the characteristic of the base field, and p is not a torsion prime for any of the almost simple components of G. However, results in this generality seem not known at this point.

4 Purity for torsors

In this section we show that unramifiedness of the sheaf $\mathcal{H}^1_{\text{ét}}(G)$ is equivalent to Nisnevich-local purity for G-torsors. In situations where local purity results are known, this allows to compute $\mathcal{H}^1_{\text{ét}}(G)$ and to reinterpret purity for torsors as surjectivity of the sheafification map $\mathrm{H}^1_{\text{ét}}(X,G) \to \mathcal{H}^1_{\text{ét}}(G)(X)$ for smooth schemes X. It will also provide a relation between global purity questions and motivic homotopy, cf. Section 6.

Definition 4.1. Let G be a connected reductive group over a field F and let X be an irreducible smooth F-scheme with function field K. We say that *purity* is satisfied for G-torsors over X if the map

$$\begin{aligned} &\operatorname{im}\left(\mathrm{H}^1_{\text{ét}}(X,G) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} K,G)\right) \\ \to \bigcap_{x\in X^{(1)}} &\operatorname{im}\left(\mathrm{H}^1_{\text{ét}}(\operatorname{Spec}\mathcal{O}_{X,x},G) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} K,G)\right)\end{aligned}$$

is surjective.

Remark 4.2. If G is a special group in the sense of Serre, i.e., G-torsors over reduced F-varieties are Zariski-locally trivial then $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} K, G) = \{*\}$ so that Definition 4.1 is trivially satisfied.

Remark 4.3. The question of purity for torsors was formulated by Colliot-Thélène and Sansuc in [CTS79, Question 6.4]. They also proved a purity theorem for the case of integral regular schemes X of dimension 2 and reductive groups G, cf. [CTS79, Theorem 6.13 resp. Corollary 6.14]. Local purity, i.e., purity for regular local rings, is known in many cases of connected reductive groups over characteristic 0 fields, e.g. most of the classical groups [Pan10].

Local purity for PGL_n-torsors whose Brauer class has invertible exponent has been proved in [AW15, Theorem 3.10]. Local purity for the orthogonal groups in characteristic $\neq 2$ has been proved in [PP10]. Local purity for G_2-torsors over fields of characteristic $\neq 2$ has been proved in [CP07] (in combination with [PP10, Remark 3.2]). Local purity for F_4 torsors with trivial g_3 invariant in characteristic $\neq 2$ has been proved in [CP13].
However, it seems at this point that local purity hasn't been proved in general (and uniformly, not based on case-by-case analysis), not even for henselian regular local rings. Most of the proofs seem to work via stabilization in infinite series (for the classical groups) combined with the additional benefits of a group structure for the stable groups, relations to quadratic forms or via cohomological invariants (for G_2 and F_4).

For the sake of completeness, let us recall the following notion from [Mor12, Definition 2.1].

Definition 4.4. Let F be a field. A finitary presheaf $\mathcal{G} : \mathrm{EssSm}_F^{\mathrm{op}} \to \mathrm{Sets}$ is said to be *unramified* if the following hold:

1. if $X \in \mathrm{Sm}_F$ is a union of irreducible components X_α then $\mathcal{G}(X) \to \prod_\alpha \mathcal{G}(X_\alpha)$ is a bijection,
2. for any $X \in \mathrm{Sm}_F$ and any dense open $U \subset X$, the map $\mathcal{G}(X) \to \mathcal{G}(U)$ is injective and,
3. if $X \in \mathrm{Sm}_F$ is furthermore irreducible and $F(X)$ is its function field, then the injection (of subsets of $\mathcal{G}(F(X))$
$$\mathcal{G}(X) \to \cap_{x \in X^{(1)}} \mathcal{G}(\mathcal{O}_{X,x})$$
is a bijection.

We note that any unramified presheaf is, in fact, a Zariski sheaf [Mor12, Remark 2.2].

Lemma 4.5. *Let $\mathcal{G}$ be a finitary presheaf on EssSm_F which is furthermore a Nisnevich sheaf and satisfies (1) and (2) of Definition 4.4. Then $\mathcal{G}$ is unramified if and only if it satisfies:*

(3') for every $X \in \mathrm{Sm}_F$ and any $x \in X$, writing $Y = \operatorname{Spec} \mathcal{O}^{\mathrm{h}}_{X,x}$, the injection (of subsets of $\mathcal{G}(\mathrm{Frac}(\mathcal{O}^{\mathrm{h}}_{X,x}))$):
$$\mathcal{G}(Y) \to \bigcap_{y \in Y^{(1)}} \mathcal{G}(\mathcal{O}_{Y,y})$$
is a bijection.

Proof. If $\mathcal{G}$ is unramified, then (3′) is clearly satisfied. We now assume (3′). Let $X \in \mathrm{Sm}_F$ be irreducible and consider the injective map:

$$\mathcal{G}(X) \to \bigcap_{x \in X^{(1)}} \mathcal{G}(\mathcal{O}_{X,x}),$$

appearing in condition (3) of Definition 4.4. We need to prove that this map is surjective. Noting that $\mathcal{G}$ is a Nisnevich sheaf it suffices to produce compatible sections of $\mathcal{G}(\mathcal{O}^{\mathrm{h}}_{X,y})$ as y ranges across all points of $y \in X$.

To this end, fix $y \in X$ and let $Y = \operatorname{Spec} \mathcal{O}^{\mathrm{h}}_{X,y}$. An element α of $\bigcap_{x \in X^{(1)}} \mathcal{G}(\mathcal{O}_{X,x})$ is a section of $\mathcal{G}$ over the function field K of X which extends over all codimension 1 points of X. Now, for every $\mathfrak{p} \in (\operatorname{Spec} \mathcal{O}^{\mathrm{h}}_{X,y})^{(1)}$, there exists a $\mathfrak{q} \in (\operatorname{Spec} \mathcal{O}_{X,y})^{(1)}$ such that the following diagram of local rings and local homomorphisms commutes:

$$\begin{array}{ccc} \mathcal{O}_{X,y} & \longrightarrow & \mathcal{O}^{\mathrm{h}}_{X,y} \\ \downarrow & & \downarrow \\ (\mathcal{O}_{X,y})_{\mathfrak{q}} & \longrightarrow & (\mathcal{O}^{\mathrm{h}}_{X,y})_{\mathfrak{p}}. \end{array}$$

Since codimension one points of $\operatorname{Spec} \mathcal{O}_{X,x}$ are just codimension one points of X in the closure of x, we obtain a section of $\mathcal{G}((\mathcal{O}^{\mathrm{h}}_{Y,y})_{\mathfrak{p}})$ from α. By construction, the restriction of this section to $\mathrm{Frac}(\mathcal{O}^{\mathrm{h}}_{Y,y})$ agrees with the restriction of α along the inclusion $K \hookrightarrow \mathrm{Frac}(\mathcal{O}^{\mathrm{h}}_{Y,y})$. Varying $\mathfrak{p} \in Y$, this defines a section of $\bigcap_{\mathfrak{p} \in Y^{(1)}} \mathcal{G}(\mathcal{O}_{Y,\mathfrak{p}})$ and thus a section of $\mathcal{G}(Y)$ by (3') whose restriction to $\mathrm{Frac}(\mathcal{O}^{\mathrm{h}}_{X,y})$ agrees with the restriction of α along the inclusion $K \hookrightarrow \mathrm{Frac}(\mathcal{O}^{\mathrm{h}}_{Y,y})$. For the compatibility of the local sections constructed this way, note that by the assumptions (1) and (2), the sections are uniquely determined by their values on generic points. Since all the local sections agree with the restriction of α along the respective inclusions $K \hookrightarrow \mathrm{Frac}(\mathcal{O}^{\mathrm{h}}_{Y,y})$, they all agree generically, hence they are compatible. This concludes the proof. ◻

The resolution of the Grothendieck–Serre conjecture, due to Fedorov and Panin [FP15], lets us reformulate local henselian purity in terms of unramifiedness.

Lemma 4.6. *Let F be a field, and let G be a reductive group over F.*

1. *For a discrete valuation ring R over F with fraction field K, there is an equality of subsets of $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} K, G)$:*

$$\operatorname{im}\left(\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} R, G) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} K, G)\right) = \mathcal{H}^1_{\text{ét}}(G)(\operatorname{Spec} R).$$

2. *For a smooth irreducible F-scheme X with function field K, there is an equality of subsets of $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} K, G)$:*

$$\bigcap_{x \in X^{(1)}} \operatorname{im}\left(\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} \mathcal{O}_{X,x}, G) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} K, G)\right) = \bigcap_{x \in X^{(1)}} \mathcal{H}^1_{\text{ét}}(G)(\mathcal{O}_{X,x})$$

Proof. We first note that for a discrete valuation ring R with fraction field K the sheafification map induces a bijection

$$\operatorname{im}\left(\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} R, G) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} K, G)\right) \xrightarrow{\cong} \mathcal{H}^1_{\text{ét}}(G)(\operatorname{Spec} R).$$

The injectivity follows from the strong Grothendieck–Serre property [FP15, Corollary 1] and the surjectivity follows from Proposition 3.3. Combined with the natural inclusion $\mathcal{H}^1_{\text{ét}}(G)(\operatorname{Spec} R) \hookrightarrow \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} K, G)$, we actually get the identification of subsets of $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} K, G)$ claimed in (1). The identification in (2) follows from (1), noting that the indexing sets for both intersections are the same. □

Lemma 4.7. *Let F be a field, and let G be a reductive group over F. Purity for henselizations $R = \mathcal{O}^{\mathrm{h}}_{X,x}$ of local rings of smooth F-schemes is equivalent to unramifiedness of $\mathcal{H}^1_{\text{ét}}(G)$.*

Proof. Let $R = \mathcal{O}^{\mathrm{h}}_{X,x}$ be the henselization of an irreducible smooth F-scheme X at a point $x \in X$. Denote $Y = \operatorname{Spec} R$ and $K = \operatorname{Frac}(R)$ and consider the diagram

$$\begin{array}{ccc}
\mathrm{H}^1_{\text{ét}}(Y, G) & \longrightarrow & \bigcap_{y \in Y^{(1)}} \operatorname{im}\left(\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} \mathcal{O}_{Y,y}, G) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} K, G)\right) \\
\downarrow & & \downarrow \\
\mathcal{H}^1_{\text{ét}}(G)(Y) & \longrightarrow & \bigcap_{y \in Y^{(1)}} \mathcal{H}^1_{\text{ét}}(G)(\mathcal{O}_{Y,y})
\end{array}$$

The diagram is commutative since the vertical sheafification maps commute with the horizontal restrictions to $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} K, G) \cong \mathcal{H}^1_{\text{ét}}(G)(\operatorname{Spec} K)$. The right vertical arrow is a bijection by Lemma 4.6, and the left vertical arrow is a bijection by the assumptions on R. Therefore, the upper horizontal map is a bijection if and only if the lower horizontal map is. But (by definition) the upper horizontal arrow is a surjection if and only if purity holds.
Now, Proposition 3.1 ensures that $\mathcal{H}^1_{\text{ét}}(G)(X) \to \mathcal{H}^1_{\text{ét}}(G)(K)$ is injective. Lemma 4.5 implies that $\mathcal{H}^1_{\text{ét}}(G)$ is unramified if and only if the lower horizontal map is an isomorphism. □

Remark 4.8. Purity for local rings $\mathcal{O}_{X,x}$ of smooth F-schemes would follow from the above if the Zariski sheafification of the presheaf $U \mapsto \mathrm{H}^1_{\text{ét}}(U, G)$ agrees with the Nisnevich sheafification $\mathcal{H}^1_{\text{ét}}(G)$.
Conversely, if purity holds for local rings, then the Zariski sheafification agrees with $\mathcal{H}^1_{\text{ét}}(G)$ (and is unramified). This follows with an argument similar to the proof of Lemmas 4.6 and 4.7, with purity providing the local sections which by Grothendieck–Serre are determined by the value on the generic point and thus compatible. Therefore, the subsets $a_{\mathrm{Zar}}\mathrm{H}^1_{\text{ét}}(X, G)$, $a_{\mathrm{Nis}}\mathrm{H}^1_{\text{ét}}(X, G) = \mathcal{H}^1_{\text{ét}}(G)(X)$ and $\bigcap_{x \in X^{(1)}} \operatorname{im}\left(\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} \mathcal{O}_{X,x}, G) \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} K, G)\right)$ of $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} K, G)$ agree.

Now at this point, we cannot prove that $\mathcal{H}^1_{\text{ét}}(G)$ is unramified in general. However, we would like to point out a few facts that strongly suggest Nisnevich local purity should be true: there is a morphism $\mathcal{H}^1_{\text{ét}}(G) \to \mathcal{T}$ to an unramified and homotopy invariant Nisnevich sheaf $\mathcal{T}$ which induces bijections over essentially smooth F-schemes of dimension ≤ 1. In the presence of group structures on the sheaves, this would be enough to imply that the morphism is an isomorphism, but since we only have sheaves of pointed sets, we don't know how to prove the surjectivity.
For the construction of the unramified sheaf $\mathcal{T}$, it is clear that the sections over a smooth F-scheme have to be the unramified sections of $\mathcal{H}^1_{\text{ét}}(G)$:

$$\mathcal{T}(X) := \bigcap_{x \in X^{(1)}} \mathcal{H}^1_{\text{ét}}(G)(\operatorname{Spec} \mathcal{O}_{X,x}) \subseteq \mathcal{H}^1_{\text{ét}}(G)(\operatorname{Spec} K).$$

For the presheaf structure, i.e., the existence of suitable restriction maps, we have to take a little detour.

Remark 4.9. Applying the above definition to an arbitrary presheaf $\mathcal{F}$ via

$$\mathcal{F}_{\text{nr}}(X) := \bigcap_{x \in X^{(1)}} \mathcal{F}(\mathcal{O}_{X,x}) \subseteq \mathcal{F}(K).$$

doesn't generally produce a presheaf. The immediate problem is with the definition of restriction maps for non-smooth morphisms of smooth schemes. Conditions for the functoriality of the unramified sections were already discussed in [CTS79, Section 6]. We will discuss another approach to the functoriality of unramified sections which makes use of purity in codimension 2 and the theory of unramified sheaves of [Mor12, Section 2].

Proposition 4.10. *Let F be a field and let $\mathcal{H}$ be a finitary Nisnevich sheaf on* Sm_F *having the strong Grothendieck–Serre property. Assume that $\mathcal{H}$ satisfies purity in dimension 2, i.e., for any irreducible essentially smooth F-scheme X of dimension* 2 *with function field K, we have an identification*

$$\mathcal{H}(X) = \bigcap_{x \in X^{(1)}} \mathcal{H}(\operatorname{Spec} \mathcal{O}_{X,x})$$

of subsets in $\mathcal{H}(\operatorname{Spec} K)$.
Then the following defines an unramified $\mathcal{F}_F$-datum in the sense of [Mor12, Definition 2.9].

(D1) We define a continuous contravariant functor $\mathcal{S}$ on the category $\mathcal{F}_F$ of separable field extensions of F by $\mathcal{S}(L/F) := \mathcal{H}(L/F)$.

(D2) For any separable field extension L/F and any discrete valuation v on L with valuation ring $\mathcal{O}_v$, we define $\mathcal{S}(\mathcal{O}_v) := \mathcal{H}(\operatorname{Spec} \mathcal{O}_v) \subseteq \mathcal{H}(\operatorname{Spec} L) = \mathcal{S}(L/F)$.

(D3) For any separable field extension L/F and any discrete valuation v on L with valuation ring $\mathcal{O}_v$ and residue field $\kappa(v)$, we define the specialization map $s\colon \mathcal{S}(\mathcal{O}_v) \to \mathcal{S}(\kappa(v))$ to be the restriction map $\mathcal{H}(\operatorname{Spec} \mathcal{O}_v) \to \mathcal{H}(\operatorname{Spec} \kappa(v))$.

Proof. We prove that $\mathcal{S}$ is an unramified $\mathcal{F}_F$-datum, i.e., we need to check the axioms (A1)-(A4) from [Mor12, Definition 2.6, 2.9]. We first note that the continuity of the functor $\mathcal{S}$ follows from the assumption that $\mathcal{H}$ is finitary.

Axiom (A1) is essentially a Nisnevich descent statement for extensions of dvrs, and it holds for $\mathcal{S}$ since $\mathcal{H}$ is a Nisnevich sheaf. In particular, the required map $\mathcal{S}(\mathcal{O}_w) \to \mathcal{S}(\mathcal{O}_v)$ is the restriction map for $\mathcal{H}$, and the sheaf property implies that the relevant square is cartesian.

Axiom (A2) follows from the assumption that $\mathcal{H}$ is finitary. If X is an irreducible smooth F-scheme with function field K, any section $\sigma \in \mathcal{S}(K)$ is already defined over some open $U \subseteq X$. Therefore, there are only finitely many $x \in X^{(1)}$ (the ones in the complement of U) such that $\sigma \notin \mathcal{S}(\mathcal{O}_{X,x})$.

Both parts of Axiom (A3) follow from the fact that $\mathcal{H}$ is a presheaf on the category of smooth schemes: $\mathcal{S}$ is defined in such a way that the commutativity of the diagram in part (i) resp. the claim about the induced maps in part (ii) follow from the identification of $\mathcal{S}$ in terms of $\mathcal{H}$ and then applying the presheaf $\mathcal{H}$ to an appropriate commutative diagram in the category of essentially smooth F-schemes.

The non-trivial input is now in the proof of Axiom (A4). Let X be any essentially smooth local F-scheme of dimension 2 with closed point z and function field K. For part (i), assume that $y_0 \in X^{(1)}$ is a codimension one point with essentially smooth closure. We need to show that the specialization map $\mathcal{S}(\mathcal{O}_{y_0}) \to \mathcal{S}(\kappa(y_0))$ maps $\bigcap_{y\in X^{(1)}} \mathcal{S}(\mathcal{O}_y)$ into $\mathcal{S}(\mathcal{O}_{\overline{y_0},z})$. By construction, the specialization map is identified with the restriction map $\mathcal{H}(\mathcal{O}_{y_0}) \to \mathcal{H}(\kappa(y_0))$, and

$$\bigcap_{y\in X^{(1)}} \mathcal{S}(\mathcal{O}_y) = \bigcap_{y\in X^{(1)}} \mathcal{H}(\mathcal{O}_y) \subset \mathcal{H}(K).$$

By the assumption on $\mathcal{H}$, this set is identified with $\mathcal{H}(X)$. In particular, the specialization map will send the set $\mathcal{H}(X) \subseteq \mathcal{H}(\mathcal{O}_{y_0})$ into $\mathcal{H}(\mathcal{O}_{\overline{y_0},z})$ which proves part (i) of (A4). For part (ii), we note that the composition

$$\bigcap_{y\in X^{(1)}} \mathcal{H}(\operatorname{Spec} \mathcal{O}_y) \to \mathcal{H}(\operatorname{Spec} \mathcal{O}_{\overline{y_0},z}) \to \mathcal{H}(\operatorname{Spec} \kappa(z))$$

is (by what we said for part (i)) simply the restriction $\mathcal{H}(X) \to \mathcal{H}(\operatorname{Spec} \kappa(z))$ and therefore independent of the choice of y_0. □

Proposition 4.11. *In the situation of Proposition 4.10, we have an unramified Nisnevich sheaf $\mathcal{H}_{\mathrm{nr}}$ corresponding to $\mathcal{S}$. The restriction of sections from irreducible schemes to their function fields defines an injective morphism of sheaves $\mathcal{H} \to \mathcal{H}_{\mathrm{nr}}$ which induces bijections on essentially smooth F-schemes* *of dimension* ≤ 1. *The sheaf* $\mathcal{H}_{\mathrm{nr}}$ *has the strong Grothendieck–Serre property. If* $\mathcal{H}$ *is homotopy invariant then so is* $\mathcal{H}_{\mathrm{nr}}$.

Proof. From Proposition 4.10 we see that $\mathcal{S}$ is an unramified $\mathcal{F}_F$-datum, and by [Mor12, Theorem 2.11] there exists an unramified sheaf $\mathcal{H}_{\mathrm{nr}}$ associated to it. More precisely, the restriction of $\mathcal{H}_{\mathrm{nr}}$ (given exactly the same way we defined $\mathcal{S}$ from $\mathcal{H}$ in the first place) is isomorphic to $\mathcal{S}$. In particular, this means that for any irreducible essentially smooth F-scheme X with function field K we have

$$\mathcal{H}_{\mathrm{nr}}(X) = \bigcap_{x \in X^{(1)}} \mathcal{S}(\mathcal{O}_{X,x}) \subseteq \mathcal{S}(K/F).$$

We define the morphism $\mathcal{H} \to \mathcal{H}_{\mathrm{nr}}$ as follows: for an irreducible essentially smooth F-scheme with function field K, any section $\sigma \in \mathcal{H}(X) \subseteq \mathcal{H}(\operatorname{Spec} K)$ already lies in

$$\bigcap_{x \in X^{(1)}} \mathcal{H}(\operatorname{Spec} \mathcal{O}_{X,x}) \subseteq \mathcal{H}(\operatorname{Spec} K),$$

and we send $\sigma \in \mathcal{H}(X)$ to its image under the identification

$$\bigcap_{x \in X^{(1)}} \mathcal{H}(\operatorname{Spec} \mathcal{O}_{X,x}) = \bigcap_{x \in X^{(1)}} \mathcal{S}(\mathcal{O}_{X,x}) = \mathcal{H}_{\mathrm{nr}}(X)$$

This defines maps $\mathcal{H}(X) \to \mathcal{H}_{\mathrm{nr}}(X)$ for any essentially smooth F-scheme X, and these maps are clearly injective since $\mathcal{H}(X) \subseteq \mathcal{H}(\operatorname{Spec} K)$. By construction, this produces bijections $\mathcal{H}(Y) \xrightarrow{\cong} \mathcal{H}_{\mathrm{nr}}(Y)$ for field extensions of F and dvrs containing F. In fact, we get such bijections for any irreducible essentially smooth F-scheme Y of dimension ≤ 1. By our assumptions, $\mathcal{H}(Y) \subseteq \mathcal{H}_{\mathrm{nr}}(Y)$ as subsets of $\mathcal{H}(\operatorname{Spec} K) = \mathcal{H}_{\mathrm{nr}}(\operatorname{Spec} K)$ for K the function field of Y. Then a section of $\mathcal{H}_{\mathrm{nr}}(Y)$ consists of local sections which by the Grothendieck–Serre property are determined on $\operatorname{Spec} K$ and therefore compatible. Since $\mathcal{H}(Y)$ is a Nisnevich sheaf, these sections glue, hence the inclusion $\mathcal{H}(Y) \subseteq \mathcal{H}_{\mathrm{nr}}(Y)$ is already an equality.

It therefore remains to show that these maps are compatible with the restriction maps for $\mathcal{H}$ resp. $\mathcal{H}_{\mathrm{nr}}$. The restriction maps of the sheaf $\mathcal{H}_{\mathrm{nr}}$ are constructed in the proofs of Proposition 2.8 (for smooth morphisms) and Theorem 2.11 (for closed immersions) of [Mor12]. The restriction maps for smooth morphisms $X \to Y$ of essentially smooth F-schemes are induced by the function field extensions and therefore compatible with the restriction maps on $\mathcal{H}$ since $\mathcal{H}$ and $\mathcal{H}_{\mathrm{nr}}$ agree on schemes of dimension ≤ 1. The definition of the restriction maps for closed immersions reduces to the codimension 1 case by factoring a closed immersion into a sequence of codimension 1 immersions in [Mor12, Lemma 2.13]. In particular, the uniqueness part of Lemma 2.13 implies that the restriction maps of $\mathcal{H}$ and $\mathcal{H}_{\mathrm{nr}}$ are compatible for closed immersion if they are compatible for codimension 1 closed immersions. But the restriction maps of $\mathcal{H}_{\mathrm{nr}}$ for codimension 1 closed immersions are defined in terms of the specialization maps for $\mathcal{S}$, hence are compatible with the restriction maps for $\mathcal{H}$. This concludes the proof.

The strong Grothendieck–Serre property for $\mathcal{H}_{\mathrm{nr}}$ is then built in by construction. Unramified sheaves are also finitary (since the associated $\mathcal{F}_F$-datum satisfies the axiom (A2), cf. the discussion before [Mor12, Proposition 2.8]). Assume that $\mathcal{H}$ satisfies homotopy invariance. Since $\mathcal{H}$ and $\mathcal{H}_{\mathrm{nr}}$ agree over one-dimensional schemes, $\mathcal{H}_{\mathrm{nr}}$ will then satisfy the $\mathbb{A}^1$-invariance over fields. Homotopy invariance for $\mathcal{H}_{\mathrm{nr}}$ follows from Proposition 3.5. □

Remark 4.12. The result applies to $\mathcal{H}^1_{\text{ét}}(G)$ for a reductive group G which is a finitary Nisnevich sheaf by construction and satisfies 2-dimensional purity by [CTS79, Theorem 6.13, Corollary 6.14]. It seems very likely that the morphism $\mathcal{H}^1_{\text{ét}}(G) \to \mathcal{T}$ should be an isomorphism whenever the group G satisfies $\mathbb{A}^1$-invariance over fields, but at this point we're unable to prove this.

Example 4.13. There exist finitary homotopy-invariant Nisnevich sheaves of pointed sets which have the strong Grothendieck–Serre property but fail to be unramified (caused by a failure of unramifiedness in codimension 2). Consider for example the variety $X = (C_1 \times C_2) \setminus \{p\}$ for C_1 and C_2 smooth projective curves of genus > 0 and p a closed point of the product $C_1 \times C_2$. This variety is $\mathbb{A}^1$-rigid. The presheaf $\mathcal{X}$ represented by X is then a finitary homotopy-invariant Nisnevich sheaf. The strong Grothendieck–Serre property is also satisfied. For a regular local ring R, the images of any two morphisms $\operatorname{Spec} R \to X$ lie in a common open affine $\operatorname{Spec} A = U \subseteq X$, and two ring homomorphisms $A \to R$ agree if their compositions with $R \subseteq \operatorname{Frac}(R)$ agree. If the sheaf $\mathcal{X}$ were unramified, then for any regular local ring R with fraction field K a map $\operatorname{Spec} K \to X$ extends to R if it extends to all codimension 1 points of $\operatorname{Spec} R$. However, by definition this fails for the regular local ring $\mathcal{O}_{C_1 \times C_2, P}$ and the map $\operatorname{Spec} \operatorname{Frac}(\mathcal{O}_{C_1 \times C_2, P}) \to X$ induced from the obvious inclusion $\operatorname{Spec} \mathcal{O}_{C_1 \times C_2, P} \hookrightarrow C_1 \times C_2$. Similar arguments apply generally to open subsets of $\mathbb{A}^1$-rigid smooth projective varieties whose complement has codimension ≥ 2. □

5 Connected components of classifying spaces

We now want to relate the sheaf $\mathcal{H}^1_{\text{ét}}(G)$ to the sheaf $a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G$ of $\mathbb{A}^1$-connected components of the classifying space $\mathrm{B}_{\text{ét}}G$ for a reductive group G. More precisely, we will show that the natural morphism of sheaves in the following construction is an isomorphism.

Construction 5.1. We construct a transformation of Nisnevich sheaves of pointed sets

$$\mathrm{can}_G : \mathcal{H}^1_{\text{ét}}(G) \to a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G).$$

To begin with, consider the stack $\mathrm{B}_{\text{ét}}G$ classifying G-torsors. We have a morphism of presheaves $\mathrm{B}_{\text{ét}}G|_{\mathrm{Sm}_F} \to L_{\mathrm{mot}}(\mathrm{B}_{\text{ét}}G|_{\mathrm{Sm}_F})$, which induces a map by taking π_0:

$$\mathrm{H}^1_{\text{ét}}(-,G) \to \pi_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G).$$

This is a transformation of presheaves of pointed sets, which then Nisnevich-sheafifies to the map can_G as above.

Remark 5.2. For $X \in \mathrm{Sm}_F$, the map $\mathrm{H}^1_{\text{ét}}(X,G) \to \pi_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G)(X)$ can be interpreted as follows: a G-torsor $\mathcal{P} \to X$ is classified by a map $X \to \mathrm{B}_{\text{ét}}G|_{\mathrm{Sm}_F}$ (here X is abusively identified with its Yoneda image) which is sent to the motivic localization

$$(L_{\mathrm{mot}}X \to L_{\mathrm{mot}}\mathrm{B}_{\text{ét}}G|_{\mathrm{Sm}_F}) \in \pi_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G)(X) = [X, \mathrm{B}_{\text{ét}}G]_{\mathbb{A}^1}.$$

In [MV99, Section 4.2], the motivic localization of the presheaf $\mathrm{B}_{\text{ét}}G|_{\mathrm{Sm}_F}$ admits a geometric model (as an ind-scheme) which goes back to the work of Morel–Voevodsky and Totaro. Roughly there exists an ind-scheme U_∞ with a G-action such that the quotient (U_∞/G) exists as an ind-scheme and we have an equivalence of presheaves:

$$L_{\mathrm{mot}}\mathrm{B}_{\text{ét}}G|_{\mathrm{Sm}_F} \simeq L_{\mathrm{mot}}(U_\infty/G)$$

We refer to [Hoy20, Theorem 2.7] for a statement in this form, though we will not need it in the sequel.

We now want to show that the map $\mathrm{can}_G\colon \mathcal{H}^1_{\text{ét}}(G) \to a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G)$ is an isomorphism of pointed sheaves of sets on Sm_F. We prove a statement of somewhat greater generality: if a space $\mathcal{X}$ has $\mathbb{A}^1$-invariant $a_{\mathrm{Nis}}\pi_0\mathcal{X}$, then this already agrees with the Nisnevich sheaf of $\mathbb{A}^1$-connected components. The key point here is that the $\mathbb{A}^1$-invariance of the simplicial connected components already ensures that the $L_{\mathbb{A}^1} = \mathrm{Sing}_\bullet^{\mathbb{A}^1}$ construction doesn't change the sheaf of connected components. This is close to results around Morel's conjecture in the work of Balwe, Hogadi and Sawant in [BHS15], but as far as we're aware, it doesn't seem to immediately follow from results in loc.cit.

Proposition 5.3. *Let $\mathcal{X}\colon \mathrm{Sm}_F^{\mathrm{op}} \to \mathrm{Spc}$ be a space such that $a_{\mathrm{Nis}}\pi_0\mathcal{X}$ is $\mathbb{A}^1$-invariant. Then the induced map $a_{\mathrm{Nis}}\pi_0\mathcal{X} \to a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathcal{X}$ is an isomorphism of sheaves of pointed sets on Sm_F.*

Proof. By definition,

$$a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathcal{X} = a_{\mathrm{Nis}}\pi_0 L_{\mathrm{mot}}\mathcal{X} = a_{\mathrm{Nis}}\pi_0\left(\mathrm{colim}_n\left(L_{\mathrm{Nis}} \circ \mathrm{Sing}_\bullet^{\mathbb{A}^1}\right)^n \mathcal{X}\right)$$

and the map $a_{\mathrm{Nis}}\pi_0\mathcal{X} \to a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathcal{X}$ we want to show is an isomorphism is induced by the natural map $\mathcal{X} \to L_{\mathrm{mot}}\mathcal{X}$. From the description of L_{mot} as a filtered colimit of the functors L_{Nis} and $L_{\mathbb{A}^1} = \mathrm{Sing}_\bullet^{\mathbb{A}^1}$ (see, for example, [AE17]), it suffices to prove that the morphism

$$a_{\mathrm{Nis}}\pi_0\mathcal{X} \to a_{\mathrm{Nis}}\pi_0\left(\left(L_{\mathrm{Nis}} \circ \mathrm{Sing}_\bullet^{\mathbb{A}^1}\right)^n \mathcal{X}\right)$$

is an isomorphism of sheaves for all natural numbers n. The claim of the proposition then follows from the fact that the filtered colimit commutes with π_0 and sheafification, and the filtered colimit of isomorphisms will be an isomorphism. Next, we note that for any space $\mathcal{X}$ the Nisnevich-local replacement map $\mathcal{X} \to L_{\mathrm{Nis}}\mathcal{X}$ is a Nisnevich-local weak equivalence and therefore induces an isomorphism $a_{\mathrm{Nis}}\pi_0\mathcal{X} \to a_{\mathrm{Nis}}\pi_0 L_{\mathrm{Nis}}\mathcal{X}$ (by checking on stalks). It therefore suffices to show that for a space $\mathcal{X}$ with $\mathbb{A}^1$-invariant $a_{\mathrm{Nis}}\mathcal{X}$ the induced morphism $a_{\mathrm{Nis}}\pi_0\mathcal{X} \to a_{\mathrm{Nis}}\pi_0 \operatorname{Sing}_\bullet^{\mathbb{A}^1}(\mathcal{X})$ is an isomorphism. The claim then follows by an induction on n.

By construction, for a smooth scheme T

$$\pi_0 \operatorname{Sing}_\bullet^{\mathbb{A}^1}(\mathcal{X})(T) = \operatorname{coeq}\left(\pi_0(\mathcal{X}(T \times \mathbb{A}^1)) \rightrightarrows \pi_0(\mathcal{X}(T))\right).$$

In particular, the morphism $a_{\mathrm{Nis}}\pi_0\mathcal{X} \to a_{\mathrm{Nis}}\pi_0 \operatorname{Sing}_\bullet^{\mathbb{A}^1}(\mathcal{X})$ is surjective. To show injectivity, we need to show that $\mathbb{A}^1$-equivalent sections of $\pi_0(\mathcal{X}(T))$ already agree in $a_{\mathrm{Nis}}\pi_0(\mathcal{X})(T)$. Let $f, g \in \pi_0\mathcal{X}(T)$ be two $\mathbb{A}^1$-equivalent sections. Without loss of generality, we can assume that there is a section $H \in \pi_0\mathcal{X}(T \times \mathbb{A}^1)$ whose restriction along 0 and 1 are f and g, respectively. From the commutative diagram

$$\begin{array}{ccc} \pi_0\mathcal{X}(T \times \mathbb{A}^1) & \longrightarrow & a_{\mathrm{Nis}}\pi_0\mathcal{X}(T \times \mathbb{A}^1) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\ \pi_0\mathcal{X}(T) & \longrightarrow & a_{\mathrm{Nis}}\pi_0\mathcal{X}(T) \end{array}$$

we see that the restrictions of the image of H in $a_{\mathrm{Nis}}\pi_0\mathcal{X}(T \times \mathbb{A}^1)$ along 0 and 1 are the images of f and g in $a_{\mathrm{Nis}}\pi_0\mathcal{X}(T)$, respectively. But now $a_{\mathrm{Nis}}\pi_0\mathcal{X}$ is $\mathbb{A}^1$-invariant by assumption, therefore $f = g$ in $a_{\mathrm{Nis}}\pi_0\mathcal{X}(T)$. This shows that $a_{\mathrm{Nis}}\pi_0\mathcal{X} \to a_{\mathrm{Nis}}\pi_0 \operatorname{Sing}_\bullet^{\mathbb{A}^1}(\mathcal{X})$ is an isomorphism, concluding the proof. □

Theorem 5.4. *Let F be a field and let G be a reductive group such that $\mathcal{H}^1_{\text{ét}}(G)$ is $\mathbb{A}^1$-invariant. Then the natural map $\mathcal{H}^1_{\text{ét}}(G) \to a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G)$ is an isomorphism of sheaves. In particular, $a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}(\mathrm{B}_{\text{ét}}G)$ is a homotopy invariant Nisnevich sheaf.*

Proof. This follows directly from Proposition 5.3 because $\pi_0(\mathrm{B}_{\text{ét}}G)(T) \cong \mathrm{H}^1_{\text{ét}}(T, G)$ for any smooth F-scheme T, hence $a_{\mathrm{Nis}}\pi_0\mathrm{B}_{\text{ét}}G \cong \mathcal{H}^1_{\text{ét}}(G)$. □

Remark 5.5. From the results in Section 3, it follows that the conditions of the theorem are satisfied for reductive groups satisfying $\mathbb{A}^1$-invariance for all finitely generated extension fields L/F. Therefore, the above result establishes Morel's conjecture for classifying spaces of such reductive groups G. If in addition local purity holds for G, the sheaf $a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G$ is also unramified, as a consequence of Theorem 5.4 and Lemma 4.7. However, it may be worth pointing out that while homotopy invariance is obviously expected for the sheaves $a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}(\mathcal{X})$ of $\mathbb{A}^1$-connected components for arbitrary motivic spaces $\mathcal{X}$, these sheaves are not expected to be unramified in general. Example 4.13 shows that the sheaves of $\mathbb{A}^1$-connected components of non-proper $\mathbb{A}^1$-rigid varieties usually fail to be unramified. In some sense, being unramified is a weak properness statement.

6 Applications

In this final section, we will discuss some consequences of our results on $\mathcal{H}^1_{\text{ét}}(G)$ and $a_{\text{Nis}}\pi_0^{\mathbb{A}^1}\text{B}_{\text{ét}}G$ and how these relate to torsor classification and purity questions. For the rest of the section we will always only talk about groups G where $\mathcal{H}^1_{\text{ét}}(G)$ is $\mathbb{A}^1$-invariant and unramified, i.e., the conditions for Theorem 5.4 are satisfied.

6.1 Computation of the sheaves of $\mathbb{A}^1$-connected components

If $\mathcal{H}^1_{\text{ét}}(G)$ is unramified, then the identification $\mathcal{H}^1_{\text{ét}}(G) \cong a_{\text{Nis}}\pi_0^{\mathbb{A}^1}\text{B}_{\text{ét}}G$ allows to compute the sheaf of $\mathbb{A}^1$-connected components of the classifying space $\text{B}_{\text{ét}}G$. For an irreducible smooth scheme X with function field K, the sections of $a_{\text{Nis}}\pi_0^{\mathbb{A}^1}\text{B}_{\text{ét}}G$ over X are given by G-torsors over $\operatorname{Spec} K$ which extend over all the codimension 1 points of X. This classification question is significantly easier than the isomorphism classification of G-torsors over X itself and can be made rather explicit in specific cases. It turns out that these sheaves are *non-trivial* in many cases, in contrast to the case of special groups.
Our first examples are orthogonal groups.

Proposition 6.1. *Let F be a field of characteristic $\neq 2$. Then for any irreducible smooth F-scheme with function field K, we have*

$$\left(a_{\text{Nis}}\pi_0^{\mathbb{A}^1}\text{B}_{\text{ét}}\text{O}(n)\right)(X) = \text{GW}_{\text{nr}}(X) \cap \text{H}^1_{\text{ét}}(K, \text{O}(n))$$

where the intersection is taken inside $\text{GW}(K)$. This means that the sheaf $a_{\text{Nis}}\pi_0^{\mathbb{A}^1}\text{B}_{\text{ét}}\text{O}(n)$ of $\mathbb{A}^1$-connected components of the classifying space of $\text{O}(n)$ is the subsheaf of the sheaf of unramified Grothendieck–Witt groups consisting of classes of quadratic forms over $\operatorname{Spec} K$ which have a rank n representative.

Proof. By Theorem 5.4 it suffices to compute

$$\mathcal{H}^1_{\text{ét}}(\text{O}(n))(X) = \bigcap_{x\in X^{(1)}} \mathcal{H}^1_{\text{ét}}(\text{O}(n))(\mathcal{O}_{X,x}) = \bigcap_{x\in X^{(1)}} \text{H}^1_{\text{ét}}(\mathcal{O}_{X,x}, \text{O}(n))$$

inside $\text{H}^1_{\text{ét}}(K, \text{O}(n))$. The first equality here is the unramifiedness of $\mathcal{H}^1_{\text{ét}}(\text{O}(n))$ from Lemma 4.7 and the purity result of Panin [Pan10] resp. its extension in [PP10], and the second equality follows from Lemma 4.6. We note that the natural map

$$\text{H}^1_{\text{ét}}(K, \text{O}(n)) \to \text{GW}(K)$$

taking an isometry class of a rank n quadratic form over K to its class in the Grothendieck–Witt ring is injective by Witt cancellation. Now a rank n

quadratic form extends over a dvr $\mathcal{O}_{X,x} \subseteq K$ if its class in the Witt ring of K is unramified by [Sch85, Theorem 2.2, Chapter 6].[1] This implies that $\mathcal{H}^1_{\text{ét}}(\mathrm{O}(n))(X)$ is the intersection of $\mathrm{H}^1_{\text{ét}}(K, \mathrm{O}(n))$ and the unramified Grothendieck–Witt group $\mathrm{GW}_{\mathrm{nr}}(X)$ inside $\mathrm{GW}(K)$. This proves the claim. □

Example 6.2. As another example for a description of the sheaf of $\mathbb{A}^1$-connected components of a classifying space, a similar result is true for PGL_n over a field F of characteristic 0. For any irreducible smooth F-scheme with function field K, we have

$$\left(a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}\mathrm{PGL}_n\right)(X) = \mathrm{Br}(X) \cap \mathrm{H}^1_{\text{ét}}(K, \mathrm{PGL}_n)$$

with the intersection taken inside $\mathrm{Br}(K) = \mathrm{H}^1_{\text{ét}}(K, \mathrm{PGL}_\infty)$ (here the colimit is taken along the divisibility poset). This uses that the Brauer group is unramified and that a PGL_n-torsor over K extends over a dvr $\mathcal{O}_{X,x} \subseteq K$ if and only if the local invariant is trivial. □

Example 6.3. We can also describe the sheaf of $\mathbb{A}^1$-connected components for the classifying space of the exceptional group G_2 over a field of characteristic $\neq 2$, using the local purity result of Chernousov and Panin [CP07] resp. its extension in [PP10]. Over a field K of characteristic $\neq 2$, we have

$$\mathrm{H}^1_{\text{ét}}(K, \mathrm{G}_2) \cong \mathrm{H}^3_{\text{ét}}(K, \mu_2)_{\mathrm{dec}},$$

i.e., G_2-torsors over K are classified by 3-fold Pfister forms over K, cf. [Ser95, Théorème 9]. Actually, there is a bijection between G_2-torsors and 3-fold Pfister forms over local rings in which 2 is invertible, cf. [CP07, Remark 10]. In particular, a G_2-torsor over a discretely valued field K extends over the valuation ring $R \subseteq K$ if and only if the norm form extends to a 3-fold Pfister form over R. Consequently, the sections of $a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}\mathrm{G}_2$ over an irreducible smooth scheme X with function field K can be identified with the isomorphism classes of unramified 3-fold Pfister forms over K. □

Remark 6.4. The above examples should convince the reader that the identification of $a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G$ allows to determine the sheaves of $\mathbb{A}^1$-connected components of classifying spaces $\mathrm{B}_{\text{ét}}G$ fairly explicitly in a number of interesting cases. All that is required is knowledge about the torsor classification over fields and dvrs.

6.2 Purity for torsors

Next, we want to discuss the relation between our results and purity questions for torsors. If Nisnevich-local purity holds for G-torsors, i.e., $\mathcal{H}^1_{\text{ét}}(G)$ is unramified, we can identify it as the target of the map in the definition of purity, cf. Definition 4.1. This way, we can reformulate purity as the surjectivity of the sheafification map for étale torsors:

[1] Note that the boundary of a class in $\mathrm{GW}(K)$ being zero means in the notation of loc.cit. that we can choose a lattice L with $L^{\#}/L = 0$. Then L with its induced form is the required form over the dvr.

Proposition 6.5. *Let F be a field and G be a reductive group over F such that $\mathcal{H}^1_{\text{ét}}(G)$ is unramified. Then purity for G-torsors over a smooth F-scheme X is equivalent to the surjectivity of the sheafification map $\mathrm{H}^1_{\text{ét}}(X,G) \to \mathcal{H}^1_{\text{ét}}(G)(X)$. If purity holds for G-torsors over X, then the sheafification map $\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X) \to a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X)$ is surjective.*

Proof. The target of the purity map in Definition 4.1 is identified with

$$\mathcal{H}^1_{\text{ét}}(G)(X) = \bigcap_{x\in X^{(1)}} \mathcal{H}^1_{\text{ét}}(G)(\mathcal{O}_{X,x}).$$

We also have an identification

$$\mathcal{H}^1_{\text{ét}}(G)(\mathcal{O}_{X,x}) \cong \operatorname{im}\left(\mathrm{H}^1_{\text{ét}}(\mathcal{O}_{X,x},G) \to \mathrm{H}^1_{\text{ét}}(K,G)\right)$$

from Lemma 4.6. With this identification, purity for G-torsors on X is equivalent to the surjectivity of the sheafification map

$$\mathrm{H}^1_{\text{ét}}(X,G) \to \mathcal{H}^1_{\text{ét}}(G)(X).$$

Now consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1_{\text{ét}}(X,G) & \longrightarrow & \pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X) \\ \downarrow & & \downarrow \\ \mathcal{H}^1_{\text{ét}}(G)(X) & \xrightarrow[\cong]{} & a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X) \end{array}$$

The lower horizontal map is an isomorphism by Theorem 5.4. As noted above, purity for torsors is equivalent to the surjectivity of the left vertical morphism. Consequently, if purity is satisfied for G-torsors over a scheme X, the sheafification map $\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X) \to a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X)$ is surjective. □

Example 6.6. Examples of the failure of global purity for torsors under PGL_p have been given in [AW15]. By the above discussion, these provide, for any prime p, examples of smooth affine complex varieties X of dimension $2p+2$ such that the sheafification map

$$\mathrm{H}^1_{\text{ét}}(X,\mathrm{PGL}_p) \to \mathcal{H}^1_{\text{ét}}(\mathrm{PGL}_p)(X) \cong \left(a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}\mathrm{PGL}_p\right)(X)$$

fails to be surjective. In particular, we also have examples of sections of the sheaf of $\mathbb{A}^1$-connected components of $\mathrm{B}_{\text{ét}}\mathrm{PGL}_p$ which fail to be represented by actual torsors. In fact, they also fail to be represented by motivic torsors as the next theorem explains. □

Theorem 6.7. *Let k be a field of characteristic zero. Then for any prime p, there exists a smooth affine k-scheme of dimension $2p+2$ such that the map*

$$\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}\mathrm{PGL}_p(X) \to \mathcal{H}^1_{\text{ét}}(\mathrm{PGL}_p)(X) \cong \left(a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}\mathrm{PGL}_p\right)(X)$$

fails to be surjective.

Proof. Let $p > 0$ be fixed. Let X be as in [AW15, Theorem 3.6]; we note that while X is over the complex numbers here, the construction works over any characteristic zero field. This is a smooth affine scheme of dimension $2p+2$ equipped with a Brauer class $\alpha \in \mathrm{Br}(X)$ such that $\alpha|_{k(X)}$ is exact degree p; this is classified by a map of presheaves

$$X \xrightarrow{\alpha} \mathrm{B}^2_{\text{ét}}\mathbb{G}_m. \tag{5}$$

By the identification in Example 6.2, α determines an element $\alpha_{\mathrm{mot}}|_K \in \left(a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}\mathrm{PGL}_p\right)(X)$. If this element does lift to an element $\alpha_{\mathrm{mot}} \in \pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}\mathrm{PGL}_p(X)$, then we have constructed a nontrivial factorization of (5) in the $\mathbb{A}^1$-homotopy category:

$$L_{\mathrm{mot}}X \to L_{\mathrm{mot}}\mathrm{B}_{\text{ét}}\mathrm{PGL}_p \to \mathrm{B}^2_{\text{ét}}\mathbb{G}_m.$$

This then Betti realizes to a factorization of the Betti realization of (5) which cannot exist as explained in [AW15, Theorem 3.6], see also [AW15, Theorem 3.11]. □

Remark 6.8. The failure of surjectivity for the sheafification map on $\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}\mathrm{PGL}_p$ is based on the fact that in the topological realization the difference between actual torsors and motivic torsors vanishes: the realization of $\mathrm{B}_{\text{ét}}\mathrm{PGL}_p$ is a classifying space for $\mathrm{PGL}_p(\mathbb{C})$ and thus counterexamples to topological purity also imply counterexamples to purity for motivic G-torsors. It is likely that similar constructions can be made in characteristic $\neq p$ using étale realization.

Remark 6.9. We also consider $G = \mathrm{PGL}_n$ where n is not necessarily a prime. Over a field of characteristic zero, we can find examples of motivic spaces X where the sheafification map is not surjective $\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X) \to a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X)$. Unfortunately, these examples are motivic spaces associated to ind-schemes (they are approximations of $\mathrm{B}_{\text{ét}}G$ in the sense of Totaro and Morel-Voeovodsky [MV99]) and thus do not lead to new counterexamples to purity in this situation. These examples also come from the work Antieau-Williams [AW15].

Proposition 6.10. *Let k be a field of characteristic zero, then there exists an ind-scheme X such that*

$$\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X) \to a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X)$$

is not surjective.

Proof. Let $m > 1$ be an integer dividing n and let H be the algebraic group SL_m/μ_n. Let X be an algebraic approximation to $\mathrm{B}_{\text{ét}}H$ as in [MV99, Proposition 4.2.6]. In particular X comes equipped with a canonical "Brauer class" α; more precisely it is a map of presheaves $\alpha\colon X \to \mathrm{B}^2_{\text{ét}}\mu_n$ induced by the exact sequence of étale sheaves of groups $1 \to \mu_n \to \mathrm{SL}_m \to H \to 1$. By the identification in Example 6.2, extended by filtered colimits to ind-schemes, the class α

determines an element of $a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X)$. If α does lift to $\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X)$, then we would obtain a factorization in the $\mathbb{A}^1$-homotopy category of the map α as

$$X \to \mathrm{B}_{\text{ét}}\mathrm{PGL}_n \to \mathrm{B}^2_{\text{ét}}\mu_n.$$

But this contradicts [AW15, Theorem 3.11] which states that no factorization can exist after taking Betti realizations. □

Example 6.11. Similarly, we can ask for the failure of global purity for $\mathrm{O}(n)$-torsors. From the description of $\mathcal{H}^1_{\text{ét}}(\mathrm{O}(n))$ in Proposition 6.1, one source for the failure of purity for $\mathrm{O}(n)$ torsors is the failure of purity for the Grothendieck–Witt group. For an irreducible smooth scheme X with function field K, any class in $\mathrm{W}_{\mathrm{nr}}(X) \cap \mathrm{H}^1_{\text{ét}}(K, \mathrm{O}(n))$ which is not in the image of the natural map $\mathrm{W}(X) \to \mathrm{W}_{\mathrm{nr}}(X)$ provides a counterexample to purity for $\mathrm{O}(n)$-torsors. The failure of surjectivity of the morphism $\mathrm{W}(X) \to \mathrm{W}_{\mathrm{nr}}(X)$ can be studied using the Gersten–Witt spectral sequence [BW02]. By the weak purity theorem of loc.cit., such phenomena cannot occur in dimensions ≤ 4, and the unique obstruction to purity of Witt groups for smooth schemes of dimension ≤ 8 is a differential $\mathrm{W}_{\mathrm{nr}}(X) \to \mathrm{H}^5(X, \mathbf{W})$ in the Gersten–Witt spectral sequence. Not much seems to be known about examples where this differential is non-trivial. □

More generally, the formulation in Proposition 6.5 now opens up the possibility to use motivic homotopy methods for investigation of counterexamples to the global purity question for G-torsors. By a conjecture of Antieau and Williams, cf. [AW15, Conjecture 1.2], purity should fail for G-torsors over some smooth affine variety if G is a non-special semisimple group. By Proposition 6.5, to get counterexamples to purity for such a group G, it would suffice to find examples of smooth affine schemes X over a field F such that the sheafification map

$$\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X) \to a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X)$$

fails to be surjective.

6.3 Towards the classification of (motivic) G-torsors

Finally, we can once more have a look at the diagram employed in the proof of Proposition 6.5 to discuss the relations between torsor classification and the homotopy theory of the classifying spaces:

$$\begin{array}{ccc} \mathrm{H}^1_{\text{ét}}(X,G) & \longrightarrow & \pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X) \\ \downarrow & & \downarrow \\ \mathcal{H}^1_{\text{ét}}(G)(X) & \xrightarrow[\cong]{} & a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X) \end{array}$$

The upper left corner is about the isomorphism classification of G-torsors on a smooth scheme X, the upper right corner is about the (motivic) homotopy classification of maps into the geometric classifying spaces $\mathrm{B}_{\text{ét}}G$. The lower part of the diagram is about the sheafified problems which we now have identified as the classification of unramified torsors over the function field.

The left-hand vertical map relates the isomorphism classification of torsors over X with the classification of unramified torsors over the function field. As noted in Proposition 6.5, surjectivity of that map is the question of purity for G-torsors over X, and that is expected to fail in general (though at this point we only have examples for PGL_p). The sheafification map $\mathrm{H}^1_{\text{ét}}(X,G) \to \mathcal{H}^1_{\text{ét}}(G)(X)$ is also not going to be injective in general: all the rationally trivial torsors over X (which need not be globally trivial) map to the trivial unramified torsor in $\mathcal{H}^1_{\text{ét}}(G)(X)$. It would certainly be interesting to classify G-torsors over X mapping to a given class in $\mathcal{H}^1_{\text{ét}}(G)(X)$, but at this point there does not seem to be methods around to approach this question.

The right-hand vertical map relates the presheaf of $\mathbb{A}^1$-connected components of $\mathrm{B}_{\text{ét}}G$ with its sheafification. By Theorem 6.7 and Proposition 6.10, the sheafification map

$$\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X) \to a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X)$$

also fails to be surjective in general. The sheafification map is generally not injective, since the representability results from [AHW20] imply that the preimage of the basepoint in $a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X)$ is given by isomorphism classes of rationally trivial torsors. Again, it would be interesting to understand the fibers of the sheafification map as well as conditions for realizability of classes in $a_{\mathrm{Nis}}\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X)$ in terms of maps $X \to \mathrm{B}_{\text{ét}}G$. Possibly, some version of obstruction theory for non-connected spaces (more precisely non-connected objects in an ∞-topos) could help here.

At last, the top horizontal map in the diagram relates the isomorphism classification of G-torsors to the $\mathbb{A}^1$-homotopy classification of maps $X \to \mathrm{B}_{\text{ét}}G$. This is essentially the question what the classifying space $\mathrm{B}_{\text{ét}}G$ actually classifies. Unfortunately, this map will not generally be a bijection. It fails to be injective because $\pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G$ is homotopy invariant whereas for non-special groups $\mathrm{H}^1_{\text{ét}}(-,G)$ is not generally homotopy invariant and therefore the map $\mathrm{H}^1_{\text{ét}}(X,G) \to \pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X)$ forgets about the counterexamples to homotopy invariance for étale torsors. At this point, it seems nothing is known regarding the surjectivity of the natural map $\mathrm{H}^1_{\text{ét}}(X,G) \to \pi_0^{\mathbb{A}^1}\mathrm{B}_{\text{ét}}G(X)$. For instance, we do not know if there exists a smooth affine scheme X and a motivic PGL_n-torsor over X which isn't represented by an actual PGL_n-torsor over X.

References

[stacks] The Stacks Project Authors. The Stacks Project. http://stacks.math.columbia.edu

[AF14a] A. Asok and J. Fasel, Algebraic vector bundles on spheres. J. Topol. 7 (2014), no. 3, 894–926.

[AF14b] A. Asok and J. Fasel. A cohomological classification of vector bundles on smooth affine threefolds. Duke Math. J. 163 (2014), no. 14, 2651–2601.

[AF15] A. Asok and J. Fasel, Splitting vector bundles outside the stable range and $\mathbb{A}^1$-homotopy sheaves of punctured affine spaces. J. Amer. Math. Soc. 28 (2015), no. 4, 1031–1062.

[AHW17] A. Asok, M. Hoyois, and M. Wendt. Affine representability results in $\mathbb{A}^1$-homotopy theory I: Vector bundles. Duke Math. J. 166 (2017), no. 10, 1923–1953.

[AHW18] A. Asok, M. Hoyois, and M. Wendt. Affine representability results in $\mathbb{A}^1$-homotopy theory II: Principal bundles and homogeneous spaces. Geom. Topol. 22 (2018), no. 2, 1181–1225.

[AHW20] A. Asok, M. Hoyois, and M. Wendt. Affine representability results in $\mathbb{A}^1$-homotopy theory III: Finite fields and complements. Algebr. Geom. 7 (2020), no. 5, 634–644.

[AW15] B. Antieau and B. Williams. Topology and purity for torsors. Doc. Math. 20 (2015), 333–355.

[AE17] B. Antieau and E. Elmanto. A primer for unstable motivic homotopy theory. Proc. Sympos. Pure Math. 95 (2017), 305–370.

[BH18] T. Bachmann and M. Hoyois. Norms in motivic homotopy theory. Astérisque 425 (2021), 207 pp.

[BHS15] C. Balwe, A. Hogadi, and A. Sawant. $\mathbb{A}^1$-connected components of schemes. Adv. Math. 282 (2015), 335–361.

[BHS22] C. Balwe, A. Hogadi, and A. Sawant. Strong $\mathbb{A}^1$-invariance of $\mathbb{A}^1$-connected components of reductive algebraic groups. Preprint, 2022. https://arxiv.org/abs/2205.07527

[BS20] C. Balwe and A. Sawant. $\mathbb{A}^1$-connected components of ruled surfaces. Geom. Topol. 26 (2022), no. 1, 321–376.

[BW02] P. Balmer and C. Walter. A Gersten–Witt spectral sequence for regular schemes. Ann. Sci. École Norm. Sup. (4) 35 (2002), no. 1, 127–152.

[Cho14] U. Choudhury. Connectivity of motivic H-spaces. Algebr. Geom. Topol. 14 (2014), no. 1, 37–55.

[CP07] V. Chernousov and I. Panin. Purity of G_2-torsors. C. R. Math. Acad. Sci. Paris 345 (2007), no. 6, 307–312.

[CP13] V. Chernousov and I. Panin. Purity for Pfister forms and F_4-torsors with trivial g_3 invariant. J. Reine Angew. Math. 685 (2013), 99–104.

[CTS79] J.-L. Colliot-Thélène and J.-J. Sansuc. Fibrés quadratiques et composantes connexes réelles, Math. Ann. 244 (1979), no. 2, 105–134.

[FP15] R. Fedorov and I. Panin. A proof of the Grothendieck–Serre conjecture on principal bundles over regular local rings containing infinite fields. Publ. Math. Inst. Hautes Études Sci. 122 (2015), 169–193.

[Gro68] A. Grothendieck. Le groupe de Brauer. II. Théorie cohomologique. Dix exposés sur la cohomologie des schémas. Adv. Stud. Pure Math. 3 (1968), 67–87.

[Hoy20] M. Hoyois. Cdh descent in equivariant homotopy K-theory. Doc. Math. 25 (2020), 457–482.

[Kne70] M. Knebusch. Grothendieck- und Wittringe von nichtausgearteten symmetrischen Bilinearformen. S.-B. Heidelberger Akad. Wiss. Math.-Natur. Kl. 1969/70 (1970), 93–157.

[KOS76] M. A. Knus, M. Ojanguren, and D. J. Saltman. On Brauer groups in characteristic p. Brauer groups (Proc. Conf., Northwestern Univ., Evanston, IL, 1975), 25-49. Lecture Notes in Mathematics, 549. Springer, Berlin, 1976.

[Lam06] T. Y. Lam. Serre's problem on projective modules. Springer Monographs in Mathematics. Springer, Berlin, 2006.

[Lin82] H. Lindel, On the Bass-Quillen conjecture concerning projective modules over polynomial rings. Invent. Math. 65 (1981/82), no. 2, 319–323.

[Mor05] F. Morel. The stable $\mathbb{A}^1$-connectivity theorems, K-Theory 35 (2005), 1–68.

[Mor12] F. Morel. $\mathbb{A}^1$-algebraic topology over a field, Lecture Notes in Mathematics, 2052. Springer, Heidelberg, 2012.

[MV99] F. Morel and V. Voevodsky. $\mathbb{A}^1$-homotopy theory of schemes. Inst. Hautes Études Sci. Publ. Math. 90 (1999), 45–143.

[Nis89] Y. A. Nisnevich. The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory. Algebraic K-theory: connections with geometry and topology (Lake Louise, AB, 1987), 241–342, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., 279. Kluwer Acad. Publ., Dordrecht, 1989.

[Pan10] I. A. Panin. Purity conjecture for reductive groups. Vestnik St. Petersburg Univ. Math. 43 (2010), no. 1, 44–48.

[Pan17] I. Panin. Proof of Grothendieck-Serre conjecture on principal bundles over regular local rings containing a finite field. Preprint, 2017, unpublished. https://arxiv.org/abs/1707.01767

[Par78] S. Parimala, Failure of a quadratic analogue of Serre's conjecture. Amer. J. Math. 5 (1978), no. 5, 913–924.

[Pop89] D. Popescu, Polynomial rings and their projective modules. Nagoya Math. J. 113 (1989), 121–128.

[PP10] I. Panin and K. Pimenov. Rationally isotropic quadratic spaces are locally isotropic: II. Doc. Math. 2010, Extra vol.: Andrei A. Suslin sixtieth birthday, 515–523.

[RR84] M. S. Raghunathan and A. Ramanathan. Principal bundles on the affine line. Proc. Indian Acad. Sci. Math. Sci. 93 (1984), no. 2/3, 137–145.

[Sch85] W. Scharlau. Quadratic and Hermitian forms. Grundlehren der Mathematischen Wissenschaften, 270. Springer, Berlin, 1985.

[Ser95] J.-P. Serre. Cohomologie galoisienne: progrès et problèmes. Séminaire Bourbaki, 1993/94. Astérisque 227 (1995), Exp. No. 783, 229–257.

[Ses63] C. S. Seshadri. Some results on the quotient space by an algebraic group of automorphisms. Math. Ann. 149 (1963), 286–301.

[Ste65] R. Steinberg. Regular elements of semisimple algebraic groups. Inst. Hautes Études Sci. Publ. Math. 25 (1965), 49–80.

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