Seshadri constants of parabolic vector bundles

Indranil Biswas, Krishna Hanumanthu, Snehajit Misra, and Nabanita Ray

Abstract. Let X be a complex projective variety, and let E_* be a parabolic vector bundle on X with parabolic structure on a divisor. We introduce the notion of *parabolic Seshadri constants* of E_*. It is shown that these constants are analogous to the classical Seshadri constants of vector bundles, in particular they have parallel definitions and properties. We prove a Seshadri criterion for parabolic ampleness of E_* in terms of the parabolic Seshadri constants of it. We also compute parabolic Seshadri constants for symmetric powers and tensor products of parabolic vector bundles.

1. Introduction

Let X be a complex projective variety, and let L be a nef line bundle on X. For a point $x \in X$, the *Seshadri constant* of L at x, denoted by $\varepsilon(X, L, x)$, is defined to be

$$\varepsilon(X, L, x) := \inf_{x \in C} \frac{L \cdot C}{\operatorname{mult}_x C},$$

where the infimum is taken over all irreducible and reduced curves $C \subset X$ passing through x; here $L \cdot C$ denotes the intersection number (equivalently, the degree of L restricted to C), while $\operatorname{mult}_x C$ denotes the multiplicity of the curve C at x.

Alternatively, we can define Seshadri constants as follows. Let $\pi : \widetilde{X} \to X$ be the blow up of X at x, and let A denote the corresponding exceptional divisor. Then

$$\varepsilon(X, L, x) = \sup \left\{ \lambda \geq 0 \mid \pi^*(L) - \lambda A \text{ is nef} \right\}.$$

This constant was first introduced by Demailly in [12] while he was studying problems related to generation by jets and Fujita's Conjecture. The name originated from an ampleness criterion of Seshadri [17, Theorem I.7.1]. Seshadri constant of line bundles blossomed into a very active area of research, especially in connection with the positivity properties of the line bundle L. Seshadri constants for vector bundles of arbitrary rank have also been studied. They were defined in [15] and studied in depth in [14]. See [6, 11, 19, 27] for some results on Seshadri constants of vector bundles.

A *parabolic vector bundle* on a curve is a vector bundle E with some additional structures given by filtrations of certain fibers of E along with weights attached to these filtrations (see Definition 2.1). We recall that parabolic vector bundles were introduced by Seshadri (see [28]) while studying certain representations of the fundamental groups of punctured Riemann surfaces; they were further studied by Mehta and Seshadri in [25].

The notion of parabolic vector bundle was generalized to higher dimensions by Maruyama and Yokogawa in [24].

There has been a lot of research on parabolic vector bundles, starting with a description of their moduli spaces in [24, 25]. In [4], certain orbifold bundles were associated to parabolic bundles (see also [9, 10]). The notion of a *ramified* principal $\mathrm{GL}_n(\mathbb{C})$-bundle, which generalizes the notion of principal $\mathrm{GL}_n(\mathbb{C})$-bundles, was introduced in [1]. It was shown there that parabolic vector bundles can be viewed as ramified $\mathrm{GL}_n(\mathbb{C})$-bundles. Using this identification, [7] constructed a projectivization of a parabolic vector bundle as well as the tautological line bundle on the projectivization.

Many positivity properties, such as ampleness of line bundles, have been generalized to vector bundles. Some of these notions have also been generalized to the context of parabolic vector bundles. For example, the notion of ample parabolic bundles was defined in [3]. The ampleness of a parabolic vector bundle is characterized by the ampleness of the tautological bundle on the projectivization associated to that parabolic bundle (see [7]).

Proceeding further in this direction, it is natural to ask whether other positivity notions can also be generalized to the set-up of parabolic vector bundles. As remarked earlier, Seshadri constant is an important invariant in the study of positivity. In this paper, we generalize the notion of Seshadri constants to parabolic vector bundles on projective varieties. Crucial ingredients in our construction are orbifold bundles associated to the parabolic bundles and projectivization of parabolic bundles. Many of our results and proofs are motivated by analogous results of Fulger and Murayama in [14].

In Section 2, we recall the definition and basic properties of parabolic vector bundles that we use. They are included for the sake of completeness and also for the convenience of the reader.

Let X be an irreducible projective variety and let E_* be a parabolic vector bundle on X. In Section 3, we give the definition of the parabolic Seshadri constant $\varepsilon_*(E_*, x)$ of E_* at any point $x \in X$; see Definition 3.1. This definition makes use of the notion of projectivization of a parabolic bundle; see Section 2.4 for a description of this construction. Alternately, one can also use the notion of orbifold bundles (described in Section 2.3) to define parabolic Seshadri constants.

Different alternative characterizations of parabolic Seshadri constants are given in Theorems 3.3 and 3.5.

We also prove an analogue of the Seshadri criterion for ampleness for parabolic bundles (see Theorem 3.6). We state this criterion here for the convenience of the reader. See Section 3 for the notation.

Theorem (Theorem 3.6). *Let E_* be a parabolic vector bundle on a smooth irreducible projective variety X such that the numerical class $\xi' \equiv \mathcal{O}_{\mathbb{P}(E')}(N(E_*))$ is τ-ample. Then E_* is parabolic ample if and only if*

$$\inf_{x \in X} \varepsilon_*(E_*, x) > 0,$$

where the infimum is taken over all points of X.

An upper bound for parabolic Seshadri constants is given in Theorem 3.7.

In Section 4, we prove several properties of parabolic Seshadri constants. We define the notion of multi-point Seshadri constants for the usual vector bundles and relate it to the parabolic Seshadri constants.

One of the results proved in this section describes how parabolic Seshadri constants can be computed using restriction to curves.

Theorem (Theorem 4.3). *Let E_* be a parabolic nef vector bundle on a smooth irreducible complex projective variety X, and let $E' \to Y$ be the corresponding orbifold bundle over Y (see Section 2.3). Then*

$$\varepsilon_*(E_*, x) = N(E_*) \cdot \inf_{C \subset Y} \left\{ \frac{\mu_{\min}(\nu^* E')}{\sum_{y \in \gamma^{-1}(x)} \operatorname{mult}_y C} \right\},$$

where the infimum is taken over all irreducible curves $C \subset Y$ such that $C \cap \gamma^{-1}(x) \neq \emptyset$, and $\nu : \bar{C} \to C$ is the normalization map.

We also compute parabolic Seshadri constants of symmetric powers of parabolic vector bundles (see Theorem 4.5) as well as tensor products of parabolic vector bundles (see Theorem 4.6). These results are stated below.

Theorem (Theorem 4.5). *Let E_* be a parabolic nef vector bundle on a smooth irreducible complex projective variety X. Then for any positive integer m*

$$\varepsilon_* \big(S^m(E_*), x \big) = m \varepsilon_*(E_*, x)$$

for every point $x \in X$.

Theorem (Theorem 4.6). *Let E_* and F_* be two parabolic nef vector bundles on a smooth irreducible complex projective variety X having a common parabolic divisor $D \subset X$. Then for every point $x \in X$,*

$$\varepsilon_*(E_* \otimes F_*, x) \geq N(E_* \otimes F_*) \cdot \left\{ \frac{\varepsilon_*(E_*, x)}{N(E_*)} + \frac{\varepsilon_*(F_*, x)}{N(F_*)} \right\}.$$

Some examples and questions are mentioned in Section 4.1.

We work throughout over the field $\mathbb{C}$ of complex numbers. The field of real numbers is denoted by $\mathbb{R}$.

2. Preliminaries

In this section, we briefly recall the definitions, properties and constructions associated to parabolic vector bundles; more details can be found in [1–4, 7, 8].

2.1. Parabolic sheaves

We start by defining the parabolic vector bundles.

Definition 2.1. Let X be a connected smooth complex projective variety of dimension d, and let $D \subset X$ be an effective divisor on X.

(1) Let E be a torsion-free coherent $\mathcal{O}_X$-module. A *quasi-parabolic structure* on E with respect to D is a filtration by $\mathcal{O}_X$-coherent subsheaves

$$E = \mathcal{F}_1(E) \supset \mathcal{F}_2(E) \supset \cdots \supset \mathcal{F}_l(E) \supset \mathcal{F}_{l+1}(E) = E(-D),$$

where $E(-D) := E \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)$. The above integer l is called the *length* of the filtration.

(2) A *parabolic structure* on E with respect to D is a quasi-parabolic structure as above together with a system of *weights* $\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$, where each α_i is a real number such that $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{l-1} < \alpha_l < 1$. The numbers $\{\alpha_i\}_{i=1}^l$ are called *parabolic weights* and we say that α_i is attached to $\mathcal{F}_i(E)$.

(3) A *parabolic sheaf* is a torsion-free coherent $\mathcal{O}_X$-module E with a parabolic structure.

   We denote a parabolic sheaf by the triple $(E, \mathcal{F}_*, \alpha_*)$, where $\mathcal{F}_*$ denotes the above filtration and α_* denotes the system of parabolic weights. When $\mathcal{F}_*$ and α_* are clear from the context, we denote $(E, \mathcal{F}_*, \alpha_*)$ by E_*.

(4) If E is a vector bundle on X, then E_* is called a *parabolic vector bundle with parabolic divisor* D. The *rank* of a parabolic vector bundle is simply the rank of the underlying vector bundle E.

For any parabolic sheaf E_* defined as above, and for any $t \in \mathbb{R}$, define the following filtration $\{E_t\}_{t \in \mathbb{R}}$ of coherent sheaves parametrized by $\mathbb{R}$:

$$E_t = \mathcal{F}_i(E)\big(-[t]D\big),$$

where $[t]$ is the integral part of t and $\alpha_{i-1} < t - [t] \leq \alpha_i$ with $\alpha_0 = \alpha_l - 1$ and $\alpha_{l+1} = 1$. The filtration $\{E_t\}_{t \in \mathbb{R}}$ evidently determines the parabolic structure $(E, \mathcal{F}_*, \alpha_*)$ uniquely. Note that any coherent subsheaf $M \subset E$ has an induced parabolic structure such that the corresponding filtration $\{M_t\}_{t \in \mathbb{R}}$ is defined by $M_t := (E_{t-[t]} \cap M)([t]D)$.

Consider the decomposition

$$D = \sum_{j=1}^n b_j D_j,$$

where every D_j is a reduced irreducible divisor, and $b_j \geq 1$. Let

$$f_j : b_j D_j \to X$$

denote the inclusion map of the subscheme $b_j D_j$. For each $1 \leq j \leq n$, choose a filtration

$$0 = F_{l_j+1}^j \subset F_{l_j}^j \subset F_{l_j-1}^j \subset \cdots \subset F_1^j = f_j^* E. \tag{2.1}$$

Fix real numbers α_k^j, $1 \le k \le l_j + 1$, such that

$$1 = \alpha_{l_j+1}^j > \alpha_{l_j}^j > \alpha_{l_j-1}^j > \cdots > \alpha_2^j > \alpha_1^j \ge 0.$$

For every $1 \le j \le n$ and $1 \le k \le l_j + 1$, define the coherent subsheaf $\bar{F}_j^i \subset E$ using the following short exact sequences:

$$0 \to \bar{F}_k^j \to E \to (f_j^* E)/F_k^j \to 0.$$

For $1 \le j \le n$ and $0 \le t \le 1$, let l_t^j be the smallest number in the set of integers

$$\{k \in \{1, 2, \ldots, l_j + 1\} \mid \alpha_k^j \ge t\}.$$

Finally, set

$$E_t := \bigcap_{j=1}^n \bar{F}_{l_t}^j \subseteq E. \tag{2.2}$$

The filtration $\{E_t\}_{t \in \mathbb{R}}$ in (2.2) defines a parabolic structure on E. It is straightforward to check that all parabolic structures on E, with D as the parabolic divisor, arise this way.

Let

$$\phi : X \setminus D \to X \tag{2.3}$$

denote the inclusion map of the complement of D. Let E_* and W_* be two parabolic sheaves on X, with D as the parabolic divisor, such that $E_0|_{X \setminus D}$ and $W_0|_{X \setminus D}$ are locally free. For any $c \in \mathbb{R}$, define M_c to be the subsheaf of $\phi_* \phi^* (E \otimes W)$ generated by all $E_s \otimes W_t$ with $s + t \ge c$, where ϕ is the map in (2.3). The parabolic sheaf given by the filtration $\{M_c\}_{c \in \mathbb{R}}$ is called the *parabolic tensor product* of E_* and W_*, and it is denoted by $E_* \otimes W_*$. The *parabolic m-fold symmetric product* $S^m(E_*)$ is the invariant subsheaf of the m-fold parabolic tensor product of E_* for the natural action of the permutation group for the factors of the tensor product. The underlying sheaf of the parabolic sheaf $S^m(E_*)$ will be denoted by $S^m(E_*)_0$.

Definition 2.2 ([3, Definition 2.3]). A parabolic bundle E_* is called *parabolic ample* if for any coherent sheaf F on X there is an integer m_0 such that for any $m \ge m_0$, the tensor product $F \otimes S^m(E_*)_0$ is globally generated.

Definition 2.3 ([8, Definition 3.2]). A parabolic vector bundle E_* is called *parabolic nef* if there is an ample line bundle L over X such that $S^m(E_*) \otimes L$ is parabolic ample for every positive integer m.

2.2. Semistability of parabolic bundles

We fix an ample line bundle L on X. For any coherent $\mathcal{O}_X$-module E, the *degree of E with respect to L* is defined to be

$$\deg(E) := \left(c_1(E) \cup c_1(L)^{d-1}\right) \cap [X] \in \mathbb{Z} \tag{2.4}$$

(see [24]). The *parabolic degree of E_* with respect to L*, denoted by par_deg(E_*), is defined as follows:

$$\text{par\_deg}(E_*) := \int_{-1}^{0} \deg(E_t)\, dt \in \mathbb{R}.$$

The *parabolic slope* of E_*, denoted by par_$\mu(E_*)$, is the quotient $\frac{\text{par\_deg}(E_*)}{\text{rank}(E)} \in \mathbb{R}$.

Definition 2.4. A parabolic sheaf E_* is called *parabolic semistable* (respectively, *parabolic stable*) if for every subsheaf M of E such that $0 < \text{rank}(M) < \text{rank}(E)$, the following inequality is satisfied:

$$\text{par\_}\mu(M_*) \leq \text{par\_}\mu(E_*) \quad \left(\text{respectively, par\_}\mu(M_*) < \text{par\_}\mu(E_*)\right).$$

Assumption 2.5. Henceforth we will always impose the following four conditions on the parabolic bundles E_*, with parabolic divisor D, that we will consider:

(a) The parabolic divisor $D = \sum_{i=1}^{n} b_i D_i$ is a normal crossing divisor, i.e., all $b_i = 1$ and D_i are smooth divisors and they intersect transversally.

(b) All F_j^i on D_i in sequence (2.1) are subbundles of $f_i^* E$ for every i.

(c) All the weights α_j^i are rational numbers; so $\alpha_j^i = m_j^i/N$, where N is some fixed integer and $m_j^i \in \{0, 1, \ldots, N-1\}$.

(d) Every point x of D has a neighborhood $U_x \subset X$, and a decomposition of $E|_{U_x}$ into a direct sum of line bundles, such that the filtration of all $E|_{U_x \cap D_i}$, $1 \leq i \leq n$, are constructed using the decomposition. To explain this condition in detail, let $D_{i_1}, \ldots, D_{i_x}$ be the irreducible components of $D = \sum_{i=1}^{n} D_i$ that contain the point x. For every $k \in i_1, \ldots, i_x$, consider the filtration $\{F_j^k\}_{j=1}^{l_k}$ of $f_k^* E = E|_{D_k}$ in sequence (2.1) (recall that $b_k = 1$ by (a)). The condition says that there is a holomorphic decomposition into a direct sum of line bundles

$$E\big|_{U_x} = \bigoplus_{\beta=1}^{r_E} L_\beta,$$

where $r_E = \text{rank}(E)$, such that every $F_j^k|_{U_x \cap D_k}$ considered above is of the form

$$F_j^k\big|_{U_x \cap D_k} = \bigoplus_{\alpha=1}^{r_{k,j}} L_{d_\alpha}\big|_{U_x \cap D_k},$$

where $r_{k,j} = \text{rank}(F_j^k)$ and $1 \leq d_1 < d_2 < \cdots < d_{r_{k,j}} \leq r_E$.

2.3. Orbifold bundles

Let Y be a smooth complex projective variety; its group of algebraic automorphisms will be denoted by Aut(Y). Let Γ be a finite group and $\psi : \Gamma \rightarrow \text{Aut}(Y)$ a homomorphism giving an action of Γ on Y.

Definition 2.6. An *orbifold bundle* on Y, with Γ as the *orbifold group*, is a vector bundle V on Y together with a lift of the action of Γ on Y to V, i.e., Γ acts on the total space of V such that the action of any $g \in \Gamma$ gives a vector bundle isomorphism between V and $\psi(g^{-1})^* V$. A subsheaf F of an orbifold bundle V is called an *orbifold subsheaf* if the action of Γ on V preserves F. A homomorphism $V \to V'$ between orbifold bundles is a homomorphism $V \to V'$ between the vector bundles that commutes with the actions of Γ on V and V'.

Let $\widetilde{L}$ be an orbifold line bundle on Y which is also ample. Then we can define the degree of any coherent sheaf on Y using $\widetilde{L}$ as done in (2.4).

Definition 2.7. An orbifold bundle V on Y is called *orbifold semistable* (respectively, *orbifold stable*) if for any orbifold subsheaf F of V with $0 < \text{rank}(F) < \text{rank}(E)$ the following inequality holds:

$$\frac{\deg(F)}{\text{rank}(F)} \leq \frac{\deg(V)}{\text{rank}(V)} \quad \left(\text{respectively, } \frac{\deg(F)}{\text{rank}(F)} < \frac{\deg(V)}{\text{rank}(V)} \right).$$

Now let X be a smooth projective complex variety, and let D be an effective divisor on X satisfying Assumption 2.5 (a). Fix an integer $N \geq 2$. From the Covering Lemma of Y. Kawamata, [21, Theorem 17] and [22, Theorem 1.1.1], we know that there is a connected smooth projective variety Y and a (ramified) Galois covering morphism

$$\gamma : Y \to X, \tag{2.5}$$

with Galois group $\Gamma = \text{Gal}(K(Y)/K(X))$, such that $\widetilde{D} := (\gamma^* D)_{\text{red}}$ is a normal crossing divisor on Y, and

$$\gamma^* D_i = k_i N (\gamma^* D_i)_{\text{red}} \tag{2.6}$$

for all $1 \leq i \leq n$, where k_i are some positive integers.

Let E_* be a parabolic bundle on X. Then using the Galois cover $\gamma : Y \to X$ in (2.5) we can construct an orbifold bundle V on Y such that the parabolic bundle E_* is recovered from it by taking Γ-invariants of the direct image of the twists of V using the irreducible components of $\widetilde{D}$ (see [4,9,10] for the explicit construction). Note that the same γ is used for all parabolic bundles E_* with the same parabolic divisor D.

For the convenience of the reader we will briefly recall the construction of an orbifold bundle on Y from a parabolic bundle on X with parabolic structure on D. First let $\mathcal{L}_*$ be a parabolic line bundle on X with parabolic structure on D. So $\mathcal{L}$ is a line bundle on X, and for each $1 \leq i \leq n$ we have $0 \leq \alpha^i = \frac{m_1^i}{N} < 1$ which is the parabolic weight over the component D_i of D; see Assumption 2.5 (c). Then the orbifold line bundle $\mathcal{L}'$ on Y corresponding to the parabolic line bundle $\mathcal{L}_*$ is

$$\mathcal{L}' = (\gamma^* \mathcal{L}) \otimes \left(\bigotimes_{i=1}^{n} \mathcal{O}_Y \left(k_i m_1^i (\gamma^* D_i)_{\text{red}} \right) \right)$$

(see (2.6) for k_i); note that each $(\gamma^* D_i)_{\mathrm{red}}$ is preserved by the action of Γ on Y and hence the above line bundle

$$\bigotimes_{i=1}^{n} \mathcal{O}_Y \left(k_i m_1^i (\gamma^* D_i)_{\mathrm{red}} \right)$$

has a natural orbifold structure. The orbifold structure on $\mathcal{L}'$ is given by the orbifold structures on $\gamma^* \mathcal{L}$ and $\bigotimes_{i=1}^{n} \mathcal{O}_Y (k_i m_1^i (\gamma^* D_i)_{\mathrm{red}})$. The action of Γ on $\mathcal{L}'$ produces an action of Γ on the direct image $\gamma_* \mathcal{L}'$. It is straightforward to check that

$$\mathcal{L} = (\gamma_* \mathcal{L}')^{\Gamma} \subset \gamma_* \mathcal{L}'.$$

Note that this construction is local in the sense that we do not need X or Y to be a projective variety. Let $S \subset D$ denote the singular locus of D, and let

$$\iota : Y \setminus \gamma^{-1}(S) \hookrightarrow Y \tag{2.7}$$

be the inclusion map. For a parabolic bundle E_* on X, consider the restriction of it to the complement $X \setminus S$. It can be locally expressed as direct sums of parabolic line bundles. Therefore, using the above construction of an orbifold line bundle from a parabolic line bundle, and patching locally defined orbifold bundles, we get an orbifold vector bundle $\mathcal{E}$ on $Y \setminus \gamma^{-1}(S)$. For the patching to work compatibly, note that if V_1 and V_2 are two orbifold bundles on $\gamma^{-1}(U)$ for two decompositions of the parabolic bundle $E_*|_U$ into direct sum of parabolic line bundles, then the natural isomorphism of orbifold bundles

$$\gamma^* \left(E|_{U \setminus (U \cap D)} \right) = V_1 |_{\gamma^{-1}(U) \cap (Y \setminus \gamma^{-1}(D))} \xrightarrow{\sim} V_2 |_{\gamma^{-1}(U) \cap (Y \setminus \gamma^{-1}(D))} = \gamma^* \left(E|_{U \setminus (U \cap D)} \right)$$

extends to an isomorphism $V_1 |_{\gamma^{-1}(U) \cap (Y \setminus \gamma^{-1}(S))} \xrightarrow{\sim} V_2 |_{\gamma^{-1}(U) \cap (Y \setminus \gamma^{-1}(S))}$. Since the codimension of $\gamma^{-1}(S) \subset Y$ is two (if $\gamma^{-1}(S)$ is nonempty), it follows that the direct image $\iota_* \mathcal{E}$ is a coherent sheaf on Y, where ι is the map in (2.7). The conditions in Assumption 2.5 ensure that the coherent sheaf $\iota_* \mathcal{E}$ is locally free. The action of Γ on $\mathcal{E}$ produces an action of Γ on $\iota_* \mathcal{E}$. This orbifold bundle $\iota_* \mathcal{E}$ is the one associated to the parabolic bundle E_*.

Let L be an ample line bundle on X. Consider $\widetilde{L} := \gamma^* L$ which is an orbifold line bundle on Y. Since γ is a finite morphism, and L is ample, the line bundle $\widetilde{L}$ is also ample. We will use L (respectively, $\widetilde{L}$) to define the degree of a coherent sheaf on X (respectively, Y).

The above mentioned correspondence between parabolic bundles and orbifold bundles satisfies the following compatibility condition with semistability.

Proposition 2.8 ([4, Lemmas 2.7 and 3.16]). *The above orbifold bundle V corresponding to the parabolic bundle E_* is orbifold semistable with respect to $\widetilde{L}$ if and only if E_* is parabolic semistable with respect to L. Also, V is semistable with respect to $\widetilde{L}$ in the usual sense if and only if E_* is parabolic semistable with respect to L.*

Proposition 2.9 ([3, Lemma 4.6], [2, Lemma 2.18], [8, Proposition 3.2]). *A parabolic bundle E_* is parabolic ample (respectively, nef) if and only if the corresponding orbifold bundle V is ample (respectively, nef) as a vector bundle (in the usual sense of [16]).*

Parabolic Chern classes, denoted by $c_i(E_*)$, of a parabolic bundle E_* have been introduced in [2] (see also [20]). Let V be the associated orbifold bundle on Y corresponding to a parabolic bundle E_* (satisfying Assumption 2.5) for a suitable Galois covering map $\gamma : Y \to X$. Then

$$\gamma^* c_i(E_*) = c_i(V)$$

for all $i \geq 0$ (see [2, Section 3]).

2.4. Parabolic bundles as ramified $GL(n, \mathbb{C})$-bundles

Let X be a smooth complex projective variety, and let D be a normal crossing divisor on X. Let

$$\phi : E_{GL(n,\mathbb{C})} \to X$$

be a *ramified principal $GL(n, \mathbb{C})$-bundle over X with ramification over D*. This means the total space $E_{GL(n,\mathbb{C})}$ is a smooth complex quasiprojective variety equipped with an algebraic right action of $GL(n, \mathbb{C})$

$$f : E_{GL(n,\mathbb{C})} \times GL(n, \mathbb{C}) \to E_{GL(n,\mathbb{C})}$$

satisfying the following five conditions:

(1) $\phi \circ f = \phi \circ p_1$, where p_1 is the natural projection of $E_{GL(n,\mathbb{C})} \times GL(n, \mathbb{C})$ to $E_{GL(n,\mathbb{C})}$,

(2) for each point $x \in X$, the action of $GL(n, \mathbb{C})$ on the reduced fiber $\phi^{-1}(x)_{\mathrm{red}}$ is transitive,

(3) the restriction of ϕ to $\phi^{-1}(X - D)$ a principal $GL(n, \mathbb{C})$-bundle over $X - D$,

(4) for each irreducible component $D_i \subset D$, the reduced inverse image $\phi^{-1}(D_i)_{\mathrm{red}}$ is a smooth divisor and

$$\widehat{D} := \sum_{i=1}^{l} \phi^{-1}(D_i)_{\mathrm{red}}$$

is a normal crossing divisor on $E_{GL(n,\mathbb{C})}$, and

(5) for any point $x \in D$, and any point $z \in \phi^{-1}(x)$, the isotropy subgroup

$$G_z \subset GL(n, \mathbb{C}),$$

for the action of $GL(n, \mathbb{C})$ on $E_{GL(n,\mathbb{C})}$, is a finite group, and if x is a smooth point of D, then the natural action of G_z on the quotient line $T_z E_{GL(n,\mathbb{C})}/T_z \phi^{-1}(D)_{\mathrm{red}}$ is faithful.

Note that the last condition implies that the isotropy subgroup $G_z \subset GL(n, \mathbb{C})$ is actually a finite cyclic group.

There is a natural bijective correspondence between the complex vector bundles of rank n on X and the principal $GL(n, \mathbb{C})$-bundles on X. This bijection sends a principal

$\mathrm{GL}(n, \mathbb{C})$-bundle F to the vector bundle $F \times^{\mathrm{GL}(n, \mathbb{C})} \mathbb{C}^n$ associated to F for the standard action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^n$. This correspondence extends to a bijective correspondence between the ramified principal $\mathrm{GL}(n, \mathbb{C})$-bundles with ramification over D and parabolic vector bundles of rank n with D as the parabolic divisor (see [1, Theorem 1.1]).

We next recall the construction of projectivization of a parabolic bundle. Let E_* be a parabolic vector bundle over X of rank n. Let

$$\phi : E_{\mathrm{GL}(n, \mathbb{C})} \to X$$

be the corresponding ramified principal $\mathrm{GL}(n, \mathbb{C})$-bundle with ramification divisor D. Consider the standard action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^n$; it induces an action of $\mathrm{GL}(n, \mathbb{C})$ on the projective space $\mathbb{P}^{n-1}$. The *projectivization* of E_*, denoted by $\mathbb{P}(E_*)$, is defined to be the associated (ramified) fiber bundle

$$\mathbb{P}(E_*) := E_{\mathrm{GL}(n, \mathbb{C})}(\mathbb{P}^{n-1}) := E_{\mathrm{GL}(n, \mathbb{C})} \times^{\mathrm{GL}(n, \mathbb{C})} \mathbb{P}^{n-1} \to X.$$

Take any point $x \in D$ and any $z \in \phi^{-1}(x)$. Let $G_z \subset \mathrm{GL}(n, \mathbb{C})$ be the isotropy subgroup for z for the action of $\mathrm{GL}(n, \mathbb{C})$ on $E_{\mathrm{GL}(n, \mathbb{C})}$. We recall that G_z is a finite group (by condition (5) above). Let n_x be the order of G_z. Note that the order of the group G_z is independent of the choice of $z \in \phi^{-1}(x)$ because $\mathrm{GL}(n, \mathbb{C})$ acts transitively on $\phi^{-1}(x)$. The number of distinct integers n_x as x varies over D is finite (see [7]). Let

$$N(E_*) = \mathrm{l.c.m.}\{n_x \mid x \in D\} \tag{2.8}$$

be the least common multiple of all these finitely many positive integers n_x.

For any point $y \in \mathbb{P}^{n-1}$, let $H_y \subset \mathrm{GL}(n, \mathbb{C})$ be the isotropy subgroup for the natural action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{P}^{n-1}$; so H_y is a maximal parabolic subgroup of $\mathrm{GL}(n, \mathbb{C})$. The group H_y then acts on the fiber of $\mathcal{O}_{\mathbb{P}^{n-1}}(1) \to \mathbb{P}^{n-1}$ over the point y. From the definition of $N(E_*)$ in (2.8) it follows immediately that for any $z \in \phi^{-1}(D)$ and any $y \in \mathbb{P}^{n-1}$, the subgroup $G_z \cap H_y \subset \mathrm{GL}(n, \mathbb{C})$ acts trivially on the fiber of the line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(N(E_*)) := \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\otimes N(E_*)}$ over the point y.

Consider the action of $\mathrm{GL}(n, \mathbb{C})$ on the total space of $\mathcal{O}_{\mathbb{P}^{n-1}}(N(E_*))$ constructed using the standard action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^n$. Let

$$E_{\mathrm{GL}(n, \mathbb{C})}\big(\mathcal{O}_{\mathbb{P}^{n-1}}(N(E_*))\big) := E_{\mathrm{GL}(n, \mathbb{C})} \times^{\mathrm{GL}(n, \mathbb{C})} \mathcal{O}_{\mathbb{P}^{n-1}}(N(E_*)) \to X$$

be the associated fiber bundle. As the natural projection $\mathcal{O}_{\mathbb{P}^{n-1}}(N(E_*)) \to \mathbb{P}^{n-1}$ intertwines the actions of $\mathrm{GL}(n, \mathbb{C})$ on $\mathcal{O}_{\mathbb{P}^{n-1}}(N(E_*))$ and $\mathbb{P}^{n-1}$, it produces a projection

$$E_{\mathrm{GL}(n, \mathbb{C})}\big(\mathcal{O}_{\mathbb{P}^{n-1}}(N(E_*))\big) \to E_{\mathrm{GL}(n, \mathbb{C})}(\mathbb{P}^{n-1}) = \mathbb{P}(E_*). \tag{2.9}$$

Using the above observation that $G_z \cap H_y$ acts trivially on the fibers of $\mathcal{O}_{\mathbb{P}^{n-1}}(N(E_*))$ over y it follows easily that the projection in (2.9) makes $E_{\mathrm{GL}(n, \mathbb{C})}(\mathcal{O}_{\mathbb{P}^{n-1}}(N(E_*)))$ an algebraic line bundle over the projectivization $\mathbb{P}(E_*)$. We denote this line bundle in (2.9) by $\mathcal{O}_{\mathbb{P}(E_*)}(1)$.

Let $E' \to Y$ be the orbifold bundle over Y corresponding to the parabolic vector bundle E_*, where $\gamma : Y \to X$ is a covering as in (2.5) with Galois group $\Gamma = \mathrm{Gal}(\gamma)$. Then the action of Γ on E' produces a left action of Γ on $\mathbb{P}(E')$. Clearly, we have

$$\mathbb{P}(E')/\Gamma = \mathbb{P}(E_*).$$

The isotropy subgroups, for the action of Γ on $\mathbb{P}(E')$, act trivially on the corresponding fibers of $\mathcal{O}_{\mathbb{P}(E')}(N(E_*))$; this follows from condition (5) above in the definition of a ramified principal $\mathrm{GL}(n, \mathbb{C})$-bundle and (2.8). Hence the quotient space $\mathcal{O}_{\mathbb{P}(E')}(N(E_*))/\Gamma$ is actually a line bundle over $\mathbb{P}(E')/\Gamma = \mathbb{P}(E_*)$. We have a natural isomorphism of line bundles

$$\mathcal{O}_{\mathbb{P}(E')}\big(N(E_*)\big)/\Gamma = \mathcal{O}_{\mathbb{P}(E_*)}(1).$$

Note that the pullback of $\mathcal{O}_{\mathbb{P}(E_*)}(1)$ to $\mathbb{P}(E')$ under the finite quotient map

$$\mathbb{P}(E') \to \mathbb{P}(E')/\Gamma = \mathbb{P}(E_*)$$

is $\mathcal{O}_{\mathbb{P}(E')}(N(E_*))$. The parabolic bundle E_* is parabolic ample (respectively, parabolic nef) if and only if the line bundle $\mathcal{O}_{\mathbb{P}(E_*)}(1)$ on $\mathbb{P}(E_*)$ is ample (respectively, nef) [7, Proposition 3.3]. Also, the following three statements are equivalent:

- the parabolic vector bundle E_* is nef,

- E' is nef, and

- $\mathcal{O}_{\mathbb{P}(E')}(N(E_*))$ is nef

(see [8, Proposition 3.2]).

3. Definition of Seshadri constant of parabolic bundles

Let E_* be a parabolic nef vector bundle of rank n on a smooth projective variety X. We fix a point $x \in X$, and let

$$\psi_x : \mathrm{Bl}_x(X) \to X$$

be the blow up of X at x with exceptional divisor $A_x = \psi_x^{-1}(x)$. Consider the following fiber product diagram:

$$
\begin{array}{ccc}
\mathrm{Bl}_{\rho^{-1}(x)}\big(\mathbb{P}(E_*)\big) = \mathbb{P}(E_*) \times_X \mathrm{Bl}_x(X) & \xrightarrow{\widetilde{\psi_x}} & \mathbb{P}(E_*) \\
\downarrow{\scriptstyle \widetilde{\rho}} & & \downarrow{\scriptstyle \rho} \\
\mathrm{Bl}_x(X) & \xrightarrow{\psi_x} & X
\end{array}
\tag{3.1}
$$

The above morphism ρ is the projectivization of the parabolic bundle E_* as constructed in Section 2.4. Hence the map ρ is flat and the fiber of ρ is a $\mathbb{P}^{n-1}$ (see [7]).

Let ξ be the numerical equivalence class of the line bundle $\mathcal{O}_{\mathbb{P}(E_*)}(1) \to \mathbb{P}(E_*)$.

Definition 3.1. Let E_* be a parabolic nef vector bundle on a smooth projective variety X. The *parabolic Seshadri constant* of E_* at a point $x \in X$, denoted by $\varepsilon_*(E_*, x)$, is defined to be

$$\varepsilon_*(E_*, x) := \sup \left\{ \lambda \in \mathbb{R}_{>0} \mid \widetilde{\psi_x}^*(\xi) - \lambda \widetilde{\rho}^* A_x \text{ is nef} \right\}$$

(see (3.1)).

Remark 3.2. As mentioned before, there is a canonical bijection between the parabolic vector bundles of rank n with parabolic divisor D and the ramified principal $\mathrm{GL}(n, \mathbb{C})$-bundles ramified over D. In particular, the constructions of the projectivization $\mathbb{P}(E_*)$ and the tautological bundle $\mathcal{O}_{\mathbb{P}(E_*)}(1)$ for a parabolic bundle E_* are uniquely determined by E_*. So the parabolic Seshadri constant $\varepsilon_*(E_*, x)$ is well-defined.

The next theorem shows that the parabolic Seshadri constant $\varepsilon_*(E_*, x)$ can be computed by examining the intersection of ξ with certain curves on $\mathbb{P}(E_*)$.

Theorem 3.3. *Let E_* be a parabolic nef vector bundle on a smooth complex projective variety X, and let $x \in X$ be a point of X. Let $\mathcal{C}_{\rho,x}$ be the set of all integral curves $C \subset \mathbb{P}(E_*)$ that intersect the fiber $\rho^{-1}(x)$ while not being contained in $\rho^{-1}(x)$. Then*

$$\varepsilon_*(E_*, x) = \inf_{C \in \mathcal{C}_{\rho,x}} \left\{ \frac{\xi \cdot C}{\mathrm{mult}_x \, \rho_* C} \right\}.$$

Proof. First note that $C \in \mathcal{C}_{\rho,x}$ if and only if $\mathrm{mult}_x \, \rho_* C > 0$. Let $\widetilde{C}$ be the strict transform of a curve $C \in \mathcal{C}_{\rho,x}$ under the map $\widetilde{\psi_x}$ in (3.1). We have

$$\widetilde{\rho}^* A_x \cdot \widetilde{C} = A_x \cdot \widetilde{\rho}_* \widetilde{C} = \mathrm{mult}_x \, \rho_* C.$$

Hence

$$\left\{ \widetilde{\psi_x}^*(\xi) - \lambda \widetilde{\rho}^* A_x \right\} \cdot \widetilde{C} \geq 0$$

if and only if $\xi \cdot C \geq \lambda \, \mathrm{mult}_x \, \rho_* C$.

Let C' be an irreducible curve in $\mathrm{Bl}_{\rho^{-1}(x)}(\mathbb{P}(E_*))$ which is contained in the exceptional locus $\widetilde{\rho}^{-1}(A_x)$ of $\widetilde{\psi_x}$. Then

$$\widetilde{\rho}^* A_x \cdot C' = A_x \cdot \widetilde{\rho}_* C' < 0.$$

Since E_* is given to be parabolic nef, the line bundle ξ is nef. Therefore, we conclude that

$$\left\{ \widetilde{\psi_x}^*(\xi) - \lambda \widetilde{\rho}^* A_x \right\} \cdot C' = \left\{ \xi \cdot \widetilde{\psi_x}_* \widetilde{C}' - \lambda A_x \cdot \widetilde{\rho}_* C' \right\} \geq 0$$

for all $\lambda \geq 0$. Every irreducible curve on $\mathrm{Bl}_{\rho^{-1}(x)}(\mathbb{P}(E_*))$ satisfies one of the following conditions:

- It is the strict transform of a curve $C \in \mathcal{C}_{\rho,x}$.
- It is contained in the exceptional locus $\widetilde{\rho}^{-1}(A_x)$ of $\widetilde{\psi_x}$.
- It does not intersect $\widetilde{\rho}^{-1}(A_x)$.

Now using the nefness of $\widetilde{\psi_x}^*(\xi)$, we obtain the result. ∎

Remark 3.4. Let $\rho : \mathbb{P}(E_*) \to X$ be the natural projection map, and let $\xi = \mathcal{O}_{\mathbb{P}(E_*)}(1)$, which is constructed in Section 2.4. For any point $x \in X$, the fiber $\rho^{-1}(x)$ is isomorphic to the projective space $\mathbb{P}^{n-1}$, and $\xi|_{\rho^{-1}(x)} = \mathcal{O}_{\mathbb{P}^{n-1}}(N(E_*))$.

Let $y \in \mathbb{P}(E_*)$. We can relate the classical Seshadri constant $\varepsilon(\xi, y)$ of ξ and the parabolic Seshadri constant $\varepsilon_*(E_*, \rho(y))$ of E_* at $\rho(y)$ as follows:

$$
\varepsilon(\xi, y) = \inf_{y \in C \subset \mathbb{P}(E_*)} \left\{ \frac{\xi \cdot C}{\mathrm{mult}_y C} \right\}
$$

$$
= \min \left\{ \inf_{C \not\subset \rho^{-1}(\rho(y)) \subseteq \mathbb{P}(E_*)} \left\{ \frac{\xi \cdot C}{\mathrm{mult}_y C} \right\}, \inf_{C \subset \rho^{-1}(\rho(y)) \subseteq \mathbb{P}(E_*)} \left\{ \frac{\xi \cdot C}{\mathrm{mult}_y C} \right\} \right\}.
$$

Hence we have

$$
\varepsilon(\xi, y) \geq \min \left\{ \inf_{C \not\subset \rho^{-1}(\rho(y)) \subseteq \mathbb{P}(E_*)} \left\{ \frac{\xi \cdot C}{\mathrm{mult}_y C} \right\}, N(E_*) \right\} \geq \min \left\{ \varepsilon_*(E_*, \rho(y)), N(E_*) \right\}.
$$

Now we give another alternative characterization of parabolic Seshadri constants. This characterization uses the Galois covering $\gamma : Y \to X$ and the orbifold bundle E' over Y associated to E_*; see Section 2.3.

Theorem 3.5. *Let E_* be a parabolic nef vector bundle on a smooth irreducible complex projective variety X, and let $E' \to Y$ be the corresponding orbifold bundle over Y, where $\gamma : Y \to X$ is a covering as in (2.5) with Galois group $\Gamma = \mathrm{Gal}(\gamma)$. Let*

$$
\phi_x : \mathrm{Bl}_{\gamma^{-1}(x)} Y \to Y
$$

be the blow up of Y along $\gamma^{-1}(x)$. Consider the following fiber product diagram:

$$
\begin{array}{ccc}
\mathbb{P}(\phi_x^* E') = \mathbb{P}(E') \times_Y \mathrm{Bl}_{\gamma^{-1}(x)} Y & \xrightarrow{\widetilde{\phi_x}} & \mathbb{P}(E') \\
\downarrow{\widetilde{\tau}} & & \downarrow{\tau} \\
\mathrm{Bl}_{\gamma^{-1}(x)} Y & \xrightarrow{\phi_x} & Y
\end{array}
$$

Denote the numerical equivalence class of the line bundle $\mathcal{O}_{\mathbb{P}(E')}(1)$ by ξ'. Let $A_{\gamma^{-1}(x)}$ be the exceptional divisor of the map ϕ_x. Then

$$
\varepsilon_*(E_*, x) = N(E_*) \cdot \sup \left\{ \lambda \in \mathbb{R}_{>0} \mid \widetilde{\phi_x}^*(\xi') - \lambda \widetilde{\tau}^*(A_{\gamma^{-1}(x)}) \text{ is nef} \right\}
$$

$$
= N(E_*) \cdot \inf_{C \in \mathcal{C}_{\tau, \gamma^{-1}(x)}} \left\{ \frac{\xi' \cdot C}{\sum_{y \in \gamma^{-1}(x)} \mathrm{mult}_y \tau_* C} \right\},
$$

where $\mathcal{C}_{\tau, \gamma^{-1}(x)}$ is the set of all irreducible curves C in $\mathbb{P}(E')$ such that

$$
C \not\subseteq \tau^{-1}\big(\gamma^{-1}(x)\big)
$$

and $\sum_{y \in \gamma^{-1}(x)} \mathrm{mult}_y \tau_* C > 0$, while $N(E_*)$ is defined as in (2.8).

Proof. Note that $\mathbb{P}(E') \times_Y \mathrm{Bl}_{\gamma^{-1}(x)} Y = \mathrm{Bl}_{\tau^{-1}(\gamma^{-1}(x))} \mathbb{P}(E')$. Consider the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Bl}_{\gamma^{-1}(x)} Y & \xrightarrow{\ \widetilde{\gamma}\ } & \mathrm{Bl}_x X \\
\downarrow{\scriptstyle \phi_x} & & \downarrow{\scriptstyle \psi_x} \\
Y & \xrightarrow{\ \gamma\ } & X
\end{array}
$$

such that $\widetilde{\gamma}^*(A_x) = A_{\gamma^{-1}(x)}$. Also, we have the following commutative diagram:

$$
\begin{array}{ccccc}
\mathbb{P}(E') \times_Y \mathrm{Bl}_{\gamma^{-1}(x)} Y & \xrightarrow{\ \widetilde{\phi_x}\ } & \mathbb{P}(E') & \xrightarrow{\ \gamma'\ } & \mathbb{P}(E_*) = \mathbb{P}(E')/\Gamma \\
\downarrow{\scriptstyle \widetilde{\tau}} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \rho} \\
\mathrm{Bl}_{\gamma^{-1}(x)} Y & \xrightarrow{\ \phi_x\ } & Y & \xrightarrow{\ \gamma\ } & X = Y/\Gamma.
\end{array}
$$

The universal property of a fiber product furnishes the following commutative diagram:

Recall that $\gamma'^*(\xi) = N(E_*)\xi'$. Hence for any $\lambda > 0$, the following three statements are equivalent:

- the line bundle $\widetilde{\psi_x}^*(\xi) - \lambda \widetilde{\rho}^* A_x$ is nef,
- $\Delta^*(\widetilde{\psi_x}^*(\xi) - \lambda \widetilde{\rho}^* A_x)$ is nef, and
- $N(E_*)\widetilde{\phi_x}^*(\xi') - \lambda \widetilde{\tau}^*(A_{\gamma^{-1}(x)})$ is nef.

Therefore, we conclude that

$$
\varepsilon_*(E_*, x) = N(E_*) \cdot \sup \left\{ \lambda \in \mathbb{R}_{>0} \mid \widetilde{\phi_x}^*(\xi') - \lambda \widetilde{\tau}^*(A_{\gamma^{-1}(x)}) \text{ is nef} \right\}.
$$

Next we claim that

$$
\sup \left\{ \lambda \in \mathbb{R}_{>0} \mid \widetilde{\phi_x}^*(\xi') - \lambda \widetilde{\tau}^*(A_{\gamma^{-1}(x)}) \text{ is nef} \right\}
$$

$$
= \inf_{C \in \mathcal{C}_{\tau, \gamma^{-1}(x)}} \left\{ \frac{\xi' \cdot C}{\sum_{y \in \gamma^{-1}(x)} \mathrm{mult}_y \, \tau_* C} \right\}. \tag{3.2}
$$

To prove (3.2), let $\widetilde{C}$ be the strict transform of a curve $C \in \mathcal{C}_{\tau, \gamma^{-1}(x)}$ under the blow up map

$$
\widetilde{\phi_x} : \mathrm{Bl}_{\tau^{-1}(\gamma^{-1}(x))} \mathbb{P}(E') \to \mathbb{P}(E').
$$

Note that

$$\widetilde{\tau}^* A_{\gamma^{-1}(x)} \cdot \widetilde{C} = A_{\gamma^{-1}(x)} \cdot \widetilde{\tau}_* \widetilde{C} = \sum_{y \in \gamma^{-1}(x)} \text{mult}_y \, \tau_* C.$$

Hence

$$\left\{ \widetilde{\phi_x}^*(\xi') - \lambda \widetilde{\tau}^* A_{\gamma^{-1}(x)} \right\} \cdot \widetilde{C} \geq 0$$

if and only if

$$\xi' \cdot C \geq \lambda \sum_{y \in \gamma^{-1}(x)} \text{mult}_y \, \tau_* C.$$

If C' is an irreducible curve in $\text{Bl}_{\tau^{-1}(\gamma^{-1}(x))} \, \mathbb{P}(E')$, and it is contained in the exceptional locus $\widetilde{\tau}^{-1}(A_{\gamma^{-1}(x)})$ of $\widetilde{\phi_x}$, then

$$\left\{ \widetilde{\phi_x}^*(\xi') - \lambda \widetilde{\tau}^* A_{\gamma^{-1}(x)} \right\} \cdot \widetilde{C} = \left\{ \xi' \cdot \widetilde{\phi_x}_* \widetilde{C} - \lambda A_{\gamma^{-1}(x)} \cdot \widetilde{\tau}_* C' \right\} \geq 0$$

for all $\lambda \geq 0$.

Any irreducible curve on $\text{Bl}_{\tau^{-1}(\gamma^{-1}(x))} \, \mathbb{P}(E')$ satisfies one of the following three conditions:

- It is the strict transform of a curve $C \in \mathcal{C}_{\tau, \gamma^{-1}(x)}$.
- It is contained in the exceptional locus $\widetilde{\tau}^{-1}(A_{\gamma^{-1}(x)})$ of $\widetilde{\phi_x}$.
- It does not intersect $\widetilde{\tau}^{-1}(A_{\gamma^{-1}(x)})$.

Now using the nefness of $\widetilde{\phi_x}^*(\xi')$ we conclude that (3.2) holds. This also completes the proof of the theorem. ∎

We use the notation of Theorem 3.5 in what follows.

Theorem 3.6 (Seshadri criterion for parabolic ampleness). *Let E_* be a parabolic vector bundle on a smooth irreducible projective variety X such that the numerical class*

$$\xi' \equiv \mathcal{O}_{\mathbb{P}(E')}\big(N(E_*)\big)$$

is τ-ample. Then E_* is parabolic ample if and only if

$$\inf_{x \in X} \varepsilon_*(E_*, x) > 0,$$

where the infimum is taken over all points of X.

Proof. Suppose that E_* is parabolic ample. Then the numerical equivalence class of the tautological line bundle $\xi = [\mathcal{O}_{\mathbb{P}(E_*)}(1)]$ is ample on $\mathbb{P}(E_*)$. Let h be an ample divisor class on X such that $\xi - \delta \rho^* h$ is ample for a sufficiently small $\delta > 0$. Now for all $C \in \mathcal{C}_{\rho,x}$, we have

$$(\xi - \delta \rho^* h) \cdot C > 0.$$

Then

$$\frac{\xi \cdot C}{\text{mult}_x \, \rho_* C} = \frac{(\xi - \delta \rho^* h) \cdot C}{\text{mult}_x \, \rho_* C} + \frac{\delta \rho^* h \cdot C}{\text{mult}_x \, \rho_* C}$$

$$\geq \frac{\delta \rho^* h \cdot C}{\mathrm{mult}_x \, \rho_* C}$$

$$\geq \delta \frac{h \cdot \rho_* C}{\mathrm{mult}_x \, \rho_* C}$$

$$\geq \delta \varepsilon(h, x)$$

$$> 0.$$

The last inequality follows using the usual Seshadri criterion for ample divisors, because h is ample; see [23, Theorem 1.4.13]. Hence

$$\inf_{C \in \mathscr{C}_{\rho,x}} \left\{ \frac{\xi \cdot C}{\mathrm{mult}_x \, \rho_* C} \right\} \geq \delta \varepsilon(h, x) > 0.$$

Consequently, from Theorem 3.3 it is deduced that

$$\varepsilon_*(E_*, x) \geq \delta \varepsilon(h, x) > 0$$

for every point $x \in X$. Thus we have

$$\inf_{x \in X} \varepsilon_*(E_*, x) \geq \delta \inf_{x \in X} \varepsilon(h, x) > 0.$$

The last inequality again follows from the usual Seshadri criterion applied to the ample divisor h.

To prove the converse, assume that $\inf_{x \in X} \varepsilon_*(E_*, x) > 0$ for all $x \in X$. Therefore, using Theorem 3.5 we have

$$\inf_{x \in X} \inf_{C \in \mathscr{C}_{\tau, \gamma^{-1}(x)}} \left\{ \frac{\xi' \cdot C}{\sum_{y \in \gamma^{-1}(x)} \mathrm{mult}_y \, \tau_* C} \right\} > 0. \tag{3.3}$$

Assume that E_* is not parabolic ample, i.e., $\xi' \equiv \mathcal{O}_{\mathbb{P}(E')}(N(E_*))$ is not ample. Note that as ξ' is τ-ample, we have ξ' is nef, i.e., E_* is parabolic nef. Then by the Seshadri criterion for ample divisors, one has

$$\inf_{z \in \mathbb{P}(E')} \varepsilon(\xi', z) = 0.$$

So for each $m \in \mathbb{N}$, there exist points $z_m \in \mathbb{P}(E')$ and irreducible curves $C_m \subset \mathbb{P}(E')$ through z_m such that

$$\xi' \cdot C_m < \frac{1}{m} \, \mathrm{mult}_{z_m} C_m.$$

We claim that the curve C_m is not contracted by τ for infinitely many m.

If the claim is not true, C_m are contracted by τ for all $m \gg 0$. Then for all $m \gg 0$

$$\inf_{y \in Y} \inf_{z \in \tau^{-1}(y)} \varepsilon\left(\xi'|_{\tau^{-1}(y)}, z\right) \leq \frac{\xi' \cdot C_m}{\mathrm{mult}_{z_m} C_m} < \frac{1}{m},$$

so that

$$\inf_{y \in Y} \inf_{z \in \tau^{-1}(y)} \varepsilon\left(\xi'|_{\tau^{-1}(y)}, z\right) = 0. \tag{3.4}$$

Choose an ample divisor h on Y so that $\xi' + \tau^* h$ is ample. Hence by the Seshadri ampleness criterion for ample divisors, we conclude that

$$\inf_{y \in Y} \inf_{z \in \tau^{-1}(y)} \varepsilon\big((\xi' + \tau^* h)|_{\tau^{-1}(y)}, z\big) = \inf_{y \in Y} \inf_{z \in \tau^{-1}(y)} \varepsilon\big(\xi'|_{\tau^{-1}(y)}, z\big) > 0,$$

which in fact contradicts (3.4). This proves our claim.

Therefore $\tau|_{C_{m_k}} : C_{m_k} \to Y$ is finite for an infinite sequence $\{m_k\}_{k=1}^{\infty}$ of distinct positive integers. Let $y_{m_k} = \tau(z_{m_k})$. Then by [13, Lemma 2.3], we have

$$\operatorname{mult}_{y_{m_k}} \tau_* C_{m_k} \geq \operatorname{mult}_{z_{m_k}} C_{m_k}.$$

This contradicts (3.3). Thus E_* is parabolic ample. ∎

The next theorem gives an upper bound for the parabolic Seshadri constants.

Theorem 3.7. *Let E_* be a parabolic nef vector bundle on X, and $x \in X$ a point. Then*

$$\varepsilon_*(E_*, x) \leq \left(\frac{N(E_*)^{\dim \tau(W)} \xi'^{\dim W} \cdot [W]}{\binom{\dim W}{\dim \tau(W)} \cdot |\Gamma| \big(\xi'^{\dim W_{\gamma^{-1}(x)}} [W_{\gamma^{-1}(x)}] \big)} \right)^{\frac{1}{\dim \tau(W)}},$$

as W ranges through the subvarieties of $\mathbb{P}(E')$ that meet $\tau^{-1}(\gamma^{-1}(x))$ without being contained in $\tau^{-1}(\gamma^{-1}(x))$, where τ and γ are as in Theorem 3.5. In the above inequality, $W_{\gamma^{-1}(x)} := \tau^{-1}(\gamma^{-1}(x)) \cap W$.

Proof. Consider the following commutative diagram

$$
\begin{array}{ccc}
\mathbb{P}(\phi_x^* E') = \mathbb{P}(E') \times_Y \mathrm{Bl}_{\gamma^{-1}(x)} Y & \xrightarrow{\ \widetilde{\phi_x}\ } & \mathbb{P}(E') \\
\downarrow{\scriptstyle \widetilde{\tau}} & & \downarrow{\scriptstyle \tau} \\
\mathrm{Bl}_{\gamma^{-1}(x)} Y & \xrightarrow{\ \ \phi_x\ \ } & Y
\end{array}
$$

where τ is the projectivization of the orbifold bundle E' over Y, and ϕ_x is the blow up of Y at $\gamma^{-1}(x)$. Recall from Theorem 3.5 that

$$\varepsilon_*(E_*, x) = N(E_*) \cdot \sup \big\{ \lambda \in \mathbb{R}_{>0} \mid \widetilde{\phi_x}^*(\xi') - \lambda \widetilde{\tau}^*(A_{\gamma^{-1}(x)}) \text{ is nef} \big\}.$$

Let $W \subseteq \mathbb{P}(E')$ be a subvariety that meets $\tau^{-1}(\gamma^{-1}(x))$ without being contained in $\tau^{-1}(\gamma^{-1}(x))$. Let $W' \subset \mathbb{P}(\phi_x^* E')$ be the strict transform of W by the blow up morphism $\widetilde{\phi_x}$. Then by the above observation we have

$$\big(N(E_*) \widetilde{\phi_x}^*(\xi') - \varepsilon_*(E_*, x) \widetilde{\tau}^*(A_{\gamma^{-1}(x)}) \big)^{\dim W'} \cdot W' \geq 0.$$

We now specialize to the situation where

$$W' = \mathbb{P}(\phi_x^* E').$$

Let $\dim Y = n$ and $\dim \mathbb{P}(E') = n + e$. Thus

$$\left(N(E_*)\widetilde{\phi_x}^*(\xi') - \varepsilon_*(E_*, x)\widetilde{\tau}^*(A_{\gamma^{-1}(x)})\right)^{n+e} \geq 0.$$

Note that the self-intersection number $A^k_{\gamma^{-1}(x)} = 0$ for $k > n$. Expanding binomially we thus get

$$\sum_{k=0}^{n} \binom{n+e}{k} \left(-\varepsilon_*(E_*, x)\widetilde{\tau}^* A_{\gamma^{-1}(x)}\right)^k N(E_*)^{n+e-k} \widetilde{\phi_x}^*(\xi'^{n+e-k}) \geq 0.$$

Note that $\widetilde{\phi}_{x_*}(-(-\widetilde{\tau}^* A_{\gamma^{-1}(x)})^k)$ are pseudoeffective since the restriction $-A_{\gamma^{-1}(x)}|_{A_{\gamma^{-1}(x)}}$ is ample. Hence using the projection formula we conclude that

$$N(E_*)^{n+e}\xi'^{n+e} + \binom{n+e}{n} N(E_*)^e\left[-\varepsilon_*(E_*, x)\widetilde{\tau}^* A_{\gamma^{-1}(x)}\right]^n \widetilde{\phi_x}^*(\xi'^e) \geq 0. \quad (3.5)$$

Now we have the following equalities:

$$\begin{aligned}
\left[-\varepsilon_*(E_*, x)\widetilde{\tau}^* A_{\gamma^{-1}(x)}\right]^n \widetilde{\phi_x}^*(\xi'^e) &= \varepsilon_*^n(E_*, x)\widetilde{\phi}_{x_*}\left((-\widetilde{\tau}^* A_{\gamma^{-1}(x)})^n\right)\xi'^e \\
&= \varepsilon_*^n(E_*, x)\tau^*\left[\left(\phi_{x_*}(-A_{\gamma^{-1}(x)})\right)^n\right]\xi'^e \\
&= \varepsilon_*^n(E_*, x)\tau^*\left(-\gamma^{-1}(x)\right)\xi'^e \\
&= -\varepsilon_*^n(E_*, x)|\Gamma|\left(\xi'^e \cdot [W_{\gamma^{-1}(x)}]\right),
\end{aligned}$$

where $W_{\gamma^{-1}(x)} = \tau^{-1}(\gamma^{-1}(x)) \cap W$.

From (3.5) it follows that

$$N(E_*)^n \cdot \xi'^{n+e} - \binom{n+e}{n}\varepsilon_*^n(E_*, x)|\Gamma|\left(\xi'^e \cdot [W_{\gamma^{-1}(x)}]\right) \geq 0.$$

Rearranging the terms we obtain the required bound on $\varepsilon_*(E_*, x)$.   ∎

For a parabolic vector bundle E_*, we define $\mu^{par}_{\min}(E_*)$ to be the parabolic slope of the minimal parabolic semistable subquotient of E_*, or in other words, $\mu^{par}_{\min}(E_*)$ is the parabolic slope of the final piece of the graded object for the Harder–Narasimhan filtration of E_*. Note that if E' is the orbifold bundle on Y corresponding to E_* for the Galois morphism $\gamma : Y \to X$ as in (2.5), then we have

$$\mu_{\min}(E') = |\mathrm{Gal}(\gamma)| \cdot \mu^{par}_{\min}(E_*).$$

Theorem 3.8. *Let E_* be a parabolic ample vector bundle over a smooth irreducible projective curve C with parabolic divisor D. Then for any point $x \in C$, the parabolic Seshadri constant satisfies the following:*

$$\varepsilon_*(E_*, x) = N(E_*) \cdot \mu^{par}_{\min}(E_*) \quad \text{when } x \notin D$$

and

$$\varepsilon_*(E_*, x) \geq N(E_*) \cdot \mu^{par}_{\min}(E_*) \quad \text{when } x \in D.$$

In particular, $\varepsilon_*(E_*, x) \geq \frac{N(E_*)}{\mathrm{rank}(E)}$ for every point $x \in C$.

Proof. As in (2.5), let $\gamma : C' \to C$ be a ramified Galois covering with Galois group $\mathrm{Gal}(\gamma) = \Gamma$. Then the cardinality of each fiber is

$$\left|\gamma^{-1}(x)\right| = |\Gamma|$$

if $x \notin D$, and it is $|\gamma^{-1}(x)| \leq |\Gamma|$ if $x \in D$.

Since C is a curve, the blow up map ϕ_x in Theorem 3.5 is the identity morphism, and hence $\widetilde{\phi_x}$ is also the identity morphism. Consequently,

$$\varepsilon_*(E_*, x) = N(E_*) \sup \left\{ \lambda \in \mathbb{R}_{>0} \mid \xi' - \lambda \tau^{-1}\left(\gamma^{-1}(x)\right) \text{ is nef} \right\}.$$

Denote the numerical class of fiber of τ by f. Then

$$\varepsilon_*(E_*, x) = N(E_*) \cdot \sup \left\{ \lambda \in \mathbb{R}_{>0} \mid \xi' - \lambda \left|\gamma^{-1}(x)\right| f \text{ is nef} \right\}.$$

Therefore,

$$\varepsilon_*(E_*, x) = N(E_*) \frac{1}{\left|\gamma^{-1}(x)\right|} \varepsilon(E', y),$$

where $y \in C'$. From [15, Theorem 3.1], it follows that $\varepsilon(E', y) = \mu_{\min}(E')$. Using [2, equation (2.15)], we also have $\mu_{\min}(E') = |\Gamma| \mu_{\min}^{\mathrm{par}}(E_*)$. Hence the theorem follows. $\blacksquare$

Remark 3.9. If E_* is a parabolic nef bundle but it is not ample, then the corresponding orbifold bundle E' is also nef but not ample. In this case

$$\varepsilon(E', y) = \mu_{\min}(E') = 0.$$

Hence $\varepsilon_*(E_*, x) = 0$.

4. Properties of parabolic Seshadri constants

In this section we define the notion of *multipoint Seshadri constants* for nef vector bundles on irreducible projective varieties, generalizing the case of a single point. We will then give a description of the parabolic Seshadri constants using restriction to curves.

Definition 4.1. Let E be a vector bundle on an irreducible complex projective variety Y, and let $Z = \{z_1, \ldots, z_n\}$ be a set of distinct points of Y. Let $\phi_z : \mathrm{Bl}_Z Y \to Y$ be the blow-up of Y along Z. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathbb{P}(\phi^* E) & \xrightarrow{\widetilde{\phi_z}} & \mathbb{P}(E) \\ \downarrow{\scriptstyle \widetilde{\tau}} & & \downarrow{\scriptstyle \tau} \\ \mathrm{Bl}_Z Y & \xrightarrow{\phi_z} & Y \end{array}$$

Let ξ and ξ' be the tautological bundles on $\mathbb{P}(E)$ and $\mathbb{P}(\phi^* E)$ respectively, and let

$$A'_{z_i} = \widetilde{\phi_z}^{-1}(\tau^{-1})(z_i)$$

for each $1 \leq i \leq n$. Then the *multipoint Seshadri constant* of E at Z, denoted by $\varepsilon(E, Z)$, is defined as follows:

$$\varepsilon(E, Z) := \sup \left\{ \lambda > 0 \mid \xi' - \lambda \sum_{i=1}^{n} A'_{z_i} \text{ is nef} \right\}.$$

It is straight-forward to verify the following equivalent formulation:

$$\varepsilon(E, Z) = \inf_{\mathcal{C}_{\tau, Z}} \left\{ \frac{\xi \cdot C}{\sum_{i=1}^{n} \mathrm{mult}_{z_i} \tau_* C} \right\},$$

where $\mathcal{C}_{\tau, Z}$ is the set of all irreducible curves $C \subset \mathbb{P}(E)$ such that

- $C \cap \tau^{-1}(Z) \neq \emptyset$, i.e., $\sum_{i=1}^{n} \mathrm{mult}_{z_i} \tau_* C > 0$, and
- $C \nsubseteq \tau^{-1}(Z)$.

Note that when Z is a singleton, the above definition recovers the usual single point Seshadri constant of a vector bundle.

The single point Seshadri constants of vector bundles over smooth curves have a particularly nice description. For a nef vector bundle E over a smooth curve Y, for any point $y \in Y$, we have

$$\varepsilon(E, y) = \inf_{C \in \mathcal{C}_{\tau, y}} \left\{ \frac{\xi \cdot C}{\mathrm{mult}_y \tau_* C} \right\} = \mu_{\min}(E),$$

where $\mathcal{C}_{\tau, y}$ is the set of curves $C \subset \mathbb{P}(E)$ such that C intersects the fiber $\tau^{-1}(y)$ but it is not contained in $\tau^{-1}(y)$.

Remark 4.2. Let C be an integral curve and $Z = \{z_1, \ldots, z_n\} \subset C$ a finite subset. Then

$$\mathrm{mult}_{z_i} C \geq 1 \quad \text{for each } 1 \leq i \leq n.$$

Set $\nu : \tilde{C} \to C$ to be the normalization. For a vector bundle E over C, consider the following fiber product diagram

$$\begin{array}{ccc}
\mathrm{Bl}_{\eta^{-1}(Z)} \mathbb{P}(E) = \mathbb{P}(\nu^* E) & \xrightarrow{\tilde{\nu}} & \mathbb{P}(E) \\
\downarrow{\tilde{\eta}} & & \downarrow{\eta} \\
\tilde{C} & \xrightarrow{\nu} & C
\end{array}$$

where $\mathbb{P}(E)$ is the projective bundle over C associated to E and η is the natural map. Let ξ and ξ' denote the numerical classes of $\mathcal{O}_{\mathbb{P}(\nu^* E)}(1)$ and $\mathcal{O}_{\mathbb{P}(E)}(1)$ respectively. Then

$$\tilde{\nu}^*(\xi) = \xi'.$$

Let $B' \subset \mathbb{P}(\nu^* E)$ (respectively, $B \subset \mathbb{P}(E)$) be a curve which is not in any fiber of $\tilde{\eta}$ (respectively, η). Suppose that B' is the strict transform of B under the blow up map $\tilde{\nu}$. Then

$$B' \cdot \xi' = B \cdot \xi.$$

Hence for any point $y \in \widetilde{C}$, we have

$$
\begin{aligned}
\varepsilon(E, Z) &= \inf_{B \in \mathcal{C}_{\eta, Z}} \left\{ \frac{\xi \cdot B}{\sum_{i=1}^{n} \mathrm{mult}_{z_i} \eta_* B} \right\} \\
&= \frac{1}{\sum_{i=1}^{n} \mathrm{mult}_{z_i} C} \left\{ \inf_{y \in \nu^{-1}(Z)} \inf_{B' \in \mathcal{C}_{\widetilde{\eta}, y}} \left\{ \frac{\xi' \cdot B'}{\mathrm{mult}_y \widetilde{\eta}_* B'} \right\} \right\} \\
&= \frac{1}{\sum_{i=1}^{n} \mathrm{mult}_{z_i} C} \left\{ \inf_{y \in \nu^{-1}(Z)} \varepsilon(\nu^* E, y) \right\} \\
&= \frac{1}{\sum_{i=1}^{n} \mathrm{mult}_{z_i} C} \left\{ \inf_{y \in \nu^{-1}(Z)} \mu_{\min}(\nu^* E) \right\} \\
&= \frac{\mu_{\min}(\nu^* E)}{\sum_{i=1}^{n} \mathrm{mult}_{z_i} C}.
\end{aligned}
$$

The next result describes how parabolic Seshadri constants can be computed using restriction to curves.

Theorem 4.3. *Let E_* be a parabolic nef vector bundle on a smooth irreducible complex projective variety X, and let $E' \to Y$ be the corresponding orbifold bundle over Y (see Section* 2.3*). Then*

$$
\varepsilon_*(E_*, x) = N(E_*) \cdot \inf_{C \subset Y} \left\{ \frac{\mu_{\min}(\nu^* E')}{\sum_{y \in \gamma^{-1}(x)} \mathrm{mult}_y C} \right\},
$$

where the infimum is taken over all irreducible curves $C \subset Y$ such that $C \cap \gamma^{-1}(x) \neq \emptyset$, and $\nu : \bar{C} \to C$ is the normalization map.

Proof. Let $C \subset Y$ be an irreducible curve such that $C \cap \gamma^{-1}(x) \neq \emptyset$. Consider the following diagram:

$$
\begin{array}{ccc}
\mathbb{P}(E'|_C) & \longrightarrow & \mathbb{P}(E') \\
\downarrow{\scriptstyle \tau_C} & & \downarrow{\scriptstyle \tau} \\
C & \longrightarrow & Y
\end{array}
$$

Recall that $\mathcal{C}_{\tau, \gamma^{-1}(x)}$ is the set of all irreducible curves W in $\mathbb{P}(E')$ such that

$$
W \not\subseteq \tau^{-1}(\gamma^{-1}(x)), \quad \text{and} \quad \sum_{y \in \gamma^{-1}(x)} \mathrm{mult}_y \tau_* W > 0.
$$

Similarly, we denote by $\mathcal{C}_{\tau_c, \gamma^{-1}(x)}$ the set of all irreducible curves $B \subset \mathbb{P}(E'|_C)$ such that

$$
B \not\subseteq \tau_c^{-1}(C \cap \gamma^{-1}(x))
$$

and $\sum_{y \in \gamma^{-1}(x)} \mathrm{mult}_y \tau_{c*} B > 0$. Note that

$$
\mathcal{C}_{\tau, \gamma^{-1}(x)} = \bigcup_{C \subset Y} \mathcal{C}_{\tau_c, \gamma^{-1}(x)}, \tag{4.1}
$$

where the union is taken over all irreducible curves C in Y such that $C \cap \gamma^{-1}(x) \neq \emptyset$. From Theorem 3.5 we have

$$\varepsilon_*(E_*, x) = N(E_*) \cdot \inf_{C \in \mathscr{C}_{\tau, \gamma^{-1}(x)}} \left\{ \frac{\xi' \cdot C}{\sum_{y \in \gamma^{-1}(x)} \text{mult}_y \, \tau_* C} \right\}.$$

Using (4.1) we have

$$\varepsilon_*(E_*, x) = N(E_*) \cdot \inf_{C \subset Y} \inf_{B \in \mathscr{C}_{\tau_c, \gamma^{-1}(x)}} \left\{ \frac{\xi' \cdot B}{\sum_{y \in \gamma^{-1}(x)} \text{mult}_y \, \tau_{c*} B} \right\},$$

where the infimum is taken over all irreducible curves C in Y such that $C \cap \gamma^{-1}(x) \neq \emptyset$.

Now observe that for an irreducible curve $C \subset Y$ with $C \cap \gamma^{-1}(x) \neq \emptyset$, using Remark 4.2 it follows that

$$\inf_{B \in \mathscr{C}_{\tau_c, \gamma^{-1}(x)}} \left\{ \frac{\xi' \cdot B}{\sum_{y \in \gamma^{-1}(x)} \text{mult}_y \, \tau_{c*} B} \right\} = \frac{\mu_{\min}(\nu^* E')}{\sum_{y \in \gamma^{-1}(x)} \text{mult}_y \, C}.$$

This completes the proof. ∎

For a parabolic vector bundle E_* of rank r we define its *parabolic discriminant*, denoted by $\triangle_{par}(E_*)$, as follows:

$$\triangle_{par}(E_*) := 2rc_2(E_*) - (r-1)c_1^2(E_*). \tag{4.2}$$

Remark 4.4. Let E_* be a parabolic nef vector bundle on X. Then, by Theorem 4.3, for any point $x \in X$

$$\varepsilon_*(E_*, x) = N(E_*) \cdot \inf_{C \subset Y} \left\{ \frac{\mu_{\min}(\nu^* E')}{\sum_{y \in \gamma^{-1}(x)} \text{mult}_y \, C} \right\},$$

where the infimum is taken over all irreducible curves $C \subset Y$ such that $C \cap \gamma^{-1}(x) \neq \emptyset$, and $\nu : \bar{C} \to C$ is the normalization map. Note that

$$\mu_{\min}(\nu^* E') \leq \mu(\nu^* E') = (\det(E') \cdot C) / \text{rank}(E').$$

Thus

$$\varepsilon_*(E_*, x) \leq \frac{N(E_*)}{\text{rank}(E_*)} \inf_{C \subset Y} \left\{ \frac{\det(E') \cdot C}{\sum_{y \in \gamma^{-1}(x)} \text{mult}_y \, C} \right\} = \frac{N(E_*)}{\text{rank}(E_*)} \varepsilon(\det(E'), \gamma^{-1}(x)).$$

Here $\varepsilon(\det(E'), \gamma^{-1}(x))$ denotes the multipoint Seshadri constants of E' at $\gamma^{-1}(x) \subset Y$. Note that the rank of the parabolic vector bundle E_* is simply the rank of the underlying vector bundle E.

Moreover, if E_* is a semistable parabolic ample vector bundle with $\triangle_{par}(E_*) = 0$ (see (4.2)), then E' is also semistable ample bundle on Y with vanishing discriminant. Hence in this case

$$\mu_{\min}(\nu^* E') = \mu(\nu^* E')$$

(see [5, Theorem 1.2]). Therefore, in this case we get the following equality:

$$\varepsilon_*(E_*, x) = \frac{N(E_*)}{\mathrm{rank}(E_*)} \varepsilon\big(\det(E'), \gamma^{-1}(x)\big)$$

for every point $x \in X$.

We now describe the parabolic Seshadri constants of symmetric powers and tensor products of parabolic vector bundles.

Theorem 4.5. *Let E_* be a parabolic nef vector bundle on a smooth irreducible complex projective variety X. Then for any positive integer m*

$$\varepsilon_*\big(S^m(E_*), x\big) = m\varepsilon_*(E_*, x)$$

for every point $x \in X$.

Proof. Let $E' \to Y$ be the corresponding orbifold bundle over Y as in Section 2.3. For any positive integer m, the parabolic bundle corresponding to the orbifold bundle $S^m(E')$ is the parabolic symmetric product $S^m(E_*)$ (see [3]). Therefore, from Proposition 2.9 it follows that $S^m(E_*)$ is nef if E_* is nef. Thus by Theorem 4.3,

$$\varepsilon_*\big(S^m(E_*), x\big) = N(E_*) \cdot \inf_{C \subset Y} \left\{ \frac{\mu_{\min}\big(\nu^*(S^m E')\big)}{\sum_{y \in \gamma^{-1}(x)} \mathrm{mult}_y\, C} \right\},$$

where the infimum is taken over all irreducible curves $C \subset Y$ such that $C \cap \gamma^{-1}(x) \neq \emptyset$, and $\nu : \bar{C} \to C$ is the normalization map.

As taking symmetric powers commutes with the operation of pullback, and

$$\mu_{\min}(S^m V) = m\mu_{\min}(V)$$

for a vector bundle V on a smooth complex projective curve $\bar{C}$, we have

$$\varepsilon_*\big(S^m(E_*), x\big) = N(E_*) \cdot \inf_{C \subset Y} \left\{ \frac{m\mu_{\min}\big(\nu^* E'\big)}{\sum_{y \in \gamma^{-1}(x)} \mathrm{mult}_y\, C} \right\} = m\varepsilon_*(E_*, x).$$

This completes the proof.   ∎

Theorem 4.6. *Let E_* and F_* be two parabolic nef vector bundles on a smooth irreducible complex projective variety X having a common parabolic divisor $D \subset X$. Then for every point $x \in X$,*

$$\varepsilon_*(E_* \otimes F_*, x) \geq N(E_* \otimes F_*) \cdot \left\{ \frac{\varepsilon_*(E_*, x)}{N(E_*)} + \frac{\varepsilon_*(F_*, x)}{N(F_*)} \right\}.$$

Proof. The correspondence between parabolic vector bundles and orbifold bundles takes tensor product of two orbifold bundles to the tensor product of the corresponding parabolic vector bundles (see [3]). Let E' and F' be the orbifold bundles on Y corresponding to E_*

and F_* respectively, where Y is the Galois cover of X as in Section 2.3. Then $E' \otimes F'$ corresponds to the parabolic bundle $E_* \otimes F_*$. By Theorem 4.3, for any point $x \in X$, we have

$$\varepsilon_*(E_* \otimes F_*, x) = N(E_* \otimes F_*) \cdot \inf_{C \subset Y} \left\{ \frac{\mu_{\min}\big(\nu^*(E' \otimes F')\big)}{\sum_{y \in \gamma^{-1}(x)} \text{mult}_y \, C} \right\},$$

where the infimum is taken over all irreducible curves $C \subset Y$ such that $C \cap \gamma^{-1}(x) \neq \emptyset$, and $\nu : \bar{C} \to C$ is the normalization map (see Theorem 4.3). For the smooth curve $\bar{C}$, we also have

$$\mu_{\min}\big(\nu^*(E' \otimes F')\big) = \mu_{\min}(\nu^* E') + \mu_{\min}(\nu^* F').$$

Therefore,

$$\varepsilon_*(E_* \otimes F_*, x) = N(E_* \otimes F_*) \cdot \inf_{C \subset Y} \left\{ \frac{\mu_{\min}(\nu^* E')}{\sum_{y \in \gamma^{-1}(x)} \text{mult}_y \, C} + \frac{\mu_{\min}(\nu^* F')}{\sum_{y \in \gamma^{-1}(x)} \text{mult}_y \, C} \right\}$$

$$\geq N(E_* \otimes F_*) \cdot \left\{ \frac{\varepsilon_*(E_*, x)}{N(E_*)} + \frac{\varepsilon_*(F_*, x)}{N(F_*)} \right\}.$$

This completes the proof. ∎

4.1. Examples and further questions

Let X be a smooth complex projective variety. Let E_* be a parabolic vector bundle on X with E as the underlying vector bundle. If X has dimension one, and E is ample, then it follows that E_* is parabolic ample; see [3, Theorem 3.1]. But in general this is not the case. We will give two examples to show that the parabolic ampleness of E_* does not imply the ampleness of E and vice versa.

Let D be an effective divisor on X and suppose that $D = \sum_{i=1}^{n} D_i$ is the decomposition of D into irreducible components. A parabolic line bundle on X with parabolic divisor D is a pair $L_* = (L, \{\alpha_1, \ldots, \alpha_n\})$, where L is a line bundle on X and the parabolic weight $0 \leq \alpha_i < 1$ corresponds to the divisor D_i. Assume that $\alpha_i \in \mathbb{Q}$ for all i. Then L_* is parabolic ample if and only if

$$c_1(L_*) := c_1(L) + \sum_{i=1}^{n} \alpha_i [D_i] \in H^{1,1}(X) \cap H^2(X, \mathbb{Q})$$

lies in the ample cone of X.

Example 4.7. Let $\pi : \mathbb{P}_C(W) \to C$ be a ruled surface over an elliptic curve C defined by the normalized rank 2 bundle W which sits in the following exact sequence:

$$0 \to \mathcal{O}_C \to W \to \mathcal{O}_C(p) \to 0$$

for some point $p \in C$. Therefore the associated invariant $e = -\deg(W)$ is -1 (see [18, Proposition 2.8]). We consider the simple normal crossing divisor $\sigma + f$, where σ denotes the normalized section of π such that $\mathcal{O}_{\mathbb{P}(W)}(\sigma) \simeq \mathcal{O}_{\mathbb{P}(W)}(1)$ and f denotes a fiber of the map π.

Let L be a line bundle on $\mathbb{P}_C(W)$ which is numerically equivalent to $\frac{1}{2}\sigma - \frac{1}{2}f$. Then L is not ample by [18, Proposition 2.21]. Next consider the parabolic bundle

$$L_* = \left(L, \{\alpha_1, \alpha_2\}\right)$$

on $\mathbb{P}_C(W)$ with parabolic weights $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = \frac{1}{2}$. Note that

$$c_1(L_*) = c_1(L) + \alpha_1\sigma + \alpha_2 f = \sigma$$

which is inside the ample cone of $\mathbb{P}_C(W)$. Thus L_* is parabolic ample, but L is not ample.

Example 4.8. Let $\pi : \mathbb{P}_C(W) \to C$ be a ruled surface over $\mathbb{P}^1_{\mathbb{C}}$ defined by the normalized rank 2 bundle $W = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}} \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)$. Then the associated invariant e is equal to 1. As done above, we consider the simple normal crossing divisor $\sigma + f$, where σ denotes the normalized section of π such that $\mathcal{O}_{\mathbb{P}(W)}(\sigma) \simeq \mathcal{O}_{\mathbb{P}(W)}(1)$ and f denotes a fiber of the map π.

Let $L \equiv \frac{1}{2}\sigma + f$ be a line bundle on $\mathbb{P}_C(W)$. Then L is ample by [18, Proposition 2.20]. Next consider the parabolic bundle

$$L_* = \left(L, \{\alpha_1, \alpha_2\}\right)$$

with parabolic weights $\alpha_1 = \frac{7}{8}$ and $\alpha_2 = \frac{1}{4}$. Note that

$$c_1(L_*) = c_1(L) + \alpha_1\sigma + \alpha_2 f = \frac{11}{8}\sigma + \frac{10}{8}f$$

which does not lie in the ample cone of $\mathbb{P}_C(W)$. Thus L_* is not parabolic ample, but L is ample.

In view of Examples 4.7 and 4.8, using the Seshadri criterion for parabolic ampleness (see Theorem 3.6) we see that there can not be an inequality between $\varepsilon_*(L_*, x)$ and $\varepsilon(L, x)$, in general. It is still interesting to ask the following question.

Question 4.9. Suppose E_* is an ample parabolic bundle on a projective variety X such that the underlying vector bundle E is also ample. Then can we compare $\varepsilon_*(E_*, x)$ and $\varepsilon(E, x)$?

Example 4.10. Let Z be a smooth complex projective variety, and let W be a vector bundle on Z. Let

$$\rho : X := \mathbb{P}(W) \to Z$$

be the associated projective bundle on Z. Take $D \subset Z$ to be a normal crossing divisor on Z, and let F_* be a semistable parabolic bundle of rank r on Z with parabolic divisor D. Then $\rho^* F_*$ is a parabolic semistable bundle on X with parabolic divisor $D' = \rho^*(D)$. Let $D' = \sum_{i=1}^n D'_i$ be the decomposition of D' into irreducible components.

Let L_* be a parabolic line bundle with parabolic divisor D' on X. So L_* is given by a pair $(L, \{\alpha_1, \ldots, \alpha_n\})$, where L is a line bundle on X and $0 \le \alpha_i < 1$ corresponds to the

divisor D_i'. Assume that $\alpha_i \in \mathbb{Q}$ for all i. Then

$$E_* = \rho^*(F_*) \otimes L_*$$

is parabolic semistable with $\triangle_{par}(E_*) = 0$. Note that

$$c_1(L_*) := c_1(L) + \sum_{i=1}^{n} \alpha_i[D_i].$$

We may choose L_* in such a way that

$$c_1(E_*) = c_1(\rho^* F_*) + rc_1(L_*)$$

lies in the ample cone of X. This way we can produce parabolic ample bundles on X with $\triangle_{par}(E_*) = 0$ (see [26, Example 22]). Using Remark 4.4 we can compute the parabolic Seshadri constants for such parabolic bundles.

In this situation, there is the following result on the relationship between the parabolic ampleness of E_* and the ampleness of its first parabolic Chern class.

Theorem 4.11 ([26, Theorem 19]). *Let E_* be a semistable parabolic vector bundle of rank r on a smooth complex projective variety X such that $\triangle_{par}(E_*) = 0$. Then E_* is parabolic ample if and only if its parabolic first Chern class $c_1(E_*)$ is in the ample cone of X.*

So one can ask the following question.

Question 4.12. Can we compare $\varepsilon_*(E_*, x)$ and $\varepsilon(c_1(E_*), x)$ in the above situation?

References

[1] V. Balaji, I. Biswas, and D. S. Nagaraj, Ramified G-bundles as parabolic bundles. *J. Ramanujan Math. Soc.* **18** (2003), no. 2, 123–138 Zbl 1095.14032 MR 1995862

[2] I. Biswas, Chern classes for parabolic bundles. *J. Math. Kyoto Univ.* **37** (1997), no. 4, 597–613 Zbl 0929.14005 MR 1625964

[3] I. Biswas, Parabolic ample bundles. *Math. Ann.* **307** (1997), no. 3, 511–529 Zbl 0877.14013 MR 1437053

[4] I. Biswas, Parabolic bundles as orbifold bundles. *Duke Math. J.* **88** (1997), no. 2, 305–325 Zbl 0955.14010 MR 1455522

[5] I. Biswas and U. Bruzzo, On semistable principal bundles over a complex projective manifold. *Int. Math. Res. Not. IMRN* **2008** (2008), no. 12, Art. ID rnn035 Zbl 1183.14065 MR 2426752

[6] I. Biswas, K. Hanumanthu, and D. S. Nagaraj, Positivity of vector bundles on homogeneous varieties. *Internat. J. Math.* **31** (2020), no. 12, article no. 2050097 Zbl 1458.14011 MR 4184429

[7] I. Biswas and F. Laytimi, Parabolic k-ample bundles. *Internat. J. Math.* **22** (2011), no. 11, 1647–1660 Zbl 1263.14041 MR 2863444

[8] I. Biswas and S. Subramanian, Parabolic ample bundles. III. Numerically effective vector bundles. *Proc. Indian Acad. Sci. Math. Sci.* **109** (1999), no. 1, 41–46 Zbl 0929.14007 MR 1687016

[9] N. Borne, Fibrés paraboliques et champ des racines. *Int. Math. Res. Not. IMRN* **2007** (2007), no. 16, article no. rnm049 Zbl 1197.14035 MR 2353089

[10] N. Borne, Sur les représentations du groupe fondamental d'une variété privée d'un diviseur à croisements normaux simples. *Indiana Univ. Math. J.* **58** (2009), no. 1, 137–180 Zbl 1186.14016 MR 2504408

[11] J. Dasgupta, B. Khan, and A. Subramaniam, Seshadri constants of equivariant vector bundles on toric varieties. *J. Algebra* **595** (2022), 38–68 Zbl 1478.14018 MR 4356829

[12] J.-P. Demailly, Singular Hermitian metrics on positive line bundles. In *Complex algebraic varieties (Bayreuth, 1990)*, pp. 87–104, Lecture Notes in Math. 1507, Springer, Berlin, 1992 Zbl 0784.32024 MR 1178721

[13] M. Fulger, Seshadri constants for curve classes. *Int. Math. Res. Not. IMRN* **2021** (2021), no. 21, 16448–16493 Zbl 1492.14006 MR 4338223

[14] M. Fulger and T. Murayama, Seshadri constants for vector bundles. *J. Pure Appl. Algebra* **225** (2021), no. 4, article no. 106559 Zbl 1445.14010 MR 4158762

[15] C. D. Hacon, Remarks on Seshadri constants of vector bundles. *Ann. Inst. Fourier (Grenoble)* **50** (2000), no. 3, 767–780 Zbl 0956.14034 MR 1779893

[16] R. Hartshorne, Ample vector bundles. *Inst. Hautes Études Sci. Publ. Math.* (1966), no. 29, 63–94 Zbl 0173.49003 MR 193092

[17] R. Hartshorne, *Ample subvarieties of algebraic varieties. Notes written in collaboration with C. Musili*. Lecture Notes in Math. 156, Springer, Berlin, 1970 Zbl 0208.48901 MR 0282977

[18] R. Hartshorne, *Algebraic geometry*. Undergrad. Texts Math. 52, Springer, New York, 1977 Zbl 0367.14001 MR 0463157

[19] M. Hering, M. Mustaţă, and S. Payne, Positivity properties of toric vector bundles. *Ann. Inst. Fourier (Grenoble)* **60** (2010), no. 2, 607–640 Zbl 1204.14024 MR 2667788

[20] J. N. N. Iyer and C. T. Simpson, A relation between the parabolic Chern characters of the de Rham bundles. *Math. Ann.* **338** (2007), no. 2, 347–383 Zbl 1132.14006 MR 2302066

[21] Y. Kawamata, Characterization of abelian varieties. *Compositio Math.* **43** (1981), no. 2, 253–276 Zbl 0471.14022 MR 622451

[22] Y. Kawamata, K. Matsuda, and K. Matsuki, Introduction to the minimal model problem. In *Algebraic geometry, Sendai, 1985*, pp. 283–360, Adv. Stud. Pure Math. 10, North-Holland, Amsterdam, 1987 Zbl 0672.14006 MR 946243

[23] R. Lazarsfeld, *Positivity in algebraic geometry. I*. Ergeb. Math. Grenzgeb. (3) 48, Springer, Berlin, 2004 Zbl 1093.14501 MR 2095471

[24] M. Maruyama and K. Yokogawa, Moduli of parabolic stable sheaves. *Math. Ann.* **293** (1992), no. 1, 77–99 Zbl 0735.14008 MR 1162674

[25] V. B. Mehta and C. S. Seshadri, Moduli of vector bundles on curves with parabolic structures. *Math. Ann.* **248** (1980), no. 3, 205–239 Zbl 0454.14006 MR 575939

[26] S. Misra and N. Ray, On ampleness of vector bundles. *C. R. Math. Acad. Sci. Paris* **359** (2021), 763–772 Zbl 1475.14080 MR 4311802

[27] S. Misra and N. Ray, Nef cones of projective bundles over surfaces and Seshadri constants. *Osaka J. Math.* **59** (2022), no. 3, 639–651 Zbl 1496.14008 MR 4450682

[28] C. S. Seshadri, Moduli of vector bundles on curves with parabolic structures. *Bull. Amer. Math. Soc.* **83** (1977), no. 1, 124–126 Zbl 0354.14005 MR 570987

Indranil Biswas
Department of Mathematics, Shiv Nadar University, NH91, Tehsil Dadri, Greater Noida,
Uttar Pradesh 201314, India; indranil.biswas@snu.edu.in, indranil29@gmail.com

Krishna Hanumanthu
Chennai Mathematical Institute, H1 SIPCOT IT Park, Siruseri, Kelambakkam 603103, India;
krishna@cmi.ac.in

Snehajit Misra
Chennai Mathematical Institute, H1 SIPCOT IT Park, Siruseri, Kelambakkam 603103, India;
snehajitm@cmi.ac.in

Nabanita Ray
Indraprastha Institute of Information Technology, Delhi, Okhla Industrial Estate, Phase III,
New Delhi, Delhi 110020, India; nabanita@iiitd.ac.in, nabanitaray2910@gmail.com