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The locus of curves with an odd subcanonical point

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Abstract. We present an explicit construction of a compactification of the locus of smooth curves whose symmetric Weierstrass semigroup at a marked point is odd. This construction extends Stöhr's techniques, which can be seen as a variant of Hauser's algorithm for computing versal deformation spaces. As an application, we prove the rationality of the locus for genus at most six.

1. Introduction

Let $\mathcal{H}_{2g-2}$ be the locus of compact Riemann surfaces (smooth projective algebraic curves) of genus $g \geq 4$ with a fixed abelian differential vanishing at a point to order $2g-2$. In a remarkable work, M. Kontsevich and A. Zorich [17, Thm. 1] showed that $\mathcal{H}_{2g-2}$ has exactly three irreducible components, namely the locus $\mathcal{H}^{\mathrm{hyp}}_{2g-2}$ of hyperelliptic points, the even $\mathcal{H}^{\mathrm{even}}_{2g-2}$ and the odd $\mathcal{H}^{\mathrm{odd}}_{2g-2}$ points. Ten years later E. Bullock [4, Thm. 2.1] characterized the general points of such components.

Theorem 1.1 (Bullock [4, Thm. 2.1]). *For $g \geq 4$, the following hold.*

- *A general point of $\mathcal{H}^{\mathrm{hyp}}_{2g-2}$ has Weierstrass gaps $\{1, 3, 5, \ldots, 2g-3, 2g-1\}$;*
- *A general point of $\mathcal{H}^{\mathrm{odd}}_{2g-2}$ has Weierstrass gaps $\{1, 2, 3, \ldots, g-1, 2g-1\}$;*
- *A general point of $\mathcal{H}^{\mathrm{even}}_{2g-2}$ has Weierstrass gaps $\{1, 2, 3, \ldots, g-2, g, 2g-1\}$.*

We recall that an abelian differential has a zero of order $2g-2$ at a point if, and only if, this point is *subcanonical*, [4, Def. 1], i.e. the associated Weierstrass semigroup is *symmetric*.

Let $\mathcal{M}^{\mathcal{S}}_{g,1}$ be the moduli space of smooth pointed curves of genus $g > 1$ with a fixed symmetric Weierstrass semigroup $\mathcal{S}$ at the marked point. In his famous Ph.D. thesis, Pinkham [20] studied the moduli $\mathcal{M}^{\mathcal{S}}_{g,1}$ relating it to the negatively graded part of the versal deformation space of the monomial curve associated to $\mathcal{S}$. Various authors have used his techniques to explicitly determine the defining equations of $\mathcal{M}^{\mathcal{S}}_{g,1}$ for low genus. For instance, in a recent paper, Stevens [23] uses computational methods to describe all the moduli spaces $\mathcal{M}^{\mathcal{S}}_{g,1}$ when $g \leq 7$, determining their dimensions, that coincide with a lower bound given by A. Contiero, A. Fontes, J. Stevens, and J. Vargas in [7].

Following a proposal made by D. Mumford [18], and also inspired by Pinkham's approach, K.-O. Stöhr [24] constructed a compactification of $\mathcal{M}^{\mathcal{S}}_{g,1}$, when $\mathcal{S}$ is symmetric,

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by allowing Gorenstein curves at the boundary. Stöhr's techniques avoid suitable classes of symmetric semigroups. Precisely, it is assumed that the multiplicity n_1 of $\mathcal{S}$ satisfies $3 < n_1 < g$, and that $\mathcal{S} \neq \langle 4, 5\rangle$, not achieving the general points of $\mathcal{H}^{\mathrm{hyp}}_{2g-2}$ and of $\mathcal{H}^{\mathrm{odd}}_{2g-2}$ of the Kontsevich–Zorich space $\mathcal{H}_{2g-2}$. The main obstruction in Stöhr's techniques to also deal with odd subcanonical points, lies in his way of constructing linear syzygies of the defining equations of the monomial curve, [24, Lem. 2.3]. Stöhr's syzygies are not explicitly given. They are constructed using the general position theorem for canonical curves and Petri's original paper, which requires the defining equations to be quadratic forms. However, the defining equations of a monomial curve associated to an odd numerical semigroup may not be quadratic forms

Since Gorenstein curves are not necessarily nodal, Stöhr's compactification does not relate to the Deligne–Munford stack of stable curves. A successful approach to study families of Weierstrass points and their limits in the moduli stack of Deligne–Mumford stable curves is to consider (generalized) Wronskians and their derivatives, we refer to [11, 12].

In this paper, we extend Stöhr's techniques to construct in a rather explicit way a compactification $\overline{\mathcal{M}}^{\mathcal{S}}_{g,1}$ of the moduli space $\mathcal{M}^{\mathcal{S}}_{g,1}$ when $\mathcal{S}$ is a symmetric semigroup different from the hyperelliptic $\langle 2, 2g+1\rangle$. Numerical semigroups of odd type tend to be realized as Weierstrass semigroups of possibly singular Gorenstein curves that are a triple covering of the projective line $\mathbb{P}^1$, i.e. 3-gonal singular curves, see Lemma 3.1. Hence the canonical ideal of the monomial Gorenstein curve associated to a numerical odd semigroup cannot be generated by only quadratic forms as required in Stöhr's paper [24], cf. Lemma 3.4 of the present work.

Given a non-hyperelliptic symmetric semigroup $\mathcal{S} \neq \langle 2, 2g+1\rangle$, following Hauser's algorithm [14,15], and also [22], we unfold the defining equations of the associated canonically embedded monomial Gorenstein curve, introducing new variables. To take care of flatness, we explore suitable syzygies that are given by purely combinatorial arguments, see Lemma 3.6, we then obtain a compactification of $\mathcal{M}^{\mathcal{S}}_{g,1}$ by allowing Gorenstein singularities at the boundary, cf. Theorem 3.9. The compactification is (by construction) a closed subset of the weighted projective space $\mathbb{P}(\mathrm{T}^{1,-}_{\mathbf{k}[\mathcal{S}]|_{\mathbf{k}}})$, where $\mathrm{T}^{1,-}_{\mathbf{k}[\mathcal{S}]|_{\mathbf{k}}}$ stands for the negatively graded part of the first module of the cotangent complex associated to the monomial curve singularity with semigroup $\mathcal{S}$, which relates to Pinkham's approach. Since our construction is completely explicit (and implementable) we can produce nontrivial examples and investigate the global geometry of the moduli spaces $\mathcal{M}^{\mathcal{S}}_{g,1}$.

In the last section of this paper we illustrate our techniques computing the equations of $\overline{\mathcal{M}}^{\mathcal{S}}_{g,1}$ when $\mathcal{S}$ is odd of genus 5, $\mathcal{S} = \langle 5, 6, 7, 8\rangle$, and of genus 6, $\mathcal{S} = \langle 6, 7, 8, 9, 10\rangle$, showing that the moduli varieties $\mathcal{M}^{\mathcal{S}}_{g,1}$ associated to these two odd numerical semigroups are rational.

2. Gorenstein subcanonical curves and Weierstrass points

Let $\mathcal{S}$ be a symmetric non-hyperelliptic numerical semigroup $\mathcal{S}$ of genus $g > 1$ with first g non-gaps $0 = n_0 < n_1 < \cdots < n_{g-1} = 2g-2$. We recall that a numerical semigroup $\mathcal{S}$

of genus g is symmetric if its Frobenius number (its largest gap) ℓ_g is the largest possible, namely $\ell_g = 2g-1$. Equivalently, $\mathcal{S}$ is symmetric if and only if $\ell_i = \ell_g - n_{g-i}$, for all $i = 1, \dots, g$, where ℓ_i stands for a gap of $\mathcal{S}$. Additionally, $\mathcal{S}$ is non-hyperelliptic if $2 \notin \mathcal{S}$.

Let us associate to $\mathcal{S}$ its canonical monomial curve

$$\mathcal{C}^{(0)} := \left\{(s^{n_0}t^{\ell_g-1} : s^{n_1}t^{\ell_{g-1}-1} : \dots : s^{n_{g-2}}t^{\ell_2-1} : s^{n_{g-1}}t^{\ell_1-1}) \mid (s:t) \in \mathbb{P}^1\right\} \subset \mathbb{P}^{g-1}.$$

It can be checked that it has a unique unbranched singular point, namely $(1:0:\dots:0)$, of singularity degree g. Since the semigroup $\mathcal{S}$ is symmetric, $C^{(0)}$ is a Gorenstein curve. The contact orders with hyperplanes at its unique point $P = (0:\dots:0:1)$ at the infinity are exactly $\ell_i - 1$, $i = 1, \dots, g$ (the vanishing sequence). Thus $\mathcal{C}^{(0)}$ has degree $2g-2$ and its Weierstrass semigroup at P is $\mathcal{S}$. Hence every symmetric non-hyperelliptic numerical semigroup can be realized as a Weierstrass semigroup at a smooth point on a canonical Gorenstein curve.

Let $\mathcal{C}$ be a complete integral Gorenstein curve of arithmetic genus $g > 1$ defined over an algebraically closed field $\mathbf{k}$. Throughout this section we assume that $\mathcal{C}$ is subcanonical, i.e. there is a rational function on $\mathcal{C}$ with pole divisor $(2g-2)P$, where P is a nonsingular point of $\mathcal{C}$. The dualizing sheaf ω of $\mathcal{C}$ is $\mathcal{O}_{\mathcal{C}}((2g-2)P)$, and the vector space of its global sections is

$$H^0(\mathcal{C}, \omega) = \mathbf{k} \cdot x_{n_0} \oplus \mathbf{k} \cdot x_{n_1} \oplus \dots \oplus \mathbf{k} \cdot x_{n_{g-1}}, \quad (n_0 = 0 < n_1 < \dots < n_{g-1} = 2g-2),$$

where each x_{n_i} is a rational function on $\mathcal{C}$ whose pole divisor is $n_i P$. Equivalently, the marked point $P \in \mathcal{C}$ is a Weierstrass point with gap sequence $1 = \ell_1 < \ell_2 < \dots < \ell_g = 2g-1$, and whose symmetric Weierstrass semigroup $\mathcal{S}$ of genus g is canonically generated by its first g non-gaps, $\langle n_0, n_1, \dots, n_{g-1}\rangle = \mathcal{S}$.

Let us assume that $\mathcal{C}$ is also non-hyperelliptic, thus its dualizing sheaf ω induces an embedding in the $(g-1)$-dimensional projective space $\mathbb{P}^{g-1}$ defined over $\mathbf{k}$,

$$(x_{n_0} : \dots : x_{n_{g-1}}) : \mathcal{C} \overset{\omega}{\hookrightarrow} \mathbb{P}^{g-1} = \mathbb{P}\big(H^0(\mathcal{C}, \omega)\big).$$

Therefore, $\mathcal{C}$ can be identified with its image under the canonical embedding. Hence $\mathcal{C} \subset \mathbb{P}^{g-1}$ is a projective curve of genus g, degree $2g-2$ and $P = (0:\dots:0:1)$.

According to Enriques–Babbage's theorem for smooth curves, if we assume that $\mathcal{C}$ is not isomorphic to a plane quintic, then its ideal can be generated by quadratic forms, when it is non-trigonal, and by quadratic and cubic forms when it is trigonal.

An extended version of Max Noether's theorem for complete integral non-hyperelliptic curves, cf. [6, 25], states the homomorphism

$$\mathrm{Sym}^r\big(H^0(\mathcal{C}, \omega)\big) \to H^0(\mathcal{C}, \omega^r)$$

is surjective for all $r \geq 1$. In the following, we recall a proof of Max Noether's theorem for subcanonical curves given by Stöhr in [24].

Let $\mathcal{C}$ be a complete non-hyperelliptic Gorenstein curve of genus g with a subcanonical point P. Since $\mathcal{C}$ is non-hyperelliptic, we must assume that the Weierstrass semigroup

$\mathcal{S}$ at P is not hyperelliptic, i.e. $2 \notin \mathcal{S}$, equivalently $\mathcal{S} \neq \langle 2, 2g+1 \rangle$. Now, for each nongap $s \leq 4g-4$, we consider a suitable partition of s as a sum of two nongaps

$$s = a_s + b_s,\ a_s \leq b_s \leq 2g-2,$$

with a_s the smallest possible nongap. From Oliveira's paper [19, Thm. 1.3] the $3g-3$ rational functions $x_{a_s} x_{b_s}$ of $\mathcal{C}$ form a P-hermitian basis for the space of the global sections of the bicanonical divisor $\omega^2 = \mathcal{O}_{\mathcal{C}}((4g-4)P)$. Now, for each integer $r \geq 3$ a P-hermitian basis for the space $H^0(\mathcal{C}, \omega^r)$ is given by the r-monomial expressions

$$\begin{array}{ll} x_{n_0}^{r-1} x_{n_i} & (i = 0, \ldots, g-1), \\ x_{n_0}^{r-2-i} x_{a_s} x_{b_s} x_{n_{g-1}}^i & (i = 0, \ldots, r-2,\ s = 2g, \ldots, 4g-4), \\ x_{n_0}^{r-3-i} x_{n_1} x_{2g-n_1} x_{n_{g-2}} x_{n_{g-1}}^i & (i = 0, \ldots, r-3). \end{array}$$

Note that the pole orders of the above $(2r-1)(g-1)$ rational functions are pairwise different, so they form a linearly independent set in $H^0(\mathcal{C}, \omega^r)$.

Let $I(\mathcal{C}) = \oplus_{r=2}^{\infty} I_r(\mathcal{C})$ be the homogeneous canonical ideal of $\mathcal{C} \subset \mathbb{P}^{g-1}$. As an immediate consequence of the existence of the above P-hermitian basis of r-monomials for $H^0(\mathcal{C}, \omega^r)$, the homomorphism

$$\mathbf{k}[X_{n_0}, \ldots, X_{n_{g-1}}]_r \to H^0(\mathcal{C}, \omega^r)$$

induced by the substitutions $X_{n_i} \mapsto x_{n_i}$ is surjective for each $r \geq 1$. Thus we obtain a proof of Max Noether's theorem for non-hyperelliptic Gorenstein curves with a subcanonical point.

Now Riemann's Theorem assures that for each $r \geq 2$, the codimension of $I_r(\mathcal{C})$ in the $\binom{r+g-1}{r}$-dimensional vector space $\mathbf{k}[X_{n_0}, \ldots, X_{n_{g-1}}]_r$ of homogeneous r-forms is equal to $(2r-1)(g-1)$. So the vector spaces of quadratic and cubic relations have dimensions

$$\dim I_2(\mathcal{C}) = \frac{(g-2)(g-3)}{2} \quad \text{and} \quad \dim I_3(\mathcal{C}) = \binom{g+2}{3} - (5g-5),$$

respectively.

For each $r \geq 2$, we define the vector subspace Λ_r of $\mathbf{k}[X_{n_0}, \ldots, X_{n_{g-1}}]_r$ spanned by the lifting, substituting $x_{n_i} \mapsto X_{n_i}$, of the above P-hermitian r-monomial basis of $H^0(\mathcal{C}, \omega^r)$. It is spanned by the r-monomials in $X_{n_0}, \ldots, X_{n_{g-1}}$ whose weights are all the nongaps $n \leq r(2g-2)$ and are pairwise different. We declare the weight of X_{n_i} to be n_i. Since $\Lambda_r \cap I_r(\mathcal{C}) = 0$ and

$$\dim \Lambda_r = \dim H^0(\mathcal{C}, \omega^r) = \operatorname{codim} I_r(\mathcal{C}),$$

we obtain

$$\mathbf{k}[X_{n_0}, \ldots, X_{n_{g-1}}]_r = I_r(\mathcal{C}) \oplus \Lambda_r, \quad \text{for each } r \geq 2.$$

Let $r\mathcal{S}$ be the set of all sums of r nongaps n_i with $i = 0, \ldots, g-1$. Oliveira showed, cf. [19, Thm. 1.5], that each nongap smaller than or equal to $r(2g-2)$ belongs to $r\mathcal{S}$.

Moreover, each sum of r nongaps $\leq 2g-2$ is a nongap $\leq r(2g-2)$. Consequently, $\#r\mathcal{S} = (2r-1)(g-1)$ and therefore

$$\#r\mathcal{S} = \dim H^0(\mathcal{C}, \omega^r).$$

Now, for each nongap $s \leq 4g-4$ we list all the partitions of s as sums of two nongaps, namely $s = a_{si} + b_{si} \in 2\mathcal{S}$ where

$$a_{si} \leq b_{si} \leq 2g-2 \ (i = 0, \dots, \nu_s) \quad \text{and} \quad a_s := a_{s0} < a_{s1} < a_{s2} < \cdots < a_{s\nu_s}.$$

Since $x_{a_{si}} x_{b_{si}} \in H^0(\mathcal{C}, \omega^2)$ and $\{x_{a_s} x_{b_s}\}$ is the above fixed basis for $H^0(\mathcal{C}, \omega^2)$, we can write

$$x_{a_{si}} x_{b_{si}} = \sum_{n=0}^{s} c_{sin} x_{a_n} x_{b_n},$$

for each $i = 0, \dots, \nu_s$, where the coefficients c_{sin} are uniquely determined constants and the summation index only varies through nongaps.

In the same way, for each nongap $\sigma \leq 6g-6$ we consider all the partitions of s as sums of three non-gaps, without repetitions, namely $\sigma = a_{\sigma j} + b_{\sigma j} + c_{\sigma j} \in 3\mathcal{S}$. Analogously, we can write

$$x_{a_{\sigma j}} x_{b_{\sigma j}} x_{c_{\sigma j}} = \sum_{n=0}^{\sigma} d_{\sigma j n} x_{a_n} x_{b_n} x_{c_n},$$

for each integer $j = 0, \dots, \nu_\sigma$, where the coefficients $d_{\sigma j n}$ are uniquely determined constants and the summation index only varies through nongaps.

Multiplying the functions $x_{n_0}, \dots, x_{n_{g-1}}$ by constants we do not change the P-hermitian property of the above basis, thus we can normalize the coefficients $c_{sis} = 1$ and $d_{\sigma j \sigma} = 1$. Therefore, the $\binom{g+1}{2} - (3g-3) = \frac{1}{2}(g-3)(g-2)$ quadratic forms

$$F_{si} = X_{a_{si}} X_{b_{si}} - X_{a_s} X_{b_s} - \sum_{n=0}^{s-1} c_{sin} X_{a_n} X_{b_n} \tag{2.1}$$

and the $\binom{g+2}{3} - (5g-5)$ cubic forms

$$G_{\sigma j} = X_{a_{\sigma j}} X_{b_{\sigma j}} X_{c_{\sigma j}} - X_{a_\sigma} X_{b_\sigma} X_{c_\sigma} - \sum_{n=0}^{\sigma-1} d_{\sigma j n} X_{a_n} X_{b_n} X_{c_n}, \tag{2.2}$$

vanish identically on $\mathcal{C}$. We attach to the coefficient c_{sin} the weight $s-n$ and to $d_{\sigma j n}$ the weight $\sigma - n$. Thus the above quadratic and cubic forms are also *isobaric* forms, recalling that the weight of X_{n_i} is n_i.

The above quadratic and cubic forms are *unfolded forms* of the initial quadratic forms $F_{si}^{(0)} = X_{a_{si}} X_{b_{si}} - X_{a_s} X_{b_s}$ and the initial cubic forms $G_{\sigma j}^{(0)} = X_{a_{\sigma j}} X_{b_{\sigma j}} X_{c_{\sigma j}} - X_{a_\sigma} X_{b_\sigma} X_{c_\sigma}$, respectively. Also, note that these initial forms vanish identically over the canonical monomial curve $\mathcal{C}^{(0)}$ and they only depend on $\mathcal{S}$.

We want to ensure that the canonical ideal of $\mathcal{C}$ can be generated by the above quadratic and cubic forms. Next, we show that there is only one remaining class of symmetric semigroups to be considered. We assume that the non-hyperelliptic symmetric semigroup $\mathcal{S}$ is non-ordinary of genus $g > 3$, that is equivalent to assume that the multiplicity n_1 of $\mathcal{S}$ satisfies $2 < n_1 \leq g$. By a theorem of Oliveira [19, Thm. 1.7], if we consider $3 < n_1 < g$, then there is at least one quadratic form, i.e. $\nu_s \geq 1$, whenever $s = n_i + 2g - 2$ for $i = 0, \ldots, g-3$. In this case, Contiero–Stöhr [9] gave an algorithmic proof that the canonical ideal of a Gorenstein curve $\mathcal{C} \subset \mathbb{P}^{g-1}$ with Weierstrass semigroup $\mathcal{S}$ at the base point is generated by only quadratic relations. If $3 \in \mathcal{S}$, then its genus has residue 1 or 0 modulo 3, hence $\mathcal{S} := \langle 3, g+1 \rangle$, implying that $\mathcal{C}^{(0)}$ is a complete intersection curve. The case where $\mathcal{C}^{(0)}$ is a local complete intersection curve is completely studied by Contiero and Mazzini in [8], where they show that $\bar{\mathcal{M}}^{\mathcal{S}}_{g,1} = \mathbb{P}(T^{1,-}_{\mathbf{k}[\mathcal{S}]|\mathbf{k}})$. If $\mathcal{S} = \langle 4, 5 \rangle$, then it is very known that $\mathcal{C}$ is isomorphic to a plane quintic contained in the Veronese surface. In addition, Contiero–Mazzini's work also achieves this last case. Hence the remaining case is just the odd case, $S = \mathbb{N}\backslash\{1, 2, \ldots, g-1, 2g-1\}$, and the curve $\mathcal{C}$ is possibly trigonal.

In the next section, we investigate the Weierstrass semigroup of trigonal complete curves and then, we will give an algorithmic proof that the canonical ideal of a complete Gorenstein curve with odd symmetric Weierstrass semigroup

$$S := \mathbb{N}\backslash\{1, 2, \ldots, g-1, 2g-1\} = \langle 0, g, g+1, \ldots, 2g-2 \rangle$$

at a smooth point is generated by quadratic and cubic forms.

3. Curves with an odd subcanonical point

Let $\mathcal{C}$ be a complete integral curve of arithmetic genus g defined over an algebraically closed field $\mathbf{k}$. A *linear system of dimension* r *on* $\mathcal{C}$ is a set of the form

$$\mathcal{L} = \mathcal{L}(\mathcal{F}, V) := \{x^{-1}\mathcal{F} \mid x \in V \setminus 0\}$$

where $\mathcal{F}$ is a coherent fractional ideal sheaf on C and V is a vector subspace of $H^0(\mathcal{C}, \mathcal{F})$ of dimension $r+1$.

The notion of linear systems on curves presented here is characterized by replacing bundles by torsion-free sheaves of rank 1. This is a meaningful approach since they may possess *non-removable* base points, see Coppens [10].

The *degree* of the linear system $\mathcal{L}$ is the integer $\deg \mathcal{F} := \chi(\mathcal{F}) - \chi(\mathcal{O}_{\mathcal{C}})$, where χ denotes the Euler characteristic. Note, in particular, that if $\mathcal{O}_{\mathcal{C}} \subset \mathcal{F}$ then

$$\deg \mathcal{F} = \sum_{P \in C} \dim(\mathcal{F}_P / \mathcal{O}_{\mathcal{C},P}).$$

The notation g^r_d stands for a linear system of degree d and dimension r. The linear system is said to be *complete* if $V = H^0(\mathcal{C}, \mathcal{F})$, in this case one simply writes $\mathcal{L} = |\mathcal{F}|$. According to E. Ballico's [1, p. 363, Def. 2.1 (3)], the gonality of $\mathcal{C}$ is the smallest d for that there exists a g^1_d on $\mathcal{C}$, or equivalently, a torsion free sheaf $\mathcal{F}$ of rank 1 on $\mathcal{C}$ with degree d and $h^0(\mathcal{C}, \mathcal{F}) \geq 2$.

The following lemma is a straightforward generalization of Kim's result [16, Thm. 2.6] characterizing the Weierstrass semigroup associated to an unramified point on a trigonal curve.

Lemma 3.1. *Let $\mathcal{C}$ be a complete integral trigonal curve of arithmetical genus $g \geq 5$ and $P \in \mathcal{C}$ a Weierstrass unramified point. The Weierstrass semigroup $\mathcal{S}$ of $\mathcal{C}$ at P is of the form*

$$\{0, m, m+1, m+2, \dots, m+(s-g), s+2, s+3, s+4, \dots\},$$

where s and m are positive integers such that $g \geq m \geq \lfloor \frac{s+1}{2} \rfloor + 1$. In particular, if $\mathcal{S}$ is symmetric, the Weierstrass semigroup is odd

$$\mathcal{S} = \{0, g, g+1, \dots, 2g-2, 2g, 2g+1, 2g+2, \dots\}.$$

Proof. Let ℓ_g be the Frobenius number of the Weierstrass semigroup $\mathcal{S}$ associated to $P \in \mathcal{C}$. Equivalently, $s := \ell_g - 1$ is the largest positive integer such that the divisor $D_0 = sP$ is special. Since P is a Weierstrass point, follows that $g \leq \ell_g - 1 \leq 2g-2$. By the maximality of s

$$\dim \left|\mathcal{O}(D_0)\right| = s - g + 1.$$

Since D_0 is a special divisor,

$$\omega_{\mathcal{C}} \simeq \mathcal{O}_{\mathcal{C}}(D_0 + P_1 + P_2 + \cdots + P_{2g-2-s})$$

is the dualizing sheaf of $\mathcal{C}$ where $P_i \in \mathcal{C}$, $P_i \neq P$ and $i = 1, \dots, 2g-2-s$. As P is an unramified point, the first nongap m is greater than 3, and so the linear system $|mP|$ is not g^1_3. Let us consider the divisor

$$D := (s-m)P + P_1 + P_2 + \cdots + P_{2g-2-s}. \tag{3.1}$$

We observe that $\omega_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}}(mP) \otimes \mathcal{O}_{\mathcal{C}}(D)$ and by [16, Lem. 2.1] follows that

$$|D| = (g-m)g^1_3 + B,$$

where B is the base locus of $|D|$, and $\dim|D| = g - m$ by Riemann–Roch theorem. For each element R of g^1_3 with $R \succeq P$, we have $R = P + Q_1 + Q_2$, with $P \neq Q_1$ and $P \neq Q_2$ because P is an unramified point of $\mathcal{C}$. Since D is the right-hand side of equation (3.1) we obtain

$$P_1 + P_2 + \cdots + P_{2g-2-s} \succeq (g-m)Q_1 + (g-m)Q_2,$$

implying $2(g-m) \leq 2g-2-s$. Therefore, $m \geq \lfloor \frac{s+1}{2} \rfloor + 1$.

On the other hand,

$$B \succeq (s-g)P,$$

that means that $(s-g)P$ is contained in the base locus of $|D|$. Consequently, each divisor iP is not in the base locus of $|mP+iP|$, $i=0,\dots,s-g$, and therefore $m, m+1,\dots,m+s-g$ are nongaps of $\mathcal{S}$. As $s+1$ is a gap and each $r\geq s+2$ is a nongap the set

$$S=\{0,m,m+1,\dots,m+(s-g),s+2,\dots\}$$

is contained in $\mathcal{S}$ and the cardinality of $\mathbb{N}-S$ is g. ∎

Let us consider a non-hyperelliptic symmetric semigroup of genus $g\geq 5$ and let us fix at once the $\frac{1}{2}(g-3)(g-2)$ initial quadratic forms as in (2.1)

$$F_{si}^{(0)} := X_{a_{si}}X_{b_{si}} - X_{a_s}X_{b_s} \tag{3.2}$$

and $\binom{g+2}{3}-(5g-5)$ initial cubic forms as in (2.2)

$$G_{\sigma j}^{(0)} := X_{a_{\sigma j}}X_{b_{\sigma j}}X_{c_{\sigma j}} - X_{a_\sigma}X_{b_\sigma}X_{c_\sigma}. \tag{3.3}$$

Remark 3.2. Many cubic initial forms are simply multiples of quadratic initial forms. Let $\wp$ denote the number of cubic initial forms that are not multiples of quadratic ones.

The next result explicitly identifies all the initial cubic forms that are not multiples of quadratic ones when $\mathcal{S}$ is an odd symmetric semigroup.

Proposition 3.3. *Let $\mathcal{S}:=\langle 0,g,g+1,\dots,2g-2\rangle$. There are $\wp=\binom{g+2}{3}-(5g-5)-\eta$, with*

$$\eta=(g-3)(g-2)+(g-2)\left\lfloor\frac{g-2}{2}\right\rfloor+\left\lfloor\frac{g-3}{2}\right\rfloor+\sum_{j=1}^{g-4}\left\lfloor\frac{g-2-j}{2}\right\rfloor,$$

initial cubic forms that are not multiples of the quadratic initial forms.

Proof. We starting by taking the fixed basis for Λ_2, that is,

$$\bigcup_{i=0,\,j=1}^{g-2}\{X_0^2, X_0X_{g+i}, X_gX_{g+i}, X_{g+j}X_{2g-2}\}.$$

The monomials of degree two that are not in Λ_2 are just $X_{g+i}X_{g+j}$ where $1\leq i\leq j$ and $j=1,\dots,g-3$. The basis for Λ_3 is

$$\bigcup_{\substack{i=0,g,g+1,\dots,2g-2\\ X_{a_s}X_{b_s}\in\Lambda_2}}\{X_0^2X_i, X_0X_{a_s}X_{b_s}, X_{a_s}X_{b_s}X_{2g-2}, X_g^2X_{2g-3}\}.$$

Set $F:=F_{sl}^{(0)}$ for an initial quadratic form. The products X_0F and $X_{2g-2}F$ are cubic forms for every F. Hence we already have $(g-3)(g-2)$ cubic forms that are multiple of quadratic initial forms.

Since $X_{g+k}X_{g+i}X_{g+j} \notin \Lambda_3$, for $k = 0, \ldots, g-3$ and $i, j = 1, \ldots, g-3$, the product $X_{g+k}F$ defines an initial cubic form when $X_{g+k}X_gX_{g+i}$ or $X_{g+k}X_{g+j}X_{2g-2}$ is in Λ_3. In the first case,

$$X_{g+k}X_gX_{g+i} \in \Lambda_3$$

for $i = g-2$, $k = 0, \ldots, g-3$ and $(i = g-3,\ k = 0)$. Hence, the initial cubic forms $X_{g+k}(X_{a_{sl}}X_{b_{sl}} - X_gX_{2g-2})$, with $k = 0, \ldots, g-3$, and $X_g(X_{a_{sl}}X_{b_{sl}} - X_gX_{2g-3})$ are multiples of quadratic initial forms, and there are $(g-2)\lfloor\frac{g-2}{2}\rfloor + \lfloor\frac{g-3}{2}\rfloor$ of them. In the remaining case, $X_{g+k}X_{g+j}X_{2g-2} \in \Lambda_3$ just for $k = 0$, $j = 1, \ldots, g-2$, obtaining $\sum_{j=1}^{g-4}\lfloor\frac{g-2-j}{2}\rfloor$ initial cubic forms $X_g(X_{a_{sl}}X_{b_{sl}} - X_{g+j}X_{2g-2})$, with $j = 1, \ldots, g-4$, and we are done. ■

It is straightforward that the quadratic $F_{si}^{(0)}$ and cubic forms $G_{\sigma j}^{(0)}$ introduced in (3.2) and (3.3), respectively, vanish identically on the monomial curve $\mathscr{C}^{(0)}$. The next lemma shows that $F_{si}^{(0)}$ and $G_{\sigma j}^{(0)}$ generate the ideal of $\mathscr{C}^{(0)}$, providing an extension of Contiero–Stoher's result [9, Lem. 2.2].

Lemma 3.4. *Let $\mathcal{S}$ be a non-hyperelliptic symmetric semigroup of genus $g \geq 4$. The canonical ideal $I(\mathscr{C}^{(0)})$ is generated by the $\frac{1}{2}(g-2)(g-3)$ quadratic forms $F_{si}^{(0)}$ and the $\wp$ (cf. Remark 3.2) cubic forms $G_{\sigma j}^{(0)}$. In particular, when $\mathcal{S}$ is odd the ideal of $\mathscr{C}^{(0)}$ is given by the $\frac{1}{2}(g-2)(g-3)$ quadratic initial forms and by the $\wp$ cubic initial forms described in Proposition 3.3.*

Proof. Since $I(\mathscr{C}^{(0)})$ is generated by homogeneous and isobaric forms, all we have to do is show that a homogeneous and isobaric form belongs to $I(\mathscr{C}^{(0)})$ if and only if it belongs to the ideal $\mathcal{J}$ generated by the binomials $F_{si}^{(0)}$ and $G_{\sigma j}^{(0)}$. It is just obvious that $\mathcal{J} \subseteq I(\mathscr{C}^{(0)})$. For the opposite inclusion we order the monomials $\prod_{k=0}^{g-1} X_{n_k}^{i_k}$ according to the lexicographic ordering of the vectors

$$\Big(\sum i_k, \sum n_k\, i_k, -i_0, -i_{g-1}, \ldots, -i_1\Big).$$

In this way, the binomials $F_{si}^{(0)}$ and $G_{\sigma j}^{(0)}$ form a Groebner basis for $\mathcal{J}$. Now, for each homogeneous form F of degree r that is also isobaric of weight ω we divide it by the Groebner basis obtaining a decomposition

$$F = \sum H_{si}F_{si}^{(0)} + \sum T_{\sigma j}G_{\sigma j}^{(0)} + R,$$

where $R \in \Lambda_r$ and H_{si} and $T_{\sigma j}$ are homogeneous of degrees $r-2$ and $r-3$, respectively, and isobaric of weights $\omega - s$ and $\omega - \sigma$, respectively. The remainder R is the only monomial in Λ_r of weight ω whose coefficient is equal to the sum of the coefficients of F. Since $F \in I(\mathscr{C}^{(0)})$ the sum of its coefficients is equal to zero, then $R = 0$. ■

A different proof of the above lemma when $\mathcal{S}$ is odd can be found in [13, Thm. 1.1] by noting that the symmetric semigroup $\mathcal{S} = \langle 0, g, g+1, \ldots, 2g-2\rangle$ is generated by a generalized arithmetic sequence. The ideal $I(\mathscr{C}^{(0)})$ is then generated by the 2×2 minors of suitable matrices, cf. [13, Thm. 1.1]. So, it can be seen immediately that the ideal given by the 2×2 minors is equal to the ideal generated by the initial forms $F_{si}^{(0)}$ and $G_{\sigma j}^{(0)}$.

As we noted above, for each numerical symmetric semigroup $\mathcal{S}$ of genus $g>4$, it is associated the canonical monomial curve $\mathscr{C}^{(0)}$ and also $\frac{1}{2}(g-2)(g-3)$ initial quadratic forms and $\wp$ initial cubic forms, $F_{si}^{(0)}$ and $G_{\sigma j}^{(0)}$, respectively, that generated the ideal $\mathrm{I}(\mathscr{C}^{(0)})$, cf. Lemma 3.4. Now we unfold the quadratic and cubic initial forms in the following way:

$$F_{si} := X_{a_{si}}X_{b_{si}} - X_{a_s}X_{b_s} - \sum_{n=0}^{s-1} c_{sin}X_{a_n}X_{b_n}$$

and

$$G_{\sigma j} := X_{a_{\sigma j}}X_{b_{\sigma j}}X_{c_{\sigma j}} - X_{a_\sigma}X_{b_\sigma}X_{c_\sigma} - \sum_{n=0}^{\sigma-1} d_{\sigma jn}X_{a_n}X_{b_n}X_{c_n},$$

where the coefficients c_{sin} and $d_{\sigma jn}$ belong to the ground field $\mathbf{k}$. Each unfolded is taken by adding to the initials forms $F_{si}^{(0)}$ (respectively $G_{\sigma j}^{(0)}$) a summation, a linear combination, of basis elements in Λ_2 (in Λ_3) whose weights are less than the weight s (respectively σ) of the respective quadratic initial forms $F_{si}^{(0)}$ (cubic initial forms $G_{\sigma j}^{(0)}$). We want to preserve the pole orders at P when we specialize $X_{n_i}\mapsto x_{n_i}$, keeping the P-hermitian properness, that was the starting point to produce the quadratic and cubic (initial) forms.

Goal 3.5. Let $\mathbb{A}^N$ be the affine space whose coordinates function are the coefficients c_{sin} and $d_{\sigma jn}$. By considering $\mathscr{X}\subset\mathbb{P}^{g-1}\times\mathbb{A}^N$ the zero locus of the unfolded forms, and taking π to be the restriction to $\mathscr{X}$ of the second projection $\mathbb{P}^{g-1}\times\mathbb{A}^N\to\mathbb{A}^N$, we obtain a commutative diagram

$$\begin{array}{ccc} \mathscr{C}^{(0)} & \hookrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \pi \\ \operatorname{Spec}\mathbf{k} & \longrightarrow & \mathbb{A}^N \end{array}$$

whose fibers are closed subsets of $\mathbb{P}^{g-1}$ and the special fiber is $\mathscr{C}^{(0)}$. Our goal is to explicitly describe the conditions on the coefficients c_{sin} and $d_{\sigma jn}$ that provide canonical Gorenstein curves with a Weierstrass semigroup $\mathcal{S}$ as fibers over π.

To achieve the Goal, we require some technical results. We begin by generalizing a result in [9, Lem. 2.3], which only considers the first syzygies of quadratic initial forms due to its assumptions. Here, we must also consider syzygies of cubic forms, which will induce nonlinear syzygies (see, for example, equations (4.1) and (4.3) of Section 4).

Syzygy lemma 3.6. *For each of the* $\frac{1}{2}(g-3)(g-4)$ *quadratic forms* $F_{s'i'}^{(0)}$ *not equal to* $F_{n_i+2g-2,1}^{(0)}(i=1,\dots,g-3)$ *there is a linear syzygy of the form*

$$X_{2g-2}F_{s'i'}^{(0)} + \sum_{nsi}\varepsilon_{nsi}^{(s'i')}X_nF_{si}^{(0)} = 0 \tag{3.4}$$

and for each cubic form $G^{(0)}_{\sigma' j'}$ not equal to $G^{(0)}_{4g-4,1}$, there is a syzygy of the form

$$X_{2g-2}G^{(0)}_{\sigma' j'} + \sum_{q\sigma j} \rho^{(\sigma' j')}_{q\sigma j} X_q G^{(0)}_{\sigma j} = 0, \tag{3.5}$$

where the coefficients $\varepsilon^{(s'i')}_{nsi}$, $\rho^{(\sigma' j')}_{q\sigma j}$ are equal to $1, -1$ or 0, and where the sum is taken over the nongaps $n, q < 2g-2$, the double indices si with $s + n = 2g-2+s'$ and σj with $q + \sigma = 2g-2+\sigma'$.

Proof. Given a quadratic form $F = F^{(0)}_{s'i'}$ or $F = -F^{(0)}_{s'i'}$, we can write

$$F = X_m X_n - X_q X_r,$$

where m, n, q, r are nongaps satisfying $m + n = q + r$ and $q < m \le n < r < 2g-2$. If $r+1$ is a gap then, by symmetry, $k := 2g-2-r+n$ is a nongap and we find the syzygy

$$X_{2g-2}(X_m X_n - X_q X_r) + X_r(X_q X_{2g-2} - X_m X_k) - X_m(X_n X_{2g-2} - X_r X_k) = 0,$$

The binomials in the brackets can be written as $F^{(0)}_{si} - F^{(0)}_{sj}$, $F^{(0)}_{si}$ or $-F^{(0)}_{sj}$. Analogously if $m+1$ is a gap then we take the nongap $k := 2g-2-m+r$ and we obtain a syzygy as above. Now we can assume that $r+1$ and $m+1$ are nongaps, hence we have the syzygy

$$\begin{aligned} &X_{2g-2}(X_m X_n - X_q X_r) + X_q(X_{2g-2}X_r - X_{2g-3}X_{r+1}) \\ &\quad - X_{2g-3}(X_{m+1}X_n - X_q X_{r+1}) - X_n(X_m X_{2g-2} - X_{2g-3}X_{m+1}) = 0. \end{aligned}$$

For a cubic form, if we set $G = G^{(0)}_{\sigma j}$ or $G = -G^{(0)}_{\sigma j}$ then we can write

$$G = X_m X_n X_p - X_q X_r X_t,$$

where m, n, p, q, r, s are nongaps satisfying $m + n + p = q + r + t$ and $q < m \le n \le r \le p < t \le 2g-2$.

If $p+1$ is a gap then, by symmetry, the integer $k := 2g-2-p+q$ is a nongap smaller than $2g-2$, hence we have the syzygy

$$\begin{aligned} &X_{2g-2}(X_m X_n X_p - X_q X_r X_t) + X_r(X_{2g-2}X_t X_q - X_t X_p X_k) \\ &\quad - X_p(X_{2g-2}X_m X_n - X_r X_t X_k) = 0, \end{aligned}$$

where the binomials in the brackets can be written as $G^{(0)}_{\sigma j} - G^{(0)}_{\sigma i}$, $G^{(0)}_{\sigma j}$ or $-G^{(0)}_{\sigma i}$. Analogously, if $r+1$ is a gap then $k := 2g-2-r+p$ is a nongap, and therefore we obtain the syzygy

$$\begin{aligned} &X_{2g-2}(X_m X_n X_p - X_q X_r X_t) + X_m(X_k X_r X_n - X_{2g-2}X_p X_n) \\ &\quad - X_r(X_k X_m X_n - X_{2g-2}X_t X_q) = 0. \end{aligned}$$

Now we can assume that $p+1$ and $r+1$ are the nongaps. We just take the syzygy

$$\begin{aligned} &X_{2g-2}(X_m X_n X_p - X_q X_r X_t) + X_{2g-3}(X_{r+1}X_q X_t - X_{p+1}X_n X_m) \\ &- X_m(X_p X_{2g-2}X_n - X_{p+1}X_{2g-3}X_n) - X_q(X_{2g-3}X_{r+1}X_t - X_{2g-2}X_r X_t) = 0. \quad \blacksquare \end{aligned}$$

The syzygies corresponding to the cubic forms that are multiples of the quadratics forms are trivial, therefore we just consider syzygies for the $\wp - 1$ cubic forms, however, these $\wp - 1$ syzygies are not necessarily linear.

Lemma 3.7. *Let I be the ideal generated by the $\frac{1}{2}(g-2)(g-3)$ unfolded quadratic forms F_{si} and by the $\wp$ unfolded cubic forms $G_{\sigma j}$. Then,*

$$\mathbf{k}[X_{n_0}, \dots, X_{n_{g-1}}]_r = I_r + \Lambda_r \quad \textit{for each } r \geq 2.$$

Proof. Let F be a homogeneous polynomial of degree r and weight w. Let S be its quasi-homogeneous component of weight w and R the unique monomial in Λ_r of weight w whose coefficient is the sum of the coefficients of S. Thus, $S - R \in I(\mathscr{C}^{(0)})$ and by Lemma 3.4 we can write the expression

$$S - R = \sum_{si} S_{si} F_{si}^{(0)} + \sum_{\sigma j} H_{\sigma j} G_{\sigma j}^{(0)}.$$

Replacing each polynomial S_{si} and $H_{\sigma j}$ with its homogeneous component of degree $r-2$ and $r-3$, respectively, we can take S_{si} and $H_{\sigma j}$ homogeneous of degree $r-2$ and $r-3$, respectively. Likewise, we can assume that S_{si} and $H_{\sigma j}$ are quasi-homogeneous of weight $w - s$ and $w - \sigma$, respectively. Then the polynomial

$$F - R - \sum_{si} S_{si} F_{si}^{(0)} - \sum_{\sigma j} H_{\sigma j} G_{\sigma j}^{(0)}$$

is homogeneous of degree r and weight smaller than w. Now, the proof follows by induction on w. ■

To achieve Goal 3.5, a key step is to establish conditions on the coefficients c_{sin} and $d_{\sigma jn}$ such that each homogeneous element in $\mathbf{k}[X_{n_0}, \dots, X_{n_{g-1}}]_r$ can be uniquely written as a sum of a homogeneous element in the ideal I_r generated by the unfolded quadratic and cubic forms and element in Δ_r. In order to do this, we start by replacing the binomials $F_{s'i'}^{(0)}$ and $F_{si}^{(0)}$ on the left-hand side of equation (3.4) of the Syzygy lemma by the unfolded quadratic forms $F_{s'i'}$ and F_{si}, we obtain for each of the $\frac{1}{2}(g-3)(g-4)$ double indexes $s'i'$ a linear combination of cubic monomials of weight less than $s' + 2g - 2$, that by Lemma 3.7 admits the decomposition

$$X_{2g-2} F_{s'i'} + \sum_{nsi} \varepsilon_{nsi}^{(s'i')} X_n F_{si} = \sum_{nsi} \eta_{nsi}^{(s'i')} X_n F_{si} + R_{s'i'},$$

where the sum on the right-hand side is taken over all the nongaps $n \leq 2g-2$, the double indexes si with $n + s < s' + 2g - 2$, the coefficients $\varepsilon_{nsi}^{(s'i')}$ are constants and where $R_{s'i'}$ is a linear combination of cubic monomials of pairwise different weights less than $s' + 2g - 2$.

Repeating the above procedure for the initial cubic forms on equation (3.5) of the Syzygy lemma, and applying the above Lemma 3.7 we obtain a decomposition

$$X_{2g-2}G_{\sigma' j'} + \sum_{q\sigma j} \rho^{(\sigma' j')}_{q\sigma j} X_q G_{\sigma j} = \sum_{mq\sigma j} \mu^{(\sigma' j')}_{mq\sigma j} X_m X_q F_{\sigma j} + \sum_{q\sigma j} \nu^{(\sigma' j')}_{q\sigma j} X_q G_{\sigma j} + R_{\sigma' j'},$$

where the sum on the right-hand side is taken over the nongaps $m, q \leq 2g-2$, the indexes $mq\sigma$ and $q\sigma$ with $m+q+\sigma < 2g-2+\sigma'$ and $q+\sigma < 2g-2+\sigma'$, the coefficients $\mu^{(\sigma' j')}_{mq\sigma j}, \nu^{(\sigma' j')}_{q\sigma j}$ are constants and where $R_{\sigma' j'}$ is a linear combination of quartic monomials of pairwise different weights less than $2g-2+\sigma'$.

For each nongap $m < s' + 2g + 2$ (resp. $r < \sigma' + 2g + 2$) let $\varrho_{s'i'm}$ (resp. $\vartheta_{\sigma' j' r}$) be the unique coefficient of $R_{s'i'}$ (resp. $R_{\sigma' j'}$) of weight m (resp. r). Note that we do not lose any information on the coefficients of $R_{s'i'}$ and $R_{\sigma' j'}$ when we replace $X_{n_i} \mapsto t^{n_i}$, where t is an indeterminate. So we introduce the polynomials

$$R_{s'i'}(t^{n_0}, \ldots, t^{n_{g-1}}) = \sum_{m=0}^{s'+2g-2} \varrho_{s'i'm} t^m,$$

$$R_{\sigma' j'}(t^{n_0}, \ldots, t^{n_{g-1}}) = \sum_{r=0}^{\sigma'+2g-2} \vartheta_{\sigma' j' r} t^r.$$

We can assume that the coefficients $\varrho_{s'i'm}$ are quasi-homogeneous polynomial expressions of weight $s' + 2g - 2 - m$ in the constants c_{sin}, while the coefficients $\vartheta_{\sigma' j' r}$ are quasi-homogeneous polynomial expressions of weight $\sigma' + 2g - 2 - r$ in the constants $d_{\sigma j n}$.

The next theorem outlines the effort required to achieve Goal 3.5.

Theorem 3.8. *Let $\mathcal{S}$ be a non-hyperelliptic numerical symmetric semigroup of genus $g > 4$. The $\frac{1}{2}(g-2)(g-3)$ unfolded quadratic forms F_{si} and the $\wp$ unfolded cubic forms $G_{\sigma j}$ cut out a canonical integral Gorenstein curve on $\mathbb{P}^{g-1}$ if and only if their coefficients $c_{sin}, d_{\sigma j n}$ satisfy the quasi-homogeneous equations $\varrho_{s'i'm} = 0$ and $\vartheta_{\sigma' j' r} = 0$. In this case, the point $P = (0 : \cdots : 0 : 1)$ is a smooth point on the canonical curve whose Weierstrass semigroup is equal to $\mathcal{S}$.*

Proof. We first assume that the $\frac{1}{2}(g-2)(g-3)$ unfolded quadratic forms F_{si} and the $\wp$ unfolded cubic forms $G_{\sigma j}$ cut out a canonical curve $\mathcal{C} \subset \mathbb{P}^{g-1}$. Since each $R_{s'i'}$ and $R_{\sigma' j'}$ belong to the ideal I generated by the unfolded quadratic and cubic forms, it follows that $R_{s'i'}(x_{n_0}, \ldots, x_{n_{g-1}}) = R_{\sigma' j'}(x_{n_0}, \ldots, x_{n_{g-1}}) = 0$ for each pair of index $s'i'$ and $\sigma' j'$. We can write

$$R_{s'i'}(x_{n_0}, \ldots, x_{n_{g-1}}) = \sum_{m=0}^{s'+2g-2} \varrho_{s'i'm} z_{s'i'm},$$

$$R_{\sigma' j'}(x_{n_0}, \ldots, x_{n_{g-1}}) = \sum_{r=0}^{\sigma'+2g-2} \vartheta_{\sigma' j' r} z_{\sigma' j' r},$$

where the $z_{s'i'm}, z_{\sigma'j'r}$ are monomial expressions of weights m and r respectively in the projective coordinates functions $x_{n_0}, \dots, x_{n_{g-1}}$, and hence $z_{s'i'm}$ has pole divisor mP while $z_{\sigma'j'r}$ has pole divisor rP. Then we conclude that $\varrho_{s'i'm} = \vartheta_{\sigma'i'r} = 0$.

Conversely, let us assume that the coefficients $c_{sin}, d_{\sigma jn}$ satisfy the equations $\varrho_{s'i'm} = 0$ and $\vartheta_{\sigma'i'r} = 0$. Since the $g-3$ quadratic hypersurfaces $V(F_{n_i+2g-2,1}) \subset \mathbb{P}^{g-1}$, for $i = 1, \dots, g-3$, and the cubic hypersurface $V(G_{4g-4,1})$ intersect transversely at P, in an open neighborhood of P their intersection has a unique irreducible component that contains P, and so this component is a projective integral algebraic curve, say $\mathcal{C}$, that is smooth at P and whose tangent line is the intersection of their tangent hyperplanes $V(X_{n_i})$, $i = 0, \dots, g-3$.

Let $y_{n_0}, \dots, y_{n_{g-1}}$ be the projective coordinate functions of $\mathcal{C}$. Let us consider the affine open set with defining equation $y_{n_{g-1}} = 1$. Since the local coordinate ring of C at P is a discrete valuation ring and $n_{g-1} - n_{g-2} = l_2 - l_1 = 1$, we have that $t := y_{n_{g-2}}$ is a local parameter of $\mathcal{C}$ at P, and $y_{n_0}, \dots, y_{n_{g-3}}$ are the power series in t of order greater than 1. More precisely, comparing coefficients in the $g-3$ equations

$$F_{n_i+2g-2}(y_{n_0}, \dots, y_{n_{g-2}}, y_{n_{g-1}}) = 0, \;\; i = 1, \dots, g-3$$

and

$$G_{4g-4}, 1(y_{n_0}, \dots, y_{n_{g-2}}, y_{n_{g-1}}) = 0$$

one sees that

$$\begin{aligned} y_{n_i} &= t^{n_{g-1}-n_i} + (\text{sum of higher orders terms}) \\ &= t^{l_{g-i}-1} + (\text{sum of higher orders terms}), \end{aligned}$$

for each integer $i = 0, \dots, g-1$. This means that the g integers $l_i - 1$ $(i = 1, \dots, g)$ are the contact orders of the curve $\mathcal{C} \subset \mathbb{P}^{g-1}$ with the hyperplanes at P. In particular, the curve $\mathcal{C}$ is not contained in any hyperplane.

By assumption, $\varrho_{s'i'm} = 0$ and $\vartheta_{\sigma'j'r} = 0$ for each pair of double indexes $s'i'$ and $\sigma'j'$, respectively. Hence, we obtain the syzygies

$$X_{2g-2} F_{s'i'} + \sum_{nsi} \varepsilon_{nsi}^{(s'i')} X_n F_{si} - \sum_{nsi} \eta_{nsi}^{(s'i')} X_n F_{si} = 0$$

and

$$X_{2g-2} G_{\sigma'j'} + \sum_{q\sigma j} \rho_{q\sigma j}^{(\sigma'j')} X_q G_{\sigma j} - \sum_{mq\sigma j} \mu_{mq\sigma j}^{(\sigma'j')} X_m X_q F_{\sigma j} - \sum_{q\sigma j} \nu_{q\sigma j}^{(\sigma'j')} X_q G_{\sigma j} = 0.$$

Replacing the variables $X_{n_0}, \dots, X_{n_{g-1}}$ by the projective coordinates functions $y_{n_0}, \dots, y_{n_{g-1}}$, two systems are provided: a system with $\frac{1}{2}(g-3)(g-4)$ linear homogeneous equations in the $\frac{1}{2}(g-3)(g-4)$ functions $F_{s'i'}(y_{n_0}, \dots, y_{n_{g-1}})$ with the coefficients in the domain $k[[t]]$ of formal power series; the second system is composed by $\wp - 1$ linear homogeneous equations in the $\wp - 1$ functions $G_{\sigma' j'}(y_{n_0}, \ldots, y_{n_{g-1}})$ with the coefficients in the domain $k[[t]]$ of formal power series. Since the triple indexes nsi of the coefficients $\varepsilon_{nsi}^{(s'i')}$, respectively, $\eta_{nsi}^{(s'i')}$, satisfy the inequalities $n < 2g-2$ and $n+s = 2g-2+s'$, respectively, $n \leq 2g-2$ and $n+s < 2g-2+s'$, the diagonal entries of the matrix of the system have constant terms 1, while the remaining entries have positive orders. Therefore, the matrix is invertible, and so the equation

$$F_{si}(y_{n_0}, \ldots, y_{n_{g-1}}) = 0$$

holds for each double index si. In the second system, the indexes $q\sigma j, mq\sigma j$ and $n\sigma j$ of the coefficients $\rho_{q\sigma j}^{(\sigma' j')}$, $\mu_{mq\sigma j}^{(\sigma' j')}$ and $\nu_{n\sigma j}^{(\sigma' j')}$, respectively, are such that satisfy the inequalities $q < 2g-2$ and $q+\sigma = 2g-2+\sigma'$, respectively, $m, q \leq 2g-2$ and $m+q+\sigma < 2g-2+\sigma'$. So the diagonal entries of the matrix of the system have constant terms 1, while the remaining entries have positive orders, hence the matrix is also invertible. This means that the equation $G_{\sigma j}(y_{n_0}, \ldots, y_{n_{g-1}}) = 0$ holds for each double index σj. Therefore we just proved that $I \subset I(\mathscr{C})$, where I is the ideal generated by the $\frac{1}{2}(g-2)(g-3)$ quadratic forms F_{si} and by the $\wp$ cubic forms $G_{\sigma j}$.

By virtue of Lemma 3.7, codim $I_r \leq \dim \Lambda_r$ for each $r \geq 2$. Since $I_r(\mathscr{C}) \cap \Lambda_r = 0$, we deduce $\dim \Lambda_r \leq \operatorname{codim} I_r(\mathscr{C})$ and we obtain

$$\operatorname{codim} I_r(\mathscr{C}) = \operatorname{codim} I_r = \dim \Lambda_r = (2g-2)r + 1 - g.$$

Thus $I(\mathscr{C}) = I$ and the curve $\mathscr{C} \subset \mathbb{P}^{g-1}$ has Hilbert polynomial $(2g-2)r+1-g$, hence $\mathscr{C}$ has degree $2g-2$ and arithmetic genus g.

Intersecting the curve $\mathscr{C}$ with the hyperplane $V(X_{2g-2})$ we obtain the divisor $D := (2g-2)P$ of degree $2g-2$, whose complete linear system $|D|$ has dimension at least $g-1$, and so by Riemann–Roch theorem for complete integral curves the Cartier divisor D is canonical, and $\mathscr{C}$ is a canonical Gorenstein curve. ■

The fixed P-hermitian basis $x_{n_0}, x_{n_1}, \ldots, x_{n_{g-1}}$ of $H^0(\mathscr{C}, (2g-2)P)$ is uniquely determined up to a linear transformation of the form

$$x_{n_i} \mapsto \sum_{j=i}^{g-1} c_{ij} x_{n_j},$$

where $(c_{ij}) \in \mathrm{GL}_g(\mathbf{k})$ is an upper triangular matrix with diagonal entries of the form $c_{ii} = c^{n_i}$, $i = 0, \ldots, g-1$, for some non-zero constant c, due to the normalizations $c_{sis} = 1$. We assume that the characteristic of the field of constants $\mathbf{k}$ is zero, or a prime not dividing any of the differences $m-n$ with n, m nongaps such that $m < n \leq 2g-2$.

If the symmetric semigroup is non-odd we can normalize $\frac{1}{2}g(g-1)$ coefficients c_{sin} of the unfolded quadratic forms to be zero, for each $i = 1, \ldots, g-1$ just transforming

$$X_{n_i} \mapsto X_{n_i} + \sum_{j=1}^{i} c_{n_i n_{i-j}} X_{n_{i-j}}$$

and proceeding by induction on the weight of the coefficients, as in [9, p. 587]. In the odd case, we can normalize $g-3$ coefficients of the cubic form $G_{4g-4,1}$ by transforming

$$X_{2g-4} \mapsto X_{2g-4} + \sum_{i=1}^{g-3} d_{2g-4,n_{g-3-i}} X_{n_{g-3-i}},$$

and

$$X_{n_i} \mapsto X_{n_i} + \sum_{j=1}^{i} c_{n_i n_{i-j}} X_{n_{i-j}}$$

with $n_i \neq n_{g-3} = 2g-4$. Next we normalize the remaining $\frac{1}{2}g(g-1)-(g-3)$ coefficients of the unfolded quadratic forms $F_{n_i+2g-2,1}$.

Due to the normalizations the only freedom left to us is to transform $x_{n_i} \mapsto c^{n_i} x_{n_i}$, $i = 0, \dots, g-1$ for some non-zero constant $c \in \mathbf{k}$. Therefore, a fine moduli space is realized as a weighted projective space, as follows.

Theorem 3.9. *Let $\mathcal{S}$ be a non-hyperelliptic symmetric semigroup of genus $g \geq 5$. The isomorphism classes of the pointed complete integral Gorenstein curves with Weierstrass semigroup $\mathcal{S}$ correspond bijectively to the orbits of the $\mathbb{G}_m(k)$-action*

$$(c, \dots, c_{sin}, \dots, d_{\sigma jm}, \dots) \mapsto (\dots, c^{s-n} c_{sin}, \dots, c^{\sigma-m} d_{\sigma jm}, \dots)$$

on the affine quasi-cone of the vectors whose coordinates are the coefficients c_{sin}, $d_{\sigma jm}$ of the normalized unfolded quadratic and cubic forms F_{si} and $G_{\sigma j}$ satisfying the quasi-homogeneous equations $\varrho_{s'i'm} = \vartheta_{\sigma' i' r} = 0$.

4. Odd numerical semigroups of genus at most six

We start this section with the following observation on the rationality of $\overline{\mathcal{M}}^{\mathcal{S}}_{g,1}$. If the symmetric semigroup $\mathcal{S}$ is generated by less than 5 elements, using Pinkham's equivariant deformation theory [20], complete intersection theory and a quasi-homogeneous version of the rather technical Buchsbaum–Eisenbud's structure theorem for Gorenstein ideals of codimension 3 (see [2, p. 466]), one can deduce that the affine monomial curve $C^{(0)}$ can be negatively smoothed without any obstructions (see [3, 26], [27, Satz 7.1]), hence

$$\overline{\mathcal{M}}^{\mathcal{S}}_{g,1} = \mathbb{P}(T^{1,-}_{\mathbf{k}[\mathcal{S}]|\mathbf{k}}).$$

Although the above observation assures that $\overline{\mathcal{M}}^{\mathcal{S}}_{g,1} = \mathbb{P}^9$ for $\mathcal{S} := \langle 5, 6, 7, 8 \rangle$, we believe it is relevant to illustrate our techniques in a simpler example without the need for complex computations. Therefore, we guess that a simpler and explicit proof of the rationality of $\overline{\mathcal{M}}^{\mathcal{S}}_{g,1}$ for symmetric semigroups generated by less than five elements can be carried forward.

4.1. Odd of genus five

Let $\mathcal{C}^{(0)}$ be the canonical monomial Gorenstein curve associated to the odd symmetric semigroup of genus 5. Up to change of coordinates, we can write

$$\mathcal{C}^{(0)} := \left\{(a^8 : a^3b^5 : a^2b^6 : a^1b^7 : b^8) \mid (a:b) \in \mathbb{P}^1\right\} \subseteq \mathbb{P}^4.$$

The symmetric Weierstrass semigroup associated to the point $P = (0:0:0:0:1)$ is $\mathcal{S} := \langle 5, 6, 7, 8\rangle$. Following Lemma 3.4 the ideal of $\mathcal{C}^{(0)}$ can be generated by the seven isobaric and homogeneous initial forms

$$\begin{aligned}
F_{12}^{(0)} &= X_6^2 - X_5X_7, & F_{13}^{(0)} &= X_6X_7 - X_5X_8,\\
F_{14}^{(0)} &= X_7^2 - X_6X_8, & G_{15}^{(0)} &= X_5^3 - X_0X_7X_8,\\
G_{16}^{(0)} &= X_5^2X_6 - X_0X_8^2, & G_{18}^{(0)} &= X_6^3 - X_5^2X_8,\\
G_{21}^{(0)} &= X_7^3 - X_5X_8^2. & &
\end{aligned}$$

Now we follow the procedure described in Theorem 3.8 to achieve Goal 3.5. We begin by unfolding the seven quadratic and cubic initial forms above, we also introduce new variables to reduce the notation

$$F_i := F_i^{(0)} - \sum_{j=1}^{i} c_{ij} Z_{i-j}, \quad (i = 12, 13, 14),$$

$$G_i := G_i^{(0)} - \sum_{j=1}^{i} d_{ij} W_{i-j}, \quad (i = 15, 16, 18, 21),$$

where Z_{i-j} (respectively W_{i-j}) stands for the basis monomial in Λ_2 (in Λ_3) of weight $i-j$, and the summation index j varies only through the integers such that $i - j \in \mathcal{S}$.

In view of Goal 3.5, counting all the coefficients c_{ij} and d_{ij} that are involved, we have the commutative diagram

$$\begin{array}{ccc}
\mathcal{C}^{(0)} & \hookrightarrow & \mathcal{X} \subset \mathbb{P}^4 \times \mathbb{A}^{74}\\
\downarrow & & \downarrow \pi\\
\operatorname{Spec}\mathbf{k} & \longrightarrow & \mathbb{A}^{74}
\end{array}$$

Due to the normalizations made before Theorem 3.9 are independent of the procedure described in Theorem 3.8, we perform them once to reduce the number of coefficients. Using the transformations

$$X_i \mapsto X_i + \sum_{j=1}^{i-1} \lambda_j X_{i-j},$$

we can normalize the ten coefficients

$$c_{12,1} = c_{12,2} = c_{12,7} = c_{13,1} = c_{13,2} = c_{13,3} = c_{13,8} = d_{16,1} = d_{16,6} = d_{21,5} = 0.$$

Now we produce the syzygies associated to the canonical monomial curve $\mathcal{C}^{(0)} \subset \mathbb{P}^4$ that are described in Syzygy lemma 3.6:

$$\begin{aligned}
&X_8F_{12}^{(0)} - X_7F_{13}^{(0)} + X_6F_{14}^{(0)} = 0,\\
&X_8G_{15}^{(0)} - X_5X_6F_{12}^{(0)} + X_5G_{18}^{(0)} - X_7G_{16}^{(0)} = 0,\\
&X_8G_{18}^{(0)} - X_5G_{21}^{(0)} + X_5X_7F_{14}^{(0)} - X_6X_8F_{12}^{(0)} = 0,\\
&X_8G_{21}^{(0)} - X_7X_8F_{14}^{(0)} + X_8^2F_{13}^{(0)} = 0.
\end{aligned} \tag{4.1}$$

Replacing the binomials $F_i^{(0)}$ and $G_j^{(0)}$ and on the left-hand side of the above syzygies by the unfolded quadratic and cubic forms F_i and G_j, respectively, and applying the division algorithm recursively, as described in Theorem 3.8, until all the monomials of these new equations belong to the basis Λ_3 or Λ_4, we obtain the four polynomial equations

$$\begin{aligned}
&X_8F_{12} - X_7F_{13} + X_6F_{14}\\
&\;= -F_{12}(c_{14,3}X_5 + c_{14,8}X_0) + F_{14}c_{13,6}X_0 - G_{16}c_{14,4}\\
&\quad + F_{13}(c_{13,7}X_0 - c_{14,2}X_5 - c_{14,7}X_0),\\
&X_8G_{15} - X_6G_{17} + X_5G_{18} - X_7G_{16}\\
&\;= (c_{12,6}X_0X_5 - d_{18,1}X_0X_8)F_{12}\\
&\quad - (c_{14,3}d_{16,4} + c_{14,3}d_{15,3}d_{18,1} + d_{18,7})X_0G_{16} + (d_{16,5}X_5 + c_{12,5}X_5)X_0F_{13}\\
&\quad + (d_{16,9}X_0 - d_{18,1}X_8 + d_{15,8}d_{18,1}X_0 + d_{15,3}d_{18,1}X_5 + d_{16,4}X_5)X_0F_{14}\\
&\quad + (d_{16,10}X_0 + d_{15,9}d_{18,1}X_0 + d_{15,1}d_{18,1}X_8 + d_{15,4}d_{18,1}X_5 + d_{16,2}X_8)X_0F_{13}\\
&\quad + (-c_{14,4}d_{16,4}X_0 - c_{14,4}d_{15,3}d_{18,1}X_0 - d_{18,1}X_7 - d_{18,8}X_0)G_{15},\\
&X_8G_{18} - X_5G_{21} - X_6X_8F_{12} + X_7X_5F_{14}\\
&\;= (-c_{14,3}^2d_{16,4} - c_{14,2}c_{14,3}d_{15,5} - c_{14,3}c_{14,4}d_{15,3} + c_{14,3}c_{14,7})G_{16}X_0\\
&\quad + (c_{14,3}d_{15,3}d_{14,4} + c_{14,2}c_{14,8} - c_{14,2}c_{14,4}d_{15,4} + c_{14,2}^2c_{14,3}d_{15,3})X_0G_{16}\\
&\quad \times (+c_{14,4}d_{15,9}X_0 - d_{15,1}c_{14,2}^2X_8 - d_{15,4}d_{14,4}X_5 - c_{14,8}X_5 + c_{12,5}X_8\\
&\quad + c_{14,2}c_{14,3}X_8 + c_{14,3}d_{16,2}X_8 - d_{15,4}c_{14,2}^2X_5 - c_{14,2}c_{14,3}d_{15,8}X_0\\
&\quad + c_{14,4}d_{15,1}X_8 - d_{15,1}d_{14,4}X_8 - d_{15,9}d_{14,4}X_0 + c_{14,3}d_{16,10}X_0 + c_{14,3}d_{16,5}X_5\\
&\quad + c_{14,4}d_{15,4}X_5 - d_{15,9}c_{14,2}^2X_0 - c_{14,2}c_{14,3}d_{15,3}X_5)X_0F_{13}\\
&\quad \times (+d_{14,11}X_0 + d_{14,4}X_7 + d_{14,3}X_8 - c_{14,4}X_7 + c_{14,2}^2X_7 - c_{14,4}c_{14,3}d_{16,4}X_0\\
&\quad + c_{14,1}c_{14,2}X_8 + d_{15,3}c_{14,2}^2c_{14,4}X_0 - c_{14,4}^2d_{15,3}X_0 + c_{14,4}d_{15,3}d_{14,4}X_0\\
&\quad - c_{14,2}c_{14,4}d_{15,5}X_0 + c_{14,2}c_{14,9}X_0 + c_{14,4}c_{14,7}X_0 + c_{14,2}c_{14,4}X_5 + c_{14,2}c_{14,3}X_6)G_{15}\\
&\quad \times (c_{14,2}^2X_0X_8 - c_{14,4}X_0X_8 + d_{14,4}X_0X_8 - c_{14,7}X_0X_5 - c_{14,2}X_5^2 + c_{14,3}d_{16,9}X_0^2\\
&\quad + c_{14,4}d_{15,3}X_0X_5 - d_{15,3}d_{14,4}X_0X_5 + c_{14,3}d_{16,4}X_0X_5 - d_{15,8}d_{14,4}X_0^2\\
&\quad + c_{14,4}d_{15,8}X_0^2 - c_{14,2}^2d_{15,3}X_0X_5 - c_{14,2}^2d_{15,8}X_0^2)F_{14} + (d_{14,2}X_8 - c_{14,3}X_7)G_{16}\\
&\quad \times (+c_{12,6}X_8 - c_{14,2}c_{14,3}d_{15,9}X_0 - c_{14,2}c_{14,3}d_{15,4}X_5 - c_{14,2}c_{14,3}d_{15,1}X_8)X_0F_{12},
\end{aligned}$$

$$
\begin{aligned}
&G_{21,2}X_8 - G_{21,1}X_8 - G_{22}X_7 \\
&= X_8(c_{14,3}X_5 + c_{14,8}X_0)F_{13} \\
&\quad + X_8\big[(c_{14,2}X_5 + c_{14,7}X_0)F_{14} - c_{14,2}c_{14,4}G_{15} - c_{14,2}c_{14,3}G_{16}\big].
\end{aligned}
$$

Now we determine the weighted vector space $T^{1,-}_{\mathbf{k}[\mathcal{S}]|\mathbf{k}}$, that is (up to an isomorphism) the locus of the *linearizations* of the above four equations. Solving in the coefficients c_{ij} and d_{ij}, the linearizations are given by the system of polynomials

$$
\begin{aligned}
&X_8F_{12} - X_7F_{13} + X_6F_{14} = 0, \\
&X_8G_{15} - X_5X_6F_{12} + X_5G_{18} - X_7G_{16} = 0, \\
&X_8G_{18} - X_5G_{21} + X_5X_7F_{14} - X_6X_8F_{12} = 0, \\
&X_8G_{21} - X_7X_8F_{14} + X_8^2F_{13} = 0,
\end{aligned}
\tag{4.2}
$$

by solving it in the coefficients c_{ij} and d_{ij}. Note that the system in (4.2) is given by substituting by zeros each right-hand side of the above four equations. To solve this system we replace $X_{n_i} \mapsto t^{n_i}$, giving rise to another 20 linear equations on the coefficients c_{ij} and d_{ij}. We can solve this linear system as follows:

$$
\begin{array}{lllll}
d_{16,10} = d_{15,10}, & d_{16,9} = d_{15,9}, & d_{16,8} = d_{15,8}, & c_{14,7} = c_{13,7}, & d_{18,7} = c_{13,7}, \\
d_{15,7} = -c_{13,7}, & d_{21,7} = 2c_{13,7}, & c_{14,6} = -c_{12,6}, & d_{21,6} = -c_{12,6}, & d_{18,6} = c_{12,6}, \\
d_{16,5} = d_{15,5}, & c_{14,4} = -c_{12,4}, & d_{16,4} = d_{15,4}, & d_{21,4} = -c_{12,4}, & d_{18,4} = c_{12,4}, \\
d_{16,3} = d_{15,3}, & d_{16,2} = d_{15,2}. & & &
\end{array}
$$

We can verify that the weighted vector space $T^{1,-}_{\mathbf{k}[\mathcal{S}]|\mathbf{k}}$ depends only on the ten coefficients $d_{15,2}, d_{15,3}, c_{12,4}, d_{15,4}, d_{15,5}, c_{12,6}, c_{13,7}, d_{15,8}, d_{15,9}, d_{15,10}$, that implies

$$
\dim T^{1,-}_{\mathbf{k}[\mathcal{S}]|\mathbf{k}} = 10.
$$

More precisely, counting the coefficients of weight s, we obtain the dimension of the graded component of $T^{1,-}_{\mathbf{k}[\mathcal{S}]|\mathbf{k}}$ of negative weight $-s$:

$$
\dim T^{1,-}_s = 1,\ (s = -10, -9, -8, -7, -6, -5, -3, -2) \quad \text{and} \quad \dim T^{1,-}_{-4} = 2.
$$

For the remaining integers the dimension of $T^{1,-}_s$ is zero. In particular, the compactified moduli space $\overline{\mathcal{M}}^{\mathcal{S}}_{5,1}$ can be realized as closed subset of the 9-dimensional weighted projective space $\mathbb{P}(T^{1,-}_{\mathbf{k}[\mathcal{S}]|\mathbf{k}}) = \mathbb{P}^9$. Note that in Diagram 4.1 we start with a large affine space of dimension 74, which was reduced to dimension 64 due to the 10 normalizations. The above computations imply that to achieve our Goal 3.5 the solution of the four equations in the coefficients c_{ij} and d_{ij} depends on 10 coefficients, i.e. there are 54 coefficients that can be expressed in terms of the other 10.

Finally, due to Theorem 3.8 and Theorem 3.9 to obtain the equations of $\overline{\mathcal{M}^{\mathcal{S}}_{5,1}}$, we must solve the four polynomial equations in the coefficients c_{ij} and d_{ij}. By replacing, again,

$X_{n_i} \mapsto t^{n_i}$ in theses four equations, the compactified moduli space $\overline{\mathcal{M}^{\mathcal{S}}_{5,1}}$ is cut out by 70 equations that depend on 64 variables. We can solve them, according to the linearizations, in the following way:

- 18 coefficients that are identically zero, namely:

$$c_{12,5} = c_{13,5} = c_{13,6} = c_{14,1} = c_{14,2} = c_{14,3} = d_{15,1} = d_{16,11} = d_{18,1} = 0,$$
$$d_{18,2} = d_{18,3} = d_{18,5} = d_{18,8} = d_{18,11} = d_{21,1} = d_{21,2} = d_{21,3} = d_{21,10} = 0,$$

- 11 linear equations:

$$\begin{aligned} &c_{14,4} = -c_{12,4}, && d_{15,7} = -c_{13,7}, && d_{16,2} = d_{15,2},\\ &d_{16,4} = d_{15,4}, && d_{16,5} = d_{15,5}, && d_{16,9} = d_{15,9},\\ &d_{18,4} = c_{12,4}, && d_{18,6} = c_{12,6}, && d_{18,7} = c_{13,7},\\ &d_{21,4} = -c_{12,4}, && d_{16,3} = d_{15,3}, \end{aligned}$$

- 17 quadratic polynomials:

$$\begin{aligned} c_{12,12} &= -c_{12,4}d_{15,8}, & c_{13,13} &= c_{12,4}d_{15,9},\\ c_{14,6} &= -c_{12,4}d_{15,2} - c_{12,6}, & c_{14,7} &= -c_{12,4}d_{15,3} + c_{13,7},\\ c_{14,8} &= -c_{12,4}d_{15,4}, & c_{14,9} &= -c_{12,4}d_{15,5},\\ c_{14,14} &= -c_{12,4}d_{15,10}, & d_{15,15} &= c_{12,6}d_{15,9} + c_{13,7}d_{15,8},\\ d_{16,8} &= -c_{12,4}d_{15,4} + d_{15,8}, & d_{16,10} &= -c_{12,6}d_{15,4} - c_{13,7}d_{15,3} + d_{15,10},\\ d_{18,12} &= -c_{12,4}d_{15,8} - c_{12,6}^2, & d_{21,6} &= -c_{12,4}d_{15,2} - c_{12,6},\\ d_{18,13} &= c_{12,4}d_{15,9}, & d_{21,7} &= -c_{12,4}d_{15,3} + 2\,c_{13,7},\\ d_{21,8} &= -c_{12,4}d_{15,4}, & d_{21,9} &= -c_{12,4}d_{15,5},\\ d_{18,10} &= -c_{12,4}c_{12,6}, \end{aligned}$$

- and the following 8:

$$\begin{aligned} d_{16,16} &= -c_{12,4}d_{15,3}d_{15,9} + c_{12,4}d_{15,4}d_{15,8},\\ d_{18,18} &= c_{12,4}c_{12,6}d_{15,8},\\ d_{21,13} &= -c_{12,4}^2d_{15,2}d_{15,3} - c_{12,4}c_{12,6}d_{15,3} + c_{12,4}c_{13,7}d_{15,2}\\ &\quad + c_{12,4}d_{15,9} + c_{12,6}c_{13,7},\\ d_{21,14} &= -c_{12,4}^2d_{15,3}^2 + 2\,c_{12,4}c_{13,7}d_{15,3} - c_{12,4}d_{15,10} - c_{13,7}^2,\\ d_{21,15} &= -c_{12,4}^2d_{15,3}d_{15,4} + 2\,c_{12,4}c_{13,7}d_{15,4},\\ d_{21,11} &= -c_{12,4}^2d_{15,3} + c_{12,4}c_{13,7},\\ d_{21,16} &= -c_{12,4}^2d_{15,3}d_{15,5} + c_{12,4}c_{13,7}d_{15,5},\\ d_{21,21} &= -c_{12,4}^2d_{15,3}d_{15,10} + c_{12,4}^2d_{15,4}d_{15,9} + c_{12,4}c_{13,7}d_{15,10}. \end{aligned}$$

We note that there are 16 missing equations from the 70 mentioned, but they are repeated. Recall that the weight of the coefficient c_{ij} (respectively d_{ij}) is j. The equations of the moduli space $\overline{\mathcal{M}^{\mathcal{S}}_{5,1}}$ are given by the first 18 equations, and by the zeros of the isobaric polynomials of the right-hand side of the last 36 equations whose weights are greater than the weights of the respective left-hand side. However, there is no isobaric polynomial satisfying this condition. Hence,

$$\overline{\mathcal{M}^{\mathcal{S}}_{5,1}} = \mathbb{P}(T^{1,-}_{\mathbf{k}[\mathcal{S}]|\mathbf{k}}) \cong \mathbb{P}^9_\alpha, \quad \text{with } \alpha = (2,3,4,4,5,6,7,8,9,10).$$

4.2. Odd of genus six

Let $\mathcal{C}^{(0)}$ be the canonical monomial Gorenstein curve of genus 6 associated to the odd symmetric semigroup $\mathcal{S} := \langle 6,7,8,9,10\rangle$. The point $P = (0:0:0:0:0:1)$ is smooth in $\mathcal{C}^{(0)}$ and its Weierstrass semigroup is $\mathcal{S}$. Applying Lemma (3.4), the generators of the ideal of $\mathcal{C}^{(0)}$ are given by the following initial 6 quadratic and 8 cubic forms

$$\begin{aligned}
F^{(0)}_{14} &= X_7^2 - X_6X_8, & F^{(0)}_{15} &= X_7X_8 - X_6X_9,\\
F^{(0)}_{16} &= X_8^2 - X_6X_{10}, & F^{(0)}_{16,1} &= X_7X_9 - X_6X_{10},\\
F^{(0)}_{17} &= X_8X_9 - X_7X_{10}, & F^{(0)}_{18} &= X_9^2 - X_8X_{10},\\
G^{(0)}_{18} &= X_6^3 - X_0X_8X_{10}, & G^{(0)}_{19} &= X_6^2X_7 - X_0X_9X_{10},\\
G^{(0)}_{20} &= X_6^2X_8 - X_0X_{10}^2, & G^{(0)}_{20,1} &= X_6X_7^2 - X_0X_{10}^2,\\
G^{(0)}_{21} &= X_7^3 - X_6^2X_9, & G^{(0)}_{22} &= X_7^2X_8 - X_6^2X_{10},\\
G^{(0)}_{26} &= X_8X_9^2 - X_6X_{10}^2, & G^{(0)}_{27} &= X_9^3 - X_7X_{10}^2.
\end{aligned}$$

Now we unfold the above 14 initial forms using the more suggestive notion of the previous subsection

$$F_i = F_i^{(0)} - \sum_{j=1}^{i} c_{ij} Z_{i-j}, \quad (i = 14,\dots,18 \text{ and } i = 16,1),$$

$$G_i = G_i^{(0)} - \sum_{j=1}^{i} d_{ij} W_{i-j}, \quad (i = 18,\dots,22,26,27 \text{ and } i = 20,1),$$

where Z_{i-j}, respectively W_{i-j}, is a monomial in Λ_2, respectively Λ_3, of weight $i-j$, whenever $i-j$ is a nongap of $\mathcal{S}$. Next, we do the normalizations described before the Theorem 3.9, transforming the variables $X_0, X_6, X_7, X_8, X_9, X_{10}$ we are able to normalize the 15 coefficients

$$c_{14,1} = c_{15,1} = c_{16,1,1} = d_{18,1} = d_{18,2} = c_{15,2} = c_{16,1,2} = c_{15,3} = 0,$$
$$c_{16,1,3} = c_{16,1,4} = c_{15,6} = c_{14,7} = c_{14,8} = c_{15,9} = c_{16,1,10} = 0.$$

We also consider the ten syzygies of the monomial curve $\mathcal{C}^{(0)}$, that are given by the Syzygy lemma (3.6)

$$\begin{aligned}
&X_{10}F_{14}^{(0)} - X_8F_{16,1}^{(0)} + X_7F_{17}^{(0)} = 0,\\
&X_{10}F_{15}^{(0)} - X_9F_{16,1}^{(0)} + X_7F_{18}^{(0)} = 0,\\
&X_{10}F_{16}^{(0)} - X_{10}F_{16,1}^{(0)} - X_9F_{17}^{(0)} + X_8F_{18}^{(0)} = 0,\\
&X_{10}G_{18}^{(0)} - X_8G_{20}^{(0)} + X_6^2F_{16}^{(0)} = 0,\\
&X_{10}G_{19}^{(0)} - X_9G_{20,1}^{(0)} + X_6X_7F_{16,1}^{(0)} = 0,\\
&X_{10}G_{20}^{(0)} - X_{10}G_{20,1}^{(0)} + X_6X_{10}F_{14}^{(0)} = 0,\\
&X_{10}G_{21}^{(0)} - X_7X_{10}F_{14}^{(0)} - X_6X_{10}F_{15}^{(0)} = 0,\\
&X_{10}G_{22}^{(0)} - X_6X_{10}F_{16}^{(0)} - X_8X_{10}F_{14}^{(0)} = 0,\\
&X_{10}G_{26}^{(0)} - X_{10}^2F_{16,1}^{(0)} - X_9X_{10}F_{17}^{(0)} = 0,\\
&X_{10}G_{27}^{(0)} - X_{10}^2F_{17}^{(0)} - X_9X_{10}F_{18}^{(0)} = 0.
\end{aligned} \tag{4.3}$$

The 10 above syzygies of the monomial curve give rise to 10 polynomial equations involving the 14 unfolded forms F_i and G_j

$$\begin{aligned}
&X_{10}F_{14} - X_8F_{16,1} + X_7F_{17},\\
&X_{10}F_{15} - X_9F_{16,1} + X_7F_{18},\\
&X_{10}F_{16} - X_{10}F_{16,1} - X_9F_{17} + X_8F_{18},\\
&X_{10}G_{18} - X_8G_{20} + X_6^2F_{16},\\
&X_{10}G_{19} - X_9G_{20,1} + X_6X_7F_{16,1},\\
&X_{10}G_{20} - X_{10}G_{20,1} + X_6X_{10}F_{14},\\
&X_{10}G_{21} - X_7X_{10}F_{14} - X_6X_{10}F_{15},\\
&X_{10}G_{22} - X_6X_{10}F_{16} - X_8X_{10}F_{14},\\
&X_{10}G_{26} - X_{10}^2F_{16,1} - X_9X_{10}F_{17},\\
&X_{10}G_{27} - X_{10}^2F_{17} - X_9X_{10}F_{18}.
\end{aligned} \tag{4.4}$$

As in the preceding subsection, we compute the linearization of the above ten polynomial equations, that is isomorphic to the weighted vector space $T^{1,-}_{\mathbf{k}[\mathcal{S}]|\mathbf{k}}$. To do this, we make the substitutions $X_i \mapsto t^i$ and solve a homogeneous linear system with 60 equations. We can solve it in a way that the solution depends only on the 15 coefficients

$$\begin{array}{cccccccc}
d_{18,12}, & d_{18,11}, & c_{15,8}, & c_{16,1,9}, & c_{16,1,8}, & c_{15,7}, & c_{14,6}, & d_{18,6},\\
d_{18,10}, & c_{14,5}, & d_{18,5}, & c_{14,4}, & d_{18,4}, & d_{18,3}, & c_{14,2}. &
\end{array}$$

Therefore the compactified moduli space $\overline{\mathcal{M}^{\mathcal{S}}_{6,1}}$ can be realized as a closed subset of the weighted projective space

$$\mathbb{P}(T^{1,-}_{\mathbf{k}[\mathcal{S}]|\mathbf{k}}) \cong \mathbb{P}^{14}_{\alpha} \quad \text{with } \alpha = (2, 3, 4, 4, 5, 5, 6, 6, 7, 8, 8, 9, 10, 11, 12).$$

Since the odd symmetric semigroup $\mathcal{S}$ is negatively graded, cf. [21], the moduli space $\mathcal{M}^{\mathcal{S}}_{6,1}$ has codimension three in $\mathcal{M}_{6,1}$, cf. [7]. Hence $\overline{\mathcal{M}^{\mathcal{S}}_{6,1}}$ has dimension 11. So it is a proper subset of $\mathbb{P}^{14}$, in contrast to the odd symmetric semigroup with genus 5.

Now we take each polynomial in (4.4) and make successive divisions until the resulting polynomial is such that all its monomials belong to the basis Λ_3 or Λ_4. This is ensured by Lemma 3.7. This procedure is completely computational and we can make it by using a suitable software on computer algebra, like Singular or Maple. Here we do not display the resulting polynomials, just because they have a large number of monomials. Then, we make the substitutions $X_i \mapsto t^i$, with $i = 6, 7, 8, 9, 10$, on the 10 polynomials whose monomials are in Λ_3 and Λ_4 and solve 188 polynomial equations. This system can be solved by increasing weights whose solution depends only on the 15 coefficients of the linearization, we rename these coefficients

$$\begin{aligned} d_{18,i} &:= b_i \quad (i = 3,4,5,6,10,11,12),\\ c_{14,j} &:= a_j \quad (j = 2,4,5,6),\\ c_{16,1,8} &:= b_8, \quad c_{15,7} := a_7, \quad c_{15,8} := a_8, \quad c_{16,1,9} := a_9. \end{aligned}$$

By Theorem 3.9 we can conclude that the moduli space $\overline{\mathcal{M}^{\mathcal{S}}_{6,1}}$ is given by the zero locus of 5 isobaric polynomials

$$\begin{aligned} \vartheta_{15} &:= 4a_5a_6 - a_2a_5b_8 + a_4a_5b_6 - a_4b_3b_8 + a_5^3 + a_5^2b_5 + a_4b_{11} + a_5b_{10} + 2a_7b_8,\\ \vartheta_{13} &:= 2a_2a_5a_6 + a_4^2a_5 + a_4^2b_5 + a_4a_5b_4 + a_4a_6b_3 + {a_5}^2b_3 - a_4a_9 + a_5a_8\\ &\quad - a_5b_8 - 2a_6a_7,\\ \vartheta_{17} &:= a_5b_{12} - a_2a_5^3 - a_2a_5^2b_5 - a_4a_5b_8 - a_4b_5b_8 + 2a_5^2a_7 + a_5a_7b_5 - a_5a_8b_4\\ &\quad - a_5a_9b_3 - a_6b_{11} + a_9b_8,\\ \vartheta_{16} &:= a_2a_4a_5^2 + a_2a_4a_5b_5 - a_2a_6b_8 - 2a_4a_5a_7 - a_4a_6^2 - a_4a_6b_6 - a_4a_7b_5\\ &\quad + b_8^2 + a_4a_8b_4 + a_4a_9b_3 - a_4b_4b_8 - a_5^2a_6 - a_5a_6b_5 - a_5b_3b_8 - a_4b_{12}\\ &\quad - a_6b_{10} - a_8b_8,\\ \vartheta_{19} &:= a_2^2a_5^3 + a_2^2a_5^2b_5 + a_2a_4a_5^2b_3 + a_2a_4a_5b_3b_5 - 4a_2a_5^2a_7 - 3a_2a_5a_7b_5 + b_8b_{11}\\ &\quad + a_2a_5a_8b_4 + a_2a_5a_9b_3 + a_4^2a_5a_6 + a_4^2a_5b_6 + a_4^2a_6b_5 + a_4^2b_5b_6 + a_4a_5^3\\ &\quad - a_9b_{10} + 2a_4a_5^2b_5 - 2a_4a_5a_7b_3 + a_4a_5b_5^2 - a_4a_7b_3b_5 + a_4a_8b_3b_4\\ &\quad + a_4a_9b_3^2 - a_2a_5b_{12} - a_2a_6b_{11} + a_4a_5b_{10} - a_4a_6a_9 - a_4a_9b_6 - a_4b_3b_{12}\\ &\quad - a_4b_4b_{11} + a_4b_5b_{10} - a_5^2a_9 + 4a_5a_7^2 - a_5a_9b_5 - a_5b_3b_{11} + 2a_7^2b_5\\ &\quad - 2a_7a_8b_4 - 2a_7a_9b_3 + 2a_7b_{12} - a_8b_{11}. \end{aligned}$$

By intersecting $\overline{\mathcal{M}^{\mathcal{S}}_{6,1}}$ with the open affine chart $\{a_5 = 1\}$ of $\mathbb{P}^{15}$, we see that $\overline{\mathcal{M}^{\mathcal{S}}_{6,1}}$ admits the local parametrization

$$\begin{aligned} b_{10} &= a_4b_3b_8 + a_2b_8 - a_4a_6 - a_4b_6 - a_4b_{11} - 2a_7b_8 - b_5 - 1,\\ b_{12} &= a_4b_5b_8 + a_2b_5 + a_4b_8 + a_6b_{11} - a_7b_5 + a_8b_4 + a_9b_3 - a_9b_8 + a_2 - 2a_7,\\ b_8 &= a_4^2b_5 + a_4a_6b_3 + 2a_2a_6 + a_4^2 - a_4a_9 + a_4b_4 - 2a_6a_7 + a_8 + b_3. \end{aligned}$$

Since $\mathcal{M}^{\mathcal{S}}_{6,1}$ is irreducible [4, Thm. 1.1], the moduli variety $\mathcal{M}^{\mathcal{S}}_{6,1}$ is rational of dimension 11. We also note that Bullock [5, Thm. 1] proved that the moduli spaces $\mathcal{M}^{\mathcal{S}}_{g,1}$ are stably rational when $2 \leq g \leq 6$, with the possible exceptions $\langle 6,7,8,9,10\rangle$ and $\langle 5,7,8,9,11\rangle$, the last one is not subcanonical and is completely studied by Stevens in [23].

For a given monomial curve $\mathcal{C}$ associated to a semigroup $\mathcal{S}$, its obstruction space lies in the second cohomological module of cotangent complex $T^2 := T^2(\mathbf{k}[\mathcal{S}]|\mathbf{k})$. As noted at the beginning of the last section of this work, if $\mathcal{S}$ is symmetric and generated by less than five elements, the monomial curve $\mathcal{C}$ can be smoothed without any obstructions, which implies that $\bar{\mathcal{M}}^{\mathcal{S}}_{g,1}$ is the weighted projective space $\mathbb{P}(T^{1,-}(\mathbf{k}[\mathcal{S}]|\mathbf{k}))$. The obstruction spaces of the two examples of this section are nonzero. To see this, we use the description of T^2 given by Buschweitz in [3, Thm. 2.3.1]. We can conclude that in genus five, $\mathcal{S} = \langle 5,6,7,8\rangle$, the homogeneous graded part of degree -9 of T^2 has dimension 1, for genus 6, $\mathcal{S} = \langle 6,7,8,9,10\rangle$, the homogeneous graded part of degree -13 has dimension 1, and in both cases T^1 and T^2 are negatively graded.

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References

[1] E. Ballico, Brill-Noether theory for rank 1 torsion free sheaves on singular projective curves. *J. Korean Math. Soc.* **37** (2000), no. 3, 359–369 Zbl 0993.14016 MR 1760368

[2] D. A. Buchsbaum and D. Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3. *Amer. J. Math.* **99** (1977), no. 3, 447–485 Zbl 0373.13006 MR 453723

[3] R.-O. Buchweitz, On deformations of monomial curves. In *Séminaire sur les Singularités des Surfaces*, edited by M. Demazure, H. C. Pinkham, and B. Teissier, pp. 205–220, Lecture Notes in Math. 777, Springer, Berlin, 1980 Zbl 0428.32016

[4] E. M. Bullock, Subcanonical points on algebraic curves. *Trans. Amer. Math. Soc.* **365** (2013), no. 1, 99–122 Zbl 1273.14069 MR 2984053

[5] E. M. Bullock, Irreducibility and stable rationality of the loci of curves of genus at most six with a marked Weierstrass point. *Proc. Amer. Math. Soc.* **142** (2014), no. 4, 1121–1132 Zbl 1299.14029 MR 3162235

[6] A. Contiero, L. Feital, and R. V. Martins, Max Noether's theorem for integral curves. *J. Algebra* **494** (2018), 111–136 Zbl 1386.14112 MR 3723173

[7] A. Contiero, A. L. Fontes, J. Stevens, and J. Q. Vargas, On nonnegatively graded Weierstrass points. 2021, arXiv:2111.07721

[8] A. Contiero and S. Mazzini, Local complete intersections and Weierstrass points. 2022, arXiv:2203.05013

[9] A. Contiero and K.-O. Stöhr, Upper bounds for the dimension of moduli spaces of curves with symmetric Weierstrass semigroups. *J. Lond. Math. Soc. (2)* **88** (2013), no. 2, 580–598 Zbl 1288.14016 MR 3106737

[10] M. Coppens, Free linear systems on integral Gorenstein curves. *J. Algebra* **145** (1992), no. 1, 209–218 Zbl 0770.14002 MR 1144668

[11] L. Gatto, On the closure in $\overline{M}_g$ of smooth curves having a special Weierstrass point. *Math. Scand.* **88** (2001), no. 1, 41–71 Zbl 1029.14009 MR 1813519

[12] L. Gatto and F. Ponza, Derivatives of Wronskians with applications to families of special Weierstrass points. *Trans. Amer. Math. Soc.* **351** (1999), no. 6, 2233–2255 Zbl 0917.14016 MR 1615963

[13] P. Gimenez, I. Sengupta, and H. Srinivasan, Minimal free resolutions for certain affine monomial curves. In *Commutative algebra and its connections to geometry*, pp. 87–95, Contemp. Math. 555, American Mathematical Society, Providence, RI, 2011 Zbl 1237.13027 MR 2882676

[14] H. Hauser, An algorithm of construction of the semiuniversal deformation of an isolated singularity. In *Singularities, Part 1 (Arcata, Calif., 1981)*, pp. 567–573, Proc. Sympos. Pure Math. 40, American Mathematical Society, Providence, RI, 1983 Zbl 0523.58012 MR 713093

[15] H. Hauser, La construction de la déformation semi-universelle d'un germe de variété analytique complexe. *Ann. Sci. École Norm. Sup. (4)* **18** (1985), no. 1, 1–56 Zbl 0583.32052 MR 803194

[16] S. J. Kim, On the existence of Weierstrass gap sequences on trigonal curves. *J. Pure Appl. Algebra* **63** (1990), no. 2, 171–180 Zbl 0712.14019 MR 1043748

[17] M. Kontsevich and A. Zorich, Connected components of the moduli spaces of Abelian differentials with prescribed singularities. *Invent. Math.* **153** (2003), no. 3, 631–678 Zbl 1087.32010 MR 2000471

[18] D. Mumford, *Curves and their Jacobians*. University of Michigan Press, Ann Arbor, MI, 1975 Zbl 0316.14010 MR 419430

[19] G. Oliveira, Weierstrass semigroups and the canonical ideal of nontrigonal curves. *Manuscripta Math.* **71** (1991), no. 4, 431–450 Zbl 0742.14029 MR 1104994

[20] H. C. Pinkham, *Deformations of algebraic varieties with G_m action*. Astérisque 20, Société Mathématique de France, Paris, 1974 Zbl 0304.14006 MR 376672

[21] D. S. Rim and M. A. Vitulli, Weierstrass points and monomial curves. *J. Algebra* **48** (1977), no. 2, 454–476 Zbl 0412.14002 MR 457447

[22] J. Stevens, Computing versal deformations of singularities with Hauser's algorithm. In *Deformations of surface singularities*, pp. 203–228, Bolyai Soc. Math. Stud. 23, János Bolyai Math. Soc., Budapest, 2013 Zbl 1315.32001 MR 3203579

[23] J. Stevens, The dimension of the moduli space of pointed algebraic curves of low genus. *Comm. Algebra* **51** (2023), no. 7, 2963–2977 Zbl 07681724 MR 4571079

[24] K.-O. Stöhr, On the moduli spaces of Gorenstein curves with symmetric Weierstrass semigroups. *J. Reine Angew. Math.* **441** (1993), 189–213 Zbl 0771.14009 MR 1228616

[25] R. Vidal Martins, A generalization of Max Noether's theorem. *Proc. Amer. Math. Soc.* **140** (2012), no. 2, 377–391 Zbl 1234.14025 MR 2846308

[26] R. Waldi, Deformation von Gorenstein-Singularitäten der Kodimension 3. *Math. Ann.* **242** (1979), no. 3, 201–208 Zbl 0426.14004 MR 545214

[27] R. Waldi, *Äquivariante Deformation monomialer Kurven*. Regensburger Math. Schriften 4, Universität Regensburg, Fachbereich Mathematik, Regensburg, 1982 Zbl 0528.14009 MR 655415

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