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Erratum to “Itô’s formula for noncommutative C^2 functions of free Itô processes”

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Abstract. The results in [Doc. Math. 27 (2022), 1447–1507] are stated for free Itô processes driven by n-tuples of freely independent semicircular Brownian motions. They should be stated instead for free Itô processes driven by n-dimensional semicircular Brownian motions. We explain and correct this error.

Fix $n \in \mathbb{N}$, a filtered W^*-probability space $(\mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, \tau)$, and two adapted processes, $x = (x_1, \ldots, x_n)\colon \mathbb{R}_+ \to \mathcal{A}_{\mathrm{sa}}^n$ and $z = (z_1, \ldots, z_n)\colon \mathbb{R}_+ \to \mathcal{A}^n$. Also, write $u_i := \sqrt{2}\,\mathrm{Re}\, z_i$ and $v_i := \sqrt{2}\,\mathrm{Im}\, z_i$ for $i \in \{1, \ldots, n\}$.

Definition 1. We say that z has *jointly ($*$-)free increments* if $0 \leq s \leq t$ implies that $\{z_i(t) - z_i(s) : 1 \leq i \leq n\}$ is ($*$-)freely independent from $\mathcal{A}_s$.

Since z is adapted, if z has jointly ($*$-)free increments and $0 \leq t_0 < \cdots < t_k$, then $(\{z_i(t_k) - z_i(t_{k-1}) : 1 \leq i \leq n\}, \ldots, \{z_i(t_1) - z_i(t_0) : 1 \leq i \leq n\}, \{z_i(t_0) : 1 \leq i \leq n\})$ is a ($*$-)freely independent family by induction and the associativity of free independence [1, Exercise 5.25].

Definition 2. We say that x is an *n-dimensional semicircular Brownian motion* (with respect to $(\mathcal{A}_t)_{t\geq 0}$) if $x(0) = 0$, x has jointly free increments, and $0 \leq s \leq t$ implies that $(x_1(t) - x_1(s), \ldots, x_n(t) - x_n(s))$ is a freely independent family of semicircular elements each with variance $t - s$. We say that z is an *n-dimensional circular Brownian motion* (with respect to $(\mathcal{A}_t)_{t\geq 0}$) if $z(0) = 0$, z has jointly $*$-free increments, and $0 \leq s \leq t$ implies that $(z_1(t) - z_1(s), \ldots, z_n(t) - z_n(s))$ is a $*$-freely independent family of circular elements each with variance $t - s$.

By definition of circular elements, the associativity of free independence, and the fact that the $*$-algebra generated by z_i is the algebra generated by $\{u_i, v_i\}$, z is an n-dimensional circular Brownian motion if and only if $(u_1, v_1, \ldots, u_n, v_n)$ is a $2n$-dimensional semicircular Brownian motion. Now, note that the components of an n-dimensional semicircular (resp., circular) Brownian motion are each one-dimensional semicircular (resp.,

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circular) Brownian motions. Moreover, these one-dimensional semicircular (resp., circular) Brownian motions are freely (resp., $*$-freely) independent.

Proposition 3. *If x is an n-dimensional semicircular Brownian motion, then the family $(\{x_i(t) : t \geq 0\})_{i=1}^n$ is freely independent, i.e. the components of x are freely independent. Similarly, if z is an n-dimensional circular Brownian motion, then the components of z are $*$-freely independent.*

Proof. Suppose that $0 = t_0 < \cdots < t_k$. If $1 \leq j \leq k$, then $(x_i(t_j) - x_i(t_{j-1}))_{i=1}^n$ is freely independent by definition. Also, $(\{x_i(t_j) - x_i(t_{j-1}) : 1 \leq i \leq n\})_{j=1}^k$ is freely independent because x has jointly free increments. Thus, by associativity of free independence,

$$\big(x_i(t_j) - x_i(t_{j-1})\big)_{(i,j)\in\{1,\ldots,n\}\times\{1,\ldots,k\}}$$

is freely independent. But then $(\{x_i(t_j) - x_i(t_{j-1}) : 1 \leq j \leq k\})_{i=1}^n$ is freely independent by another appeal to associativity of free independence. Since $x_i(0) = 0$, $t_0 = 0$, and

$$x_i(t_j) = \sum_{p=1}^{j} \big(x_i(t_p) - x_i(t_{p-1})\big)$$

for $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, k\}$, we conclude that $(\{x_i(t_j) : 1 \leq j \leq k\})_{i=1}^n$ is freely independent, as desired. The second statement may be proven similarly or deduced from the first via the observation following Definition 2. ∎

Next, we show that the converse of Proposition 3 holds when the filtrations are natural. The author believes the converse of Proposition 3 is false for general filtrations despite being unable to find a counterexample as of yet.

Definition 4. We define $(\mathcal{A}_t^z)_{t\geq 0} := (W^*(z_i(s) : 1 \leq i \leq n,\ 0 \leq s \leq t))_{t\geq 0}$ to be the *natural filtration* of z.

Proposition 5. *Suppose that for each $i \in \{1, \ldots, n\}$, x_i is a semicircular Brownian motion with respect to $(\mathcal{A}_t^{x_i})_{t\geq 0}$. Then x is an n-dimensional semicircular Brownian motion with respect to $(\mathcal{A}_t^{x})_{t\geq 0}$ if and only if the components of x are freely independent. Similarly, suppose that for each $i \in \{1, \ldots, n\}$, z_i is a circular Brownian motion with respect to $(\mathcal{A}_t^{z_i})_{t\geq 0}$. Then z is an n-dimensional circular Brownian motion with respect to $(\mathcal{A}_t^{z})_{t\geq 0}$ if and only if the components of z are $*$-freely independent.*

Proof. The "only if" direction is addressed by Proposition 3. We now prove the "if" direction. Suppose that

$$0 = t_0 < t_1 < \cdots < t_{k-1} = s < t_k = t.$$

If $1 \leq i \leq n$, then $(x_i(t_j) - x_i(t_{j-1}))_{j=1}^k$ is freely independent by definition. By assumption, $(\{x_i(t_j) - x_i(t_{j-1}) : 1 \leq j \leq k\})_{i=1}^n$ is freely independent. By associativity of free independence,

$$\big(x_i(t_j) - x_i(t_{j-1})\big)_{(i,j)\in\{1,\ldots,n\}\times\{1,\ldots,k\}}$$

is freely independent. By another appeal to associativity of free independence,

$$\{x_i(t) - x_i(s) = x_i(t_k) - x_i(t_{k-1}) : 1 \le i \le n\}$$

is freely independent from $\{x_i(t_j) - x_i(t_{j-1}) : 1 \le i \le n,\ 1 \le j \le k-1\}$. Since $t_0 = 0$, $x_i(0) = 0$, and $x_i(t_j) = \sum_{p=1}^{j}(x_i(t_p) - x_i(t_{p-1}))$ for $(i, j) \in \{1, \dots, n\} \times \{1, \dots, k-1\}$, we conclude that $\{x_i(t) - x_i(s) : 1 \le i \le n\}$ is freely independent from

$$\{x_i(t_j) : 1 \le i \le n,\ 1 \le j \le k-1\}.$$

The desired result follows. The second statement may be proven similarly or deduced from the first via the observation following Definition 2. ■

All the main results in [2] are stated in terms of n freely independent semicircular Brownian motions $x_1, \dots, x_n$ and n $*$-freely independent circular Brownian motions $z_1, \dots, z_n$. However, the correct standing assumptions should be the (seemingly) stronger statements that $(x_1, \dots, x_n)$ is an n-dimensional semicircular Brownian motion and $(z_1, \dots, z_n)$ is an n-dimensional circular Brownian motion. When one uses only natural filtrations, the assumptions in the previous two sentences agree, so no changes to [2] are necessary in this case. In general, the change to [2] is necessary because the joint free increments property is used in the proof of Lemma 3.2.9, upon which all the main results rest. (This proof is the only place joint free increments are invoked.)

References

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