Publ. RIMS, Kyoto Univ.
14 (1978), 731-739

Extensions of the Inner Automorphism Group of a Factor

By

Marie CHODA*

1. Introduction

Let M be the crossed product $R(G, A, \alpha)$ of a von Neumann algebra A by a locally compact group G under a continuous action α. By $\mathrm{Aut}(M, A)$ we shall denote the group of all automorphisms of M, each of which is an extension of an automorphism of A. A systematic attempt to study $\mathrm{Aut}(M, A)$ for a finite factor M by the group measure space construction has been made in [11]. For the crossed product M of a von Neumann algebra A by a discrete countable group G of freely acting automorphisms of A, some results concerning the structure of an elment of $\mathrm{Aut}(M, A)$, which is inner on M, have been obtained in [2], [3], [8] and [9], and generalized in [1]. Some relations between elements in $\mathrm{Aut}(M, A)$ and two-cocycles on G have been studied for a general crossed product of a von Neumann algebra A by a discrete countable group, or a locally compact group G under an action ([6], [10], [12], [14]).

In this paper, we consider this generalized crossed product in the form $M=R(G, A, \alpha, v)$ of a factor A by a locally compact group G under an action α with a factor set $\{v(g, h)\,;\, g, h\in G\}$ (cf. Definition in below). In § 2, we shall study the structure of the normal subgroup K of $\mathrm{Aut}(M, A)$, each element of which acts on A as an inner automorphism. Under a certain condition, the group K is isomorphic to the direct product of $\mathrm{Int}(A)$ and $\chi(G)$, where $\mathrm{Int}(A)$ is the group of natural extensions $\mathrm{Ad}u\,(u\in A)$ and $\chi(G)$ is the character

* Communicated by H. Araki, October 31, 1977.
Department of Mathematics, Osaka Kyoiku University, Osaka 543, Japan.

group of G (Theorem 1 and Corollary 4). In § 3, we shall restrict our interest to a discrete countable group G and study the structure of the normal subgroup $\mathrm{Int}(M, A)$ of $\mathrm{Aut}(M, A)$, each element of which is an inner automorphism of M. If the action under α of all elements in G except the identity is outer on A, then $\mathrm{Int}(M, A)$ is isomorphic to an extension group of $\mathrm{Int}(A)$ by G (Theorem 7).

2. Extensions of Inner Automorphisms

Let A be a von Neumann algebra acting on a separable Hilbert space H. By $\mathrm{Aut}(A)$ we shall denote the group of all automorphisms (*-preserving) of A and by $\mathrm{Int}(A)$ the group of all inner automorphisms of A. For a locally compact group G, we denote by $K(H; G)$ the vector space of all continuous H-valued functions on G with compact support. Considering the inner product in $K(H; G)$ defined by

$$(\xi, \eta)=\int_G (\xi(g), \eta(g))dg, \qquad \xi, \eta\in K(H; G),$$

$K(H; G)$ is a pre Hilbert space, where dg is a fixed left Haar measure of G. The completion of $K(H; G)$ with respect to this inner product is denoted by $L^2(H; G)$. A map α of G into $\mathrm{Aut}(A)$ is called an action of G on A, if for each fixed a in A, the map: $g\in G\rightarrow\alpha_g(a)\in A$ is σ-strongly *continuous and α satisfies the following condition (1);

$$(1) \qquad i(g, h)=\alpha_{gh}^{-1}\alpha_g\alpha_h\in\mathrm{Int}(A), \qquad g, h\in G.$$

For such a map α, a family $\{v(g, h); g, h\in G\}$ of unitaries in A is called a factor set associated with the action α, if the map: $(g, h)\in G\times G\rightarrow v(g, h)\in A$ is σ-strongly *continuous and the following conditions (2) and (3) are satisfied;

$$(2) \qquad i(g, h)=Adv(g, h), \qquad g, h\in G,$$

$$(3) \qquad v(g, hk)v(h, k)=v(gh, k)\alpha_k^{-1}(v(g, h)), \qquad g, h, k\in G,$$

where $\mathrm{Ad}u$ is an automorphism of A such that $\mathrm{Ad}u(a)=uau^*$ for a in A. In the sequel, we assume that $\alpha_1=\iota$, where 1 is the identity of G and ι is the identity automorphism of A. On the Hilbert space $L^2(H; G)$, we shall denote by π_α the representation of A such that

$$(\pi_\alpha(a)\xi)(h) = \alpha_h^{-1}(a)\xi(h), \qquad h \in G,\ \xi \in L^2(H; G). \tag{4}$$

By ρ, we shall denote a map of G into the unitary group on $L^2(H; G)$ such that

$$(\rho(g)\xi)(h) = v(g, g^{-1}h)\xi(g^{-1}h), \qquad h \in G,\ \xi \in L^2(H; G). \tag{5}$$

By the direct computation, we have that

$$\rho(g)\rho(h) = \rho(gh)\pi_\alpha(v(g, h)), \qquad g, h \in G \tag{6}$$

and π_α and ρ satisfiy the covariance relation;

$$\rho(g)\pi_\alpha(a)\rho(g)^* = \pi_\alpha(\alpha_g(a)), \qquad g \in G,\ a \in A. \tag{7}$$

The von Neumann algebra on $L^2(H; G)$ generated by $\pi_\alpha(A)$ and $\rho(G)$ is called the *crossed product of* A *by* G *with the factor set* $\{v(g, h); g, h \in G\}$ respect to α and denoted by $R(G, A, \alpha, v)$. If the action α is a representation of G into $\mathrm{Aut}(A)$ and the factor set $\{v(g, h); g, h \in G\}$ associated with the action α is the trivial set, that is, $v(g, h)$ is the identity for every g, h in G, then $R(G, A, \alpha, v)$ is the usual crossed product ([16]), which we shall denote by $R(G, A, \alpha)$.

At first, we shall be concerned with the group of all extensions to $R(G, A, \alpha, v)$ of the inner automorphism group of a factor. Fix a von Neumann algebra A equipped with an action α of a locally compact group G and a factor set $\{v(g, h); g, h \in G\}$ associated with the action α. Throughout this paper, we shall denote by M the crossed product $R(G, A, \alpha, v)$. By $\mathrm{Aut}(M, A)$, we shall denote the group of automorphisms of M sending $\pi_\alpha(A)$ onto itself:

$$\mathrm{Aut}(M, A) = \{\beta \in \mathrm{Aut}(M);\ \beta(\pi_\alpha(A)) = \pi_\alpha(A)\}.$$

It is clear that all inner automorphisms of A admit natural extensions $\mathrm{Ad}u\,(u \in u(A))$ to M and the automorphisms α_G admit natural liftings $\mathrm{Ad}\rho(g)\,(g \in G)$, where $u(A)$ is the group of unitaries in A. By the same notation $\mathrm{Int}(A)$ and $\alpha(G)$ we shall denote the set of such automorphisms of M:

$$\mathrm{Int}(A) = \{\mathrm{Ad}u \in \mathrm{Aut}(M)\ ;\ u \in u(\pi_\alpha(A))\}.$$

Let K be the group of all extensions to M of the inner automorphism group of A:

$$K = \{\beta \in \mathrm{Aut}(M, A)\ ;\ \beta \text{ is inner on } \pi_\alpha(A)\}.$$

Theorem 1. *Let A be a factor equipped with an action α of a locally compact group G and a factor set $\{v(g, h)\ ; g, h \in G\}$ associated with the action α. If α is such that $\pi_\alpha(A)' \cap M$ is the scalar multiples of the identity, then K is isomorphic to the direct product of* $\mathrm{Int}(A)$ *and $\chi(G)$, where $\chi(G)$ is the group of all continuous characters of G.*

Proof. Take a β in K. Let u be a unitary in $\pi_\alpha(A)$ such that $\beta(a) = uau^*$ for all a in $\pi_\alpha(A)$. Then, for each a in $\pi_\alpha(A)$ and g in G, we have that

$$\begin{aligned} u\rho(g)a\rho(g)^*u^* &= \beta(\rho(g)a\rho(g)^*) = \beta(\rho(g))\beta(a)\beta(\rho(g))^* \\ &= \beta(\rho(g))uau^*\beta(\rho(g))^*, \end{aligned}$$

so that $\rho(g)^*u^*\beta(\rho(g))u$ is contained in $\pi_\alpha(A)' \cap M$. Since $\pi_\alpha(A)' \cap M$ is the scalar multiples of the identity I, we have a χ in $\chi(G)$ such that

(8) $$\beta(\rho(g)) = \chi(g)u\rho(g)u^*.$$

In fact, put $\chi(g)I = \rho(g)^*u^*\beta(\rho(g))u$, then we have that

$$\begin{aligned} \chi(gh)I &= \rho(gh)^*u^*\beta(\rho(gh))u = \rho(gh)^*u^*\beta(\rho(g)\rho(h)\pi_\alpha(v(g,h)))u \\ &= \rho(gh)^*u^*\beta(\rho(g))\beta(\rho(h))u\pi_\alpha(v(g, h)) \\ &= \chi(g)\rho(gh)^*\rho(g)u^*\beta(\rho(h))u\pi_\alpha(v(g, h)) \\ &= \chi(g)\chi(h)I. \end{aligned}$$

For each character χ of G, put

$$(u(\chi)\xi)(g) = \overline{\chi(g)}\xi(g), \qquad g \in G,\ \xi \in L^2(H\ ;\ G),$$

where $\overline{\chi(g)}$ is the complex conjugate of $\chi(g)$. Then $u(\chi)$ is a unitary satisfying

(9) $$u(\chi)au(\chi)^* = a, \qquad a \in \pi_\alpha(A),$$

and

$$u(\chi)(\rho(g))u(\chi)^* = \overline{\chi(g)}\rho(g), \qquad g \in G. \tag{10}$$

Let $\delta(\chi)$ be an automorphism of M induced by $u(\chi)$, then $\delta(\chi)$ belongs to the group K.

For a β in K, let u be a unitary in $\pi_\alpha(A)$ such that $\beta(a) = uau^*$ for all a in $\pi_\alpha(A)$. Take a χ in $\chi(G)$ satisfying the property (8), then we have that

$$(\beta\delta(\chi))(a) = \beta(a) = \mathrm{Ad}u(a), \qquad a \in \pi_\alpha(A) \tag{11}$$

and

$$(\beta\delta(\chi))(\rho(g)) = \overline{\chi(g)}\beta(\rho(g)) = \mathrm{Ad}u(\rho(g)), \qquad g \in G, \tag{12}$$

so that $\beta\cdot\delta(\chi)$ belongs to $\mathrm{Int}(A)$. Thus every β in K has a form $\beta = \mathrm{Ad}u\cdot\delta(\chi)$ for some u in $\pi_\alpha(A)$ and χ in $\chi(G)$. Such a decomposition is unique. In fact, if

$$\mathrm{Ad}u\cdot\delta(\chi) = \mathrm{Ad}w\cdot\delta(\chi'), \qquad u,\ w \in \pi_\alpha(A),\ \chi,\ \chi' \in \chi(G),$$

then we have that on $\pi_\alpha(A)$, $\mathrm{Ad}w^*u$ is the identity automorphism. Since A is a factor, it follows that w is a scalar multiple of u, which implies that $\mathrm{Ad}u = \mathrm{Ad}w$ on M and that $\delta(\chi) = \delta(\chi')$.

By the property (10), we have that, for χ and χ' in $\chi(G)$, $\delta(\chi) = \delta(\chi')$ if and only if $\chi = \chi'$.

Therefore, defining a map σ of the direct product of $\mathrm{Int}(A)$ and $\chi(G)$ onto K by $\sigma(\gamma, \chi) = \gamma\cdot\delta(\chi)$, $(\gamma \in \mathrm{Int}(A),\ \chi \in \chi(G))$, we have an isomorphism of K onto the direct product of $\mathrm{Int}(A)$ and $\chi(G)$.

Let K_0 be the group of all extensions to M of the identity automorphism of A:

$$K_0 = \{\beta \in \mathrm{Aut}(M, A)\ ;\ \beta \text{ is the identity on } \pi_\alpha(A)\}.$$

Corollary 2. *Let A, α, G and $\{v(g, h)\ ;\ g, h \in G\}$ be as in Theorem* 1. *The group K_0 is isomorphic to $\chi(G)$.*

Denote by $[G, G]$ the commutator group of G, that is, $[G, G]$

is the closed group generated by $\{ghg^{-1}h^{-1}; g, h \in G\}$. A group G is called perfect if $[G, G]$ coincides with G.

Corollary 3. *Let A, G, α and $\{v(g, h); g, h \in G\}$ be as in Theorem* 1. *The following three statements are equivalent:*

(a) *K coincides with* $\mathrm{Int}(A)$;

(b) *K_0 is the trivial group* $\{\iota\}$;

(c) *G is perfect.*

Proof. By Theorem 1 and Corollary 2, it is clear that the statements (a) and (b) are equivalent and that they are equivalent to the condition that $\chi(G) = \{1\}$. On the other hand, $\chi(G)$ is the group $\mathrm{Hom}(G, T)$ of all continuous homomorphism of G into T, where T is the unit circle of the complex plane. Since T is an abelian group, it follows that for each χ in $\chi(G)$, $[G, G]$ is contained in the kernel of χ. Hence $\chi(G)$ is isomorphic to $\mathrm{Hom}(G/[G, G], T)$. Thus the condition that $\chi(G) = \{1\}$ is equivalent to $G = [G, G]$, which is statement (c).

Especially, assume that G is a discrete countable group. If α_g is an outer automorphism of A for all g in G except the unit, then by [5, Corollary 3], we have that $\pi_\alpha(A)' \cap M$ is the scalar multiples of the identity. Therefore, we have the following corollary:

Corollary 4. *Let A be a factor equipped with an action α of a discrete countable group G and a factor set $\{v(g, h); g, h \in G\}$ associated with the action α. Assume that α_g is an outer automorphism of A for all g in G except the unit. Then K is isomorphic to the direct product of* $\mathrm{Int}(A)$ *and* $\chi(G)$, *so that K_0 is isomorphic to* $\chi(G)$. *The three statements in Corollary* 3 *are equivalent.*

3. Extensions as Inner Automorphisms.

In this section, we shall be concerned with extensions of automorphisms of A to M which are inner on M.

Throughout this section, we shall treat a factor equipped with an action of a discrete countable group G and a factor set $\{v(g, h)\,;\, g, h \in G\}$ associated with the action α. For $M = R(G, A, \alpha, v)$, we shall denote by $\mathrm{Int}(M, A)$ the group of inner automorphisms of M sending $\pi_\alpha(A)$ into iteself and by $u(M, A)$ the group of unitaries in M normalizing $\pi_\alpha(A)$:

$$\mathrm{Int}(M, A) = \{\beta \in \mathrm{Int}(M)\,;\, \beta(\pi_\alpha(A)) = \pi_\alpha(A)\},$$

and

$$u(M, A) = \{u \in u(M)\,;\, u\pi_\alpha(A)u^* = \pi_\alpha(A)\}.$$

We shall determine a relation among $\mathrm{Int}(M, A)$, $\mathrm{Int}(A)$ and G.

Theorem 5. *Let A be a factor equipped with an action α of a discrete countable group G and a factor set $\{v(g, h)\,;\, g, h \in G\}$ associated with the action α. Then each u in $u(M, A)$ has a form*;

$$(13) \qquad u = ww'\rho(g), \qquad w \in u(\pi_\alpha(A)),\ w' \in u(\pi_\alpha(A)' \cap M),\ g \in G.$$

By the same technique as [8; Corollary 1] or [9; Theorem], we can prove this theorem. For the sake of completeness, we shall give a proof of Theorem 5.

Proof. Take a u in $u(M, A)$. Let

$$u = \sum_{g \in G} a(g)\rho(g) \qquad a(g) \in \pi_\alpha(A), \text{ (in the } \sigma\text{-strong topology)}$$

be the Fourier expansion of u([5; Lemma 1]). By the property that $u\pi_\alpha(A)u^* = \pi_\alpha(A)$, we have that

$$\sum_{g \in G} a(g)\alpha_g(a)\rho(g) = \sum_{g \in G} uau^*a(g)\rho(g), \qquad a \in \pi_\alpha(A),$$

so that

$$a(g)\alpha_g(a) = uau^*a(g), \qquad a \in \pi_\alpha(A),\ g \in G.$$

If $\alpha_g^{-1}\,\mathrm{Ad}u$ is an outer automorphism of $\pi_\alpha(A)$, then we have that $a(g) = 0$. Since u is unitary, it follows that there exists a g in G such that $\alpha_g^{-1}\mathrm{Ad}u$ is inner on $\pi_\alpha(A)$. Let w be a unitary in $\pi_\alpha(A)$ such that on $\pi_\alpha(A)$, $\alpha_g^{-1}\,\mathrm{Ad}u = \mathrm{Ad}w$. Put $w' = \rho(g)^*uw^*$, then w' belongs to $\pi_\alpha(A)' \cap M$.

Corollary 6. *Let A, G, α and $\{v(g, h)\,;\, g, h\in G\}$ be as in Theorem* 5. *Each β in* $\mathrm{Int}(M, A)\cap \mathrm{Aut}(A)$ *has a form*:

$$\beta=\gamma\alpha_g, \qquad \gamma\in \mathrm{Int}(A),\ g\in G. \tag{14}$$

Theorem 7. *Let A, G, α and $\{v(g, h)\,;\, g, h\in G\}$ be as in Theorem 5. Assume that α_g is an outer automorphism of A for all g in G except the identity. Then $u(M, A)$ is isomorphic to an extension group of $u(A)$ by G and* $\mathrm{Int}(M, A)$ *is isomorphic to an extension group of* $\mathrm{Int}(A)$ *by G. If M is the usual crossed product $R(G, A, \alpha)$, then these extensions are a semi-direct product.*

Proof. If α_g is an outer automorphism of A for all g in G except the unit, then $\pi_\alpha(A)'\cap M$ is the scalar multiples of the identity ([5; Corollary 3]). Hence, by Theorem 5, each u in $u(M, A)$ has a form:

$$u=w\rho(g), \qquad w\in u(\pi_\alpha(A)),\ g\in G. \tag{15}$$

If

$$w\rho(g)=w'\rho(h), \qquad w,\ w'\in u(\pi_\alpha(A)),\ g,\ h\in G, \tag{16}$$

then we have that

$$\begin{aligned} w'^*w &=\rho(h)\rho(g)^*=\rho(h)\rho(g^{-1})\pi_\alpha(v(g, g^{-1})^*) \\ &=\rho(hg^{-1})\pi_\alpha(v(h, g^{-1})v(g, g^{-1})^*). \end{aligned} \tag{17}$$

On the other hand, by [4; Theorem 6], there exists a faithful normal expectation e of M onto $\pi_\alpha(A)$ such that $e(\rho(g))=0$ for all g in G except the unit. Therefore, if the relation (16) is satisfied for g and h in G such that $g\neq h$, then we have that $w'^*w=0$ by (17), which is a contradiction. Thus the decomposition of u in $u(M, A)$ with the form (15) is unique. We shall define a map σ on the set $u(\pi_\alpha(A))\times G$ by $\sigma(w, g)=w\rho(g)$, $w\in u(\pi_\alpha(A))$, $g\in G$. Define a multiplication on $u(\pi_\alpha(A))\times G$ by

$$(w, g)(w', h)=(w\alpha_g(w')\alpha_{gh}(\pi_\alpha(v(g, h))),\ gh), \tag{18}$$

then σ is an isomorphism of the extension group $E(G, u(\pi_\alpha(A)), \alpha, v)$ of $u(\pi_\alpha(A))$ by G under the multiplication (18) onto $u(M, A)$. If

M is the usual crossed product $R(G, A, \alpha)$, then we may always take $v(g, h)=I$ for all g, h in G, so that mapping σ gives an isomorphism of a semi direct product of $u(\pi_\alpha(A))$ by G onto $u(M, A)$.

Similarly, define a multiplication in the set $\mathrm{Int}(A)\times G$ by

$$(\mathrm{Ad}u, g)(\mathrm{Ad}w, h)=(\mathrm{Ad}(u\alpha_g(w)\alpha_{gh}(\pi_\alpha(v(g, h)))), gh). \tag{19}$$

The group $\mathrm{Int}(A)$ is isomorphic to the factor group $u(\pi_\alpha(A))/TI$ of $u(\pi_\alpha(A))$ by the normal subgroup $\{\mu I ; \mu\in T\}$. The extension group $E(G, \mathrm{Int}(A), \alpha, v)$ of $\mathrm{Int}(A)$ by G under the multiplication (19) is isomorphic to the factor group $E(G, u(\pi_\alpha(A)), \alpha, v)/TI\times\{1\}$ of $E(G, u(\pi_\alpha(A)), \alpha, v)$ by the normal subgroup $TI\times 1=\{(\mu I, 1) ; \mu\in T\}$. On the other hand, $E(G, u(\pi_\alpha(A)), \alpha, v)/TI\times 1$ is isomorphic to the factor group $u(M, A)/TI$ of $u(M, A)$ by the normal subgroup TI, which is isomorphic to $\mathrm{Int}(M, A)$. Thus $\mathrm{Int}(M, A)$ is isomorphic to the extension group $E(G, \mathrm{Int}(A), \alpha, v)$ of $\mathrm{Int}(A)$ by G under the multiplication (19).

References

[1] Behencke, H., Automorphisms of crossed product *Tohoku, Math.J.*, **21** (1969), 580-600.
[2] Choda, H. and Choda, M., On extensions of automorphisms of abelian von Neumann algebras, *Proc.Japan Acad.*, **43** (1967), 295-299.
[3] Choda, H., On freely acting automorphisms of operator algebras, *Kodai Math. Sem. Report*, **26** (1974), 1-21.
[4] Choda, M., Some relations of II_1-factors on free groups, *Math. Japonicae*, 22 (1977) 383-394.
[5] Choda, M., A characterization of crossed products of factors by discrete outer automorphism groups, To appear in J. Math. Soc. Japan, 31.
[6] Feldman, J., and Moore, C. C., Ergodic equivalence relations, cohomology and von Neumann algebras, II, *Preprint.*
[7] Hewitt, E. and Ross, K. A., *Abstract Harmonic Analysis*, I, Springer-Verlag.
[8] Haga, Y. and Takeda, Z., Correspondence between subgroups and subalgebras in a cross product von Neumann algebra, *Tohoku Math.* J., **19** (1967), 315-323.
[9] Nakamura, M. and Takeda, Z., On inner automorphisms of certain finite factors, *Proc. Japan Acad.*, **37** (1961), 31-32.
[10] Nakamura, M. and Takeda, Z., On extensions of finite factors, I, *Proc.Japan Acad.*, **35** (1959), 149-154.
[11] Singer, I. M., Automorphisms of finite factors, *Amer.J.Math.*, **17** (1955), 117-183.
[12] Sutherland, C. E., Cohomology and extensions of von Neumann algebras, I, *Preprint.*
[13] Takesaki, M., Duality for crossed products and the structure of von Neumann algebras of type III, *Acta Math.*, **131** (1974) 249-310.
[14] Zeller-Meier, G., Produits croisés d'une C*-algèbra par un groupe d'automorphisms, *J.Math.Pures et Appl.*, **47** (1968), 101-239.