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Automatically presented groups

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Abstract. We introduce the notions of automatically presented groups and piecewise automatically presented groups. We show that if G is a piecewise automatically presented group satisfying the property T of Kazhdan, then G is finite. We prove that if G is amenable and finitely presented, then G is virtually abelian. We give further restrictions for a group to be piecewise automatically presented and study properties of such groups. We also give examples of automatically presented groups.

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1. Introduction

We introduce the notion of an automatically presented group and the more general notion of a piecewise automatically presented group. Such a group is defined in a natural way by a finite state automaton (or a sequence of finite state automata in the general case) and admits the group generated by the finite state automaton (or, respectively, the piecewise automatic group defined by the sequence of automata) as a quotient. Informally speaking, a word in such a group is equivalent to the identity if it becomes trivial on some level of the tree on which the corresponding (piecewise) automatic group acts (see the next section for a formal definition).

Groups similar to automatically presented groups appear implicitly already in [14] (where Grigorchuk constructs torsion-free groups of intermediate growth). However, not much is known about this class of groups.

In this paper we provide examples of automatically presented groups. We show that, on one hand, these groups resemble groups generated by finite state automata in some aspects, and, on the other hand, these two classes of groups are basically different. We show, in particular, that an infinite piecewise automatically presented group cannot have the property T of Kazhdan and that every finitely presented amenable group in this class is virtually abelian. This is in contrast with the fact that groups generated by finite state automata may have the property T and that there are many solvable (e.g. nilpotent) finitely presented groups generated by finite state automata. We recall also that all known restrictions for a group to be generated by a finite state automaton are related to residual finiteness properties and properties of the word problem (see the last section of this paper for more details).

Finally, we show that piecewise automatic presentations help in establishing properties of some groups that were obtained earlier by other constructions.

2. Finite state automata, automatically presented groups and piecewise automatically presented groups

Definition 1. *A finite state automaton* over a finite alphabet X consists of a finite set of states A and a map $\tau\colon A \times X \to A \times X$.

Given a state a, denote by $\tau_a\colon X \to X$ the composition of $\tau(a,-)$ with the projection of $A \times X$ to X.

If, for every state $a \in A$, the map τ_a is bijective, the automaton is said to be *invertible*.

We recall the definition of a piecewise automatic group.

Definition 2. Consider an ascending sequence of sets $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ and a sequence of finite state invertible automata (A_n, τ_n), $1 \le n < \infty$ defined over a common alphabet X (an important example is $A_1 = A_2 = A_3 \ldots$, with τ_i not necessarily equal to each other).

For every $a \in A_1$, define a transformation $\bar{a}$ of the set of onesided infinite words X^∞ (or the set of finite words X^*) in the following way. Take $x = x_1x_2x_3\ldots$ in X^∞ (or in X^*). Set

$$(a_2, y_1) = \tau_1(a, x_1),$$

and, for all $j \ge 2$,

$$(a_{j+1}, y_j) = \tau_j(a_j, x_j).$$

Finally, define

$$\bar{a}(x) = y_1y_2y_3\ldots.$$

Note that if all (A_n, τ_n) are invertible automata then, for every state $a \in A_1$, $\bar{a}$ is a bijection on X^∞ (or X^*). Hence in this case we can consider the group generated by the transformations $\bar{a}$, $a \in A_1$. This group is called *piecewise automatic group*.

If all automata in the above definition are equal to some automaton τ, we obtain the group generated by the finite state automaton τ.

Piecewise automatic groups were defined in [10] and [29] (in [10] we only considered the case when there is only a finite number of mutually distinct automata among τ_n).

In the sequel we always assume that $A \subset e \cup S \cup S^{-1}$, $e \in A$, $\tau(e,x) = (e,x)$ and that $\tau(s^{-1}, y) = (a^{-1}, x)$ whenever $s, s^{-1} \in A$ and $\tau(s,x) = (a,y)$.

Consider an automaton A, τ over the alphabet X and take a word w over A. Note that w defines a map $\tau_w \colon X \to X$. For $w = a_1 a_2 \dots a_k$, $a_i \in A$,

$$\tau_w = \tau_{a_1} \circ \tau_{a_2} \circ \cdots \circ \tau_{a_k}.$$

Here, as before, $\tau_{a_i}(x)$ is the projection of $\tau(a_i, x)$ onto X.

Let $\tau \colon A \times X \to A \times X$ be an invertible automaton and let w be a word over the alphabet A. Note that, for every $x \in X$, w defines a restriction w_x, which is again a word over A: for $w = a_1 a_2 \dots a_k$, $a_i \in A$,

$$w_x = b_1 b_2 \dots b_k,$$

where the state b_i is the projection of $\tau(a_i, \tau_{a_{i+1} a_{i+2} \dots a_k}(x))$ onto A.

Also for a branch of the first level of the form $\gamma_1 = x_1 \dots$ we say that w_{x_1} is the restriction of w to the branch γ_1 and for a branch γ_m of level m, $\gamma_m = x_1 \dots x_m \dots$, we define by induction on m the restriction of w to γ_m. This restriction is equal to the restriction to x_m of the restriction of w to the branch $\gamma_{m-1} = x_1 \dots x_{m-1} \dots$ of level $m-1$.

Definition 3 (Automatically presented group). Let $\tau \colon A \times X \to A \times X$ ($A \subset e \cup S \cup S^{-1}$, $e \in A$) be an invertible automaton. The automatically presented group $G^*(\tau)$ over τ is the group given by the following presentation. The generating set is S and a word w over $S \cup S^{-1}$ is in the set of the defining relations if the following holds. There exists i such that w represents an element from the stabilizer of level i (in the group generated by the finite state automaton τ) and for every branch γ of level i the restriction of w to γ is freely equivalent to the identity in the group freely generated by S.

It is clear that the group generated by τ is a quotient of the automatically presented group over τ. In the next section we see that in some cases these groups are equal and in some cases they are essentially different.

Given a sequence $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$, a sequence of automata $\tau_i \colon A_i \times X \to A_i \times X$ and a word in the alphabet A_1, we define the restrictions of w to the branches of the rooted tree in a similar way as for a single automaton τ. For a branch γ_m of level m, $\gamma_m = x_1 \dots x_m \dots$, we define by induction on m the restriction of w to γ_m. This restriction is equal to the restriction to x_m with respect to the automaton τ_m of the restriction of w to the branch $\gamma_{m-1} = x_1 \dots x_{m-1} \dots$ of level $m-1$. By definition, the restriction of w to a branch of level m is a word over the alphabet A_m.

Definition 4 (Piecewise automatically presented group). Consider an ascending sequence of sets $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ and a sequence of finite state invertible automata A_n, τ_n, $1 \le n < \infty$, defined over a common alphabet X (as before, an important case is $A_1 = A_2 = A_3 \dots$, with τ_i not necessarily equal to each other). We assume that $A_i \subset e \cup S_i \cup S_i^{-1}$ and that $e \in A_i$ for each i. The *piecewise automatically presented group* defined by the sequence (A_n, τ_n) is the group generated by A_1 in the following way. A word w over A_1 is a defining relation in this group if there exists i such that w represents an element from the stabilizer of level i (in the piecewise automatic group defined by $\tau_1, \tau_2 \dots$) and for every branch γ of level i the restriction of w to γ (the restriction is taken with respect to $\tau_1, \tau_2, \dots, \tau_i$) is freely equivalent to the identity in the group freely generated by S_i.

If all (A_n, τ_n) coincide, then the piecewise automatically presented group defined above is equal to the automatically presented group over τ_1.

It is clear that piecewise automatic groups are quotients of the corresponding piecewise automatically presented groups.

Note that, under the assumption of the previous definition, the piecewise automatically presented group is generated by S_1.

3. Examples of automatically presented groups

Example 1. Every finite group is automatically presented.

Indeed, let G be a finite group, set $A = X = G$, and consider the automaton $\tau\colon A \times X \to A \times X$ given by $\tau(g_1, g_2) = (e, g_1 g_2)$. It is clear that the automatically presented group over τ is equal to the group generated by this finite state automaton and is isomorphic to G.

Example 2. $\mathbb{Z}$ is automatically presented. Indeed, take any finite state automaton with a single non-identity state generating $\mathbb{Z}$. Such automata do exist: consider for example $\tau\colon \{a, e\} \times \{0, 1\} \to \{a, e\} \times \{0, 1\}$ defined by $\tau(a, 0) = (a, 1)$, $\tau(a, 1) = (e, 0)$ and $\tau(e, x) = (e, x)$ for $x = 0, 1$. It is clear that the automatically presented group over τ is isomorphic to $\mathbb{Z}$.

Note that the class of automatically presented groups is closed under taking direct products. Indeed, let G_1, G_2 be automatically presented groups over τ_1 and τ_2, respectively, where $\tau_1\colon A_1 \times X_1 \to A_1 \times X_1$, $\tau_2\colon A_2 \times X_2 \to A_2 \times X_2$, $A_1 = \bar{A}_1 \sqcup e$, $A_2 = \bar{A}_2 \sqcup e$. Put $X = X_1 \sqcup X_2$, $A = \bar{A}_1 \sqcup \bar{A}_2 \cup e$ and define $\tau\colon A \times X \to A \times X$ by $\tau(e, x) = (e, x)$, for all x, $\tau(a_1, x) = \tau_1(a_1, x)$ for $a_1 \in \bar{A}_1$, $x \in X_1$, $\tau(a_1, x) = (e, x)$ for $a_1 \in \bar{A}_1$, $x \in X_2$ and similarly $\tau(a_2, x) = \tau_2(a_2, x)$ for $a_2 \in \bar{A}_2$, $x \in X_2$, $\tau(a_2, x) = (e, x)$ for $a_2 \in \bar{A}_2$, $x \in X_1$. It is clear that the automatically presented group over τ is $G_1 \times G_2$.

Thus we see that, for every n, $\mathbb{Z}^n$ is automatically presented.

Example 3. Let τ be the standard automaton generating the first Grigorchuk group. Recall that this is the finite state automaton (A, τ) over the alphabet $X = \{0, 1\}$, where $A = \{a, b, c, d, e\}$ and $\tau\colon A \times X \to A \times X$ is defined by

$$\tau(e, 0) = (e, 0), \quad \tau(e, 1) = (e, 1), \quad \tau(a, 0) = (e, 1), \quad \tau(a, 1) = (e, 0)$$

and

$$\begin{aligned} \tau(b, 0) &= (a, 0), & \tau(c, 0) &= (a, 0), & \tau(d, 0) &= (e, 0), \\ \tau(b, 1) &= (c, 1), & \tau(c, 1) &= (d, 1), & \tau(d, 1) &= (b, 1). \end{aligned}$$

Then the automatically presented group over τ contains a free non-abelian subgroup. In particular, this group is not equal to the group generated by the finite state automaton τ.

Let us show that the subgroup generated by b, c and d is free. To see this take any word w in b, c, d representing a non-trivial element in the free group on these three generators. Observe that for every n that is divisible by 3 the restriction of w to the branch $11\ldots1\ldots$ of level n is again the word w. Therefore, w represents a non-trivial element in the automatically presented group over τ.

Example 4. Let τ be the automaton from [17], generating the Basilica group. Recall that this is the finite state automaton (A, τ) over the alphabet $X = \{0, 1\}$, where $A = \{a, b, e\}$ and $\tau\colon A \times X \to A \times X$ is defined by

$$\tau(a, 0) = (e, 0), \quad \tau(a, 1) = (b, 1), \quad \tau(b, 0) = (e, 1), \quad \tau(b, 1) = (a, 0),$$

(and $\tau(e, x) = (e, x)$ for $x = 0, 1$). We will prove bellow that the automatically presented group over τ is equal to the group generated by this automaton.

It was shown in [5] that the Basilica group is amenable. Therefore, we see that the class of automatically presented groups contains amenable groups that are not virtually abelian.

Lemma 1. (i) *Let w be a word of length l, $l \geq 3$, in the alphabet a, b, a^{-1}, b^{-1}, e and let γ be a branch of level* 2 *in the tree, corresponding to $X = \{0, 1\}$* (*that is, γ is equal to* $00\ldots$, $01\ldots$, $10\ldots$ *or* $11\ldots$). *Then the restriction w_γ of w to γ defined by τ* (*from Example* 4) *is freely equivalent to a word of length at most $l - 1$.*

(ii) *Let w be a word in the alphabet a, b, a^{-1}, b^{-1}, e of length* 2 *representing a trivial word in the group, generated by the finite state automaton τ. Then w is freely equivalent to a trivial word.*

Proof. (i) Note that for every automaton the restriction of a word of length m to any branch has length at most m. Thus it is sufficient to consider the case $l = 3$. Moreover, without loss of generality we can assume that w is freely irreducible word (of length 3).

First assume that w contains as a subword one of the following words b^2, b^{-2}, ba, ba^{-1}, ab^{-1}, $a^{-1}b^{-1}$. Note that the restriction of each of these words to any branch of the first level is freely equivalent to a word of length at most 1. Therefore, the restriction of w to any branch of the first level, and consequently, to any branch of the second level has length at most $l-1$

Observe also that if w contains as a subword a^2 or a^{-2}, then w_γ has length at most $l-1$, because the restriction of a^2 and of a^{-2} to any branch of level 2 is freely equivalent to a word of length at most 1.

Therefore, it suffices to consider the case when w is freely irreducible word of length 3 that does not contain as a subword any of the words a^2, a^{-2}, b^2, b^{-2}, ba, ba^{-1}, ab^{-1}, or $a^{-1}b^{-1}$. Note that in this case $w = bab^{-1}$ or $ba^{-1}b^{-1}$. One can check that the restriction of each of these two words to any branch of the second level is freely equivalent to a word of length at most 2.

(ii) It is sufficient to consider the case when w is equal to a^2, a^{-2}, ae, ea, b^2, b^{-2}; Otherwise w would either be freely equivalent to a trivial word, or its action of the first level would be non-trivial (and, therefore, w would represent a non-trivial element in the group generated by the finite state automaton τ). Each of the words a^2, a^{-2}, ae, ea, b^2, b^{-2} acts non-trivially on the second level, and thus w does not represent the trivial element in the group generated by τ. □

Note that the lemma above implies that the automatically presented group over τ (from Example 4) is equal to the group generated by this automaton. Indeed, take a word w representing the identity in the group generated by τ. The first part of the lemma implies that there exists a level l such that each restriction of w to a branch of this level has length at most two. Since w represents the identity in the group generated by τ, the action on the level l of w is trivial and all the restrictions represent the identity in the group generated by τ. Combining this with the claim of the second part of the lemma, we see that each of these restrictions is freely equivalent to the identity word. Therefore, w represents the identity element in the automatically presented group defined by τ.

4. Properties of piecewise automatically presented groups

We recall that a finitely generated group Γ has the property T of Kazhdan if for some (and hence for all) finite generating set S of Γ there exists a positive constant $\varepsilon(S)$ such that for every unitary representation (π, H) of Γ with no invariant vectors and for every $u \in H$ there exists $s \in S$ such that $\|\pi(s)u - u\| \geq \varepsilon(S)\|u\|$. For more on the property T see, for example, [20].

Theorem 1. *Let G be a piecewise automatically presented group.*

(i) *If G has the property T of Kazhdan, then G is finite.*

(ii) *If G is amenable and finitely presented, then G is virtually abelian.*

(iii) *More generally, if G is a quotient of a finitely presented group J without free subgroups, then G is virtually abelian.*

We recall that a group Γ is polycyclic if there exists a sequence $e = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_M = \Gamma$ such that each Γ_i is a normal subgroup in Γ_{i+1} and Γ_{i+1}/Γ_i is a (finite or infinite) cyclic group. It is known that every polycyclic group is finitely presented (see for example [25]). Therefore, the second part of the theorem shows in particular that every polycyclic piecewise automatically presented group is virtually abelian.

The theorem is in contrast with the fact that among groups generated by finite state automata there are groups having the property T (see Mozes and Glasner, [22]) and there are many amenable finitely presented groups that are not virtually abelian (for example any nilpotent group admitting an expanding map is generated by some finite state automaton, [23]).

It is known that the Schreier graph of any contracting action on a regular rooted tree is of polynomial growth ([2], see also [23]), and thus groups admitting a faithful contracting action cannot have the property T (unless they are finite). There is a conjecture that any group admitting a faithful contracting action is amenable. Note that we cannot replace the assumption in (i) by non-amenability.

Remark. The third part of the theorem implies that if J is a group without free-subgroups admitting as a quotient the Basilica group from Example 3, then J is infinitely presented.

Let G be a finitely generated group and S a finite generating set of G. Recall that the labeled Cayley graph of (G, S) is the graph whose vertices are the elements of G and in which two vertices g_1 and g_2 are joined by an edge whenever there exists $s \in S$ such that $g_1 s = g_2$. In this case this oriented edge is labeled by s.

The Grigorchuk topology (also called the Cayley topology or the Chabauty topology) on the space of d-generated groups is the topology in which two groups G_1 and G_2 generated by S, ($\#S = d$) are close whenever the labeled Cayley graphs of (G_1, S) and (G_2, S) coincide in the ball of radius R, for large R. This space is metrizable and the associated metric is $m_G((G_1, S), (G_2, S)) = (1/2)^R$, where R is the maximal radius such that the labeled Cayley graphs of (G_1, S) and (G_2, S) coincide in the ball of radius R. It is easy to see that the space of groups generated by a finite set S is compact in this topology. See [19] for further properties of this topology.

Lemma 2. *Let G be the piecewise automatically presented group defined by the sequence (A_1, τ_1), (A_2, τ_2), ... Consider a piecewise automatically presented group G^+, defined by the sequence (A_1, τ_1^+), (A_2, τ_2^+), ..., such that, for all $i \le N$, we have $\tau_i = \tau_i^+$.*

If N is large enough, then G^+ is a quotient of some group G' which is close to G in the Grigorchuk topology. That is, for every $\varepsilon > 0$ there exists N_ε and a group G', generated by S, satisfying $m_G((G', S)(G, S)) < \varepsilon$ and such that G^+ is a quotient of G', whenever $N > N_\varepsilon$. (Here in G and G^+ we consider the system of generators S coming from A_1).

Proof. Let R be a finite set of words representing the identity in G. Note that if N is large enough, then by definition of piecewise automatically presented groups all words in R represents the identity in G^+ as well.

Now take $L > 0$. Put R_L to be the set of words of length at most L (in the alphabet A_1) that represent the identity in the group G. Let G' be the group generated by A_1 with R_L as the set of defining relations. By construction, the group G' is finitely presented. If L is large enough, then G' is close to G in the Grigorchuk topology. Note that there exists N_L such that any group G^+ satisfying the assumption of the lemma with $N > N_L$ is the quotient of the group G'. □

Proposition 1. *Let G be a piecewise automatically presented group. Suppose that G is not virtually abelian. Then there exist sequences of groups G_i, H_i, such that, for every i, H_i is a quotient of G_i, each H_i contains a free non-abelian subgroup and G_i tends to G in the Grigorchuk topology.*

Moreover, the sequences above can be chosen in such a way that H_i admits a finite index subgroup which is a subgroup in a direct product of free groups.

Proof. Let $A_i \subset e \cup S_i \cup S_i^{-1}$. Let σ_i be some automaton, such that the set of states of this automaton contains $S_i \cup S_i^{-1}$ and the subgroup generated by S_i in the group generated by σ_i (and hence in the automatically presented group over σ_i as well) is free. (Such automata do exist, see e.g. [23], [24] or [28].)

Let H_i be the piecewise automatically presented group defined at each level $j < i$ by the automaton for the level j in the definition of G and by σ_i for $j \geq i$. Let us show that each H_i contains a free group on two generators as a subgroup.

Indeed, for any level i there are two words w_1 and w_2 over S_1 such that the action on this level defined by these words is trivial and such that for some branch γ of level i the restriction to this branch w_1^γ and w_2^γ represent two non-commuting words in the free group over S_i. (Otherwise the stabilizer of the level i in G would be abelian and G would be virtually abelian.) We see that w_1 and w_2 freely generate a free subgroup in H_i.

Now, we apply Lemma 2 and see that there exists a sequence of groups G_i, such that, for every i, H_i is a quotient of G_i and G_i tends to G in Grigorchuk topology. (In this special case one can take $G' = G^+$ and it is not necessary to take a quotient, but we do not need it for the proof).

To prove the second claim of the proposition observe that the stabilizer of the level i in H_i is a subgroup in the direct product of free groups. Indeed, observe that a word in this stabilizer represents the identity in H_i if and only if it represents the identity in the corresponding piecewise automatic group. □

Proof of Theorem 1. To prove (i) we assume the contrary and suppose that G is infinite and has the property T. Then G is not virtually abelian. Therefore, we can apply Proposition 1 and conclude that there exist a sequence of groups G_i, tending to G in the Grigorchuk topology, and a sequence of groups H_i such that each H_i is an infinite quotient of G_i, admitting a finite index subgroup H_i', which is a subgroup in a direct product of free groups.

We know that G_i tends to G and that G has the property T. Recall that if G has the property T and G' is close enough to H in the Grigorchuk topology, then G' also has the property T ([26]). Therefore, there exists N such that, for all $i \geq N$, G_i has the property T. The property T of Kazhdan is stable with respect to taking quotients, and therefore, for every $i \geq N$, the group H_i has the property T. The property T is also stable with respect to taking finite index subgroups, and so, for every $i \geq N$, H_i' has the property T.

On the other hand, free groups have the Haagerup property, which is stable with respect to taking direct products and subgroups (see e.g. [7]). Thus, for every i, the group H_i' has the Haagerup property.

In particular, H_N' is an infinite group that satisfies both the property T and Haagerup property. We have arrived at a contradiction.

(ii) Since any group admitting a free non-abelian subgroup is non-amenable, (ii) is a particular case of (iii).

Now we prove (iii). We assume the contrary and suppose that J is finitely presented group without free non-abelian subgroups and G is a quotient of J. We suppose also that G is not virtually abelian.

By Proposition 1 we know that there exists a sequence of groups G_i, tending to G in the Grigorchuk topology, and a sequence of groups H_i such that each H_i contains a free non-abelian subgroup and such that H_i is a quotient of G_i, for all i.

We know that G_i tends to G and that G is a quotient of a finitely presented group J. Hence there exists N such that, for every $i \geq N$, the group G_i is a quotient of J. Since J contains no free non-abelian subgroup and since this property is stable by taking a quotient, this implies that, for all $i \geq N$, G_i contains no free non-abelian subgroups. Since each H_i is a quotient of G_i, this shows that, for all $i \geq N$, H_i contains no free non-abelian subgroups.

In particular, H_N is a quotient of a group without free non-abelian subgroups, but H_N itself contains a free non-abelian subgroup. We arrived at a contradiction. □

5. Word problem

The proof of the first part of the following proposition represents a well-known algorithm that solves the word problem in groups generated by finite state automata. The proof of the second part represents an algorithm that solves the word problem in automatically presented groups.

Proposition 2. (i) *Every group G generated by a finite state automaton is recursively presented and the word problem in such a group is solvable in exponential time: there exists an algorithm which, for a word w of word length l, performs* $\exp(Cl)$ *steps and decides whether w represents the identity element in G.*

(ii) *Every automatically presented group G (defined by a finite state automaton) is recursively presented and, moreover, the word problem in such a group is solvable in exponential time.*

Proof. (i) Take $\tau\colon A\times X \to A\times X$ over an alphabet X of cardinality d, and consider a word w of length l over the alphabet A in the group G generated by this finite state automaton.

Consider the action of w on the first level of the tree. If it is non-trivial, the algorithm stops and answers that w represents a non-identity element in G. If the action is trivial, consider the restrictions of w to the branches of the first level. We denote these restrictions by $w_1, w_2, \dots, w_d$.

Observe that the length of each w_i is at most l. Apply recursively the same procedure to $w_1, \dots, w_d$, having in mind that if, at some point, we get as a restriction a word w that has already appeared, then we do not apply the algorithm to this additional occurrence of w. If at some point the algorithm stops producing new words (all restrictions of all considered words already appeared before) and all obtained words act trivially on the first level of the tree, then the algorithm stops and answers that w represents the identity in G. Note that the algorithm makes at most $v_{G,A}(l)$ steps (where $v_{G,A}(l)$ is the growth function of G with respect to the generating set A). At each step one performs at most Kl operations, where K is some positive constant. Therefore, the total number of operations is at most $\exp(Cl)$, where C is some positive constant.

(ii) Let $\tau\colon A \times X \to A \times X$, $A \subset S \cup S^{-1} \cup e$, be an automaton which gives an automatic presentation of G. Consider the set W of non-empty freely irreducible words over the alphabet S of length at most l. Consider the oriented graph Γ_W whose vertices are the words in W and in which two words w_1 and w_2 are joined by an edge in the direction from w_1 to w_2 if the restriction of w_1 to some branch of the first level is freely equivalent to w_2. In addition, draw a loop at every vertex w that acts as a non-trivial permutation on the first level of the tree.

Observe that a word w represents a non-identical element in the group G if and only if there is an oriented path starting at w and ending in a vertex which belongs to an oriented cycle of the graph Γ_W (to see this we use again the fact that the restriction to a branch does not increase the length of the word).

For an oriented graph of bounded valency and of cardinality s this problem can be solved in time, which is polynomial in s. Indeed, first we observe that we can determine in polynomial time which of the vertices belong to an oriented cycle. If a vertex belongs to some cycle, then it belongs to a cycle of length at most s. For every m with $1 \leq m \leq s$ and for all pairs of vertices w_1 and w_2, we can determine whether there exists an oriented path of length m joining these two vertices, and this can be done in polynomial time in s. To see this we apply an inductive argument and observe, that if we can do this for some $m < s$, then we can do it also for $m + 1$. Indeed, w_1 and w_2 are joined by an oriented path of length $m + 1$ if and only if there exists w_3 such that there is an oriented path of length m from w_1 to w_3 and there is an oriented edge from w_3 to w_2. This shows, that we can determine which vertices belong to some oriented cycle in polynomial time in s. After this we take the vertex w in our graph and for each vertex u belonging to some oriented cycle we check whether there exists an oriented path of length at most s from w to u.

Now we return to the proof of (ii). Note that the cardinality of Γ_W is at most $\exp(Cl)$ and the valency of each vertex is bounded by the cardinality of X. Therefore, in time $\exp(Kl)$ (K is a positive constant not depending on l) we can determine which vertices correspond to a word, representing a non-identity element in G. □

1) For a given automaton, it seems interesting to understand the structure of the graphs described in the proof of the second part of the proposition. In particular, additional information about these graphs can lead to better estimates for the word problem in the corresponding groups.

2) If the sequence of automata τ_n is recursive, then the piecewise automatically presented group defined by this sequence is recursively presented.

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