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On the first L^p-cohomology of discrete groups

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Abstract. For finitely generated groups Γ, the isomorphism between the first ℓ^p-cohomology $H^1_{(p)}(\Gamma)$ and the reduced 1-cohomology with coefficients in $\ell^p(\Gamma)$ is exploited to obtain vanishing results for $H^1_{(p)}(\Gamma)$. The following cases are treated: groups acting on trees, groups with infinite center, wreath products, and lattices in product groups.

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1. Introduction

L^p-cohomology for discrete groups Γ, in its simplicial version, was introduced by Gromov in Chapter 8 of [Gro93] as a useful group invariant.

Assume first that Γ admits a classifying space X which is a simplicial complex, finite in every dimension; let $\tilde{X}$ be the universal cover of X. Denote by $\ell^p C^k$ the space of p-summable complex k-cochains on $\tilde{X}$, i.e. the ℓ^p-functions on the set C^k of k-simplices of $\tilde{X}$. The *L^p-cohomology of* Γ is the reduced cohomology of the complex

$$d_k \colon \ell^p C^k \to \ell^p C^{k+1},$$

where d_k is the simplicial coboundary operator; we denote it by

$$\bar{H}^k_{(p)}(\Gamma) = \operatorname{Ker} d_k / \overline{\operatorname{Im} d_{k-1}}.$$

As explained at the beginning of Section 8 of [Gro93], this definition only depends on Γ.[1]

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[1] Of course, L^2-cohomology had been considered much earlier, the use of the von Neumann algebra of Γ allowing to define, for $k \geq 0$, the k-th L^2-Betti number, i.e. the von Neumann dimension of $\bar{H}^k_{(2)}(\Gamma)$ (see [CG86]).

For $k = 1$, one can define $\bar{H}^1_{(p)}(\Gamma)$ under the mere assumption that Γ is finitely generated. Indeed, denote by λ_Γ the left regular representation of Γ on functions on Γ. For $1 \leq p < \infty$, denote by $D_p(\Gamma)$ the space of functions f on Γ such that $\lambda_\Gamma(g)f - f \in \ell^p(\Gamma)$ for every $g \in \Gamma$: this is the space of *p-Dirichlet finite* functions on Γ. If S is a finite generating subset of Γ, define a norm on $D_p(\Gamma)/\mathbb{C}$ by $\|f\|^p_{D_p} = \sum_{s\in S} \|\lambda_\Gamma(s)f - f\|^p_p$, and denote by $i : \ell^p(\Gamma) \to D_p(\Gamma)$ the inclusion. The *first L^p-cohomology* of Γ is then

$$\bar{H}^1_{(p)}(\Gamma) = D_p(\Gamma)/\overline{i(\ell^p(\Gamma)) + \mathbb{C}}.$$

The compatibility between this definition and the previous one, was checked e.g. in [BMV05].

This paper is mainly devoted to vanishing results for the first L^p-cohomology of a finitely generated group. Among motivations for studying these, we quote:

(1) vanishing of the first L^2-Betti number has impact in geometric group theory and topology (see e.g. Eckmann's paper [Eck97]);

(2) spaces of L^p-cohomology are quasi-isometry invariants for finitely generated groups (see [BP03], [Pan]);

(3) it was shown in [BMV05] that, whenever a non-amenable group Γ acts properly isometrically on a proper CAT(-1) space X, then for p larger than the critical exponent $e(\Gamma)$ in X, the first L^p-cohomology of Γ is *not* zero; on the other hand, a result of Burger–Mozes [BM96] states that if X is a proper CAT(-1) space such that the full isometry group Isom(X) acts co-compactly, then every group acting properly isometrically on X has finite critical exponent; so a group whose first L^p-cohomology vanishes for every $p > 1$, cannot act properly isometrically on such a CAT(-1)-space X.

The following theorem is our main result (it subsumes Theorems 4.1, 4.2, 4.3, 4.6, 4.7, 4.8).

Theorem. *Fix* $p \in]1, +\infty[$. (i) *Let Γ be a finitely generated group acting* (*without inversion*) *on a tree with non-amenable vertex stabilizers and infinite edge stabilizers. If all vertex stabilizers have vanishing first L^p-cohomology, then so does Γ.*

(ii) *Let N be a normal, infinite, finitely generated subgroup of a finitely generated group Γ. Assume that N is non-amenable and that its centralizer $Z_\Gamma(N)$ is infinite. Then $\bar{H}^1_{(p)}(\Gamma) = 0$.*

(iii) *Let Γ be a finitely generated group. If the centre of Γ is infinite, then $\bar{H}^1_{(p)}(\Gamma) = 0$.*

(iv) *Let H, Γ be* (*non-trivial*) *finitely generated groups, and let $H \wr \Gamma$ be their wreath product. If H is non-amenable, then $\bar{H}^1_{(p)}(H \wr \Gamma) = 0$.*

(v) *Let $G = G_1 \times \cdots \times G_n$ be a direct product of non-compact, second countable locally compact groups* ($n \geq 2$). *Let Γ be a finitely generated, cocompact lattice in G. If Γ is non-amenable* (*equivalently, if some G_i is non-amenable*), *then $\bar{H}^1_{(p)}(\Gamma) = 0$.*

(vi) *Fix $n \geq 2$. For $i = 1, \dots, n$, let G_i be the group of k_i-rational points of some k_i-simple, k_i-isotropic linear algebraic group, for some local field k_i. Let Γ be an irreducible lattice in $G_1 \times \cdots \times G_n$. Then $\overline{H}^1_{(p)}(\Gamma) = 0$.*

Moreover, for $p = 2$, the results in (ii), (iv), (v) *above hold without the non-amenability assumption.*[2]

Part (iii) of this Theorem extends a result of Gromov (Corollary on p. 221 of [Gro93]): if the center of Γ contains an element of infinite order, then $\overline{H}^1_{(p)}(\Gamma) = 0$. A very short proof of this fact was recently given by Tessera (Proposition 3 in [Tes]).

Part (vi) is a modest contribution to a conjecture of Gromov (see p. 253 in [Gro93]): if Γ is a co-compact lattice of isometries of a Riemannian symmetric space (of non-compact type) or a Euclidean building X, then one should have $\overline{H}^k_{(p)}(\Gamma) = 0$ for $k < \operatorname{rank}(X)$.

We now describe our approach to $\overline{H}^1_{(p)}$, which is to appeal on the one hand to an identification between the first L^p-cohomology and the (reduced) first group cohomology with coefficients in $\ell^p(\Gamma)$ (the relevant cohomological background being presented in Section 2), on the other hand to p-harmonic functions: if S is a finite, symmetric generating subset of Γ, we say, following [Pul06], that a function f on Γ is *p-harmonic* if

$$\sum_{s \in S} |f(s^{-1}x) - f(x)|^{p-2}(f(s^{-1}x) - f(x)) = 0$$

for every $x \in \Gamma$. We denote by $HD_p(\Gamma)$ the set (not a linear space, if $p \neq 2$) of harmonic, p-Dirichlet finite functions on Γ. It was observed by B. Bekka and the second author [BV97] for $p = 2$, and by M. Puls [Pul06] in general, that for Γ an infinite, finitely generated group, the following are equivalent:

i) the first L^p-cohomology $\overline{H}^1_{(p)}(\Gamma)$ is zero;

ii) $HD_p(\Gamma) = \mathbb{C}$;

iii) $\ell^p(\Gamma)$ is dense in $D_p(\Gamma)/\mathbb{C}$;

iv) $\overline{H}^1(\Gamma, \ell^p(\Gamma)) = 0$, where $\overline{H}^1(\Gamma, \ell^p(\Gamma))$ denotes the reduced 1-cohomology of Γ with coefficients in the Γ-module $\ell^p(\Gamma)$.

In Section 3 we add a fifth characterization to this list, giving much flexibility:

Corollary 3.2. *For an infinite, finitely generated group Γ, the above properties are still equivalent to: $\overline{H}^1(\Gamma, \ell^p(H)|_\Gamma) = 0$ for every group H containing Γ as a subgroup.*

[2] W. Lück informed us that, in the case $p = 2$, it is possible to prove part (i) of the Theorem without the non-amenability assumption, using his algebraic version of L^2-Betti numbers (see [Lue02]). The case of amalgamated products is treated in [Lue02], Theorem 7.2 (4), p. 294.

Section 4 contains our vanishing results for $\overline{H}^1_{(p)}$, while Section 5 has a somewhat different flavor: using the Cheeger–Gromov vanishing result for L^2-cohomology of amenable groups [CG86], we obtain a new characterization of amenability for finitely generated groups:

Proposition 5.3. *Let* Γ *be an infinite, finitely generated group. The following are equivalent*:

(i) Γ *is amenable*;

(ii) $\ell^2(\Gamma)$ *is a dense, proper subspace of* $D_2(\Gamma)/\mathbb{C}$.

This paper can be viewed a sequel to [BMV05], although it can be read independently.

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2. 1-cohomology versus reduced 1-cohomology

2.1. 1-cohomology. Let G be a topological group and let V be a topological G-module, i.e. a real or complex topological vector space endowed with a continuous linear representation $\pi\colon G \times V \to V$; $(g, v) \mapsto \pi(g)v$. If H is a closed subgroup we denote by $V|_H$ the space V viewed as an H-module for the restricted action, and by V^H the set of H-fixed points

$$V^H = \{v \in V \mid \pi(h)v = v \text{ for all } h \in H\}.$$

We say that V is a *Banach G-module* if V is a Banach space and π is a representation of G by isometries of V. A G-module is *unitary* if V is a Hilbert space and π a unitary representation.

We now introduce the space of 1-cocycles and 1-coboundaries on G, and the 1-cohomology with coefficients in V:

$$\begin{aligned} Z^1(G,V) &= \{b\colon B \to V \text{ continuous} \mid b(gh) = b(g) + \pi(g)b(h) \text{ for all } g,h \in G\},\\ B^1(G,V) &= \{b \in Z^1(G,V) \mid \text{there exists } v \in V \text{ such that } b(g) = \pi(g)v - v\\ &\qquad\qquad \text{for all } g \in G\},\\ H^1(G,V) &= Z^1(G,V)/B^1(G,V). \end{aligned}$$

If N is a closed normal subgroup of G and V is a G-module, there is a well-known action of G on $H^1(N, V|_N)$. On $Z^1(N, V|_N)$, this action is given by:

$$(g \cdot b)(n) = \pi(g)(b(g^{-1}ng)) \tag{1}$$

($b \in Z^1(N, V|_N)$, $g \in G$, $n \in N$). Clearly this action leaves $B^1(N, V|_N)$ invariant, so it defines an action of G on $H^1(N, V|_N)$. We have for $m \in N$,

$$(m \cdot b)(n) = b(n) + (\pi(n)b(m) - b(m)) \tag{2}$$

showing that the N-action on $H^1(N, V|_N)$ is trivial, hence the action of G on $H^1(N, V|_N)$ factors through G/N. The following result is well known (see e.g. Corollary 6.4 in [Bro82]) and usually proved using the Hochschild–Serre spectral sequence in group cohomology.[3]

Proposition 2.1. 1) *There is an exact sequence*

$$0 \to H^1(G/N, V^N) \xrightarrow{i_*} H^1(G, V) \xrightarrow{\mathrm{Rest}_G^N} H^1(N, V|_N)^{G/N} \to \cdots \tag{3}$$

where $i : V^N \to V$ denotes the inclusion.

2) *If $V^N = 0$ then the restriction map*

$$\mathrm{Rest}_G^N : H^1(G, V) \to H^1(N, V|_N)^{G/N}$$

is an isomorphism.

Let X be a set with a Γ-action. The space $\mathcal{F}(X)$ of all functions $X \to \mathbb{C}$ is endowed with the permutation representation, i.e. the Γ-module structure given by $(\gamma f)(x) = f(\gamma^{-1}x)$ ($f \in \mathcal{F}(X)$ for all $x \in X$). The following lemma is well known; the proof is given for completeness.

Lemma 2.2. *Let Γ be a (discrete) group and let X be a set on which Γ acts freely. Then $H^1(\Gamma, \mathcal{F}(X)) = 0$.*

Proof. Let $(s_i)_{i \in I}$ be a set of representatives for Γ-orbits in X. For $x \in X$, there exists a unique $i \in I$ and $\gamma \in \Gamma$ such that $x = \gamma s_i$. For $b \in Z^1(\Gamma, \mathcal{F}(X))$, define then $f(x) = (b(\gamma^{-1}))(s_i)$. It is readily verified that $\gamma f - f = b(\gamma)$ for every $\gamma \in \Gamma$. □

[3] For a proof without spectral sequences, see 8.1 in Chapter 1 of [Gui80].

2.2. Reduced 1-cohomology. Since G is a topological group and V is a topological G-module, we may endow $Z^1(G,V)$ with the topology of uniform convergence on compact subsets of G. The closure of $B^1(G,V)$ for this topology is denoted by $\overline{B^1(G,V)}$, and the quotient space $Z^1(G,V)/\overline{B^1(G,V)}$, called the *reduced first cohomology* of G with coefficients in V, is denoted by $\overline{H}^1(G,V)$. We will use the abuse of notation $H^1(G,V) = \overline{H}^1(G,V)$ to mean "the canonical epimorphism $H^1(G,V) \to \overline{H}^1(G,V)$ is an isomorphism". We recall without proof the following result of Guichardet (Théorème 1 in [Gui72]):

Proposition 2.3. *Let G be a locally compact, second countable group and let V be a Banach module such that $V^G = 0$. The following are equivalent*:

(i) $H^1(G,V) = \overline{H}^1(G,V)$;

(ii) *V does not have almost invariant vectors (this means that there exists a compact subset K of G and $\varepsilon > 0$ such that* $\sup_K \|\pi(g)v - v\| \geq \varepsilon\|v\|$ *for every $v \in V$).*

Let λ_G denote the left regular representation of G on $L^p(G)$ $(1 \leq p < \infty)$. Since λ_G has almost invariant vectors if and only if G is amenable (see [Eym72]), we immediately deduce (see Corollaire 1 in [Gui72]):

Corollary 2.4. *Fix $1 \leq p < \infty$. Let G be a locally compact, non compact, second countable group. The following are equivalent*:

(i) $H^1(G, L^p(G)) = \overline{H}^1(G, L^p(G))$;

(ii) *G is not amenable.*

Reduced 1-cohomology behaves well with respect to inductive limits:

Lemma 2.5. *Let G be a locally compact group which is the union of a directed system of open subgroups $(G_i)_{i\in I}$. Let (V,π) be a Banach G-module, with $b \in Z^1(G,V)$. If $b|_{G_i} \in \overline{B^1}(G_i, V|_{G_i})$ for all $i \in I$, then $b \in \overline{B^1}(G,V)$. In particular, if $\overline{H}^1(G_i, V|_{G_i}) = 0$ for all $i \in I$, then $\overline{H}^1(G,V) = 0$.*

Proof. Let K be a compact subset of G and $\varepsilon > 0$. By compactness K is covered by a finite union $G_{i_1} \cup \cdots \cup G_{i_n}$; with $i \geq i_1, \ldots, i_n$, we get $K \subset G_i$. Since $b|_{G_i} \in \overline{B^1}(G_i, V|_{G_i})$, we find a vector $v \in V$ such that $\sup_K \|b(g) - (\pi(g)v - v)\| < \varepsilon$, i.e. $b \in \overline{B^1(G,V)}$. □

The next result will be used to characterize vanishing of the first L^p-cohomology in Corollary 3.2.

Proposition 2.6. *Fix $1 \leq p < \infty$. Let H be a subgroup of the countable, discrete group Γ. Consider the following properties*:

(i) $\overline{H}^1(H, \ell^p(H)) = 0$,

(ii) $\overline{H}^1(H, \ell^p(\Gamma)|_H) = 0$,

(i′) $H^1(H, \ell^p(H)) = 0$,

(ii′) $H^1(H, \ell^p(\Gamma)|_H) = 0$.

Then (i) $\Longleftrightarrow$ (ii) *and* (i′) $\Longleftrightarrow$ (ii′).

Proof. Choosing representatives $(s_n)_{n\geq 1}$ for the right cosets of H in Γ, we may identify $\ell^p(\Gamma)|_H$, in an H-equivariant way, with the ℓ^p-direct sum of $[\Gamma : H]$ copies of $\ell^p(H)$.

(ii) $\Longrightarrow$ (i) and (ii′) $\Longrightarrow$ (i′): The continuous map

$$Z^1(H, \ell^p(H)) \to Z^1(H, \ell^p(\Gamma)|_H), \quad b \mapsto (b, 0, 0, \dots)$$

induces inclusions

$$H^1(H, \ell^p(H)) \to H^1(H, \ell^p(\Gamma)|_H) \quad \text{and} \quad \overline{H}^1(H, \ell^p(H)) \to \overline{H}^1(H, \ell^p(\Gamma)|_H).$$

(i) $\Longrightarrow$ (ii): The result is obvious for $[\Gamma : H] < \infty$, so we assume $[\Gamma : H] = \infty$. For $b \in Z^1(H, \ell^p(\Gamma)|_H)$, let $b_n \in Z^1(H, \ell^p(H))$ be its projection on the n-th factor $\ell^p(Hs_n)$. So, for $h \in H$, one has $b(h) = \oplus b_n(h)$. Fix K a finite subset of H, and $\varepsilon > 0$. Let $N > 0$ be such that $\sum_{n>N} \|b_n(h)\|^p < \frac{\varepsilon}{2}$ for every $h \in K$. For $i = 1, \dots, N$, using the assumption we find a function $v_i \in \ell^p(H)$ such that $\|b_i(h) - (\lambda_H(h)v_i - v_i)\|^p < \frac{\varepsilon}{2N}$ for every $h \in K$. Set $v_n = 0$ for $n > N$, and define $v = \oplus v_n \in \ell^p(\Gamma)$. Then by construction $\|b(h) - [\lambda_\Gamma(h)v - v]\|^p < \varepsilon$ for every $h \in K$, i.e., b is a limit of 1-coboundaries.

(i′) $\Longrightarrow$ (ii′): We consider two cases:

a) If H is finite then $H^1(H, \ell^p(H)) = H^1(H, \ell^p(\Gamma)|_H) = 0$.

b) If H is infinite then the assumption $H^1(H, \ell^p(H)) = 0$ implies, by Corollary 2.4, that H is not amenable. By Lemma 2 in [BMV05], this implies that $\ell^p(\Gamma)|_H$ does not almost have invariant vectors. By Proposition 2.3, we have $H^1(H, \ell^p(\Gamma)|_H) = \overline{H}^1(H, \ell^p(\Gamma)|_H)$, so that the result follows from the implication (i) $\Rightarrow$ (ii). □

Remark. Let G be a locally compact second countable group and let V be a Banach G-module with $V^G = 0$. Fix $p \in]1, +\infty[$, and denote by $\infty_p V$ the ℓ^p-direct sum of countably many copies of V. Consider the following properties:

(i) $\overline{H}^1(G, V) = 0$,

(ii) $\overline{H}^1(G, \infty_p V) = 0$,

(i′) $H^1(G, V) = 0$,

(ii′) $H^1(G, \infty_p V) = 0$.

Then the same proof as in Proposition 2.6 shows that (i) $\iff$ (ii) and (ii′) $\implies$ (i′). However, the implication (i′) $\implies$ (ii′) is not clear in general (as Lemma 2 in [BMV05] is very special to ℓ^p-spaces). A proof of that implication, using a different approach and assuming that V is a uniformly convex Banach space, has been communicated to us by N. Monod.

3. First L^p-cohomology

3.1. *p*-Dirichlet finite functions. Let Γ be a finitely generated group; fix a finite generating set S. Let Γ act on a set X. Denote by λ_X the permutation representation of Γ on $\mathcal{F}(X)$. Fix $p \in [1, \infty[$, and denote by $\| \,.\, \|_p$ the ℓ^p-norm with respect to counting measure on X.

The space of *p-Dirichlet finite functions* on X (relative to the Γ-action) is

$$\begin{aligned} D_p(X) &= \{ f \in \mathcal{F}(X) \mid \|\lambda_X(g)f - f\|_p < \infty \text{ for all } g \in \Gamma \} \\ &= \{ f \in \mathcal{F}(X) \mid \|\lambda_X(s)f - f\|_p < \infty \text{ for all } s \in S \}. \end{aligned}$$

Then $D_p(X)^\Gamma$ is the space of functions on X which are constant on Γ-orbits of X (it does not depend on p). Define a semi-norm on $D_p(X)$ by

$$\|f\|_{D_p(X)} = \Big[\sum_{s \in S} \|\lambda_X(s)f - f\|_p^p \Big]^{\frac{1}{p}}.$$

The kernel of this semi-norm is precisely $D_p(X)^\Gamma$, and the quotient $\mathcal{D}_p(X) = D_p(X)/D_p(X)^\Gamma$ is a Banach space (the norm on $\mathcal{D}_p(X)$ depends on the choice of S, but the underlying topology does not).

Define a linear map $\tilde{\alpha}\colon D_p(X) \to Z^1(\Gamma, \ell^p(X))$ by $\tilde{\alpha}(f)(\gamma) = \lambda_X(\gamma)f - f$. The kernel of this map is $D_p(X)^\Gamma$, so $\tilde{\alpha}$ descends to a continuous injection $\alpha\colon \mathcal{D}_p(X) \to Z^1(\Gamma, \ell^p(X))$.

Let $\tilde{i} : \ell^p(X) \to D_p(X)$ be the canonical inclusion. Clearly $\ell^p(X)^\Gamma$ is the space of ℓ^p-functions which are constant on Γ-orbits and zero on infinite orbits. Set $l^p_\Gamma(X) = \ell^p(X)/\ell^p(X)^\Gamma$ (so that $l^p_\Gamma(X) = \ell^p(X)$ if all orbits are infinite). The map $\tilde{i}$ induces a continuous inclusion $i : l^p_\Gamma(X) \to \mathcal{D}_p(X)$. Note that the image of $\alpha \circ i$ is exactly the space $B^1(\Gamma, \ell^p(X))$ of 1-coboundaries. This shows that:

- if i is not onto, then $H^1(\Gamma, \ell^p(X)) \neq 0$;
- if the image of i is not dense, then $\overline{H}^1(\Gamma, \ell^p(X)) \neq 0$.

Theorem 3.1. *Let X be a free Γ-space. Then $\alpha\colon \mathcal{D}_p(X) \to Z^1(\Gamma, \ell^p(X))$ is a topological isomorphism, and consequently*:

$$H^1(\Gamma, l^p(X)) \simeq \mathcal{D}_p(X)/i(l^p_\Gamma(X));$$
$$\overline{H}^1(\Gamma, \ell^p(X)) \simeq \mathcal{D}_p(X)/\overline{i(l^p_\Gamma(X))}.$$

Proof. We already know that α is continuous and injective. Since the Γ-action on X is free, we have $H^1(\Gamma, \mathcal{F}(X)) = 0$ by Lemma 2.2. So for $b \in Z^1(\Gamma, \ell^p(X))$ there exists $f \in \mathcal{F}(X)$ such that $b(g) = \lambda_X(g)f - f$ for every $g \in \Gamma$. Clearly f belongs to $D_p(X)$, so that $\tilde{\alpha}(f) = b$ and α is onto. It is then clear that α^{-1} is continuous. □

When Γ is infinite and $X = \Gamma$, we have $l^p_\Gamma(X) = \ell^p(\Gamma)$ and $\mathcal{D}_p(X) = D_p(X)/\mathbb{C}$. It was already observed (see Lemma 1 in [BMV05]; end of Section 2 in [Pul06]) that:

- $H^1(\Gamma, \ell^p(\Gamma))$ is isomorphic to $D_p(\Gamma)/(\ell^p(\Gamma) + \mathbb{C})$;
- the first ℓ^p-cohomology $\overline{H}^1_{(p)}(\Gamma)$ is isomorphic to $D_p(\Gamma)/\overline{(\ell^p(\Gamma) + \mathbb{C})}$.

So we get, using Proposition 2.6:

Corollary 3.2. *Let Γ be an infinite, finitely generated group. The following are equivalent*:

(i) $\overline{H}^1_{(p)}(\Gamma) = 0$;
(ii) $\ell^p(\Gamma)$ *is dense in* $D_p(\Gamma)/\mathbb{C}$;
(iii) $\overline{H}^1(\Gamma, \ell^p(\Gamma)) = 0$;
(iv) $\overline{H}^1(\Gamma, \ell^p(H)|_\Gamma) = 0$ *for every group H containing Γ as a subgroup.*

From this and Corollary 2.4, we get immediately:

Corollary 3.3. *Let Γ be an infinite, finitely generated group. The following are equivalent*:

(i) $H^1(\Gamma, \ell^p(\Gamma)) = 0$;
(ii) $\overline{H}^1_{(p)}(\Gamma) = 0$ *and Γ is non-amenable.*

3.2. *p*-harmonic functions (after M. Puls). This section is essentially borrowed from Section 3 in Puls' paper [Pul06]. We chose to include it mainly for the sake of completeness, but also to make sure that Puls' results hold for any Γ-action (not only for simply transitive ones). Our presentation, emphasizing the role of Gâteaux-differentials, is slightly different from the one in [Pul06].

So we come back to the general setting of a finitely generated group Γ (with a given, finite, symmetric, generating set S) acting on a countable set X. For $f \in \mathcal{F}(X)$ and $p > 1$, define

$$(\Delta_p f)(x) = \sum_{s \in S} |f(s^{-1}x) - f(x)|^{p-2}(f(s^{-1}x) - f(x))$$

with the convention, if $p < 2$, that $|f(s^{-1}x) - f(x)|^{p-2}(f(s^{-1}x) - f(x)) = 0$ if $f(s^{-1}x) = f(x)$. Say that f is *p-harmonic* if $\Delta_p f = 0$, and denote by $HD_p(X)$ the set of p-harmonic functions in $D_p(X)$. For $p \neq 2$, the set $HD_p(X)$ is not necessarily a linear subspace in $D_p(X)$. Note however that it contains the linear subspace $D_p(X)^\Gamma$.

For $f \in D_p(\Gamma)$, define a linear form on $\mathcal{D}_p(X)$ by

$$d_f(g) = \sum_{x \in X} \sum_{s \in S} |f(s^{-1}x) - f(x)|^{p-2}(f(s^{-1}x) - f(x))(g(s^{-1}x) - g(x))$$

(where $g \in D_p(X)$; but clearly $d_f(g)$ only depends on the image of g in $\mathcal{D}_p(X)$, and d_f only depends on the image of f in $\mathcal{D}_p(X)$). Let q be the conjugate exponent of p (so that $\frac{1}{p} + \frac{1}{q} = 1$); by Hölder's inequality, we get

$$\begin{aligned} |d_f(g)| &\leq \Big[\sum_{x \in X} \sum_{s \in S} |f(s^{-1}x) - f(x)|^{(p-1)q}\Big]^{\frac{1}{q}} \Big[\sum_{x \in X} \sum_{s \in S} |g(s^{-1}x) - g(x)|^p\Big]^{\frac{1}{p}} \\ &\leq \|f\|_{D_p(X)}^{p-1} \|g\|_{D_p(X)}, \end{aligned}$$

proving continuity of d_f as a linear form on $D_p(X)$.

This linear form d_f can be understood as follows. Let us identify a function $f \in D_p(X)$ with its image in $\mathcal{D}_p(X)$. Consider the strictly convex, continuous, non-linear functional on $\mathcal{D}_p(X)$ given by

$$F(f) = \|f\|_{\mathcal{D}_p(X)}^p.$$

The Gâteaux-differential of F at $f \in \mathcal{D}_p(X)$ (see [ET74], Def. 5.2 in Chapter I) is given by

$$F'_f(g) = \lim_{t \to 0+} \frac{F(f + tg) - F(f)}{t}$$

($g \in \mathcal{D}_p(X)$). An easy computation shows that

$$F'_f = p\, d_f.$$

The following lemma extends Lemma 3.1 in [Pul06].

Lemma 3.4. *For $f_1, f_2 \in D_p(X)$, the following are equivalent*:

(i) $f_1 - f_2 \in D_p(X)^\Gamma$;

(ii) $d_{f_1}(f_1 - f_2) = d_{f_2}(f_1 - f_2)$.

Proof. If $f_1 - f_2 \in D_p(X)^\Gamma$, then $d_f(f_1 - f_2) = 0$ for every $f \in D_p(X)$, in particular $d_{f_1}(f_1 - f_2) = 0 = d_{f_2}(f_1 - f_2)$.

Conversely, if $f_1 - f_2 \notin D_p(X)^\Gamma$, then f_1, f_2 define distinct elements in $\mathcal{D}_p(X)$. As F is strictly convex on $\mathcal{D}_p(X)$, by Proposition 5.4 in Chapter I of [ET74], we have

$$F(f_1) > F(f_2) + F'_{f_2}(f_1 - f_2) = F(f_2) + p\, d_{f_2}(f_1 - f_2).$$

Similarly

$$F(f_2) > F(f_1) + F'_{f_1}(f_2 - f_1) = F(f_1) - p\, d_{f_1}(f_1 - f_2).$$

So $d_{f_1}(f_1 - f_2) > d_{f_2}(f_1 - f_2)$. □

The next lemma generalizes Lemma 3.2 and Proposition 3.4 in [Pul06].

Lemma 3.5. *For $f \in D_p(X)$, the following are equivalent*:

(i) *f is p-harmonic*;

(ii) *$d_f(\delta_y) = 0$ for every $y \in X$ (where δ_y denotes the characteristic function of $\{y\}$)*;

(iii) *$d_f(g) = 0$ for every $g \in \overline{i(l^p_\Gamma(X))}$ (where the closure is in $\mathcal{D}_p(X)$).*

Proof. (i) $\Longleftrightarrow$ (ii): We compute

$$\begin{aligned} d_f(\delta_y) &= \sum_{s \in S} |f(y) - f(sy)|^{p-2}(f(y) - f(sy)) \\ &\quad - \sum_{s \in S} |f(s^{-1}y) - f(y)|^{p-2}(f(s^{-1}y) - f(y)) \\ &= -2(\Delta_p f)(y), \end{aligned}$$

as S is symmetric.

(ii) $\Longleftrightarrow$ (iii): The linear span of the δ_y's ($y \in X$) is dense in $\ell^p(X)$. By continuity of $i : l^p_\Gamma(X) \to \mathcal{D}_p(X)$, the linear span of the δ_y's is dense in $\overline{i(l^p_\Gamma(X))}$. This shows the desired equivalence. □

The following result extends Theorem 3.5 in [Pul06].

Theorem 3.6. *Every $f \in D_p(X)$ can be decomposed as $f = g + h$, where $g \in \overline{i(\ell^p(X))}$ and $h \in HD_p(X)$. This decomposition is unique, up to an element of $D_p(X)^\Gamma$.*

Proof. We start with uniqueness. So assume that $f = g_1 + h_1 = g_2 + h_2$. Then $d_{h_1}(h_1 - h_2) = d_{h_1}(g_2 - g_1) = 0$ by appealing to Lemma 3.5 (since h_1 is p-harmonic). Similarly $d_{h_2}(h_1 - h_2) = 0$. By Lemma 3.4, it follows that $h_1 - h_2$ is in $D_p(X)^\Gamma$.

To prove existence, we denote by g the projection of f on the closed convex subset $\overline{i(l^p(X))}$ in $\mathcal{D}_p(X)$ (this projection is well-defined by uniform convexity and reflexivity of $\mathcal{D}_p(X)$, see Theorem 2.8 in [BL00] or Lemma 6.2 in [BFGM]). Setting $h = f - g$, we must show that h is p-harmonic. For every $j \in i(l^p(X))$, consider the smooth function

$$G_j : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \|f - g + tj\|^p_{\mathcal{D}_p(X)}.$$

Since g minimizes the distance between f and $\overline{i(l^p(X))}$, the function $G_j(t)$ assumes its minimal value at $t = 0$, hence $G_j'(0) = 0$. The same computation as for the Gâteaux-differential of F shows that $G_j'(0) = p\, d_h(j) = 0$. Since this holds for every $j \in i(l^p(X))$, we conclude, by Lemma 3.5, that h is harmonic. □

Comparing Theorem 3.6 with Theorem 3.1, we immediately get:

Corollary 3.7. *Let X be a free Γ-space. Then $\overline{H}^1(\Gamma, \ell^p(X))$ identifies with $HD_p(X)/D_p(X)^\Gamma$ (where two functions in $HD_p(X)$ are identified if and only if they differ by an element in $D_p(X)^\Gamma$).*

4. Vanishing of first L^p-cohomology

4.1. Groups acting on trees

Theorem 4.1. *Fix $p \in [1, +\infty[$. Let G be a finitely generated group acting (without inversion) on a tree with non-amenable vertex stabilizers and infinite edge stabilizers. If all vertex stabilizers have vanishing first L^p-cohomology, then so does G.*

Proof. By Bass–Serre theory (see [Ser77]), G is the fundamental group of a graph of groups $(\mathcal{G}, Y)$. So Y is a graph and $\mathcal{G}$ is a system of groups attached to edges and vertices of Y in such a way that the edge groups are infinite and the vertex groups are non-amenable and have vanishing first L^p-cohomology. Consider the following cases:

1) If Y is a segment, then G is an amalgamated product $G = \Gamma_1 \star_A \Gamma_2$ with A infinite and Γ_1, Γ_2 non-amenable. The first cohomology of a G-module V is computed by means of the Mayer–Vietoris sequence (see [Bro82], formula (9.1), p. 81):

$$0 \to V^G \to V^{\Gamma_1} \oplus V^{\Gamma_2} \to V^A$$

$$\to H^1(G, V) \to H^1(\Gamma_1, V|_{\Gamma_1}) \oplus H^1(\Gamma_2, V|_{\Gamma_2}) \xrightarrow{\mathrm{Rest}^A_{\Gamma_1} - \mathrm{Rest}^A_{\Gamma_2}} H^1(A, V|_A) \to \cdots$$

We apply this to $V = \ell^p(G)$. By Corollary 3.3, we have $H^1(G, \ell^p(G)) = 0$. Therefore $\bar{H}^1_{(p)}(G) = 0$.

2) If Y is a loop, then G is a HNN-extension $G = \mathrm{HNN}(\Gamma, A, \theta)$, with A infinite and Γ non-amenable. The first cohomology of a G-module V is computed by means of the Mayer–Vietoris sequence (see [Bro82], formula (9.2)):

$$0 \to V^G \to V^\Gamma \to V^A \to H^1(G, V) \to H^1(\Gamma, V|_\Gamma) \to H^1(A, V|_A) \to \cdots .$$

This is applied to $V = \ell^p(G)$. By Corollary 3.3, we have $H^1(G, \ell^p(G)) = 0$, so again $\bar{H}^1_{(p)}(G) = 0$.

3) If Y is finite, we can argue by induction on the number n of edges. If $n = 1$, the result follows from the first two cases. For arbitrary n, we choose an edge and contract it. If this edge is a segment with vertex groups Γ_1, Γ_2 and edge group A, we replace it by a vertex whose group is $\Gamma_1 *_A \Gamma_2$; if the edge is a loop with vertex group Γ and edge group A, we replace it by a vertex whose group is $\mathrm{HNN}(\Gamma, A, \theta)$. This operation does not change the fundamental group and we obtain a graph with $n - 1$ edges, so the induction assumption applies.

4) In the general case, Y is the increasing union of finite subgraphs, so we may apply Lemma 2.5. □

The converse of Theorem 4.1 fails. We give two examples for $p = 2$, one for amalgamated products, one for HNN-extensions.

Example 1. Let q be a prime, and consider $\Gamma = \mathrm{SL}_2(\mathbb{Z}[\frac{1}{q}])$. It follows from Example 4 below that $\bar{H}^1_{(2)}(\Gamma) = 0$. But (see [Ser77])

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) \star_A \mathrm{SL}_2(\mathbb{Z})$$

(with $A = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \bmod q\}$ and $\bar{H}^1_{(2)}(\mathrm{SL}_2(\mathbb{Z})) \neq 0$ (see Example 5.1.1 in [Pan96]).

Example 2. Let M be a closed, hyperbolic 3-manifold fibering over S^1; the fiber is a hyperbolic surface Σ_g. Then $\Gamma = \pi_1(M)$ is a semi-direct product (hence a particular case of an HNN-extension):

$$\Gamma = \pi_1(\Sigma_g) \rtimes \mathbb{Z};$$

then $\bar{H}^1_{(2)}(\Gamma) = 0$, but $\bar{H}^1_{(2)}(\pi_1(\Sigma_g)) \neq 0$ (see Example 5.1.2 in [Pan96]).

4.2. Normal subgroups with large commutant

Theorem 4.2. *Let N be a normal, infinite, finitely generated subgroup of a finitely generated group Γ. Assume that N is non-amenable and its centralizer $Z_\Gamma(N)$ is infinite. Then $\bar{H}^1_{(p)}(\Gamma) = 0$ for $1 < p < +\infty$. If $p = 2$ this holds without the non-amenability assumption on N.*

Proof. We consider Γ as a free N-space. Let $D_p(\Gamma)$ be the space of p-Dirichlet finite functions *with respect to* N on Γ and $\mathcal{D}_p(\Gamma) = D_p(\Gamma)/D_p(\Gamma)^N$ as in the previous section. It is clear that $\mathcal{D}_p(\Gamma)$ is a Banach $Z_\Gamma(N)$-module, where $Z_\Gamma(N)$ acts by left translations.

Claim: $\mathcal{D}_p(\Gamma)^{Z_\Gamma(N)} = 0$.

Indeed, let a class $[f] \in \mathcal{D}_p(\Gamma)^{Z_\Gamma(N)}$ be represented by the function $f \in D_p(\Gamma)$; then $\lambda_\Gamma(g)f - f \in D_p(\Gamma)^N$ for every $g \in Z_\Gamma(N)$. This means that for every $n \in N$:

$$\lambda_\Gamma(n)(\lambda_\Gamma(g)f - f) = \lambda_\Gamma(g)f - f$$

or else (using $gn = ng$)

$$\lambda_\Gamma(g)(\lambda_\Gamma(n)f - f) = \lambda_\Gamma(n)f - f.$$

Since $f \in D_p(\Gamma)$, we have $\lambda_\Gamma(n)f - f \in \ell^p(\Gamma)$, hence

$$\lambda_\Gamma(n)f - f \in \ell^p(\Gamma)^{Z_\Gamma(N)}.$$

As $Z_\Gamma(N)$ is infinite, this shows that $\lambda_\Gamma(n)f - f$ vanishes identically, so that $f \in D_p(\Gamma)^N$, hence $[f] = 0$. This proves the claim.

Consider the map $\alpha : \mathcal{D}_p(\Gamma) \to Z^1(N, \ell^p(\Gamma)|_N)$ from Theorem 3.1. Let Γ act by translations on $\mathcal{D}_p(\Gamma)$, and let it act on $Z^1(N, \ell^p(\Gamma)|_N)$ by the action of formula (1) in Section 2.1. A simple computation shows that α is Γ-equivariant. In view of Corollary 3.7, α induces a $Z_\Gamma(N)$-equivariant bijection between $HD_p(\Gamma)/D_p(\Gamma)^N$ and $\overline{H}^1(N, \ell^p(\Gamma)|_N)$. We now separate two cases:

i) $N \cap Z_\Gamma(N)$ is infinite. Since we know that the action of N on $\overline{H}^1(N, \ell^p(\Gamma)|_N)$ is trivial (by equation (2)), every element of $\overline{H}^1(N, \ell^p(\Gamma)|_N)$ is $(N \cap Z_\Gamma(N))$-fixed. So every element of $HD_p(\Gamma)/D_p(\Gamma)^N$ is $(N \cap Z_\Gamma(N))$-fixed. Now by the claim (noticing that we may replace there $Z_\Gamma(N)$ by $N \cap Z_\Gamma(N)$ since the latter is infinite), the only $(N \cap Z_\Gamma(N))$-fixed point in $\mathcal{D}_p(\Gamma)$ is 0. So $HD_p(\Gamma)/D_p(\Gamma)^N = \{0\}$, hence $\overline{H}^1(N, \ell^p(\Gamma)|_N) = 0$. By Corollary 3.2, we get that $\overline{H}^1_{(p)}(N) = 0$. By Theorem 1 in [BMV05] (in which the non-amenability of N is used) we conclude that $\overline{H}^1_{(p)}(\Gamma) = 0$.

ii) $N \cap Z_\Gamma(N)$ is finite. Since N is non-amenable, we have $\overline{H}^1(N, \ell^p(\Gamma)|_N) = H^1(N, \ell^p(\Gamma)|_N)$. By the claim, there is no fixed point in $HD_p(\Gamma)/D_p(\Gamma)^N$ under $Z_\Gamma(N)/(N \cap Z_\Gamma(N))$. So we have $H^1(N, \ell^p(\Gamma)|_N)^{Z_\Gamma(N)/(N \cap Z_\Gamma(N))} = 0$. In particular $H^1(N, \ell^p(\Gamma)|_N)^{\Gamma/N} = 0$. By equation (3), we have $H^1(\Gamma, \ell^p(\Gamma)) = 0$, so $\overline{H}^1_{(p)}(\Gamma) = 0$ by Corollary 3.2.

If $p = 2$ we may assume that N is amenable. Then Γ contains an infinite, amenable, normal subgroup, so by the Cheeger–Gromov vanishing theorem [CG86], all the L^2-cohomology of Γ does vanish. □

Remarks. a) It was observed by Marc Bourdon that part ii) in the above proof can be obtained differently in case $Z_\Gamma(N)$ contains an element z of *infinite* order. Indeed let

H be the subgroup generated by $N \cup \{z\}$. Then z is central in H, so $\bar{H}^1_{(p)}(H) = 0$ by the Corollary on p. 221 in [Gro93]. The result then follows from Theorem 1 in [BMV05].

b) Among the recent vanishing results for L^2-cohomology, the most striking is probably Gaboriau's (Théorème 6.8 in [Gab02]): assume that Γ contains a normal subgroup N which is infinite, has infinite index, and is finitely generated (as a group): then $\bar{H}^1_{(2)}(\Gamma) = 0$. To prove this, Gaboriau needs a substantial part of his theory of L^2-Betti numbers for measured equivalence relations and group actions. It is very tempting to try to get a simpler proof of that result, and this is what motivated this section. More precisely, one possible line of attack for Gaboriau's result is the following: if the normal subgroup N is amenable, then all of the L^2-cohomology of Γ does vanish by the Cheeger–Gromov vanishing theorem [CG86]. So we may assume that N, hence also Γ, is non-amenable. We must then prove that $H^1(\Gamma, \ell^2(\Gamma)) = 0$; by the exact sequence (3), this is still equivalent to $H^1(N, \ell^2(\Gamma)|_N)^{\Gamma/N} = 0$.

Although a proof of Gaboriau's result along these lines remains elusive so far, this line of attack opened the possibility of replacing L^2-cohomology by L^p-cohomology, which resulted in Theorem 4.2 above.

Theorem 4.3. *Let Γ be a finitely generated group. If the centre of Γ is infinite, then $\bar{H}^1_{(p)}(\Gamma) = 0$ for $1 < p < \infty$.*

Proof. We apply the first case of the proof of Theorem 4.2, with $N = \Gamma$. It yields $\bar{H}^1_{(p)}(\Gamma) = 0$, in full generality (i.e. without appealing to non-amenabilility). □

The following example, kindly provided by M. Bourdon, shows that the previous Theorem 4.3 does *not* hold for $p = 1$.

Example 3. One has $\bar{H}^1_{(1)}(\mathbb{Z}) \neq 0$. To see this, first observe that every function $f \in D_1(\mathbb{Z})$ admits a limit at $+\infty$ and $-\infty$. Indeed, the sequence $(f(n))_{n\geq 1}$ is a Cauchy sequence since for $n > m$,

$$|f(n) - f(m)| = |\sum_{k=m}^{n-1}(f(k+1) - f(k))| \leq \sum_{k=m}^{n-1} |(f(k+1) - f(k))|$$

and the RHS goes to zero for $m, n \to +\infty$ as $f \in D_1(\mathbb{Z})$. Similarly, the sequence $(f(-n))_{n\geq 1}$ is Cauchy.

Next consider the linear form τ on $D_1(\mathbb{Z})$ defined by

$$\tau(f) = (\lim_{n\to+\infty} f(n)) - (\lim_{n\to-\infty} f(n))$$

$(f \in D_1(\mathbb{Z}))$. For χ the characteristic function of $\mathbb{N}$, we get $\chi \in D_1(\mathbb{Z})$ and

$\tau(\chi) = 1$. The form τ is continuous on $D_1(\mathbb{Z})$ because

$$|\tau(f)| = |\lim_{n\to+\infty}(f(n)-f(-n))| = |\lim_{n\to+\infty}\sum_{k=-n}^{n-1}(f(k+1)-f(k))| \leq \|f\|_{D_1(\mathbb{Z})}.$$

Since τ is a continuous non-zero linear form on $D_1(\mathbb{Z})$ which vanishes on $\mathbb{C}+\ell^1(\mathbb{Z})$, we conclude that $\overline{H}^1_{(1)}(\mathbb{Z}) \neq 0$.

4.3. Wreath products

Lemma 4.4. *Let G_1, G_2 be non-compact, locally compact groups. Let $N \lhd G_1 \times G_2$ be a closed normal subgroup such that $N \cap (G_1 \times \{1\})$ (resp. $N \cap (\{1\} \times G_2)$) is not co-compact in $G_1 \times \{1\}$ (resp. $\{1\} \times G_2$). Set $G = (G_1 \times G_2)/N$. Then*

1) $\overline{H}^1(G, L^2(G)) = 0$;
2) *if moreover G is non-amenable, then $H^1(G, L^p(G)) = 0$ for $1 < p < \infty$.*

Proof. 1) We appeal to a result of Shalom ([Sha00], Theorem 3.1): if (V, π) is a unitary $(G_1 \times G_2)$-module, and $b \in Z^1(G_1 \times G_2, V)$, then b is cohomologous in $\overline{H}^1(G_1 \times G_2, V)$ to a sum $b_0 + b_1 + b_2$, where b_0 takes values in $V^{G_1 \times G_2}$ and, for $i = 1, 2$, b_i factors through G_i and takes values in a $(G_1 \times G_2)$-invariant subspace on which π factors through G_i. This implies the following alternative: either $\overline{H}^1(G_1 \times G_2, V) = 0$, or there exists in V a non-zero vector fixed by some restriction $\pi|_{G_i}$.

We apply this to the regular representation λ_G, viewed as a representation of $G_1 \times G_2$. The assumption ensures that the restriction $\lambda_G|_{G_i}$ $(i = 1, 2)$ does not have non-zero invariant vectors. Therefore, $\overline{H}^1(G_1 \times G_2, L^2(G)) = 0$, hence also $\overline{H}^1(G, L^2(G)) = 0$.

2) We replace Shalom's result by a recent result of Bader–Furman–Gelander–Monod ([BFGM], Theorem 7.1): let (V, π) be a Banach $(G_1 \times G_2)$- module, with V uniformly convex, such that π does not almost have invariant vectors, and $H^1(G_1 \times G_2, V) \neq 0$; then for some $i \in \{1, 2\}$, there exists a non-zero $\pi(G_i)$-fixed vector.

We apply this to the regular representation λ_G on $L^p(G)$, viewed as a representation of $G_1 \times G_2$. Our assumptions ensures that the restriction $\lambda_G|_{G_i}$ $(i = 1, 2)$ does not have non-zero invariant vectors and λ_G does not almost have invariant vectors. So $H^1(G_1 \times G_2, L^p(G)) = 0$, hence also $H^1(G, L^p(G)) = 0$. □

Lemma 4.5. *Fix $n \geq 2$. Let $G_1, \ldots, G_n$ be non-compact, locally compact groups. Assume that at least one G_i is non-amenable. Set $G = G_1 \times \cdots \times G_n$. Then $H^1(G, L^p(G)) = 0$ for $1 < p < \infty$.*

Proof. Renumbering the groups if necessary, we may assume that G_1 is non-amenable. The result then follows from Lemma 4.4 by induction over n (the case $n = 2$ being Lemma 4.4, with $N = \{1\}$). □

As an application, we show the vanishing of the first L^p-cohomology for wreath products. For $p = 2$, that fact can also be deduced from Theorem 7.2 (2) in [Lue02].

Theorem 4.6. *Let H, Γ be (non-trivial) finitely generated groups. Then*

(i) $\overline{H}^1_{(2)}(H \wr \Gamma) = 0$;

(ii) *if H is non-amenable, then $\overline{H}^1_{(p)}(H \wr \Gamma) = 0$ for $1 < p < \infty$.*

Proof. Let $N = \bigoplus_\Gamma H$. Note that N is amenable exactly when H is. We separate two cases:

1) Proof of (i) when N is amenable. If N is finite, then so are H, Γ, $H \wr \Gamma$ and the result is clear. If N is infinite, then the result follows from the Cheeger–Gromov vanishing theorem [CG86].

2) Proof of (i) and (ii) when N is non-amenable. Then N can be written as the direct product of two non-amenable groups, e.g., $N = H \times (\bigoplus_{\Gamma - \{1\}} H)$. By Lemma 4.5, we have $H^1(N, \ell^p(N)) = 0$, hence also $H^1(N, \ell^p(H \wr \Gamma)|_N) = 0$ by Proposition 2.6. The result then follows from equation (3). □

4.4. Lattices in products

Theorem 4.7. *Let $G = G_1 \times \cdots \times G_n$ be a direct product of non-compact, second countable locally compact groups ($n \geq 2$). Let Γ be a finitely generated, cocompact lattice in G. Then*

(i) $\overline{H}^1_{(2)}(\Gamma) = 0$;

(ii) *if Γ is non-amenable (equivalently, if some G_i is non-amenable), then $\overline{H}^1_{(p)}(\Gamma) = 0$ for $1 < p < \infty$.*

Proof. By the version of Shapiro's lemma proved in Proposition 4.5 of [Gui80], since Γ is cocompact, there exists a topological isomorphism $H^1(\Gamma, \ell^p(\Gamma)) \simeq H^1(G, {}_p\mathrm{Ind}_\Gamma^G\, \ell^p(\Gamma))$, where ${}_p\mathrm{Ind}_\Gamma^G\, V$ denotes the induced module in the L^p-sense, i.e., ${}_p\mathrm{Ind}_\Gamma^G\, V = (L^p(G, V))^\Gamma$. But ${}_p\mathrm{Ind}_\Gamma^G\, \ell^p(\Gamma)$ is G-isomorphic to $L^p(G)$, so we get $\overline{H}^1(\Gamma, \ell^p(\Gamma)) \simeq \overline{H}^1(G, L^p(G)) = 0$ by Lemma 4.4. □

Theorem 4.8. *Fix $n \geq 2$. For $i = 1, \ldots, n$, let G_i be the group of k_i-rational points of some k_i-simple, k_i-isotropic linear algebraic group for some local field k_i. Let Γ be an irreducible lattice in $G_1 \times \cdots \times G_n$. Then $\overline{H}^1_{(p)}(\Gamma) = 0$ for $1 < p < \infty$.*

Proof. We need some terminology. A lattice Λ in a locally compact group G is *p-integrable* if either it is cocompact, or it is finitely generated and for some finite generating set $S \subset \Lambda$, there is a Borel fundamental domain $D \subset G$ such that

$$\int_D |\chi(g^{-1}h)|_S^p \, dh < \infty$$

for every $g \in G$; here $|\cdot|_S$ denotes word length, and $\chi\colon G \to \Gamma$ is defined by $\chi(\gamma g) = \gamma$ for $\gamma \in \Gamma$, $g \in D$.

We then appeal to a result of Bader–Furman–Gelander–Monod (see Section 8.2 in [BFGM], especially the few lines preceding Proposition 8.7): if Λ is a p-integrable lattice in G and V is a Banach Λ-module, then there is a topological isomorphism

$$H^1(\Lambda, V) \simeq H^1(G, {}_p\mathrm{Ind}_\Lambda^G V).$$

In our case, set $G = G_1 \times \cdots \times G_n$. Then Γ is p-integrable for every $p \geq 1$ by a result of Shalom (Section 2 in [Sha00]; this is where irreducibility of Γ is used). With $V = \ell^p(\Gamma)$, we get ${}_p\mathrm{Ind}_\Gamma^G V \simeq L^p(G)$, so, using Lemma 4.5, the result follows.[4] □

Example 4. Let q be a prime; $\Gamma = \mathrm{SL}_2(\mathbb{Z}[\frac{1}{q}])$ is an irreducible non-uniform lattice in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_q)$. Theorem 4.8 applies to give $\overline{H}^1_{(p)}(\Gamma) = 0$.

5. Application to amenable groups

Proposition 5.1. *Let Γ be a finitely generated group. If Γ has an infinite amenable normal subgroup (in particular if Γ is infinite amenable), then $\overline{H}^1(\Gamma, \ell^2(\Gamma)) = 0$.*

Proof. By the Cheeger–Gromov vanishing result [CG86], the assumptions imply that $\overline{H}^1_{(2)}(\Gamma) = 0$. So the result follows from Corollary 3.2. □

When Γ is itself amenable, the finite generation assumption can be removed:

Corollary 5.2. *Let Γ be an amenable discrete group. Then $\overline{H}^1(\Gamma, \ell^2(\Gamma)) = 0$.*

Proof. Let $(\Gamma_i)_{i \in I}$ be the directed system of finitely generated subgroups of Γ (so that $\Gamma = \bigcup_{i \in I} \Gamma_i$). By Proposition 5.1, we have $\overline{H}^1(\Gamma_i, \ell^2(\Gamma_i)) = 0$ for every $i \in I$. By Proposition 2.6, this implies that $\overline{H}^1(\Gamma_i, \ell^2(\Gamma)|_{\Gamma_i}) = 0$ for every $i \in I$. The conclusion then follows from Lemma 2.5. □

[4] To apply Section 8.2 in [BFGM], only the p-integrability condition is needed. Our vanishing result for $\overline{H}^1_{(p)}(\Gamma)$ is therefore valid in more general situations; it holds, for example, for the Kac–Moody lattices studied by B. Rémy in [Rem05].

We also get a new characterization of amenability for finitely generated, infinite groups.

Proposition 5.3. *Let Γ be an infinite, finitely generated group. The following are equivalent:*

(i) Γ *is amenable*;

(ii) $\ell^2(\Gamma)$ *is a dense, proper subspace of* $D_2(\Gamma)/\mathbb{C}$.

Proof. In view of Theorem 3.1, $\ell^2(\Gamma)$ is a dense, proper subspace of $D_2(\Gamma)/\mathbb{C}$ if and only if $H^1(\Gamma, \ell^2(\Gamma)) \neq 0$ and $\overline{H}^1(\Gamma, \ell^2(\Gamma)) = 0$. If this happens, then Γ is amenable by Corollary 3.3. Conversely, if Γ is amenable, then $H^1(\Gamma, \ell^2(\Gamma)) \neq \overline{H}^1(\Gamma, \ell^2(\Gamma))$ by the converse of Corollary 3.3, and the latter space is zero by Proposition 5.1. □

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