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The ℓ^2-cohomology of hyperplane complements

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Abstract. We compute the ℓ^2-Betti numbers of the complement of a finite collection of affine hyperplanes in $\mathbb{C}^n$. At most one of the ℓ^2-Betti numbers is nonzero.

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1. Introduction

Suppose X is a finite CW complex with universal cover $\widetilde{X}$. For each $p \geq 0$, one can associate to X a Hilbert space, $\mathcal{H}^p(\widetilde{X})$, the p-dimensional "reduced ℓ^2-cohomology," cf. [3]. Each $\mathcal{H}^p(\widetilde{X})$ is a unitary $\pi_1(X)$-module. Using the $\pi_1(X)$-action, one can attach a nonnegative real number called "von Neumann dimension" to such a Hilbert space. The "dimension" of $\mathcal{H}^p(\widetilde{X})$ is called the p^{th} *ℓ^2-Betti number of X*.

Here we are interested in the case where X is the complement of a finite number of affine hyperplanes in $\mathbb{C}^n$. (Technically, in order to be in compliance with the first paragraph, we should replace the complement by a homotopy equivalent finite CW complex. However, to keep from pointlessly complicating the notation, we shall ignore this technicality.) Let $\mathcal{A}$ be the finite collection of hyperplanes, $\Sigma(\mathcal{A})$ their union and $M(\mathcal{A}) := \mathbb{C}^n - \Sigma(\mathcal{A})$. The *rank* of $\mathcal{A}$ is the maximum codimension l of any nonempty intersection of hyperplanes in $\mathcal{A}$. It turns out that the ordinary (reduced) homology of $\Sigma(\mathcal{A})$ vanishes except in dimension $l-1$ (cf. Proposition 2.1). Let $\beta(\mathcal{A})$ denote the rank of $\bar{H}_{l-1}(\Sigma(\mathcal{A}))$. Our main result, proved as Theorem 6.2, is the following.

Theorem A. *Suppose $\mathcal{A}$ is an affine hyperplane arrangement of rank l. Only the l^{th} ℓ^2-Betti number of $M(\mathcal{A})$ can be nonzero and it is equal to $\beta(\mathcal{A})$.*

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This is reminiscent of a well-known result about the cohomology of $M(\mathcal{A})$ with coefficients in a generic flat line bundle ("generic" is defined in Section 5). This result is proved as Theorem 5.3. We state it below.

Theorem B. *Let L be a generic flat line bundle over $M(\mathcal{A})$. Then $H^*(M(\mathcal{A}); L)$ vanishes except in dimension l and* $\dim_{\mathbb{C}} H^l(M(\mathcal{A}); L) = \beta(\mathcal{A})$.

Both theorems have similar proofs. In the case of Theorem A the basic fact is that the ℓ^2-Betti numbers of S^1 vanish. (In other words, if the universal cover $\mathbb{R}$ of S^1 is given its usual cell structure, then $\mathcal{H}^*(\mathbb{R}) = 0$.) Similarly, for Theorem B, if L is a flat line bundle over S^1 corresponding to an element $\lambda \in \mathbb{C}^*$, with $\lambda \neq 1$, then $H^*(S^1; L) = 0$. By the Künneth Formula, there are similar vanishing results for any central arrangement. To prove the general results, one considers an open cover of $M(\mathcal{A})$ by "small" open sets each homeomorphic to the complement of a central arrangement. The E_1-page of the resulting Mayer–Vietoris spectral sequence is nonzero only along the bottom row, where it can be identified with the simplicial cochains with constant coefficients on a pair $(N(\mathcal{U}), N(\mathcal{U}_{\text{sing}}))$, which is homotopy equivalent to $(\mathbb{C}^n, \Sigma)$. It follows that the E_2-page can be nonzero only in position $(l, 0)$. (Actually, in the case of Theorem A, technical modifications must be made to the above argument. Instead of reduced ℓ^2-cohomology one takes local coefficients in the von Neumann algebra associated to the fundamental group and the vanishing results only hold modulo modules which do not contribute to the ℓ^2-Betti numbers.)

In [2] the first and third authors proved a similar result for the ℓ^2-cohomology of the universal cover of the Salvetti complex associated to an arbitrary Artin group (as well as a formula for the cohomology of the Salvetti complex with generic, 1-dimensional local coefficients). This can be interpreted as a computation of the ℓ^2-cohomology of universal covers of hyperplane complements associated to infinite reflection groups. Although the main argument in [2] uses an explicit description of the chain complex of the Salvetti complex, an alternative argument, similar to the one outlined above, is given in [2, Section 10].

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2. Hyperplane arrangements

A *hyperplane arrangement* $\mathcal{A}$ is a finite collection of affine hyperplanes in $\mathbb{C}^n$. A *subspace* of $\mathcal{A}$ is a nonempty intersection of hyperplanes in $\mathcal{A}$. Denote by $L(\mathcal{A})$ the poset of subspaces, partially ordered by inclusion, and let $\overline{L}(\mathcal{A}) := L(\mathcal{A}) \cup \{\mathbb{C}^n\}$. An arrangement is *central* if $L(\mathcal{A})$ has a minimum element. Given $G \in L(\mathcal{A})$, its *rank*, $\mathrm{rk}(G)$, is the codimension of G in $\mathbb{C}^n$. The minimal elements of $L(\mathcal{A})$ are a family of parallel subspaces and they all have the same rank. The *rank* of an arrangement $\mathcal{A}$ is the rank of a minimal element in $L(\mathcal{A})$. $\mathcal{A}$ is *essential* if $\mathrm{rk}(\mathcal{A}) = n$.

The *singular set* $\Sigma(\mathcal{A})$ of the arrangement is the union of hyperplanes in $\mathcal{A}$ (so that $\Sigma(\mathcal{A})$ is a subset of $\mathbb{C}^n$). The complement of $\Sigma(\mathcal{A})$ in $\mathbb{C}^n$ is denoted $M(\mathcal{A})$. When there is no ambiguity we will drop the "$\mathcal{A}$" from our notation and write L, Σ or M instead of $L(\mathcal{A})$, $\Sigma(\mathcal{A})$ or $M(\mathcal{A})$.

Proposition 2.1. *Σ is homotopy equivalent to a wedge of $(l-1)$-spheres, where $l = \mathrm{rk}(\mathcal{A})$. (So, if $\mathcal{A}$ is essential, the spheres are $(n-1)$-dimensional.)*

Proof. The proof follows from the usual "deletion-restriction" argument and induction. If the rank l is 1, then Σ is the disjoint union of a finite family of parallel hyperplanes. Hence, Σ is homotopy equivalent to a finite set of points, i.e., to a wedge of 0-spheres. Similarly, when $l = 2$, it is easy to see that Σ is homotopy equivalent to a connected graph; hence, a wedge of 1-spheres. So, assume by induction that $l > 2$. Choose a hyperplane $H \in \mathcal{A}$, let $\mathcal{A}' = \mathcal{A} - \{H\}$ and let $\mathcal{A}''$ be the restriction of $\mathcal{A}$ to H (i.e., $\mathcal{A}'' := \{H' \cap H \mid H' \in \mathcal{A}'\}$). Put $\Sigma' = \Sigma(\mathcal{A}')$, $\Sigma'' = \Sigma(\mathcal{A}'')$, $l' = \mathrm{rk}(\mathcal{A}')$ and $l'' = \mathrm{rk}(\mathcal{A}'')$. We can also assume by induction on $\mathrm{Card}(\mathcal{A})$ that Σ' and Σ'' are homotopy equivalent to wedges of spheres. If $l' < n$ and H is transverse to the minimal elements of $L(\mathcal{A}')$, then $l'' = l$, the arrangement splits as a product, $\Sigma = \Sigma'' \times \mathbb{C}$, and we are done by induction. In all other cases $l' = l$ and $l'' = l - 1$. We have $\Sigma = \Sigma' \cup H$ and $\Sigma' \cap H = \Sigma''$. H is simply connected and since $l > 2$, Σ' is simply connected and Σ'' is connected. By van Kampen's Theorem, Σ is simply connected. Consider the exact sequence of the pair (Σ, Σ'):

$$\to H_*(\Sigma') \to H_*(\Sigma) \to H_*(\Sigma, \Sigma') \to .$$

There is an excision isomorphism, $H_*(\Sigma, \Sigma') \cong H_*(H, \Sigma'')$. Since H is contractible it follows that $H_*(H, \Sigma'') \cong \bar{H}_{*-1}(\Sigma'')$. By induction, $\bar{H}_*(\Sigma')$ is concentrated in dimension $l-1$ and $\bar{H}_*(\Sigma'')$ in dimension $l-2$. So, $\bar{H}_*(\Sigma)$ is also concentrated in dimension $l-1$. It follows that Σ is homotopy equivalent to a wedge of $l-1$ spheres. □

3. Certain covers and their nerves

Suppose $\mathcal{U} = \{U_i\}_{i \in I}$ is a cover of some space X (where I is some index set). Given a subset $\sigma \subset I$, put $U_\sigma := \bigcap_{i \in \sigma} U_i$. Recall that the *nerve* of $\mathcal{U}$ is the simplicial complex $N(\mathcal{U})$, defined as follows. Its vertex set is I and a finite, nonempty subset $\sigma \subset I$ spans a simplex of $N(\mathcal{U})$ if and only if U_σ is nonempty.

We shall need to use the following well-known lemma several times in the sequel, see [4, Cor. 4G.3 and Ex. 4G(4)]

Lemma 3.1. *Let $\mathcal{U}$ be a cover of a paracompact space X and suppose that either* (a) *each U_i is open, or* (b) *X is a CW complex and each U_i is a subcomplex. Further* *suppose that for each simplex σ of $N(\mathcal{U})$, U_σ is contractible. Then X and $N(\mathcal{U})$ are homotopy equivalent.*

Suppose $\mathcal{A}$ is a hyperplane arrangement in $\mathbb{C}^n$. An open convex subset U in $\mathbb{C}^n$ is *small* (with respect to $\mathcal{A}$) if the following two conditions hold:

(i) $\{G \in \overline{L}(\mathcal{A}) \mid G \cap U \neq \emptyset\}$ has a unique minimum element $\mathrm{Min}(U)$.

(ii) A hyperplane $H \in \mathcal{A}$ has nonempty intersection with U if and only if $\mathrm{Min}(U)$ lies in H.

The intersection of two small convex open sets is also small; hence, the same is true for any finite intersection of such sets.

Now let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of $\mathbb{C}^n$ by small convex sets. Put

$$\mathcal{U}_{\mathrm{sing}} := \{U \in \mathcal{U} \mid U \cap \Sigma \neq \emptyset\}.$$

Lemma 3.2. *$N(\mathcal{U})$ is a contractible simplicial complex and $N(\mathcal{U}_{\mathrm{sing}})$ is a subcomplex homotopy equivalent to Σ. Moreover, $H_*(N(\mathcal{U}), N(\mathcal{U}_{\mathrm{sing}}))$ is concentrated in dimension l, where $l = \mathrm{rk}\,\mathcal{A}$.*

Proof. $\mathcal{U}_{\mathrm{sing}}$ is an open cover of a neighborhood of Σ which deformation retracts onto Σ. For each simplex σ of $N(\mathcal{U})$, U_σ is contractible (in fact, it is a small convex open set). By Lemma 3.1, $N(\mathcal{U})$ is homotopy equivalent to $\mathbb{C}^n$ and $N(\mathcal{U}_{\mathrm{sing}})$ is homotopy equivalent to Σ. The last assertion of the lemma follows from Proposition 2.1. □

Remark 3.3. Lemma 3.1 can also be used to show that the geometric realization of L is homotopy equivalent to Σ.

Definition 3.4. $\beta(\mathcal{A})$ is the rank of $H_l(N(\mathcal{U}), N(\mathcal{U}_{\mathrm{sing}}))$.

Equivalently, $\beta(\mathcal{A})$ is the rank of $H_l(\mathbb{C}^n, \Sigma(\mathcal{A}))$ (or of $\overline{H}_{l-1}(\Sigma(\mathcal{A}))$. Also, it is not difficult to see that $(-1)^l \beta(\mathcal{A}) = \chi(\mathbb{C}^n, \Sigma) = 1 - \chi(\Sigma) = \chi(M)$, where $\chi()$ denotes the Euler characteristic.

Remark 3.5. Suppose $\mathcal{A}_{\mathbb{R}}$ is an arrangement of real hyperplanes in $\mathbb{R}^n$ and $\Sigma_{\mathbb{R}} \subset \mathbb{R}^n$ is the singular set. Then $\mathbb{R}^n - \Sigma_{\mathbb{R}}$ is a union of open convex sets called *chambers* and $\beta(\mathcal{A}_{\mathbb{R}})$ is the number of bounded chambers. If $\mathcal{A}$ is the complexification of $\mathcal{A}_{\mathbb{R}}$, then $\Sigma(\mathcal{A}) \sim \Sigma(\mathcal{A}_{\mathbb{R}})$. Hence, $\beta(\mathcal{A}) = \beta(\mathcal{A}_{\mathbb{R}})$.

For any small open convex set U, put

$$\widehat{U} := U - \Sigma(\mathcal{A}) = U \cap M(\mathcal{A}).$$

Since U is convex, $(U, U \cap \Sigma(\mathcal{A}))$ is homeomorphic to $(\mathbb{C}^n, \Sigma(\mathcal{A}_G))$, where $G = \operatorname{Min}(U)$ and $\mathcal{A}_G$ is the central subarrangement defined by

$$\mathcal{A}_G := \{H \in \mathcal{A} \mid G \subset H\}.$$

(G might be $\mathbb{C}^n$, in which case $\mathcal{A}_G = \emptyset$.) Hence, $\widehat{U}$ is homeomorphic to $M(\mathcal{A}_G)$, the complement of a central subarrangement.

The next lemma is well known.

Lemma 3.6. *Suppose U is a small open convex set. Then $\pi_1(\widehat{U})$ is a retract of $\pi_1(M(\mathcal{A}))$.*

Proof. The composition of the two inclusions $\widehat{U} \hookrightarrow M(\mathcal{A}) \hookrightarrow M(\mathcal{A}_G)$ is a homotopy equivalence, where $G = \operatorname{Min}(U) \in L(\mathcal{A})$. $\square$

By intersecting the elements of $\mathcal{U}$ with M $(= \mathbb{C}^n - \Sigma)$ we get an induced cover $\widehat{\mathcal{U}}$ of M. An element of $\widehat{\mathcal{U}}$ is a deleted small convex open set $\widehat{U}$ for some $U \in \mathcal{U}$. Similarly, by intersecting $\mathcal{U}_{\mathrm{sing}}$ with M we get an induced cover $\widehat{\mathcal{U}}_{\mathrm{sing}}$ of a deleted neighborhood of Σ. The key observation is the following.

Observation 3.7. $N(\widehat{\mathcal{U}}) = N(\mathcal{U})$ *and* $N(\widehat{\mathcal{U}}_{\mathrm{sing}}) = N(\mathcal{U}_{\mathrm{sing}})$.

4. The Mayer–Vietoris spectral sequence

Let X be a space, $\pi = \pi_1(X)$ and $r\colon \widetilde{X} \to X$ the universal cover. Given a left π-module A, define

$$C^*(X; A) := \operatorname{Hom}_\pi(C_*(\widetilde{X}), A),$$

the cochains with *local coefficients in A*. Taking cohomology gives $H^*(X; A)$.

Let $\mathcal{U}$ be an open cover of X and $N = N(\mathcal{U})$ its nerve. Let $N^{(p)}$ denote the set of p-simplices in N. There is an induced cover $\widetilde{\mathcal{U}} := \{r^{-1}(U)\}_{U \in \mathcal{U}}$ with the same nerve. There is a Mayer–Vietoris double complex

$$C_{p,q} = \bigoplus_{\sigma \in N^{(p)}} C_q(r^{-1}(U_\sigma))$$

(cf. [1, §VII.4]) and a corresponding double cochain complex with local coefficients:

$$C^{p,q}(A) := \operatorname{Hom}_\pi(C_{p,q}; A).$$

The cohomology of the total complex is $H^*(X; A)$. Now suppose that for each simplex σ of N, U_σ is connected and that $\pi_1(U_\sigma) \to \pi_1(X)$ is injective. (This implies that $r^{-1}(U_\sigma)$ is a disjoint union of copies of the universal cover $\widetilde{U}_\sigma$.) We get a spectral sequence with E_1-page

$$E_1^{p,q} = \bigoplus_{\sigma \in N^{(p)}} H^q(U_\sigma; A). \tag{1}$$

Here $H^q(U_\sigma; A)$ means the cohomology of $\mathrm{Hom}_\pi(C_*(r^{-1}(U_\sigma)), A)$ or equivalently, of $\mathrm{Hom}_{\pi_1(U_\sigma)}(C_*(\widetilde{U}_\sigma); A)$. The E_2-page has the form $E^{p,q} = H^p(N; \mathfrak{H}^q)$, where $\mathfrak{H}^q$ means the functor $\sigma \to H^q(U_\sigma; A)$. The spectral sequence converges to $H^*(X; A)$.

In the next two sections we will apply this spectral sequence to the case where X is $M(\mathcal{A})$ and the open cover is $\widehat{\mathcal{U}}$ from the previous section. By Lemma 3.6 $\pi_1(\widehat{U}_\sigma) \to \pi_1(M(\mathcal{A}))$ is injective, so we get a spectral sequence with E_1-page given by (1). Moreover, the π-module A will be such that for any simplex σ in $N(\widehat{\mathcal{U}}_{\mathrm{sing}})$, $H^q(\widehat{U}_\sigma; A) = 0$ for all q (even for $q = 0$) while for a simplex σ of $N(\widehat{\mathcal{U}})$ which is not in $N(\widehat{\mathcal{U}}_{\mathrm{sing}})$, $H^q(U_\sigma; A) = 0$ for all $q > 0$ and is constant (i.e., independent of σ) for $q = 0$. Thus $E_1^{p,q}$ will vanish for $q > 0$ and $E_1^{*,0}$ can be identified with the cochain complex $C^*(N(\mathcal{U}), N(\mathcal{U}_{\mathrm{sing}}))$ with constant coefficients.

5. Generic coefficients

Here we will deal with 1-dimensional local coefficient systems. We begin by considering such local coefficients on S^1. Let α be a generator of the infinite cyclic group $\pi_1(S^1)$. Suppose k is a field of characteristic 0 and $\lambda \in k^*$. Let A_λ be the $k[\pi_1(S^1)]$-module which is a 1-dimensional k-vector space on which α acts by multiplication by λ.

Lemma 5.1. *If $\lambda \neq 1$, then $H^*(S^1; A_\lambda)$ vanishes identically.*

Proof. If S^1 has its usual CW structure with one 0-cell and one 1-cell, then in the chain complex for its universal cover both C_0 and C_1 are identified with the group ring $k[\pi_1(S^1)]$ and the boundary map with multiplication by $1 - \alpha$, where α is the generator of $\pi_1(S^1)$. Hence, the coboundary map $C^0(S^1; A_\lambda) \to C^1(S^1; A_\lambda)$ is multiplication by $1 - \lambda$. □

Next, consider $M(\mathcal{A})$. Its fundamental group π is generated by loops a_H for $H \in \mathcal{A}$, where the loop a_H goes once around the hyperplane H in the "positive" direction. Let α_H denote the image of a_H in $H_1(M(\mathcal{A}))$. Then $H_1(M(\mathcal{A}))$ is free abelian with basis $\{\alpha_H\}_{H\in\mathcal{A}}$. So, a homomorphism $H_1(M(\mathcal{A})) \to k^*$ is determined by an $\mathcal{A}$-tuple $\Lambda \in (k^*)^{\mathcal{A}}$, where $\Lambda = (\lambda_H)_{H\in\mathcal{A}}$ corresponds to the homomorphism sending α_H to λ_H. Let $\psi_\Lambda : \pi \to k^*$ be the composition of this homomorphism with the abelianization map $\pi \to H_1(M(\mathcal{A}))$. The resulting local coefficient system on $M(\mathcal{A})$ is denoted A_Λ. The next lemma follows from Lemma 5.1.

Lemma 5.2. *Suppose $\mathcal{A}$ is a nonempty central arrangement and Λ is such that $\prod_{H\in\mathcal{A}} \lambda_H \neq 1$. Then $H^q(M(\mathcal{A}))$ vanishes for all q.*

Proof. Without loss of generality we can suppose that the elements of $\mathcal{A}$ are linear hyperplanes. The Hopf bundle $M(\mathcal{A}) \to M(\mathcal{A})/S^1$ is trivial (cf. [6, Prop. 5.1, p. 158]); so, $M(\mathcal{A}) \cong B \times S^1$, where $B = M(\mathcal{A})/S^1$. Let $i : S^1 \to M(\mathcal{A})$ be inclusion of the fiber. The induced map on $H_1(\,)$ sends α to $\sum \alpha_H$. Thus, if we pull back A_Λ to S^1, we get A_λ, where $\lambda = \prod_{H\in\mathcal{A}} \lambda_H$. The condition on Λ is $\lambda \neq 1$, which by Lemma 5.1 implies that $H^*(S^1; A_\lambda)$ vanishes identically. By the Künneth Formula $H^*(M(\mathcal{A}); A_\Lambda)$ also vanishes identically. $\square$

Returning to the case where $\mathcal{A}$ is a general arrangement, for each simplex σ in $N(\widehat{\mathcal{U}})$, let $\mathcal{A}_\sigma := \mathcal{A}_{\mathrm{Min}(U_\sigma)}$ be the corresponding central arrangement (so that $\hat{U}_\sigma \cong M(\mathcal{A}_\sigma)$). Given $\Lambda \in (k^*)^{\mathcal{A}}$, put

$$\lambda_\sigma := \prod_{H\in\mathcal{A}_\sigma} \lambda_H .$$

Call Λ *generic* if $\lambda_\sigma \neq 1$ for all $\sigma \in N(\mathcal{U}_{\mathrm{sing}})$.

Theorem 5.3 (Compare [7, Thm. 4.6, p. 160]). *Let $\mathcal{A}$ be an affine arrangement of rank l and Λ a generic $\mathcal{A}$-tuple in k^*. Then $H^*(M(\mathcal{A}); A_\Lambda)$ is concentrated in degree l and*

$$\dim_k H^l(M(\mathcal{A}); A_\Lambda) = \beta(\mathcal{A}).$$

Proof. We have an open cover of $\widetilde{M}(\mathcal{A})$, $\{r^{-1}(\hat{U})\}_{U\in\mathcal{U}}$. By Observation 3.7, its nerve is $N(\mathcal{U})$. By Lemma 5.2 and the last paragraph of Section 4, the E_1-page of the Mayer–Vietoris spectral sequence is concentrated along the bottom row where it can be identified with $C^*(N(\mathcal{U}), N(\mathcal{U}_{\mathrm{sing}}); k)$. So, the E_2-page is concentrated on the bottom row and $E_2^{p,0} = H^p(N(\mathcal{U}), N(\mathcal{U}_{\mathrm{sing}}); k)$. By Lemma 3.2, this group is nonzero only for $p = l$ and

$$\dim_k E_2^{l,0} = \dim_k H^l(N(\mathcal{U}), N(\mathcal{U}_{\mathrm{sing}}); k) = \beta(\mathcal{A}). \qquad \square$$

Remark 5.4. When $k = \mathbb{C}$, a 1-dimensional local coefficient system on X is the same thing as a flat line bundle over X.

6. ℓ^2-cohomology

For a discrete group π, $\ell^2\pi$ denotes the Hilbert space of complex-valued, square integrable functions on π. There are unitary π-actions on $\ell^2\pi$ by either left or right multiplication; hence, $\mathbb{C}\pi$ acts either from the left or right as an algebra of operators.

The *associated von Neumann algebra* $\mathcal{N}\pi$ is the commutant of $\mathbb{C}\pi$ (acting from, say, the right on $\ell^2\pi$).

Given a finite CW complex X with fundamental group π, the space of ℓ^2-cochains on its universal cover $\widetilde{X}$ is the same as $C^*(X;\ell^2\pi)$, the cochains with local coefficients in $\ell^2\pi$. The image of the coboundary map need not be closed; hence, $H^*(X;\ell^2\pi)$ need not be a Hilbert space. To remedy this, one defines the *reduced* ℓ^2-cohomology $\mathcal{H}^*(\widetilde{X})$ to be the quotient of the space of cocycles by the closure of the space of coboundaries. We shall also use the notation $\mathcal{H}^*(X;\ell^2\pi)$ for the same space.

The von Neumann algebra admits a trace. Using this, one can attach a "dimension," $\dim_{\mathcal{N}\pi} V$, to any closed, π-stable subspace V of a finite direct sum of copies of $\ell^2\pi$ (it is the trace of orthogonal projection onto V). The nonnegative real number $\dim_{\mathcal{N}\pi}(\mathcal{H}^p(X;\ell^2\pi))$ is the p^{th} *ℓ^2-Betti number* of X.

A technical advance of Lück [5, Ch. 6] is the use of local coefficients in $\mathcal{N}\pi$ in place of the previous version of ℓ^2-cohomology. He shows there is a well-defined dimension function on $\mathcal{N}\pi$-modules, $A \to \dim_{\mathcal{N}\pi} A$, which gives the same answer for ℓ^2-Betti numbers, i.e., for each p one has that $\dim_{\mathcal{N}\pi} H^p(X;\mathcal{N}\pi) = \dim_{\mathcal{N}\pi} \mathcal{H}^p(X;\ell^2\pi)$. Let $\mathcal{T}$ be the class of $\mathcal{N}\pi$-modules of dimension 0. The dimension function is additive with respect to short exact sequences. This allows one to define ℓ^2-Betti numbers for spaces more general than finite complexes. The class $\mathcal{T}$ is a Serre class of $\mathcal{N}\pi$-modules [8], which allows one to compute ℓ^2-Betti numbers by working with spectral sequences modulo $\mathcal{T}$.

Lemma 6.1. *Suppose $\mathcal{A}$ is a nonempty central arrangement. Then, for all $q \geq 0$, $H^q(M(\mathcal{A});\mathcal{N}\pi)$ lies in $\mathcal{T}$. In other words, all ℓ^2-Betti numbers of $M(\mathcal{A})$ are zero.*

Proof. The proof is along the same line as that of Lemma 5.2. It is well known that the reduced ℓ^2-cohomology of $\mathbb{R}$ vanishes. Since $M(\mathcal{A}) = S^1 \times B$, the result follows from the Künneth Formula for ℓ^2-cohomology in [5, 6.54 (5)]. □

Theorem 6.2. *Suppose $\mathcal{A}$ is an affine hyperplane arrangement. Then*

$$H^*(M(\mathcal{A});\mathcal{N}\pi) \cong H^*(N(\mathcal{U}), N(\mathcal{U}_{\text{sing}})) \otimes \mathcal{N}\pi \pmod{\mathcal{T}}.$$

Hence, for $l = \operatorname{rk}(\mathcal{A})$, the ℓ^2-Betti numbers of $M(\mathcal{A})$ vanish except in dimension l, where $\dim_{\mathcal{N}\pi} \mathcal{H}^l(\widetilde{M}(\mathcal{A})) = \beta(\mathcal{A})$.

Proof. For each $\sigma \in N(\mathcal{U}_{\text{sing}})$, let $\pi_\sigma := \pi_1(U_\sigma)$. By Lemma 6.1,

$$\dim_{\mathcal{N}\pi_\sigma} H^*(M(\mathcal{A}_\sigma);\mathcal{N}\pi_\sigma) = 0.$$

Since the $\mathcal{N}_\pi$-module $H^*(M(\mathcal{A}_\sigma),\mathcal{N}\pi)$ is induced from $H^*(M(\mathcal{A}_\sigma),\mathcal{N}\pi)$,

$$\dim_{\mathcal{N}\pi} H^*(M(\mathcal{A}_\sigma);\mathcal{N}\pi) = \dim_{\mathcal{N}\pi_\sigma} H^*(M(\mathcal{A}_\sigma);\mathcal{N}\pi_\sigma) = 0.$$

As in the proof of Theorem 5.3, it follows that the E_1-page of the spectral sequence consists of modules in $\mathcal{T}$, except that $E_1^{*,0}$ is identified with $C^*(N(\mathcal{U}), N(\mathcal{U}_{\mathrm{sing}})) \otimes \mathcal{N}(\pi)$. Similarly, the E_2-page consists of modules in $\mathcal{T}$, except that $E_2^{*,0}$ is identified with $H^*(N(\mathcal{U}), N(\mathcal{U}_{\mathrm{sing}})) \otimes \mathcal{N}\pi$. For each subsequent differential, either the source or the target is a module in $\mathcal{T}$, and hence for each i and j one has that $E_\infty^{i,j} \cong E_2^{i,j}$ (mod $\mathcal{T}$). The claim follows since the filtration of $H^*(M(\mathcal{A}); \mathcal{N}\pi)$ given by the E_∞-page of the spectral sequence is finite. $\square$

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