Groups Geom. Dyn. 1 (2007), 311–328 **Groups, Geometry, and Dynamics**

Aperiodic colorings and tilings of Coxeter groups

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Abstract. We construct a limit aperiodic coloring of hyperbolic groups. Also we construct limit strongly aperiodic strictly balanced tilings of the Davis complex for all Coxeter groups.

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Introduction

In [BDS] we constructed a quasi-isometric embedding of hyperbolic groups into a finite product of binary trees. First we implemented such construction for hyperbolic Coxeter groups [DS]. As a byproduct we obtained aperiodic tilings with finitely many tiles of the Davis complex for these groups. Our tilings are limit strongly aperiodic and the set of tiles can be taken to be aperiodic. As a result we obtain limit strongly aperiodic tilings of those hyperbolic spaces $\mathbb{H}^n$ which admit cocompact reflection groups. Thus 2-dimensional examples come from regular p-gons, $p \geq 5$ in the hyperbolic plane. In dimension 3 there exists a right-angled regular hyperbolic dodecahedron ([A]). In dimension 4 there exists a right-angled hyperbolic 120-cell ([C], [D2], [PV]). The examples exist up to dimension 8 [VS]. Of course the dimension of the hyperbolic spaces is limited by Vinberg's theorem (≤ 29) [V]. Existence of aperiodic sets of tiles for these cases also follows from results of Block and Weinberger [BW]. A new part of our results is the limit strong aperiodicity of tilings. Also the Block–Weinberger aperiodic tilings are unbalanced. In this paper we construct strongly balanced tilings which are limit strongly aperiodic. A strongly aperiodic tiling of $\mathbb{H}^2$ was recently constructed by Goodman-Strauss [G] (his tiling even has a finite strongly aperiodic set of tiles).

First we obtain our tiling of the Davis complex as a tiling by coloring with geometrically the same tile (the chamber). Then we take a geometric resolution of the tiling by coloring. If a discrete group G acts by isometries properly cocompactly on a metric

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space X, there is a universal way to construct an aperiodic tiling of X by means of an aperiodic coloring of G. We consider an orbit Gx and the Voronoi cells V_{gx} for $g \in G$ where

$$V_y = \{x \in X \mid d(y,x) \leq d(y, Gx)\}.$$

Clearly, all cells are isometric to each other and an aperiodic coloring of G defines an aperiodic tiling of X by color. In the case of Coxeter groups one can consider colorings of the walls instead of groups. This allows us to construct strictly balanced aperiodic tilings.

This paper is arranged as follows. First we consider a coloring of discrete groups (Section 1). Then we extend this to a coloring of spaces, in particular trees, on which the group acts (Section 2). Then we apply this to the case of trees of walls in a Coxeter group (Section 3). Finally we construct a strictly balanced limit aperiodic tiling (Section 5). In Section 4 we give a brief account of the topology on the space of tilings.

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1. Aperiodic coloring of groups

Definition. A coloring of a set X by the set of colors F is a map $\phi\colon X \to F$.

We consider the product topology on the set of all colorings F^X of X where F is taken with the discrete topology.

A group G acts (from the left) on F^G by $(g\phi)(x) = \phi(g^{-1}x)$.

Definition. A coloring $\phi\colon \Gamma \to F$ of a discrete group is called *aperiodic* if $\phi \neq g\phi$ for all $g \in \Gamma \setminus e$. This means that the orbit $\Gamma\phi$ of ϕ under the left action of Γ on F^Γ is full i.e., the isotropy group of ϕ is trivial.

If $\phi = g\phi$ for some $g \in \Gamma$ we call ϕ *g-periodic*. Clearly, the g-periodicity is equivalent to the g^{-1}-periodicity.

Note that every group admits an aperiodic coloring $\delta_e\colon \Gamma \to \{0,1\}$ by two elements with $\delta_e(e) = 1$ and $\delta_e(g) = 0$ for $g \neq e$. This coloring is not interesting since it fails to satisfy the following condition.

Definition. A coloring $\phi\colon \Gamma \to F$ of a discrete group Γ is called *limit aperiodic* if the action of Γ on the closure $\overline{\Gamma\phi}$ of the orbit $\Gamma\phi$ is free, i.e., every coloring $\psi \in \overline{\Gamma\phi}$ is aperiodic.

We consider the question whether every finitely generated group admits a limit aperiodic coloring by finitely many colors.

Remark. There is a weaker version of this question. We say that a coloring $\phi \in F^\Gamma$ is *weakly aperiodic* if the orbit $\Gamma\phi$ is infinite. A coloring ϕ is called *limit weakly aperiodic* if every coloring in the closure $\psi \in \overline{\Gamma\phi}$ is weakly aperiodic. We note that the weakly aperiodic version of this question has an affirmative answer. Namely V. Uspenskii proved [U] that for every discrete group Γ the topological dynamical system (F^Γ, Γ) has a compact infinite Γ-invariant set $X \subset F^\Gamma$ with a minimal action on it. We recall that an action is *minimal* if every orbit is dense. Thus no orbit in X can be finite, and hence every element $\phi \in X$ is limit weakly aperiodic.

We do not deal with weakly aperiodic colorings in this paper. We just note that weakly aperiodic colorings correspond to aperiodic tilings and aperiodic coloring correspond to strongly aperiodic tilings (see Section 4).

Proposition 1. *Let $H \subset G$ be a finite index subgroup. Then the group G admits a limit aperiodic coloring by finitely many colors if and only if H does.*

Proof. Let $\phi\colon G \to F$ be a limit aperiodic coloring of G. Let $n = [G : H]$ and let $G = \coprod_{i=1}^n Hy_i$ be the decomposition of G into right H-cosets. Let ϕy denote the result of the right y-action, that is $(\phi y)(x) = \phi(xy)$.

We define $\psi\colon H \to F^n$ by the formula $\psi(x) = ((\phi y_1)(x), \dots, (\phi y_n)(x))$. Assume that $\lim_k(h_k\psi) = \psi'$ for a sequence $h_k \in H$ and $\psi'(ax) = \psi'(x)$ for some $a \in H$ and for all $x \in H$. Taking a subsequence we may assume that the sequence $h_k\phi$ is convergent. Since $\lim_k(h_k\phi)$ is not a-periodic, there is $z \in G$ such that $(h_k\phi)(az) \neq (h_k\phi)(z)$ for infinitely many k. Let $z \in Hy_i$. Then for all sufficiently large k we have the equality $(h_k\psi)(ax) = (h_k\psi)(x)$ for $x = zy_i^{-1} \in H$. Hence $(h_k\phi y_i)(ax) = (h_k\phi y_i)(x)$. Therefore we have a contradiction:

$$\begin{aligned}(h_k\phi)(az) &= \phi(h_k^{-1}azy_i^{-1}y_i)\\ &= (h_k\phi y_i)(ax)\\ &= (h_k\phi y_i)(x)\\ &= \phi(h_k^{-1}xy_i) = (h_k\phi)(z).\end{aligned}$$

In the other direction we may assume that H is normal. If it is not, we take a smaller normal subgroup of finite index H'. By the above H' admits a limit aperiodic coloring. Let $\psi\colon H \to F$ be a limit aperiodic coloring of H and let $n = [G : H]$. We define a coloring $\phi\colon G \to F \times \{1, \dots, n\}$ by the formula $\phi(x) = (\psi(y_i^{-1}x), i)$ for $x \in Hy_i$ where $G = \coprod_{i=1}^n Hy_i$ is the decomposition of G into right H-cosets. Assume that $\phi' = \lim_k g_k\phi$ is a-periodic for some a. Since all H-cosets are colored by different colors, a must be in H. We may assume that all $g_k \in y_jH$ for some fixed j. Let $a' = y_j^{-1} a y_j$. Since ψ is limit aperiodic, there is $z \in H$ such that $(y_j^{-1} g_k \psi)(z) \neq (y_j^{-1} g_k \psi)(a'z)$. We take $x = y_j z$. Then for infinitely many k, $g_k \phi(x) = g_k \phi(ax)$. Note that $g_k \phi(x) = \phi(g_k^{-1} y_j z) = \psi(g_k^{-1} y_j z)$ by the definition of ψ and the choice $y_1 = e$. Thus $g_k \phi(x) = (y_j^{-1} g_k \psi)(z)$. On the other hand $g_k \phi(x) = g_k \phi(ax) = \phi(g_k^{-1} a y_j z) = \phi(g_k^{-1} y_j a' z) = (y_j^{-1} g_k \psi)(a'z)$. Contradiction. □

Example. The following coloring of $\mathbb{Z}$ is not limit aperiodic:

$$\phi(n) = \begin{cases} 1 & \text{if } n = \pm k^2, k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

We consider the sequence of colorings $\psi_m(x) = \phi(x + m^2)$, $\psi_m \in \mathbb{Z}\phi$, and show that the constant 0-coloring is the limit of ψ_m. We need to show that $\lim_{m\to\infty} \psi_m(x) = 0$ for every $x \in \mathbb{N}$. Since the equation $x + m^2 = \pm k^2$ has only finitely many integral solutions (m, k), the result follows.

Since every weakly aperiodic coloring of $\mathbb{Z}$ is aperiodic, the existence of a limit aperiodic coloring of $\mathbb{Z}$ follows from the existence of a non-periodic minimal set for the shift action of $\mathbb{Z}$ on $\{0, 1\}^{\mathbb{Z}}$ (see the above remark).

An explicit example of a limit aperiodic coloring of $\mathbb{Z}$ can be given by means of the Morse–Thue sequence $m \colon \mathbb{N} \to \{0, 1\}$.

Definition ([Mor], [T], Morse–Thue sequence $m(i)$). Consider the substitution rule $0 \to 01$ and $1 \to 10$. Then start from 0 to perform these substitutions:

$$0 \to 01 \to 0110 \to 01101001 \to \cdots .$$

By taking the limit we obtain a sequence of 0's and 1's called the *Morse–Thue sequence*.

Theorem 1 ([HM], [T]). *The Morse–Thue sequence contains no string of type WWW where W is any word in 0 and 1.*

We consider the coloring $\phi \colon \mathbb{Z} \to \{0, 1\}$ defined as $\phi(x) = m(|x|)$.

Proposition 2. *The coloring ϕ of $\mathbb{Z}$ is limit aperiodic.*

Proof. Assume the contrary: there is a sequence $\{n_k\}$ tending to infinity and $a \in \mathbb{N}$ such that $\psi(x + a) = \psi(x)$ where $\psi(x) = \lim_{k\to\infty} \phi(x + n_k)$ for all $x \in \mathbb{Z}$. We may assume that all $n_k > 0$. Then there is k_0 such that for all $x \in [1, 3a]$ and all $k > k_0$, we have $\psi(x) = \phi(x + n_k)$. Let $W = \psi(1)\psi(2)\dots\psi(a)$. Note that $\psi(1)\psi(2)\dots\psi(3a) = WWW$. On the other hand $\psi(x) = \phi(x + n_k) = m(x + n_k)$ for $x \in [1, 3a]$. Thus we have a "cube" WWW in the Morse–Thue sequence. Contradiction. □

Actually this coloring of $\mathbb{Z}$ has the following stronger property: given $n \in \mathbb{Z} \setminus 0$ and $m \in \mathbb{Z}$ there exists a $q \in \mathbb{Z}$ with $|q - m| \leq 3|n|$ such that $\phi(q) \neq \phi(q + n)$.

One can ask whether every finitely generated group has a finite coloring with a similar property.

Question. Let G be a finitely generated group, and let d be the word metric with respect to a finite generating set. Does there exists a finite coloring $\phi \colon G \to F$ and a constant $\lambda > 0$ such that for every element $g \in G \setminus e$ and every $h \in G$ there exists $b \in B_{\lambda d_g(h)}(h)$ with $\phi(gb) \neq \phi(b)$? Here $d_g(h) = d(gh, h)$ is the displacement of g at h and $B_r(h)$ is the distance ball of radius r with center h.

Such a coloring ϕ is not g-invariant and one can see this aperiodicity already by considering the coloring only on a distance ball $B_r(h)$, where h is an arbitrary point and the radius is a fixed constant times the displacement $d_g(h)$. A coloring with this property can be considered as a natural generalization of the Morse–Thue coloring to the group G. Such a coloring is in some sense "as aperiodic as possible" and in particular limit aperiodic.

2. Aperiodic coloring of hyperbolic groups

It is very plausible that every finitely generated group has limit aperiodic colorings by finitely many colors. On the other hand a random coloring is not limit aperiodic. In this section we construct such colorings for torsion-free finitely generated hyperbolic groups.

In the sequel G is a finitely generated group, and $\|\cdot\|$ the norm with respect to a finite set of generators.

Lemma 1. *Let $a \in G$ be an element of infinite order in a hyperbolic group. Then there is $n = n(a)$ such that for every $g \in G$ there is $k \leq n$ with $\|ga^k\| \neq \|g\|$.*

Proof. It is known that the sequences $\{a^k\}$ and $\{a^{-k}\}$ define two different points on the Gromov boundary $\partial_\infty G$ of G. Let $\xi \colon \mathbb{R} \to K$ be a geodesic in the Cayley graph K joining these two points. Note that the action of a on K leaves these points at infinity invariant. Let $d = d(e, \operatorname{im} \xi)$, then $d(a^k, \operatorname{im} \xi) = d(e, a^{-k} \operatorname{im} \xi) \leq d + \delta$ for every k where G is δ-hyperbolic. The last inequality follows from the fact that a degenerated triangle in K defined by the geodesics $\operatorname{im} \xi$ and $a^{-k}(\operatorname{im} \xi)$ is δ-thin. Take n such that $\|a^n\| > 2\|a\| + 10d + 14\delta$. Assume that there is g such that $\|g\| = \|ga\| = \|ga^2\| = \cdots = \|ga^n\|$. Consider the geodesic $g(\operatorname{im} \xi)$. Let $w \in g(\operatorname{im} \xi)$ and $w' \in g(\operatorname{im} \xi)$ be the closest points in $g(\operatorname{im} \xi)$ to g and ga^n respectively.

Since the triangle $\langle e, w, w'\rangle$ is δ-thin, the geodesic segment $[w, w']$ contains a point z such that $d(z, y) < \delta$ and $d(z, y') < \delta$ where $y \in [e, w]$ and $y' \in [e, w']$. Thus

$$2\|z\| - \|y\| - \|y'\| \le 2\delta. \tag{1}$$

Denote by z_k, $k = 0, \dots, n$, a point in $g(\operatorname{im}\xi)$ such that $d(z_k, ga^k) \le d + \delta$. There is i such that $z \in [z_i, z_{i+1}] \subset g(\operatorname{im}\xi)$. Then $d(z, ga^i) \le 3d + 3\delta + \|a\|$. Thus

$$|g\| = \|ga^i\| \le \|z\| + 3d + 3\delta + \|a\|. \tag{2}$$

In view of (1) and (2) and the facts $|\|w\| - \|g\||, |\|w'\| - \|g\|| \le d + \delta$ we obtain a contradiction:

$$\begin{aligned}\|a^n\| = d(g, ga^n) &\le d(w, w') + 2d + 2\delta = d(w, z) + d(z, w') + 2d + 2\delta \\ &\le d(w, y) + d(y', w') + 2d + 4\delta = \|w\| - \|y\| + \|w'\| - \|y'\| + 2d + 4\delta \\ &\le 2\|g\| - \|y\| - \|y'\| + 4d + 6\delta \\ &\le 2\|z\| - \|y\| - \|y'\| + 2\|a\| + 10d + 12\delta \\ &\le 2\|a\| + 10d + 14\delta.\end{aligned}$$

□

Example. The group $\mathbb{Z}^2$ does not have the above property with respect to the generators $(0, \pm 1)$ and $(\pm 1, 0)$. Take $a = (1, -1)$ and $g_n = (n, n)$. Then $\|(n, n) - k(1, -1)\| = 2n$ for all $k \le n$.

A geodesic segment in a finitely generated group is the corresponding sequence of vertices in a geodesic segment in the Cayley graph. A geodesic segment $[x_1, \dots, x_k]$ is called *radial* if $\|x_1\| < \|x_2\| < \dots < \|x_k\|$.

To construct limit aperiodic colorings we consider a square free sequence $\nu\colon \mathbb{N} \to \{0, 1, 2\}$, i.e., a sequence which does not contain any subsequence of the form WW, where W is a word in $0, 1, 2$. It is possible to construct a square free sequence in the following way. Take the Thue–Morse sequence 0110100110010110… and look at the sequence of words of length 2 that appear: 01, 11, 10, 01, 10, 00, 01, 11, 10…. Replace 01 by 0, 10 by 1, 00 by 2 and 11 by 2 to get the following: 021012021…. Then this sequence is square-free [HM].

Theorem 2. *Every finitely generated torsion-free hyperbolic group admits a limit aperiodic coloring by* 9 *colors.*

Proof. The set of colors will be the set of pairs (m, n) where $m, n \in \{0, 1, 2\}$. Let G be a group taken with the word metric with respect to a finite generating set S. We define $\phi(g) = (\nu(\|g\|), \|g\| \bmod 3)$ for every $g \in \Gamma$.

Claim: *Let* $[x_1, x_2, \dots, x_k]$ *be a radial geodesic segment. Let* $g \in G$ *be such that* $\phi(x_i) = \phi(gx_i)$ *for all* $i \in \{1, \dots k\}$. *Assume in addition that* $d(gx_{i_0}, x_{i_0}) < k/2$ *for some* i_0, $1 \le i_0 \le k$. *Then* $\|gx_i\| = \|x_i\|$ *for all* i.

We first show that $[gx_1, \dots, gx_k]$ is also a radial geodesic segment. Since multiplication from the left is an isometry, $[gx_1, \dots, gx_k]$ is a geodesic segment and in particular $-1 \leq \|gx_{i+1}\| - \|gx_i\| \leq 1$. Since g preserves the coloring we have $\|gx_{i+1}\| \equiv \|gx_i\| + 1 \bmod 3$. These two relations imply $\|gx_{i+1}\| = \|gx_i\| + 1$, hence $[gx_1, \dots, gx_k]$ is a radial geodesic segment and thus $\|gx_i\| = \|x_i\| + q$ for some fixed integer q. By our assumption $|q| < k/2$. Assume $0 < q$. Since g preserves the colors we obtain the equality

$$(\nu(\|x_1\|), \dots, \nu(\|x_q\|)) = (\nu(\|x_{q+1}\|), \dots, \nu(\|x_{2q}\|)),$$

a contradiction to the square freeness of ν. In a similar way we obtain a contradiction if $q < 0$. This implies $q = 0$ and hence the claim.

Now assume that there is a sequence $g_l \in G$ with $\|g_l\| \to \infty$ such that the limit $\lim_{l\to\infty} \phi(g_l x) = \psi(x)$ exists for every $x \in G$.

Assume that there is $b \in G \setminus \{e\}$ such that $\psi(x) = \psi(bx)$ for all x.

Let n be taken from Lemma 1 for $a = b$. We may assume that there is l_0 such that for $l > l_0$, $\psi(y) = \phi(g_l y)$ for all $y \in B_{4n\|b\|}(e)$. Consider a radial geodesic segment $x_1, \dots, x_k$ of length $k - 1$ with $k = 3n\|b\|$ and with endpoint $x_k = g_l$. Such a segment clearly exists for all l large enough.

Let s be the smallest natural number such that $\|g_l b^s\| \neq \|g_l\|$, thus $s \leq n$. Both segments $g_l^{-1}([x_1, \dots x_k])$ and $g_l^{-1}(g_l b^s g_l^{-1}[x_1, \dots, x_k])$ lie in $B_{4n\|b\|}(e)$. Thus $\phi(x) = \phi(g_l g_l^{-1}(x)) = \psi(g_l^{-1}(x)) = \psi(b^s g_l^{-1}(x)) = \phi(g_l b^s g_l^{-1} x)$ for all $x \in [x_1, \dots, x_k]$. Furthermore we compute $d(g_l b^s g_l^{-1} x_k, x_k) = d(g_l b^s, g_l) = \|b^s\| \leq n\|b\| < k/2$. We apply the claim to $g = g_l b^s g_l^{-1}$ to obtain the contradiction: $\|g_l b^s\| = \|g_l b^s g_l^{-1} x_k\| = \|x_k\| = \|g_l\|$. □

Remark. It is still an open problem whether every hyperbolic group contains a torsion-free subgroup of finite index.

3. Aperiodic coloring of G-spaces

We note that in the definitions from the beginning of Section 1 one can replace a group G by a space X with a G-action. Thus G acts on the space of colorings F^X also via $(g\phi)(x) = \phi(g^{-1}x)$. Let K be the kernel of the action. We call a coloring ϕ of X *G-aperiodic* if $\phi \neq g\phi$ for all $g \in G \setminus K$. Similarly one can define *limit G-aperiodic* colorings of X as those colorings whose orbit $G\phi$ has only G-aperiodic colorings in its closure $\overline{G\phi}$ with respect to the product topology F^X.

The following is an analog of Proposition 1.

Proposition 3. *Let $H \subset G$ be a finite index subgroup and let G act on X. Then X admits a limit G-aperiodic coloring by finitely many colors if and only if it admits a limit H-aperiodic coloring by finitely many colors.*

It is possible to extend Theorem 2 to the case of an isometric action on a hyperbolic space X. We consider only the case when X is a simplicial tree. Thus every edge in X has length equal to 1.

If x_0 is a root of X, we denote $\|x\| = d(x, x_0)$ for $x \in X$. We prove the following analog of Lemma 1.

Lemma 2. *Let G act on a simplicial rooted tree X and let $a \in G$ operate without fixed points. Then for every $g \in G$ there is $k \leq 2$ with $\|ga^k x_0\| \neq \|gx_0\|$.*

Proof. Assume that $\|ga^2x_0\| = \|gax_0\| = \|gx_0\|$. Because $d(gx_0, gax_0) = d(hgx_0, hgax_0) = d(gax_0, ga^2x_0)$ for $h = gag^{-1}$, the points $z = gx_0$, $h(z) = gax_0$, and $h^2(z) = ga^2x_0$ have a common predecessor y in the tree which is the common midpoint of the geodesic segments $[z, h(z)]$ and $[h(z), h^2(z)]$. Then $hy = y$ and hence $g^{-1}y$ is a fixed point for a: $a(g^{-1}y) = g^{-1}y$. This contradicts the assumption. □

Let $x_0 \in X$ be a base point in a tree X. We consider the coloring of the set of vertices of X defined as in the proof of Theorem 2: $\phi(x) = (\nu(\|x\|), \|x\| \bmod 3)$.

Proposition 4. *Suppose that a group G acts by isometries on a simplicial tree X with the above coloring ϕ and let $\psi \in \overline{G\phi}$. Then $b\psi \neq \psi$ for every $b \in G$ with unbounded orbit $\{b^k x_0 \mid k \in \mathbb{N}\}$. Moreover, $b\psi \neq \psi$ on the orbit Gx_0.*

Proof. First we note that a similar claim as in the proof of Theorem 2 holds:

Let $[z_1, \dots, z_k]$ be a radial geodesic segment in X. Let $g \in G$ such that $\phi(gz_i) = \phi(z_i)$ for all i and $d(gz_{i_0}, z_{i_0}) < k/2$ for some i_0, $1 \leq i_0 \leq k$. Then $\|gz_i\| = \|z_i\|$.

The proof is exactly the same as in Theorem 2.

Assume that there is a sequence $g_l \in G$ with $\|g_l\| \to \infty$ with the limit $g_l^{-1}\phi$ equal to a b-periodic coloring ψ such that $\{b^k x_0\}$ is infinite. That is the limit $\lim_{k\to\infty} \phi(g_k x) = \psi(x)$ exists for every $x \in X$.

Since the orbit $b^m x_0$ is infinite we can apply Lemma 2 for $a = b$. Let $[y_1, \dots, y_{8k}]$ be a radial segment with $y_1 = x_0$ where $k = 6\|b(x_0)\|$. By the definition of the pointwise limit we may assume that there is l_0 such that for $l > l_0$, $\psi(y) = \phi(g_l y)$ for all $y \in [y_1, \dots, y_{8k}] \cup b[y_1, \dots, y_{8k}] \cup b^2[y_1, \dots, y_{8k}]$. Then the image $g_l[y_1, \dots, y_{8k}]$ contains either a radial segment $[z_1, \dots, z_k]$ of length $k - 1$ with $z_k = g_l(x_0)$ (first case) or it contains a radial segment $[z_1, \dots, z_{6k}]$ with $d(z_1, g_l(x_0)) < k$ (second case).

Let s be the smallest natural number such that $\|g_l b^s x_0\| \neq \|g_l x_0\|$. Thus $s \leq 2$.

Then $\phi(x) = \phi(g_l g_l^{-1}(x)) = \psi(g_l^{-1}(x)) = \psi(b^s g_l^{-1}(x)) = \phi(g_l b^s g_l^{-1} x)$ for all $x \in [z_1, \dots, z_k]$. Furthermore, in the first case we compute $d(g_l b^s g_l^{-1} z_k, z_k) = d(b^s x_0, x_0) = \|b^s x_0\| \leq 2\|bx_0\| < k/2$. We apply the claim to $g = g_l b^s g_l^{-1}$ with $i_0 = k$ to obtain the contradiction: $\|g_l b^s x_0\| = \|g_l b^s g_l^{-1} z_k\| = \|z_k\| = \|g_l x_0\|$.

In the second case

$$\begin{aligned} d(z_1, g_l b^s g_l^{-1} z_1) &\le d(z_1, g_l(x_0)) + d(g_l(x_0), g_l b^s(x_0)) \\ &\qquad + d(g_l b^s(x_0), g_l b^s g_l^{-1} z_1) \\ &\le 2k + 2\|b(x_0)\| \le 3k. \end{aligned}$$

We apply the claim with $g = g_l b^s g_l^{-1}$, $i_0 = 1$, and $6k$ instead of k. Let i be such that $g_l y_i = z_1$. Then $\|z_i\| = \|g_l x_0\|$. Since $\phi(\|g_l b^s y_{i-1}\|) = \phi(\|g_l y_{i-1}\|) \neq \phi(\|g_l y_i\| - 1)$, we obtain that $\|g_l b^s y_{i-1}\| \neq \|g_l y_i\| - 1 = \|z_1\| - 1$. Hence $\|g_l b^s y_{i-1}\| = \|z_1\| + 1$ and $\|g_l b^s x_0\| = \|z_1\| + i - 1$. Then we obtain a contradiction: $\|g_l b^s x_0\| = \|z_1\| + i - 1 = \|g_l b^s g_l^{-1} z_1\| + i - 1 = \|g_l b^s g_l^{-1} z_i\| = \|z_i\| = \|g_l x_0\|$. □

As a consequence we obtain the following.

Corollary 1. *Suppose that a group G acts on a rooted simplicial tree X such that Gx_0 is a full orbit. Then there is a limit G-aperiodic coloring of X by 9 colors. Moreover, the restriction of this coloring to the orbit Gx_0 is also limit G-aperiodic.*

Theorem 3. *Suppose that a group G acts by isometries on simplicial trees $X_1, \dots, X_n$ in such a way that the induced action on the product $\prod X_i$ is free. Then G admits a limit aperiodic coloring by $9n$ colors.*

Proof. First we note that the fixed point theorem for CAT(0) spaces implies that G must be torsion-free.

Let K_i be the kernel of the representation $G \to \mathrm{Aut}(X_i)$ and let $G_i = G/K_i$. We fix a base point x_0^i in each tree and consider colorings $\phi^i : X_i \to F_i$, $|F_i| = 9$ from Proposition 4. This defines a map $\phi\colon \prod X_i \to \prod F_i$. Let $\phi'\colon G \to \prod F_i$ be the restriction to the orbit: $\phi'(g) = \phi(gx_0)$ where $x_0 = (x_0^1, \dots, x_0^n)$.

Assume that $\psi = \lim g_k^{-1}\phi'$ is a-periodic: $a\psi(x) = \psi(x)$, $a \in G \setminus \{e\}$. Then $\psi = (\psi_1, \dots, \psi_n)$ where $\psi_i = \lim g_k^{-1}\phi_i'$ and $\phi_i' = \phi_i|_{Gx_0^i}$. Since a operates without fixed points on X, there exists i such that a has no fixed points on X_i. Then $a\psi \neq \psi$ by Proposition 4, which contradicts our assumption. □

We recall the definition of Coxeter groups. A *Coxeter matrix* $M = (m_{ij})$ is a symmetric square matrix with 1's on the diagonal and all other entries from $\mathbb{N}_+ = \{0\} \cup \mathbb{N}$. A *Coxeter group* Γ with a generating set S is a group with a presentation

$$\langle S \mid (uv)^{m_{ij}} = 1,\ m_{ij} \in M \rangle$$

where M is a Coxeter matrix. Here we use the convention $a^0 = 1$. Traditionally the literature on Coxeter groups uses ∞ instead of 0.

A Coxeter group is called *right-angled* if all m_{ij} with $i \neq j$ are 0 or 2.

Theorem 4. *Every Coxeter group* Γ *admits a limit aperiodic coloring by finitely many colors.*

Proof. Since every Coxeter group contains a finite index torsion-free subgroup, in view of Proposition 1 it suffices to prove it for a finite index subgroup. We apply Januszkiewicz's construction of equivariant isometric embedding of a torsion-free finite index normal subgroup $\Gamma' \subset \Gamma$ into the finite product of trees [DJ] and then apply Theorem 3. □

Let K be the Cayley graph of a Coxeter group (Γ, S). For every generator $c \in S$, every conjugate $w = gcg^{-1}$, $g \in \Gamma$, acts on K by reflection. Let M_w be the set of fixed points of w. We call it the *wall* (*or mirror*) of the reflection w. Clearly, uM_w is a wall for all u and w. Therefore Γ acts on the set of all walls $\mathcal{W}$. According to [DJ] all walls can be partitioned in finitely many classes $\mathcal{W} = \mathcal{W}_1 \cup \dots \cup \mathcal{W}_m$ such that the walls from each class $\mathcal{W}_i$ form a vertex set of a simplicial tree T_i. Moreover, all sets $\mathcal{W}_i$ are invariant under a normal finite index subgroup Γ' which acts by isometries on each T_i in such a way that the Γ'-action on the product $\prod_{i=1}^m T_i$ is free.

Theorem 5. *The set* $\mathcal{W}$ *of all walls in a Coxeter group admits a limit* Γ*-aperiodic coloring by finitely many colors.*

Proof. On every tree T_i we consider a coloring from Proposition 4 with 9 each time different colors. Thus we use $9m$ colors. This defines a coloring $\phi = \bigcup \phi_i$ of $\mathcal{W}$. Let ψ be a limit coloring of $\mathcal{W}$. Then, clearly, $\psi = \bigcup \psi_i$ where each ψ_i is a limit coloring for ϕ_i. Let $b \in \Gamma'$. Since the action of Γ' on the product $\prod_{i=1}^m T_i$ is free and b is of infinite order, we obtain that b has an infinite orbit on some T_i. By Proposition 4 $b\psi_i \neq \psi_i$ and hence $b\psi \neq \psi$. Proposition 3 completes the proof. □

4. Space of tilings

We recall a definition of tiling of a metric space from [BW]. Let X be a metric space. A set of tiles $(\mathcal{T}, \mathcal{F})$ is a finite collection of n-dimensional complexes $t \in \mathcal{T}$ and a collection of subcomplexes $f \in \mathcal{F}$ of dimension $< n$, together with an opposition function $o\colon \mathcal{F} \to \mathcal{F}$, $o^2 = \mathrm{id}$. A space X is tiled by the set $(\mathcal{T}, \mathcal{F})$ if

(1) $X = \bigcup_\lambda t_\lambda$, where each t_λ is isometric to one of the tiles in $\mathcal{F}$;

(2) $t_\lambda \setminus \bigcup_{f \in t_\lambda} = \mathrm{Int}(t_\lambda)$ in X for every λ;

(3) If $\mathrm{Int}(t_\lambda \cup t_{\lambda'}) \neq \mathrm{Int}(t_\lambda) \cup \mathrm{Int}(t_{\lambda'})$, then t_λ and $t_{\lambda'}$ intersect along $f \in t_\lambda$ and $o(f) \in t_{\lambda'}$;

(4) there are no free faces of t_λ.

Strictly speaking a tiling of X is a collection $\alpha = \{\phi_\lambda\}$ of isometries $\phi_\lambda : t_\lambda \to t$, $t \in \mathcal{T}$, satisfying the above axioms. For every tiling α there is a minimal (or reduced) set of tiles $(\mathcal{T}_\alpha, \mathcal{F}_\alpha) \subset (\mathcal{T}, \mathcal{F})$.

Let X be a metric space with a base point x_0. Assume that $\operatorname{diam} t \leq 1$ for all $t \in \mathcal{T}$. Let α be a tiling $X = \bigcup_\lambda t_\lambda$ of X. We denote by α_n the set $\{t_\lambda \mid t_\lambda \subset B_n(x_0)\}$ where $B_r(x)$ stands for the closed ball of radius r centered at x. For a metric space Y we denote by $\exp Y$ the space of compact subsets of Y taken with the Hausdorff metric. Also for $m \in \mathbb{N}$ denote by $\exp_m Y$ the m-th hyperpower of Y, i.e., the subspace of $\exp Y$ that consists of subsets of cardinality $\leq m$. Note that α_n defines a point in $\exp(\exp B_n(x_0))$. Actually, there is $k = k(n)$ such that α_n lies in $\exp_k(\exp B_n(x_0))$. Clearly, the sequence α_n completely defines the tiling α.

The space of tilings was defined by many authors (see for example [BBG], [S], [SW]). Here we give an alternative definition. Let $\operatorname{tl}(X, \mathcal{T})$ denote the set of all $\mathcal{T}$-tilings of X. We introduce the topology on $\operatorname{tl}(X, \mathcal{T})$ as a subspace topology:

$$\operatorname{tl}(X, \mathcal{T}) \subset \prod_{n=1}^{\infty} \exp(\exp B_n(x_0)).$$

Let $\mathcal{F}$ be a finite family of compact subsets in a metric space Y. We denote by $\exp^{\mathcal{F}} Y$ the subspace of $\exp Y$ whose points are isometric copies of elements of $\mathcal{F}$.

Note that $\operatorname{tl}(X, \mathcal{T}) \subset \prod_{n=1}^{\infty} \exp_{k(n)} \exp^{\mathcal{T}}(B_n)$.

The following proposition is well known [S].

Proposition 5. *The space* $\operatorname{tl}(X, \mathcal{T})$ *is compact.*

Proof. Since $\exp^{\mathcal{T}}(B_n)$ is compact, it suffices to show that $\operatorname{tl}(X, \mathcal{T})$ is a closed subset in $\prod_{n=1}^{\infty} \exp_{k(n)} \exp^{\mathcal{T}}(B_n)$. For that it suffices to show that the set $\{\alpha_n \mid \alpha \in \operatorname{tl}(X, \mathcal{T})\}$ is closed in $\exp_{k(n)} \exp^{\mathcal{T}}(B_n)$ for every n. We leave this to the reader. □

Let $G \subset \operatorname{Iso}(X)$ be a subgroup of the group of isometries of X. Clearly G acts on $\operatorname{tl}(X, \mathcal{T})$. We say that a tiling $\alpha \in \operatorname{tl}(X, \mathcal{T})$ is *strongly G-aperiodic* if $g\alpha \neq \alpha$ for all $g \in G \setminus \{e\}$. A tiling α is called *aperiodic* if the group $\operatorname{Iso}(\alpha) \subset \operatorname{Iso}(X)$ of isometries of α does not act cocompactly on X. A tiling α is *limit strongly G-aperiodic* if every tiling $\beta \in \overline{G\alpha}$ is strongly aperiodic. If $G = \operatorname{Iso}(X)$ we use the terms *strongly aperiodic* and *limit strongly aperiodic*.

5. Aperiodic tiling of the Davis complex

Here we recall the definition of the Davis complex [D1]. Let Γ be a Coxeter group with generating set S. The nerve $N = N(\Gamma, S)$ is the simplicial complex defined in the following way. The vertices of N are elements of S. Different vertices $s_1, \dots, s_k$

span a simplex σ if and only if the set $\{s_1, \dots, s_k\}$ generates a finite subgroup Γ_σ of Γ. By N' we denote the barycentric subdivision of N. The cone $C = \operatorname{Cone} N'$ over N' is called a *chamber* for Γ. The Davis complex $X = X(\Gamma, S)$ is the image of a simplicial map $q\colon \Gamma \times C \to X$ defined by the following equivalence relation on the vertices: $a \times v_\sigma \sim b \times v_\sigma$ provided that $a^{-1}b \in \Gamma_\sigma$ where σ is a simplex in N and v_σ is the barycenter of σ. We identify C with the image $q(e \times C)$. The group Γ acts simplicially on X with the orbit space equivalent to the chamber. Thus the Davis complex is obtained by gluing the chambers γC, $\gamma \in \Gamma$ along their boundaries. Note that X admits an equivariant cell structure with the vertices $X^{(0)}$ equal to the cone points of the chambers and with the 1-skeleton $X^{(1)}$ isomorphic to the Cayley graph of Γ. A conjugate $r = wsw^{-1}$ of every generator $s \in S$ is a reflection. The fixed point set M_r of a reflection r is called the wall of r. Note that walls defined in Section 3 are obtained from the walls in Davis' complex by the restriction to the Cayley graph.

Proposition 6. *Every finite coloring $\phi\colon \mathcal{W} \to F$ of the set of walls of the Davis complex X defines a tiling $\bar\phi$ of X with $o(f) = f$.*

Proof. The set of tiles $\mathcal{T}$ of $\bar\phi$ is the set of chambers with all possible colorings of their faces. The set of faces $\mathcal{F}$ is the set of all possible colored faces of the chambers. Set $o(f) = f$. Then all conditions hold. □

We call the tiling $\bar\phi$ a *tiling by the coloring* ϕ.

Let $(\mathcal{T}, \mathcal{F})$ be a set of tiles. A function $w\colon \mathcal{F} \to \mathbb{Z}$ is called *a weight function* if $w(o(f)) = -w(f)$ for every $f \in \mathcal{F}$. We recall a definition from [BW].

Definition. A finite set of tiles $(\mathcal{T}, \mathcal{F})$ is unbalanced if there is a weight function w such that $\sum_{f\in t} w(f) > 0$ for all $t \in \mathcal{T}$.

It is called semibalanced if $\sum_{f\in t} w(f) \geq 0$ for all $t \in \mathcal{T}$.

We call a set of tiles *strictly balanced* if for every nontrivial weight function w there are tiles t_+ and t_- such that $\sum_{f\in t_+} w(f) > 0$ and $\sum_{f\in t_-} w(f) < 0$.

A tiling is called *strictly balanced* (*unbalanced*) if its minimal set of tiles is strictly balanced (unbalanced).

We now associate to every wall in the Davis complex an orientation. A wall divides the Davis complex into two components. Roughly speaking the orientation says which of the components is left and which is right. Let $\bar\phi$ be a tiling of the Davis complex X by the coloring ϕ of the walls with the set of tiles $(\mathcal{T}, \mathcal{F})$. The orientations of the walls define a new tiling ϕ' of X with the set of tiles $(\mathcal{T}', \mathcal{F}')$, where $\mathcal{F}' = \mathcal{F}_+ \cup \mathcal{F}_-$ and $\mathcal{F}_+$ and $\mathcal{F}_-$ are copies of $\mathcal{F}$. The face $f \in t_\lambda$ has sign $+$ if $\operatorname{Int}(t_\lambda)$ is left of the wall and has sign $-$ if $\operatorname{Int}(t_\lambda)$ is right of the wall. The opposition function $o\colon \mathcal{F}' \to \mathcal{F}'$ maps f_+ to f_-. We call such tiling a *geometric resolution* of a tiling by coloring. This new tiling is not a tiling by coloring anymore. A geometric meaning of this resolution is that we deform all faces of a given color and a given sign in the same direction by the same pattern. For the faces of the same color but of opposite sign we take the opposite deformation.

The following is obvious.

Lemma 3. *Assume that a coloring $\phi\colon \mathcal{W} \to F$ is limit aperiodic. Then the tiling by coloring $\overline{\phi}$ as well as any of its geometric resolutions is limit strongly aperiodic.*

Note that in the Davis complex every wall has a canonical orientation, by deciding that the base chamber C is in the left component. Thus we can indicate the chosen orientation itself by a sign. A wall gets the sign $+$ if the orientation of the wall is the canonical one and the sign $-$ otherwise.

In [BW] unbalanced tilings of some nonamenable spaces are constructed. In particular all hyperbolic Coxeter groups admit such tilings. We can derive this fact using geometric resolutions.

Proposition 7. *Every coloring of the walls for a hyperbolic Coxeter group admits an unbalanced geometric resolution.*

Proof. We assign $+$ to every wall. The hyperbolicity implies that for every chamber C' the numbers of faces of C' whose walls separate C' from the base chamber C is strictly less than the number of faces whose walls do not separate C' and C. Then for every chamber C' the faces whose walls do not separate C' from C obtain the sign $+$, all other faces obtain the sign $-$. We define a weight function by sending a positive face to $+1$ and a negative face to -1. □

We note that every unbalanced tiling is aperiodic. This fact can be derived formally from Proposition 4.1 [BW]. Since the proof there has some omissions we present a proof below.

Proposition 8. *Let $(\mathcal{T}, \mathcal{F})$ be the set of tiles of a geometric realization of a tiling by coloring of the Davis complex X of a Coxeter group Γ. Suppose that the set of tiles $(\mathcal{T}, \mathcal{F})$ is unbalanced. Then any $(\mathcal{T}, \mathcal{F})$-tiling α is aperiodic.*

Proof. Let G be a group of isometries of α. Then $G \subset \Gamma$. Hence G is a matrix group. By Selberg's lemma it contains a torsion-free subgroup G' of finite index. Then the orbit space X/G' is compact and admits a $(\mathcal{T}, \mathcal{F})$-tiling. (Note that by taking X/G as in [BW] we cannot always obtain a tiling because of free faces.) Then we obtain a contradiction:

$$0 < \sum_{t \in X/G'} \sum_{f \in t} w(f) = \sum_{f \in X/G'} (w(f) + w(o(f))) = 0. \qquad \square$$

Theorem 6. *For every Coxeter group Γ and for every coloring $\phi\colon \mathcal{W} \to F$ with the property that walls of the same color do not intersect, there is a strictly balanced geometric resolution. Additionally, every limit tiling of this resolution is strictly balanced.*

Proof. First we construct a strictly balanced geometric resolution of ϕ. Consider the set of walls $\mathcal{W}_c = \phi^{-1}(c)$ of the same color $c \in F$. Since walls of the same color do not intersect, they are ordered by level from the base chamber. (The level 'lev' is defined by induction. If one removes the walls $\mathcal{W}_c$ from X, the space is divided into components. Walls from $\mathcal{W}_c$ that bound the component of the base chamber are of level one. Then drop the walls of level one and repeat the procedure to get new walls of level one and call them to be of level two and so on.) We give the walls $\mathcal{W}_c$ signs in an alternating fashion by the level $(-1)^{\mathrm{lev}(M)}$: $-+-+-+-+-\cdots$.

We show that this geometric resolution is strictly balanced. Let $w\colon F_+ \cup F_- \to \mathbb{Z}$ be a nontrivial weight function with $w(f_+) = -w(f_-)$. We show that there are chambers C_+ and C_- such that

$$\sum_{f\in C_+} w(f) > 0 \quad \text{and} \quad \sum_{f\in C_-} w(f) < 0.$$

Because of the symmetry it suffices to show the first inequality. Since w is nontrivial, there exists a face f^0 which is the common face of two adjacent tiles t_λ and $t_{\lambda'}$ such that $w(f^0) \neq 0$. Let M_0 denote the wall that contains f^0. Now there are four cases corresponding to the parity of the sign of $w(f^0_+)$ and the sign of M_0. We discuss only one, and to be fair not the easiest of the cases: $w(f^0_+) > 0$ and the orientation of M_0 is negative.

We assume that f^0 in t_λ has the sign $-$ and in $t_{\lambda'}$ has the sign $+$. We take a number k larger than the number of walls separating $t_{\lambda'}$ and the base chamber C. Let $c \in F$ be a color. We call c *even* if $w(c_+) > 0$, *odd* if $w(c_+) < 0$, and *neutral* if $w(c_+) = 0$. We define

$$\begin{aligned}
\mathcal{W}^{2k}_{\mathrm{ev}} &= \{M \in \mathcal{W} \mid \phi(M) \text{ is even and } \mathrm{lev}(M) = 2k\},\\
\mathcal{W}^{2k+1}_{\mathrm{odd}} &= \{M \in \mathcal{W} \mid \phi(M) \text{ is odd and } \mathrm{lev}(M) = 2k+1\},\\
\mathcal{W}^{k+1}_{0} &= \{M \in \mathcal{W} \mid \phi(M) \text{ is neutral and } \mathrm{lev}(M) = k+1\}.
\end{aligned}$$

Claim 1: *The set of walls $\mathcal{W}^{2k}_{\mathrm{ev}} \cup \mathcal{W}^{2k+1}_{\mathrm{odd}} \cup \mathcal{W}^{k+1}_{0} \cup \{M_0\}$ bounds a bounded set D containing the chamber $t_{\lambda'}$.*

Clearly, it bounds a convex set in the Hadamard space X. If it is unbounded, then there is a geodesic ray from $t_{\lambda'}$ to the visual boundary which does not intersect any of our mirrors. Since we have only finitely many colors, there is a color c such that this ray intersects infinitely many walls of this color. By the choice of k the first of this crossed walls has level $\leq k+1$. To get to infinity the ray must cross walls of color c with all levels $\geq k+1$. Thus one of the intersected walls is contained in our set.

Claim 2: *If f occurs as a face of a tile $t_\mu \subset D$ such that $f \subset \partial D$, then $w(f) \geq 0$.*

We consider four cases.

(i) If $M_f = M_0$, then t_μ lies on the same side of M_0 as $t_{\lambda'}$. Then $w(f) = w(f_+^0) > 0$.

(ii) If f is a neutral face, then $w(f) = 0$ anyway.

(iii) If f is of even color, then f is contained in a wall M from $\mathcal{W}_{\text{ev}}^{2k}$ and t_μ lies on the same side of the wall M_f as the base chamber since M has orientation $+$. Hence f as a face of t_μ gets the sign $+$. Therefore $w(f) = w(f_+) > 0$.

(iv) A similar argument applies in the case that f is of odd color.

According to Claim 1 we have $D = \bigcup_{i=1}^k C_i$ where $C_1, \dots, C_k$ is a finite collection of chambers. Then

$$\sum_{i=1}^{k} \sum_{f' \in C_i} w(f') = \sum_{f' \in \partial D} w(f') \geq 0$$

by Claim 2. Since f_+^0 is in the last set of faces, we see that the expression is indeed > 0. Therefore, $\sum_{f' \in C_i} w(f') > 0$ for some i.

This finishes the proof of the first step. Thus we have constructed a strictly balanced geometric resolution of ϕ.

Actually the above proof shows more: If we choose for any given color c an orientation of walls $\mathcal{W}_c$ in the alternating way $+-+-+-\cdots$ or $-+-+-+\cdots$ (and maybe for different colors in a different way), then the resulting geometric resolution is strictly balanced. Let us call such choice of orientations *allowed*. The levels of walls depend on the base chamber. If we define levels with respect to a different chamber, all parities of the levels will be either preserved or changed to the opposite. As a consequence we obtain the following: if the orientation of the tiling by the coloring ϕ is allowed, then also the orientation of the tiling by the coloring $g\phi$ is allowed for every $g \in \Gamma$. Thus also all limit tilings of the tiling constructed in the step 1 are strictly balanced. □

Corollary 2. *For every Coxeter group Γ there is a strictly balanced strictly aperiodic tiling of the Davis complex such that every limit tiling is strictly balanced and strictly aperiodic.*

Proof. We apply Theorem 6 to a coloring from Theorem 5. □

Note that in the proof of Corollary 2 we used that Γ is the isometry group of the Davis complex.

In 2-dimensional jigsaw tiling puzzles a geometric resolution is usually realized by adding rounded tabs out on the sides of the pieces with corresponding blank cuts into intervening sides to receive the tabs of adjacent pieces. This procedure destroys the convexity of the pieces. We show that in the case of the hyperbolic plane $\mathbb{H}^2$ we can modify this construction to obtain aperiodic and strictly balanced tilings with convex tiles. Compare also the papers [MM], [Moz].

Theorem 7. (1) *For every $n \geq 3$ there is a strictly balanced limit strongly aperiodic tiling of $\mathbb{H}^2$ by convex $2n$-gons.*

(2) *For every $n \geq 3$ there is a finite set of tiles $(\mathcal{T}, \mathcal{F})$ that consists of convex $2n$-gons with a limit strongly aperiodic tiling of $\mathbb{H}^2$ such that every $(\mathcal{T}, \mathcal{F})$-tiling of $\mathbb{H}^2$ is aperiodic.*

Proof. (1) Identify $\mathbb{H}^2$ with the Davis complex for the right-angled Coxeter group Γ generated by reflections at a regular right-angled $2n$-gon. Coloring the sides of the $2n$-gon in two colors a and b in an alternating fashion induces a coloring of the walls $\psi\colon \mathcal{W} \to \{a, b\}$ such that the walls of the same color do not intersect. The walls of the same color c define a tree T_c with an action of Γ on it such that the induced Γ-action on the product $T_a \times T_b$ is free (see [BDS] or [DJ]). By Theorem 3 we can refine ψ to a limit aperiodic coloring $\psi\colon \mathcal{W} \to \{a_i, b_i\}_{i=1,\dots,9}$. We apply Theorem 6 to obtain a limit strictly balanced geometric resolution ϕ'. By Lemma 3 it will be a limit strongly Γ-aperiodic tiling. It is easy to see that the tiling is limit strongly aperiodic with respect to the entire isometry group of $\mathbb{H}^2$.

Now we define a modification of the tiling ϕ'. Consider a vertex of a translate of the $2n$-gon. It is the intersection point of an a_i-wall with a b_j-wall. Denote it by O_{ij}. The orientations on these walls define a local coordinate system. We move the vertex O_{ij} by a small amount using these coordinates. We choose small different numbers $d_{ij}, i, j \in \{1, \dots, 9\}$ and move O_{ij} to the distance d_{ij} in the direction of the diagonal of the positive quadrant. After this deformation we obtain a finite number of new convex tiles, which (for generic deformations) only allow tilings of $\mathbb{H}^2$ compatible with the matching rule defined by ϕ'. The new tiling has all desired properties.

(2) We take the above coloring ϕ of the walls and take a geometric resolution from Proposition 7. Apply Proposition 8 to complete the proof. □

An interesting question is under what conditions the set of tiles $(\mathcal{T}, \mathcal{F})$ of a geometric resolution ϕ' of a tiling by coloring of the Davis complex X is (strongly) aperiodic. Clearly, it is strongly aperiodic whenever its orbit $\Gamma\phi'$ in $\mathrm{tl}(X, \mathcal{T})$ is dense.

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