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The size of the solvable residual in finite groups

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Dedicated to our colleague and friend Avinoam Mann
on the occasion of his retirement

Abstract. Let G be a finite group. The *solvable residual* of G, denoted by $\mathrm{Res}(G)$, is the smallest normal subgroup of G such that the respective quotient is solvable. We prove that every finite non-trivial group G with a trivial Fitting subgroup satisfies the inequality $|\mathrm{Res}(G)| > |G|^{\beta}$, where $\beta = \log(60)/\log(120(24)^{1/3}) \approx 0.700265861$. The constant β in this inequality can not be replaced by a larger constant.

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1. Introduction

All groups in this paper are finite. We use the standard notation $Z(G)$, $\Phi(G)$, $F(G)$ and $k(G)$ for the centre, the Frattini subgroup, the Fitting subgroup and the number of conjugacy classes of G, respectively. The *solvable residual* of G, $\mathrm{Res}(G)$, is the smallest normal subgroup of G such that the respective quotient is solvable; this is the minimal term in the derived series of G. The *nilpotent residual*, $U(G)$, is defined similarly, and, of course, the commutator subgroup G' is the *abelian residual* of G. As usual, we denote by S_n and A_n the symmetric and alternating groups on n letters.

In [5] and [7] the authors obtained the following bounds for the commutator subgroup and the nilpotent residual:

Theorem A. *Let G be a non-abelian group such that $\Phi(G) = 1$. Then*

$$|G'| > [G : Z(G)]^{1/2}.$$

Theorem B. *Let G be a non-abelian group such that $\Phi(G) = 1$ and let $U = U(G)$, the nilpotent residual of G. If G is of odd order, then*

$$|U| > (2[G : Z(G)])^{1/2},$$

and if G is of even order not divisible by a Mersenne or a Fermat prime, then

$$|U| > [G : Z(G)]^{1/2}.$$

Other related results may be found in [1], [5], [7], [8] and [9]. In their recent paper [6], Guralnick and Robinson proved that $k(G) \leq (|F(G)||G|)^{1/2}$, which implies the inequality of Theorem A under the assumption that $F(G) = 1$.

In this paper we prove a respective result for $\mathrm{Res}(G)$:

Theorem C. *Let $G \neq 1$ be a group such that $F(G) = 1$. Then*

$$|\mathrm{Res}(G)| > |G|^{2/3}.$$

More precisely,

$$|\mathrm{Res}(G)| > |G|^{\beta},$$

where $\beta = \log(60)/\log(120(24)^{1/3}) \approx 0.700265861$.

Theorem C improves the result of Proposition 2.4 in [7] ($|\mathrm{Res}(G)| > |G|^{1/2}$ if $F(G) = 1$). In the proof of Theorem C we use the method of that proposition and the classification of the finite non-abelian simple groups. Both proofs rely heavily on J. D. Dixon's Theorem 3 in [3], where it is shown that the order of a solvable subgroup of S_n is bounded by $24^{(n-1)/3}$. The proof of this and related results can be also found in Section 5.8 of [4]; see also [11].

Remark 1. The assumption $F(G) = 1$ can not be omitted in Theorem C. Furthermore, it is not enough to assume $\Phi(G) = 1$ in Theorem C (unlike Theorem A and Theorem B). Indeed, by taking a direct product of a "large" elementary abelian group and a simple non-abelian group, one can find examples of non-solvable groups G satisfying $\Phi(G) = 1$ and $|\mathrm{Res}(G)| < |G|^{\varepsilon}$, for any $\varepsilon > 0$.

Remark 2. The choice of the constant β given in Theorem C is best possible, as the following example shows.

Example 1 (see [4], Exercise 5.8.6). Let $H = A_5 \cong \mathrm{PSL}(2,4)$. Then $|H| = 60$ and $|\mathrm{Aut}(H)| = 120$. Denote $L_4 = S_4$, $L_{16} = L_4 \operatorname{wr} L_4$, and by induction: $L_{4^k} = L_{4^{k-1}} \operatorname{wr} L_4$ for every integer $k \geq 2$. Then L_{4^k} is a solvable subgroup of S_{4^k} of order $24^{(4^k-1)/3}$.

Denote $m = 4^k$ and let $G_m = \mathrm{Aut}(H) \,\mathrm{wr}\, L_m$. Then G_m is a subgroup of $\mathrm{Aut}(H^m)$ (see [12], Lemma 9.24), and one can verify that $F(G_m) = 1$. Furthermore, we have $\mathrm{Res}(G_m) = \mathrm{Inn}(H)^m$, and so $|\mathrm{Res}(G_m)| = 60^m$, while $|G_m| = 120^m 24^{(m-1)/3}$.

Let $\beta_m = \log(60)/\log(120(24)^{\frac{m-1}{3m}})$. Then $|\mathrm{Res}(G_m)| = |G_m|^{\beta_m}$. Since $\{\beta_m\}$ is a decreasing sequence and since $\lim_{m\to\infty} \beta_m = \beta$, we have that for every $\varepsilon > 0$ there exists an integer m such that $|\mathrm{Res}(G_m)| < |G_m|^{\beta+\varepsilon}$. Thus the choice of β in Theorem C is best possible.

2. Proof of Theorem C

We will make use of the following proposition:

Proposition D. *Let S be a non-abelian simple group. Then, for every positive integer α,*

$$((24)^{\frac{\alpha-1}{3\alpha}} |\mathrm{Out}(S)|)^{\frac{\beta}{1-\beta}} < |S|,$$

where $\beta = \log(60)/\log(120(24)^{(1/3)})$.

Proof. Since the sequence $\{\frac{\alpha-1}{3\alpha}\}$ is an increasing sequence which converges to $1/3$, it suffices to show that

$$((24)^{1/3} |\mathrm{Out}(S)|)^{\frac{\beta}{1-\beta}} \leq |S|.$$

We note first that if $|\mathrm{Out}(S)| = 2, 3$ or 4, then, for the above inequality to hold, it suffices that the value of $|S|$ will be at least 60, 155 and 304, respectively. Note further that $24^{1/3} < 2.9$ and $\frac{\beta}{1-\beta} < 2.4$. We will use the above values in the sequel.

In the following, we refer to tables 5.1.A, 5.1.B and 5.1.C in [10].

First, observe that if S is sporadic or alternating, then $|\mathrm{Out}(S)| \leq 2$, with the exception $|\mathrm{Out}(A_6)| = 4$ (see [2], remark on p. ix and Table 1). Then, since $|S| \geq 60$ and $|A_6| = 360$, we are done in this case.

Hence, we are left with the simple groups of Lie type over a field of q elements. As in [10], we denote $q = p^f$, where p is a prime. We start with the simple classical groups and consider the following cases:

I) $S = A_{n-1}(q)$ (i.e. $S = \mathrm{PSL}_n(q)$), $n \geq 2$. Here

$$|S| = \frac{1}{d} q^{\frac{n(n-1)}{2}} \prod_{i=2}^{n} (q^i - 1) \quad \text{and} \quad |\mathrm{Out}(S)| = \begin{cases} df & \text{if } n = 2 \\ 2df & \text{if } n \geq 3 \end{cases}$$

where $d = (n, q-1)$.

We consider first the case $n \geq 3$. Using the fact that $d \leq q-1$ and by the remark at the beginning of the proof, it suffices to show that $(2.9 \cdot 2(q-1)f)^{2.4} \leq |S|$.

Since the left hand side of the inequality is independent of n and since the right hand side is an increasing function of n, it suffices to show that $(2.9 \cdot 2(q-1)f)^{2.4} \leq \frac{1}{q-1}q^3(q^2-1)(q^3-1) = \frac{1}{q-1}(q-1)^2q^3(q+1)(q^2+q+1)$, and in particular, it is enough to show that $(2.9 \cdot 2(q-1)f)^{2.4} \leq (q-1)q^6$. Hence, it suffices to show that $(5.8f)^{2.4} \leq q^{4.6} = p^{4.6f}$. This inequality holds for all integers $p \geq 2$ and $f \geq 1$ with the only exception $p = 2$ and $f = 1$ (i.e. $q = 2$). However, in that case we have $|\mathrm{Out}(S)| = 2$, $|S| = 168$ and the result follows.

Consider now the case $n = 2$. Since S is not simple for $q = 2, 3$, we consider the case $q \geq 4$. Since $|\mathrm{Out}(S)| = 2$ for $q = 4, 5, 7$ (while of course $|S| \geq 60$), and since $|\mathrm{Out}(S)| \leq 4$ for $q \leq 25$ (while $|S| \geq 304$ for $q \geq 8$), it is enough to check the case $q \geq 27$. As before, it suffices to show that $(2.9 \cdot df)^{2.4} \leq |S| = \frac{1}{d}q(q^2-1) = \frac{1}{d}p^f(p^{2f}-1)$. If $p = 2$ then $d = 1$ and (by the above considerations) $f \geq 5$, while the last inequality reduces to $(2.9f)^{2.4} \leq 2^f(2^{2f}-1)$ which holds for every $f \geq 5$. Therefore, we may assume that $p \geq 3$. Furthermore, if $p = 3$ or 5 then $f \geq 3$, while $f \geq 1$ otherwise. Since the inequality $(5.8f)^{2.4} \leq \frac{1}{2}p^f(p^{2f}-1)$ holds when either $p = 3$ or 5 and $f \geq 3$ or $p \geq 7$ and $f \geq 1$, the result follows.

II) $S = {}^2A_{n-1}(q) = U_n(q)$ $(n \geq 2)$. Here we have

$$|S| = \frac{1}{d}q^{\frac{n(n-1)}{2}}\prod_{i=2}^{n}(q^i - (-1)^i) \quad \text{and} \quad |\mathrm{Out}(S)| = \begin{cases} df & \text{if } n = 2 \\ 2df & \text{if } n \geq 3 \end{cases}$$

where $d = (n, q+1)$.

If $n = 2$, then $|S|$ and $|\mathrm{Out}(S)|$ are the same as in the previous case. Assume now that $n = 3$. Then $|S| = \frac{1}{d}q^3(q^2-1)(q^3+1)$, where $d \in \{1, 3\}$. If $q = 2$ then $|S| = 72$ and S is not simple. When $q \geq 3$ it is enough to set $d = 3$ (the "worst case") and check the inequality $(2.9 \cdot 2 \cdot 3f)^{2.4} \leq \frac{1}{3}q^7 = \frac{1}{3}p^{7f}$, which is easily verified for every $p \geq 3$, $f \geq 2$. When $p = 3$ and $f = 1$ we have $|\mathrm{Out}(S)| = 2$, and so we may apply the remark at the beginning of the proof.

If $n \geq 4$, then it is enough to check (the "worst case" $d = q+1$) that $(2.9 \cdot 2 \cdot (q+1)f)^{2.4} \leq \frac{1}{q+1}q^6(q^2-1)(q^3+1)(q^4-1)$. Thus it certainly suffices to check that $(5.8 \cdot 2qf)^{2.4} \leq q^{12}$, i.e. $(5.8 \cdot 2p^f f)^{2.4} \leq p^{12f}$ for every $p \geq 2$, $f \geq 1$. This inequality is easily verified and so we are done in this case.

III) Either $S = \mathrm{PSp}_{2m}(q) = C_m(q)$ (where $m \geq 2$) or $S = \Omega_{2m+1}(q) = B_m(q)$ (where $m \geq 2$ and q is odd). Here we have

$$|S| = \frac{1}{d}q^{m^2}\prod_{i=1}^{m}(q^{2i}-1) \quad \text{and} \quad |\mathrm{Out}(S)| = \begin{cases} 2f & \text{if } m = 2 \\ df & \text{if } m \geq 3 \end{cases}$$

with $d = (2, q-1)$ if $S = \mathrm{PSp}_{2m}(q)$, while $d = 2$ if $S = \Omega_{2m+1}(q)$. Except the case $S = \mathrm{PSp}_4(q)$ (where $m = 2$) with q even, it is clear that $|S|$ exceeds $|\mathrm{PSL}_m(q)|$, while $|\mathrm{Out}(S)|$ does not exceed $|\mathrm{Out}(\mathrm{PSL}_m(q))|$. Thus, with the above exception, the validity of the required inequality follows by the arguments given in case I. However, in the case $S = \mathrm{PSp}_4(q)$ we have $|\mathrm{Out}(S)| = 2f$ and $|S| = p^{4f}(p^{2f}-1)(p^{4f}-1)$. Since the inequality $(2.9 \cdot 2f)^{2.4} \le p^{4f}(p^{2f}-1)(p^{4f}-1)$ holds for every $p \ge 2$ and $f \ge 1$ (it suffices to check the case $p = 2$ and $f = 1$), the checking of this case is completed.

IV) Either $S = P\Omega^+_{2m}(q) = D_m(q)$, where $m \ge 3$ and

$$|S| = \frac{1}{d} q^{m(m-1)}(q^m - 1)\prod_{i=1}^{m-1}(q^{2i}-1) \quad \text{and} \quad |\mathrm{Out}(S)| = \begin{cases} 2df & \text{if } m \ne 4 \\ 6df & \text{if } m = 4 \end{cases}$$

with $d = (4, q^m - 1)$, or $S = P\Omega^-_{2m}(q) = {}^2D_m(q)$, where $m \ge 2$ and

$$|S| = \frac{1}{d} q^{m(m-1)}(q^m + 1)\prod_{i=1}^{m-1}(q^{2i}-1) \quad \text{and} \quad |\mathrm{Out}(S)| = 2df$$

with $d = (4, q^m + 1)$.

We notice that in this case some of the groups are isomorphic to groups discussed in previous cases (see [2], p. xii). However, for the convenience of the reader we provide a direct argument for all the groups considered.

Consider first the case that $q = 2$. Then, except the case $S = P\Omega^+_{2m}(q), m = 4$, we have $|\mathrm{Out}(S)| = 2$, and so we are done by the remark at the beginning of the proof. If $S = P\Omega^+_{2m}(q), m = 4$, then $|\mathrm{Out}(S)| = 6$, while $|S| = 2^{12}(2^4-1)\cdot(2^2-1)(2^4-1)(2^6-1)$. Since $(2.9\cdot 6)^{2.4} \le |S|$, the case that $q = 2$ is completed.

The case $S = P\Omega^+_{2m}(q), m = 4$ for $q \ge 3$ is checked by verifying the inequality $(2.9{\cdot}24f)^{2.4} \le \frac{1}{4}p^{12f}(p^{4f}-1)(p^{2f}-1)(p^{4f}-1)(p^{6f}-1)$ for every $p \ge 3$, $f \ge 1$ and for $p = 2$, $f \ge 2$.

Assume now $q = 3$, $S = P\Omega^-_{2m}(q)$, $m = 2$. Then $|\mathrm{Out}(S)| = 4$ and $|S| = 360 \ge 304$, hence the result follows by the remark at the beginning of the proof. The case $q = 3$, $m \ge 3$ for $P\Omega^+_{2m}(q)$ and $P\Omega^-_{2m}(q)$ (excluding the possibility $S = P\Omega^+_{2m}(q), m = 4$, which was already done) is done by verifying the inequality $(2.9\cdot 8)^{2.4} \le \frac{1}{4}3^6\cdot(3^3-1)\cdot(3^2-1)\cdot(3^4-1)$ (the right hand side is a lower bound for $|S|$ in this case).

In the case $q = 4$, for $P\Omega^+_{2m}(q)$ and $P\Omega^-_{2m}(q)$ (excluding the possibility $S = P\Omega^+_{2m}(q), m = 4$, which was already done), we have $d = 1$, $|\mathrm{Out}(S)| = 4$, and so it suffices to verify the inequality $(2.9\cdot 4)^{2.4} \le 4^2\cdot(4^2+1)\cdot(4^2-1)$ (the right hand side is a lower bound for $|S|$).

It is left to consider the case $q \ge 5$. Since the inequality $(2.9\cdot 8f)^{2.4} \le \frac{1}{4}p^{fm(m-1)}(p^{fm}-1)\prod_{i=1}^{m-1}(p^{2fi}-1)$ holds for every $p \ge 5, m \ge 2, f \ge 1$ (it suffices to check for $p = 5, m = 2, f = 1$), we are done. Hence case IV is completed.

It remains to consider the simple exceptional groups. Assume first that $|\mathrm{Out}(S)| = f$ (see [10], Table 5.1.B). Then $|S| \geq q^4 = p^{4f}$. Hence, we have only to consider the inequality $(2.9f)^{2.4} \leq p^{4f}$. Since this inequality holds for every $p \geq 2$ and $f \geq 1$, we are done in this case. Assume now that $|\mathrm{Out}(S)| > f$. Then we observe (Table 5.1.B again) that $|\mathrm{Out}(S)| \leq 6f$ and $|S| \geq q^{12}$ in this case. Hence, since the inequality $(1.9 \cdot 6f)^{2.4} \leq q^{12} = p^{12f}$ holds for every $p \geq 2$ and $f \geq 1$, the proof of Proposition D is completed. □

Proof of Theorem C. Apply induction on $|G|$. Let N be a minimal normal subgroup of G and let N_1/N be the maximal normal solvable subgroup of G/N. Then $F(G/N_1) = 1$. Suppose first that $N_1 < G$. Then from $F(N_1) \leq F(G) = 1$ it follows by induction that

$$|\mathrm{Res}(G) \cap N_1| \geq |\mathrm{Res}(N_1)| > |N_1|^{\beta}. \tag{1}$$

By applying further the inductive hypothesis to G/N_1, we obtain

$$[\mathrm{Res}(G) : \mathrm{Res}(G) \cap N_1] = |\mathrm{Res}(G)N_1/N_1| = |\mathrm{Res}(G/N_1)| > |G/N_1|^{\beta}. \tag{2}$$

Now from (1) and (2) follows $|\mathrm{Res}(G)| > |G|^{\beta}$, as claimed.

Thus we may assume from now on that $N_1 = G$, i.e., G/N is solvable and $\mathrm{Res}(G) = N = T^{\alpha}$, a direct product, where T is a simple non-abelian group (recall that $F(G) = 1$) and $\alpha \geq 1$ is an integer. We notice that N is the unique minimal normal subgroup of G. Indeed, suppose on the contrary that there exists another minimal normal subgroup, say M, of G. Then $M \cap N = 1$ and M is embedded in the solvable group G/N, contradicting $F(G) = 1$.

We deduce that $C_G(N) = 1$ and $N \leq G \leq \mathrm{Aut}(N) = \mathrm{Aut}(T) \text{ wr } S_{\alpha}$ (see [12], Lemma 9.24). Thus G/N is a solvable group embedded in $\mathrm{Out}(T) \text{ wr } S_{\alpha}$. Any element of G/N has the form (b, σ), where b belongs to the base subgroup of $\mathrm{Out}(T) \text{ wr } S_{\alpha}$ and $\sigma \in S_{\alpha}$. Then the function $(b, \sigma) \mapsto \sigma$ is a homomorphism from G/N into S_{α}. Denote the image of this homomorphism by D. Then D is a solvable subgroup of S_{α} and thus, by [3], Theorem 3, $|D| \leq f(\alpha) := 24^{(\alpha-1)/3}$. Since $|G/N| \leq |\mathrm{Out}(T)|^{\alpha} \cdot |D|$, it follows that $|G/N| \leq |\mathrm{Out}(T)|^{\alpha} \cdot f(\alpha)$. Since $\mathrm{Res}(G) = N$ and since we want to show that $|N| > |G|^{\beta}$, it suffices to show that $|G/N| < |N|^{\frac{1-\beta}{\beta}} = |T|^{\alpha\frac{1-\beta}{\beta}}$. For that, it is enough to check that $|\mathrm{Out}(T)| f(\alpha)^{1/\alpha} < |T|^{\frac{1-\beta}{\beta}}$ for each simple non-abelian group T. Thus, it suffices to show that $(24^{\frac{\alpha-1}{3\alpha}} |\mathrm{Out}(T)|)^{\frac{\beta}{1-\beta}} < |T|$ holds for every positive integer α and every non-abelian simple group T. Since this inequality holds by Proposition D, the proof of the theorem is complete. □

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