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Weak amenability of hyperbolic groups

Narutaka Ozawa*

Abstract. We prove that hyperbolic groups are weakly amenable. This partially extends the result of Cowling and Haagerup showing that lattices in simple Lie groups of real rank one are weakly amenable. We take a combinatorial approach in the spirit of Haagerup and prove that for the word length distance d of a hyperbolic group, the Schur multipliers associated with the kernel r^d have uniformly bounded norms for $0 < r < 1$. We then combine this with a Bożejko–Picardello type inequality to obtain weak amenability.

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1. Introduction

The notion of weak amenability for groups was introduced by Cowling and Haagerup [CH]. (It has almost nothing to do with the notion of weak amenability for Banach algebras.) We use the following equivalent form of the definition. See Section 2 and [BO], [CH], [Pi] for more information.

Definition. A countable discrete group Γ is said to be *weakly amenable with constant* C if there exists a sequence of finitely supported functions φ_n on Γ such that $\varphi_n \to 1$ pointwise and $\sup_n \|\varphi_n\|_{\mathrm{cb}} \le C$, where $\|\varphi\|_{\mathrm{cb}}$ denotes the (completely bounded) norm of the Schur multiplier on $\mathbb{B}(\ell_2\Gamma)$ associated with $(x, y) \mapsto \varphi(x^{-1}y)$.

In the pioneering paper [Ha], Haagerup proved that the group C*-algebra of a free group has a very interesting approximation property. Among other things, he proved that the graph distance d on a tree Γ is conditionally negatively definite; in particular, the Schur multiplier on $\mathbb{B}(\ell_2\Gamma)$ associated with the kernel r^d has (completely bounded) norm 1 for every $0 < r < 1$. For information of Schur multipliers and completely bounded maps, see Section 2 and [BO], [CH], [Pi]. Bożejko and Picardello [BP] proved that the Schur multiplier associated with the characteristic

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function of the subset $\{(x, y) : d(x, y) = n\}$ has (completely bounded) norm at most $2(n + 1)$. These two results together imply that a group acting properly on a tree is weakly amenable with constant 1. Recently, this result was extended to the case of finite-dimensional CAT(0) cube complexes by Guentner and Higson [GH]. See also [Mi]. Cowling and Haagerup [dCH], [Co], [CH] proved that lattices in simple Lie groups of real rank one are weakly amenable and computed explicitly the associated constants. It is then natural to explore this property for hyperbolic groups in the sense of Gromov [GdH], [Gr]. We prove that hyperbolic groups are weakly amenable, without giving estimates of the associated constants. The results and proofs are inspired by and partially generalize those of Haagerup [Ha], Pytlik–Szwarc [PS] and Bożejko–Picardello [BP]. We denote by $\mathbb{N}_0$ the set of non-negative integers, and by $\mathbb{D}$ the unit disk $\{z \in \mathbb{C} : |z| < 1\}$.

Theorem 1. *Let Γ be a hyperbolic graph with bounded degree and d be the graph distance on Γ. Then, there exists a constant C such that the following are true.*

(1) *For every $z \in \mathbb{D}$, the Schur multiplier θ_z on $\mathbb{B}(\ell_2\Gamma)$ associated with the kernel*

$$\Gamma \times \Gamma \ni (x, y) \mapsto z^{d(x,y)} \in \mathbb{C}$$

has (completely bounded) norm at most $C|1 - z|/(1 - |z|)$. Moreover, $z \mapsto \theta_z$ is a holomorphic map from $\mathbb{D}$ into the space $V_2(\Gamma)$ of Schur multipliers.

(2) *For every $n \in \mathbb{N}_0$, the Schur multiplier on $\mathbb{B}(\ell_2\Gamma)$ associated with the characteristic function of the subset*

$$\{(x, y) \in \Gamma \times \Gamma : d(x, y) = n\}$$

has (completely bounded) norm at most $C(n + 1)$.

(3) *There exists a sequence of finitely supported functions $f_n : \mathbb{N}_0 \to [0, 1]$ such that $f_n \to 1$ pointwise and that the Schur multiplier on $\mathbb{B}(\ell_2\Gamma)$ associated with the kernel*

$$\Gamma \times \Gamma \ni (x, y) \mapsto f_n(d(x, y)) \in [0, 1]$$

has (completely bounded) norm at most C for every n.

Let Γ be a hyperbolic group and d be the word length distance associated with a fixed finite generating subset of Γ. Then, for the sequence f_n as above, the sequence of functions $\varphi_n(x) = f_n(d(e, x))$ satisfy the properties required for weak amenability. Thus we obtain the following as a corollary.

Theorem 2. *Every hyperbolic group is weakly amenable.*

This solves affirmatively a problem raised by Roe at the end of [Ro]. We close the introduction with a few problems and remarks. Is it possible to construct a family of uniformly bounded representations as it is done in [Do], [PS]? Is it true that a group which is hyperbolic relative to weakly amenable groups is again weakly amenable? There is no serious difficulty in extending Theorem 1 to (uniformly) fine hyperbolic graphs in the sense of Bowditch [Bo]. Ricard and Xu [RX] proved that weak amenability with constant 1 is closed under free products with finite amalgamation. The author is grateful to Professor Masaki Izumi for conversations and encouragement.

2. Preliminary on Schur multipliers

Let Γ be a set and denote by $\mathbb{B}(\ell_2\Gamma)$ the Banach space of bounded linear operators on $\ell_2\Gamma$. We view an element $A \in \mathbb{B}(\ell_2\Gamma)$ as a $\Gamma \times \Gamma$-matrix: $A = [A_{x,y}]_{x,y\in\Gamma}$ with $A_{x,y} = \langle A\delta_y, \delta_x\rangle$. For a kernel $k\colon \Gamma \times \Gamma \to \mathbb{C}$, the Schur multiplier associated with k is the map m_k on $\mathbb{B}(\ell_2\Gamma)$ defined by $m_k(A) = [k(x,y)A_{x,y}]$. We recall the necessary and sufficient condition for m_k to be bounded (and everywhere-defined). See [BO], [Pi] for more information of completely bounded maps and the proof of the following theorem.

Theorem 3. *Let a kernel $k\colon \Gamma \times \Gamma \to \mathbb{C}$ and a constant $C \geq 0$ be given. Then the following are equivalent.*

(1) *The Schur multiplier m_k is bounded and $\|m_k\| \leq C$.*

(2) *The Schur multiplier m_k is completely bounded and $\|m_k\|_{\mathrm{cb}} \leq C$.*

(3) *There exist a Hilbert space $\mathcal{H}$ and vectors $\zeta^+(x)$, $\zeta^-(y)$ in $\mathcal{H}$ with norms at most $\sqrt{C}$ such that $\langle \zeta^-(y), \zeta^+(x)\rangle = k(x,y)$ for every $x, y \in \Gamma$.*

We denote by $V_2(\Gamma) = \{m_k : \|m_k\| < \infty\}$ the Banach space of Schur multipliers. The above theorem says that the sesquilinear form

$$\ell_\infty(\Gamma, \mathcal{H}) \times \ell_\infty(\Gamma, \mathcal{H}) \ni (\zeta^-, \zeta^+) \mapsto m_k \in V_2(\Gamma),$$

where $k(x,y) = \langle \zeta^-(y), \zeta^+(x)\rangle$, is contractive for any Hilbert space $\mathcal{H}$.

Let $\mathfrak{P}_f(\Gamma)$ be the set of finite subsets of Γ. We note that the empty set $\emptyset$ belongs to $\mathfrak{P}_f(\Gamma)$. For $S \in \mathfrak{P}_f(\Gamma)$, we define $\tilde{\xi}_S^+$ and $\tilde{\xi}_S^- \in \ell_2(\mathfrak{P}_f(\Gamma))$ by

$$\tilde{\xi}_S^+(\omega) = \begin{cases} 1 & \text{if } \omega \subset S, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\tilde{\xi}_S^-(\omega) = \begin{cases} (-1)^{|\omega|} & \text{if } \omega \subset S, \\ 0 & \text{otherwise.} \end{cases}$$

We also set $\xi_S^+ = \tilde{\xi}_S^+ - \delta_\emptyset$ and $\xi_S^- = -(\tilde{\xi}_S^- - \delta_\emptyset)$. Note that $\xi_S^\pm \perp \xi_T^\pm$ if $S \cap T = \emptyset$. The following lemma is a trivial consequence of the binomial theorem.

Lemma 4. *One has* $\|\xi_S^\pm\|^2 + 1 = \|\tilde{\xi}_S^\pm\|^2 = 2^{|S|}$ *and*

$$\langle \xi_T^-, \xi_S^+ \rangle = 1 - \langle \tilde{\xi}_T^-, \tilde{\xi}_S^+ \rangle = \begin{cases} 1 & \text{if } S \cap T \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

for every $S, T \in \mathfrak{P}_f(\Gamma)$.

3. Preliminary on hyperbolic graphs

We recall and prove some facts of hyperbolic graphs. We identify a graph Γ with its vertex set and equip it with the graph distance:

$$d(x, y) = \min\{n : \exists x = x_0, x_1, \dots, x_n = y \text{ such that } x_i \text{ and } x_{i+1} \text{ are adjacent}\}.$$

We assume the graph Γ to be connected so that d is well-defined. For a subset $E \subset \Gamma$ and $R > 0$, we define the R-neighborhood of E by

$$\mathfrak{N}_R(E) = \{x \in \Gamma : d(x, E) < R\},$$

where $d(x, E) = \inf\{d(x, y) : y \in E\}$. We write $B_R(x) = \mathfrak{N}_R(\{x\})$ for the ball with center x and radius R. A geodesic path $\mathfrak{p}$ is a finite or infinite sequence of points in Γ such that $d(\mathfrak{p}(m), \mathfrak{p}(n)) = |m - n|$ for every m, n. Most of the time, we view a geodesic path $\mathfrak{p}$ as a subset of Γ. We note the following fact (see e.g., Lemma E.8 in [BO]).

Lemma 5. *Let Γ be a connected graph. Then, for any infinite geodesic path $\mathfrak{p}\colon \mathbb{N}_0 \to \Gamma$ and any $x \in \Gamma$, there exists an infinite geodesic path $\mathfrak{p}_x$ which starts at x and eventually flows into $\mathfrak{p}$ (i.e., the symmetric difference $\mathfrak{p} \triangle \mathfrak{p}_x$ is finite).*

Definition. We say a graph Γ is hyperbolic if there exists a constant $\delta > 0$ such that for every geodesic triangle each edge is contained in the δ-neighborhood of the union of the other two. We say a finitely generated group Γ is hyperbolic if its Cayley graph is hyperbolic. Hyperbolicity is a property of Γ which is independent of the choice of the finite generating subset [GdH], [Gr].

From now on, we consider a hyperbolic graph Γ which has bounded degree: $\sup_x |B_R(x)| < \infty$ for every $R > 0$. We fix $\delta > 1$ satisfying the above definition. We fix once for all an infinite geodesic path $\mathfrak{p}\colon \mathbb{N}_0 \to \Gamma$ and, for every $x \in \Gamma$, choose an infinite geodesic path $\mathfrak{p}_x$ which starts at x and eventually flows into $\mathfrak{p}$. For $x, y, w \in \Gamma$, the Gromov product is defined by

$$\langle x, y\rangle_w = \frac{1}{2}(d(x,w) + d(y,w) - d(x,y)) \geq 0.$$

See [BO], [GdH], [Gr] for more information on hyperbolic spaces and the proof of the following lemma which says every geodesic triangle is "thin".

Lemma 6 (Proposition 2.21 in [GdH]). *Let $x, y, w \in \Gamma$ be arbitrary. Then, for any geodesic path $[x, y]$ connecting x to y, one has $d(w, [x, y]) \leq \langle x, y\rangle_w + 10\delta$.*

Lemma 7. *For $x \in \Gamma$ and $k \in \mathbb{Z}$, we set*

$$T(x,k) = \{w \in \mathfrak{N}_{100\delta}(\mathfrak{p}_x) : d(w,x) \in \{k-1, k\}\},$$

where $T(x,k) = \emptyset$ if $k < 0$. Then, there exists a constant R_0 satisfying the following: For every $x \in \Gamma$ and $k \in \mathbb{N}_0$, if we denote by v the point on $\mathfrak{p}_x$ such that $d(v,x) = k$, then

$$T(x,k) \subset B_{R_0}(v).$$

Proof. Let $w \in T(x,k)$ and choose a point w' on $\mathfrak{p}_x$ such that $d(w,w') < 100\delta$. Then, one has $|d(w',x) - d(w,x)| < 100\delta$ and

$$d(w,v) \leq d(w,w') + d(w',v) \leq 100\delta + |d(w',x) - k| < 200\delta + 1.$$

Thus the assertion holds for $R_0 = 200\delta + 1$. □

Lemma 8. *For $k, l \in \mathbb{Z}$, we set*

$$W(k,l) = \{(x,y) \in \Gamma \times \Gamma : T(x,k) \cap T(y,l) \neq \emptyset\}.$$

Then, for every $n \in \mathbb{N}_0$, one has

$$E(n) := \{(x,y) \in \Gamma \times \Gamma : d(x,y) \leq n\} = \bigcup_{k=0}^{n} W(k, n-k).$$

Moreover, there exists a constant R_1 such that

$$W(k,l) \cap W(k+j, l-j) = \emptyset$$

for all $j > R_1$.

Proof. First, if $(x, y) \in W(k, n-k)$, then one can find $w \in T(x, k) \cap T(y, n-k)$ and $d(x, y) \leq d(x, w) + d(w, y) \leq n$. This proves that the right hand side is contained in the left hand side. To prove the other inclusion, let (x, y) and $n \geq d(x, y)$ be given. Choose a point p on $\mathfrak{p}_x \cap \mathfrak{p}_y$ such that $d(p, x) + d(p, y) \geq n$, and a geodesic path $[x, y]$ connecting x to y. By Lemma 6, there is a point a on $[x, y]$ such that $d(a, p) \leq \langle x, y\rangle_p + 10\delta$. It follows that

$$\langle x, p\rangle_a + \langle y, p\rangle_a = d(a, p) - \langle x, y\rangle_p \leq 10\delta.$$

We choose a geodesic path $[a, p]$ connecting a to p and denote by $w(m)$ the point on $[a, p]$ such that $d(w(m), a) = m$. Consider the function $f(m) = d(w(m), x) + d(w(m), y)$. Then, one has that $f(0) = d(x, y) \leq n \leq d(p, x) + d(p, y) = f(d(a, p))$ and that $f(m+1) \leq f(m) + 2$ for every m. Therefore, there is $m_0 \in \mathbb{N}_0$ such that $f(m_0) \in \{n-1, n\}$. We claim that $w := w(m_0) \in T(x, k) \cap T(y, n-k)$ for $k = d(w, x)$. First, note that $d(w, y) = f(m_0) - k \in \{n-k-1, n-k\}$. Since

$$\begin{aligned}\langle x, p\rangle_w &\leq \frac{1}{2}(d(x, a) + d(a, w) + d(p, w) - d(x, p)) \\ &= \frac{1}{2}(d(x, a) + d(p, a) - d(x, p)) \\ &= \langle x, p\rangle_a \\ &\leq 10\delta,\end{aligned}$$

one has that $d(w, \mathfrak{p}_x) \leq 20\delta$ by Lemma 6. This proves that $w \in T(x, k)$. One proves likewise that $w \in T(y, n-k)$. Therefore, $T(x, k) \cap T(y, n-k) \neq \emptyset$ and $(x, y) \in W(k, n-k)$.

Suppose now that $(x, y) \in W(k, l) \cap W(k+j, l-j)$ exists. We choose $v \in T(x, k) \cap T(y, l)$ and $w \in T(x, k+j) \cap T(y, l-j)$. Let v_x (resp. w_x) be the point on $\mathfrak{p}_x$ such that $d(v_x, x) = k$ (resp. $d(w_x, x) = k + j$). Then, by Lemma 7, one has $d(v, v_x) \leq R_0$ and $d(w, w_x) \leq R_0$. We choose v_y, w_y on $\mathfrak{p}_y$ likewise for y. It follows that $d(v_x, v_y) \leq 2R_0$ and $d(w_x, w_y) \leq 2R_0$. Choose a point p on $\mathfrak{p}_x \cap \mathfrak{p}_y$. Then, one has $|d(v_x, p) - d(v_y, p)| \leq 2R_0$ and $|d(w_x, p) - d(w_y, p)| \leq 2R_0$. On the other hand, one has $d(v_x, p) = d(w_x, p) + j$ and $d(v_y, p) = d(w_y, p) - j$. It follows that

$$2j = d(v_x, p) - d(w_x, p) - d(v_y, p) + d(w_y, p) \leq 4R_0.$$

This proves the second assertion for $R_1 = 2R_0$. □

Lemma 9. *We set*

$$Z(k, l) = W(k, l) \cap \bigcap_{j=1}^{R_1} W(k+j, l-j)^c.$$

Then, for every $n \in \mathbb{N}_0$, one has

$$\chi_{E(n)} = \sum_{k=0}^{n} \chi_{Z(k,n-k)}.$$

Proof. We first note that Lemma 8 implies $Z(k,l) = W(k,l) \cap \bigcap_{j=1}^{\infty} W(k+j, l-j)^c$ and $\bigcup_{k=0}^{n} Z(k, n-k) \subset \bigcup_{k=0}^{n} W(k, n-k) = E(n)$. It is left to show that for every (x, y) and $n \geq d(x, y)$, there exists one and only one k such that $(x, y) \in Z(k, n-k)$. For this, we observe that $(x, y) \in Z(k, n-k)$ if and only if k is the largest integer that satisfies $(x, y) \in W(k, n-k)$. □

4. Proof of Theorem 1

Proposition 10. *Let Γ be a hyperbolic graph with bounded degree and define $E(n) = \{(x, y) : d(x, y) \leq n\}$. Then, there exist a constant $C_0 > 0$, subsets $Z(k,l) \subset \Gamma$, a Hilbert space $\mathcal{H}$ and vectors $\eta_k^+(x)$ and $\eta_l^-(y)$ in $\mathcal{H}$ which satisfy the following properties:*

(1) $\eta_m^{\pm}(w) \perp \eta_{m'}^{\pm}(w)$ *for every $w \in \Gamma$ and $m, m' \in \mathbb{N}_0$ with $|m - m'| \geq 2$.*

(2) $\|\eta_m^{\pm}(w)\| \leq \sqrt{C_0}$ *for every $w \in \Gamma$ and $m \in \mathbb{N}_0$.*

(3) $\langle \eta_l^-(y), \eta_k^+(x) \rangle = \chi_{Z(k,l)}(x, y)$ *for every $x, y \in \Gamma$ and $k, l \in \mathbb{N}_0$.*

(4) $\chi_{E(n)} = \sum_{k=0}^{n} \chi_{Z(k,n-k)}$ *for every $n \in \mathbb{N}_0$.*

Proof. We use the same notations as in the previous sections.

Let $\mathcal{H} = \ell_2(\mathfrak{P}_f(\Gamma))^{\otimes(1+R_1)}$ and define $\eta_k^+(x)$ and $\eta_l^-(y)$ in $\mathcal{H}$ by

$$\eta_k^+(x) = \xi_{T(x,k)}^+ \otimes \tilde{\xi}_{T(x,k+1)}^+ \otimes \cdots \otimes \tilde{\xi}_{T(x,k+R_1)}^+$$

and

$$\eta_l^-(y) = \xi_{T(y,l)}^- \otimes \tilde{\xi}_{T(y,l-1)}^- \otimes \cdots \otimes \tilde{\xi}_{T(y,l-R_1)}^-.$$

If $|m - m'| \geq 2$, then $T(w,m) \cap T(w,m') = \emptyset$ and $\xi_{T(w,m)}^{\pm} \perp \xi_{T(w,m')}^{\pm}$. This implies the first assertion. By Lemma 7 and the assumption that Γ has bounded degree, one has $C_1 := \sup_{w,m} |T(w,m)| \leq \sup_v |B_{R_0}(v)| < \infty$. Now the second assertion follows from Lemma 4 with $C_0 = 2^{C_1(1+R_1)}$. Finally, by Lemma 4, one has

$$\langle \eta_l^-(y), \eta_k^+(x) \rangle = \chi_{W(k,l)}(x, y) \prod_{j=1}^{R_1} \chi_{W(k+j,l-j)^c}(x, y) = \chi_{Z(k,l)}(x, y).$$

This proves the third assertion. The fourth is nothing but Lemma 9. □

Proof of Theorem 1. Take $\eta_m^{\pm} \in \ell_\infty(\Gamma, \mathcal{H})$ as in Proposition 10 and set $C = 2C_0$. For every $z \in \mathbb{D}$, we define $\zeta_z^{\pm} \in \ell_\infty(\Gamma, \mathcal{H})$ by the absolutely convergent series

$$\zeta_z^+(x) = \overline{\sqrt{1-z}} \sum_{k=0}^{\infty} \bar{z}^k \eta_k^+(x) \quad \text{and} \quad \zeta_z^-(y) = \sqrt{1-z} \sum_{l=0}^{\infty} z^l \eta_l^-(y),$$

where $\sqrt{1-z}$ denotes the principal branch of the square root. The construction of $\zeta_z^{\pm}$ draws upon [PS]. We note that the map $\mathbb{D} \ni z \mapsto (\zeta_z^{\pm}(w))_w \in \ell_\infty(\Gamma, \mathcal{H})$ is (anti-)holomorphic. By Proposition 10, one has

$$\begin{aligned}
\langle \zeta_z^-(y), \zeta_z^+(x) \rangle &= (1-z) \sum_{k,l} z^{k+l} \chi_{Z(k,l)}(x,y) \\
&= (1-z) \sum_{n=0}^{\infty} z^n \chi_{E(n)}(x,y) \\
&= (1-z) \sum_{n=d(x,y)}^{\infty} z^n \\
&= z^{d(x,y)}
\end{aligned}$$

for all $x, y \in \Gamma$, and

$$\begin{aligned}
\|\zeta_z^{\pm}(w)\|^2 &\le 2|1-z| \sum_{j=0,1} \Big\| \sum_{m=0}^{\infty} (z^{\pm})^{2m+j} \eta_{2m+j}^{\pm}(w) \Big\|^2 \\
&= 2|1-z| \sum_{j=0,1} \sum_{m=0}^{\infty} |z|^{4m+2j} \|\eta_{2m+j}^{\pm}(w)\|^2 \\
&\le 2|1-z| \frac{1}{1-|z|^2} C_0 \\
&< C \frac{|1-z|}{1-|z|}
\end{aligned}$$

for all $w \in \Gamma$. Therefore the Schur multiplier θ_z associated with the kernel z^d has (completely bounded) norm at most $C|1-z|/(1-|z|)$ by Theorem 3. Moreover, the map $\mathbb{D} \ni z \mapsto \theta_z \in V_2(\Gamma)$ is holomorphic.

For the second assertion, we simply write $\|Z\|$ for the (completely bounded) norm of the Schur multiplier associated with the characteristic function χ_Z of a subset $Z \subset \Gamma \times \Gamma$. By Proposition 10 and Theorem 3, one has

$$\|E(n)\| \le \sum_{k=0}^{n} \|Z(k, n-k)\| \le C_0(n+1).$$

and $\|\{(x,y) : d(x,y) = n\}\| = \|E(n) \setminus E(n-1)\| \leq C(n+1)$. This proves the second assertion. The third assertion follows from the previous two, by choosing $f_n(d) = \chi_{E(K_n)}(d) r_n^d$ for suitable $0 < r_n < 1$ and $K_n \in \mathbb{N}_0$ with $r_n \to 1$ and $K_n \to \infty$. We refer to [BP], [Ha] for the proof of this fact. $\square$

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N. Ozawa, Department of Mathematical Sciences, University of Tokyo, Komaba, 153-8914, Japan, and Department of Mathematics, UCLA, Los Angeles, CA 90095-1555, U.S.A.

E-mail: narutaka@ms.u-tokyo.ac.jp