Groups Geom. Dyn. 2 (2008), 263–269 **Groups, Geometry, and Dynamics**

Kleinian groups with ubiquitous surface subgroups

Joseph D. Masters

Abstract. We show that every finitely generated free subgroup of a right-angled, co-compact Kleinian reflection group is contained in a surface subgroup.

Mathematics Subject Classification (2000). 57M50, 20E07.

Keywords. Coxeter group, surface subgroup, reflection group, hyperbolic polyhedron, 3-manifold.

1. Introduction

It is conjectured that every co-compact Kleinian group contains a surface subgroup. We show that, for some special examples, much more is true.

Theorem 1.1. *Let P be a right-angled, compact Coxeter polyhedron in $\mathbb{H}^3$, and let $\Gamma(P) \subset \mathrm{Isom}(\mathbb{H}^3)$ be the group generated by reflections in the faces of P. Then every finitely generated free subgroup of $\Gamma(P)$ is contained in a surface subgroup of $\Gamma(P)$.*

Remarks. 1. It is well known that every such $\Gamma(P)$ contains a surface subgroup. Indeed, it was shown in [6] that the number of "inequivalent" surface subgroups of $\Gamma(P)$ grows factorially with the genus.

2. Lewis Bowen has recently applied Theorem 1.1 to show that every such $\Gamma(P)$ contains a sequence of surface subgroups for which the Hausdorff dimensions of the limit sets approach two (see [3]).

2. Outline of the proof

Given a free subgroup G, we look at the convex core $\mathrm{Core}(G) = \mathrm{Hull}(\Lambda(G))/G$, which will be homeomorphic to a handlebody. Replacing $\mathrm{Hull}(\Lambda(G))$ with a suitable nighborhood in $\mathbb{H}^3$, we can expand the handlebody to make it polyhedral, so that the boundary is a union of copies of the faces of P. By expanding further, we can make the induced decomposition of the boundary finer and finer. If we expand enough, it becomes possible to attach mirrors to certain faces along the boundary (see Figure 1), in such a way that the resulting 3-orbifold is the product of a compact 2-orbifold with an interval. The desired surface group is a finite-index subgroup of the 2-orbifold group.

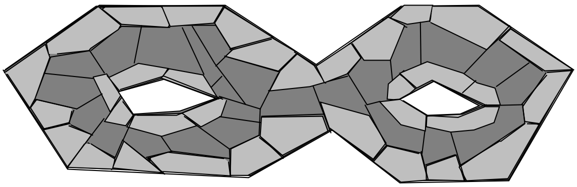


Figure 1. Mirrors are attached to the lightly-shaded faces.

3. Proof

Proof. The first ingredient is the Tameness Theorem. Let G be a free subgroup of $\Gamma(P)$. Then by [1] and [5], the (infinite-volume) hyperbolic manifold $\mathbb{H}^3/G$ is *topologically tame*, i.e., homeomorphic to the interior of a compact 3-manifold. Then work of Canary ([4]) implies that G is geometrically finite – i.e., if C is an ϵ-neighborhood of the convex hull of the limit set of G, then C/G is a compact 3-manifold.

The next step, based on the ideas of [7], is to give a polyhedral structure to C_ϵ/G. Let $\mathcal{T}$ be the tesselation of $\mathbb{H}^3$ by copies of P, and let C^+ be the *tiling hull* of G – this is the intersection of half-spaces containing C, with the restriction that each half-space must be a union of faces in $\mathcal{T}$. Then C^+ is convex and invariant under G, and C^+/G is compact (see [2], 3.1). It follows that C^+/G is a compact irreducible 3-manifold with free fundamental group, and thus C^+/G is homeomorphic to a handlebody W.

The tesselation $\mathcal{T}$ induces a tesselation of ∂W. Since all dihedral angles of P are $\pi/2$, then every pair of adjacent faces in ∂W will meet at an angle of either $\pi/2$ or π. However, if two faces meet at an angle of π, then we actually consider them as part of a single face. Thus, every face in ∂W can be decomposed as a union $F = X_1 \cup \cdots \cup X_m$, where each X_i is congruent to a face of the original polyhedron P.

Lemma 3.1. *Each face in ∂W is an embedded disk.*

Proof. Suppose not. Then there is a face $F \subset \partial W$ with non-trivial π_1. Let γ be a non-trivial loop in F. Since F is a totally geodesic sub-manifold, there is a loop in

F which is freely homotopic to γ, and which represents a geodesic in W. But this is impossible, since the convex core of G is contained in the interior of W. $\square$

Along each X_i, we may attach to W a copy of P, to obtain a handlebody with convex boundary containing W, called the *expansion of W along F*. More generally, we define an *expansion* of W to be a handlebody $W' \supset W$, obtained from W by a finite sequence of such operations.

Let g be the genus of W, and represent W as $S \times I$, for a planar surface S. Let $\alpha_1, \ldots, \alpha_{n=g+1}$ be the boundary curves of $S \times \{0\}$. Say that a collection of faces $\mathcal{F}$ of ∂W forms a *face annulus* if the faces can be indexed $F_1, \ldots, F_m$, where F_i is adjacent to F_j if and only if $|i - j| = 1 \pmod m$ and $\bigcap_i F_i = \emptyset$. The last condition excludes the case of three faces meeting at a vertex.

The following lemma is the key to proving Theorem 1.1.

Lemma 3.2. *There is an expansion W' of W and a collection $\mathcal{F}$ of disjoint face annuli $A_1, \ldots, A_n \subset \partial W'$, so that the core curve of A_i is freely homotopic to α_i in W'.*

Proof. Let $\mathcal{A} = \bigcup_i \alpha_i$. Our first claim is that there is an expansion W' of W so that, after an isotopy of the α_i's to $\partial W'$, we have $F \cap \mathcal{A}$ being connected for each $F \in \partial W'$.

We may assume, after an isotopy, that each face in ∂W meets $\mathcal{A}$ in a collection of disjoint, properly embedded arcs. Let

$$k = k(\mathcal{A}) = \mathrm{Max}_{F \in \partial W} |F \cap \mathcal{A}|.$$

Suppose $k > 1$. Let $n(\mathcal{A})$ be the number of faces in ∂W which meet $\mathcal{A}$ in k components. Let $F \in \partial W$ such that $|F \cap \mathcal{A}| = k$, and let W' be the expansion of W along F. Note that $W' - W$ is a polyhedron P' (made up of copies of P) with dihedral angles $\pi/2$. Let F' be the face of P' which is identified to F, and let $F'_1, \ldots, F'_n$ be the faces in P' which are adjacent to F', in cyclic order.

Let $N_1(F') = F' \cup F'_1 \cup \cdots \cup F'_n$, and let $N_2(F')$ be the union of $N_1(F')$ together with all faces in P' which meet faces in $N_1(F')$. Since P' is a Coxeter polyhedron in $\mathbb{H}^3$, it follows that int $N_2(F')$ is an embedded disk.

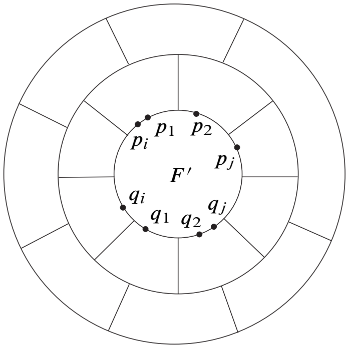
Recall that $\mathcal{A} \cap F$ consists of k disjoint arcs; let $\beta_1, \ldots, \beta_k$ be the images of these arcs in F', and let (p_i, q_i) be the endpoints of β_i.

Lemma 3.3. *There are disjoint arcs γ_i in $\partial P' - F'$ with endpoints (p_i, q_i) so that:*

1. $\left|F^* \cap \left(\bigcup \gamma_i\right)\right| < k$ *for all faces F^* in $\partial P' - N_1(F')$.*
2. $\left|F'_j \cap \left(\bigcup \gamma_i\right)\right| = \left|F'_j \cap \left(\bigcup \partial\beta_i\right)\right|$ *for all j.*

Proof. Case 1: There are four endpoints (say (p_1, q_1), (p_2, q_2)) which lie on four distinct sides of F'.

In this case, we let δ be a properly embedded arc in $N_1(F')$, disjoint from $\bigcup_i \beta_i$, which separates β_1 and β_2 (See Figure 2). For each i, let β_i^+ (resp. β_i^-) be an arc, properly embedded in some F_j, so that one endpoint is on $\partial N_1(F)$, the other is the point p_i (resp. q_i) and so that the arcs $\beta_1^\pm, \beta_2^\pm, \ldots$ are all disjoint from each other and from δ. Let β_i^* be the component of $\partial N_1(F') - (\beta_i^+ \cup \beta_i^-)$ which is disjoint from δ. Let $\gamma_i = \beta_i^+ \cup \beta_i^- \cup \beta_i^*$. After an isotopy (supported in a neighborhood of β_i^* in $N_2(F') - \operatorname{int} N_1(F')$) the arcs γ_i satisfy the hypotheses of the lemma.



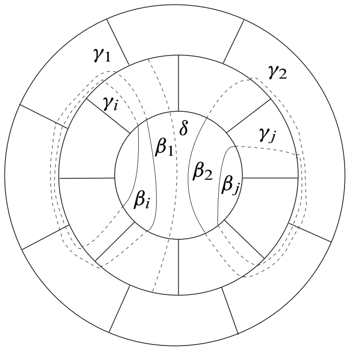


Figure 2. a. The polyhedron F'. b. Construction of γ_i's (Case 1).

Case 2: Suppose that some edge of F meets every arc β_i.

We repeat the construction from Case 1 (i.e., pick an arc δ in $N_1(F')$ disjoint from the β_i's, separating β_1 and β_2; then construct $\beta_i^\pm$'s, β_i^*'s, and γ_i's). The only difference is that we must arrange that the arcs $\beta_1^+, \beta_2^+, \ldots$ are not all parallel (i.e., their union meets at least three distinct sides) and that the arcs $\beta_1^-, \beta_2^-, \ldots$ are not all parallel. This can be done, since, P' being a right-angled Coxeter polyhedron in $\mathbb{H}^3$, each F_i' has at least five edges (see Figure 3). $\square$

Now we return to the proof of Lemma 3.2. We obtain a loop α_i' in W' by replacing each $\beta_j \subset \alpha_i$ with γ_j. Let $\mathcal{A}' = \bigcup \alpha_i'$. Since the face F has been removed and replaced by faces which meet $\mathcal{A}'$ in fewer than k components, we have $n(\mathcal{A}') < n(\mathcal{A})$.

Similarly, we see that, by enlarging W repeatedly, $n(\mathcal{A})$ can be reduced until it reaches 0. By further enlargements, we may suppose that $k(\mathcal{A}) = 1$. So we may assume that $F \cap \mathcal{A}$ is connected for each F.

Let A_i be the union of the faces which meet α_i. For each face F in $\bigcup A_i$, let us define the *overlap* of F by the formula:

$$o(F) = (\text{Number of faces in } \textstyle\bigcup A_i \text{ which are adjacent to } F) - 2.$$

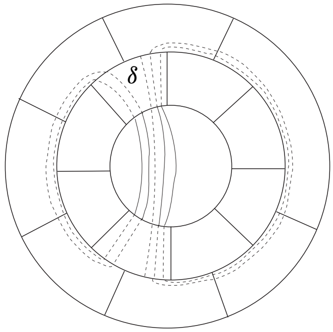


Figure 3. Construction of γ_i's (Case 2).

Since the core curve of A_i is essential in W, no point in ∂W meets every face in A_i. Thus, if $o(F) = 0$ for all $F \in \bigcup A_i$, then the A_i's are the disjoint face annuli we are looking for.

Let F be a face in A_i, let F_1 and F_2 be the two faces in A_i which are consecutive to F, and let $e_i = F \cap F_i$. Let γ_1 and γ_2 be the components of $\partial F - \{e_1 \cup e_2\}$. We say that F is *good* if one of the γ_i's is disjoint from the interior of $\bigcup A_i$.

Case 3: Every face in $\bigcup A_i$ is good.

Let F be a face in some A_i, and let $\beta = F \cap (\cup\alpha_i)$. By previous assumption, β is connected. Let p and q be the endpoints of β. As before, let W' be the enlargement of W along F, let $P' = W' - \operatorname{int} W$, and let F' be the face of W' which is identified to F. Let $F'_1, \dots, F'_n$ be the faces adjacent to F' in P', labeled consecutively, so that $p \in \partial F'_1$ and $q \in \partial F'_i$. Since F is good, then we may assume that none of the faces $F'_2, \dots, F'_{i-1}$ is glued to a face in $\bigcup A_i$.

As in the proof of Lemma 3.3, we replace β with an appropriate arc $\gamma \subset \partial P' - F'$. In this case, we choose arcs β^+ (resp. β^-) from p (resp. q) to $\partial N_1(F')$, so that β^+ and β^- each meet only one face of $\partial P'$. We let β^* be the component of $\partial N_1(F') - (\beta_1^+ \cup \beta_1^-)$ contained in $F'_1, \dots, F'_i$; then we perturb β^* so that it is a properly embedded arc in $N_2(F') - N_1(F')$. See Figure 4.

A complication is that $\partial N_2(F')$ may not be an embedded circle in P', and thus there may be pairs of adjacent faces in P' which meet β^* non-consecutively. In this case, we perform "shortcut" operations on β^*, as indicated in Figure 5.

Let $\gamma = \beta^+ \cup \beta^- \cup \beta^*$. Then we have the required arc γ and a new loop α'. The number of faces with positive overlap decreases, so eventually we may eliminate them all.

Case 4: Suppose there is a face F in $\bigcup A_i$ which is not good.

Here the construction is similar to the construction of Case 3. In this case, we choose β^* to be either of the two components of $\partial N_1(F') - \beta_1^+ \cup \beta_1^-$; then we push β^* off of $\partial N_1(F')$; and then, as in Case 3, we perform shortcuts if possible. The result

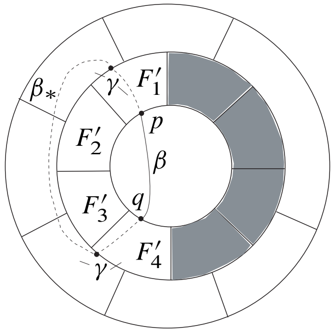


Figure 4. Construction of γ in Case 3; shaded faces may possibly glue to $\bigcup A_i$; however F'_2 and F'_3 do not.

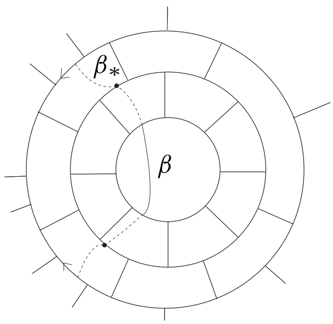


Figure 5. If the edges with arrows are actually the same, then it is possible to shorten the arc β^*.

is that the face F is removed and replaced with good faces. Repeating this operation along all faces which are not good, we may reduce to Case 3.

Thus, we have shown that, after a sequence of enlargements, every face in $\bigcup A_i$ has zero overlap. Thus we have constructed the required A_i's, completing the proof of Lemma 3.2. $\square$

Returning to the proof of Theorem 1.1, we let H be the group generated by G, together with the reflections in the lifts to $\mathbb{H}^3$ of the faces of the face annuli $A_1, \dots, A_n$. Then we claim that H is the group of a closed, hyperbolic 2-orbifold.

Indeed, let V be the orbifold with underlying space W and with mirrors on the faces of $A_1, \dots, A_n$. Then V is a hyperbolic 3-orbifold with convex boundary, and there is a local isometry $i : V \to \mathbb{H}^3/\Gamma(P)$, with induced map $i_* : \pi_1^{\text{orb}}(V) \to \Gamma(P)$, so that image$(i_*) = H$. Since V has convex boundary, every element in $\pi_1^{\text{orb}}(V)$ is represented by a closed geodesic, and since i takes geodesics to geodesics, it follows that i is π_1-injective.

Just as a handlebody is homeomorphic to a planar surface times I, one may check that the 3-orbifold V is equivalent to a product orbifold $X \times I$, where X is the 2-orbifold with reflector edges corresponding to one of the components of $\partial W - \bigcup A_i$. The underlying space of X is a planar surface, and there are cycles of reflector edges on the frontier. Thus $H = \mathrm{image}(i_*)$ is isomorphic to the orbifold fundamental group of X.

The orientable double cover of X is a 2-orbifold, $\widetilde{X}$, where the underlying space is an orientable surface of genus g and the cone points of $\widetilde{X}$ all have order 2. If we identify H with $\pi_1^{\mathrm{orb}} X$, then the loops generating G all lift to $\widetilde{X}$, and so $G \subset \pi_1 \widetilde{X}$. The group $\pi_1 \widetilde{X}$ has a torsion-free subgroup of index two (if the number of cone points is even) or four (if the number of cone points is odd), containing G. This is the surface subgroup we were looking for. $\square$

References

[1] I. Agol, Tameness of hyperbolic 3-manifolds. Preprint 2004. arXiv:math/0405568

[2] I. Agol, D. D. Long, and A. W. Reid, The Bianchi groups are separable on geometrically finite subgroups. *Ann. of Math.* (2) **153** (2001), 599–621. Zbl 1067.20067 MR 1836283

[3] L. Bowen, Free groups in lattices. Preprint 2008. arXiv:0802.0185

[4] R. D. Canary, A covering theorem for hyperbolic 3-manifolds and its applications. *Topology* **35** (1996), 751–778. Zbl 0863.57010 MR 1396777

[5] D. Calegari and D. Gabai, Shrinkwrapping and the taming of hyperbolic 3-manifolds. *J. Amer. Math. Soc.* **19** (2006), 385–446. Zbl 1090.57010 MR 2188131

[6] J. D. Masters, Counting immersed surfaces in hyperbolic 3-manifolds. *Algebr. Geom. Topol.* **5** (2005), 835–864. Zbl 1082.57013 MR 2153105

[7] P. Scott, Subgroups of surface groups are almost geometric. *J. London Math. Soc.* (2) **17** (1978), 555–565; Correction to ‘Subgroups of surface groups are almost geometric’, *ibid.* **32** (1985), 217–220. Zbl 0412.57006 MR 0494062; Zbl 0581.57005 MR 0811778

Received June 13, 2007; revised December 19, 2007

J. D. Masters, Department of Mathematics, SUNY Buffalo, Buffalo, NY 14260-2900, U.S.A.

E-mail: jdmaster@buffalo.edu