

Iterated monodromy groups of quadratic polynomials, I

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Abstract. We describe the iterated monodromy groups associated with post-critically finite quadratic polynomials, and make explicit their connection to the ‘kneading sequence’ of the polynomial.

We then give recursive presentations by generators and relations for these groups, and study some of their properties, like torsion and ‘branchness’.

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1. Introduction

Symbolic dynamics of quadratic complex polynomials are traditionally studied by their “kneading sequence”, an infinite sequence over the alphabet $\{0, 1, *\}$ that encodes symbolically the dynamics of the map (see [9], [16]). This sequence is (pre)periodic if the polynomial has finite post-critical set, and we shall make that assumption here.

A construction by the second author associates a finitely generated group, defined by automata, with such a polynomial. This group is the iterated monodromy group of the polynomial, and acts on a binary rooted tree.

In this paper, we show that the automata defining the group may be chosen in a particularly simple manner. These automata are “bounded”, i.e., their activity is trivial away from a ray in the tree. If the tree’s rays are labeled by infinite $\{0, 1\}$ sequences, then the rays on which activity is non-trivial are the translates of the periodic part of the kneading sequence of the polynomial.

Actually, we study a class of groups containing all the iterated monodromy groups of post-critically finite quadratic polynomials as a proper subset. The first set of such groups $\mathfrak{K}(v)$ corresponds to kneading sequences of the form $(v^*)^\omega$, and the second set of groups $\mathfrak{K}(w, v)$ corresponds to kneading sequences of the form $w(v^\omega)$.

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Since not every sequence of this form is realizable as a kneading sequence, not all groups $\mathfrak{K}(v)$ and $\mathfrak{K}(w, v)$ are iterated monodromy groups of quadratic polynomials. In particular, if v is periodic (is a proper power of a word), then $\mathfrak{K}(w, v)$ is not an iterated monodromy group of any polynomial (they are iterated monodromy groups of *obstructed topological polynomials*). In all the other cases the corresponding groups are the iterated monodromy groups of some post-critically finite polynomials of degree 2^n . For more details see a general description of the iterated monodromy groups of post-critically finite polynomials in [17].

We describe elementary properties of the groups $\mathfrak{K}(v)$ and $\mathfrak{K}(w, v)$. Unsurprisingly, the groups are of a very different nature depending on whether the kneading sequence is periodic (for $\mathfrak{K}(v)$) or pre-periodic (for $\mathfrak{K}(w, v)$):

- if the sequence is periodic, these groups are weakly branch, but not branch; they are torsion-free;
- if the sequence is pre-periodic, these groups are branch, and contain elements of arbitrarily high 2-power order, as well as elements of infinite order.

We give in all cases recursive presentations for these groups.

The paper is organised as follows: Section 2 contains necessary recollections on iterated monodromy groups; Section 3 describes the groups $\mathfrak{K}(v)$ associated with periodic kneading sequences; Section 4 describes the groups $\mathfrak{K}(w, v)$ associated with pre-periodic kneading sequences; and Section 5 proves that these groups are indeed the iterated monodromy groups of polynomials with prescribed kneading sequence.

Our aim in this paper was to show that kneading sequences and bounded automata are really equivalent descriptions for post-critically finite quadratic polynomials and to describe the basic algebraic properties of the iterated monodromy groups.

Some of the groups we consider already appeared in the literature, and the present paper extends or unifies algebraic results concerning them. The groups $\mathfrak{K}(v)$ and $\mathfrak{K}(w, v)$ that coincide with previously known ones are

- $\mathfrak{K}() = \mathbb{Z}$;
- $\mathfrak{K}(0, 1) = \mathfrak{K}(1, 0) =$ the infinite dihedral group;
- $\mathfrak{K}(0)$, an “amenable non-subexponentially-amenable” group considered in [7], where it is called the “Basilica group”;
- $\mathfrak{K}(0, 11)$, a group whose growth is studied in [12];
- $\mathfrak{K}(0, 111)$, a group whose Lie algebra is studied in [2], and whose presentation and growth are studied in [3], where it is called the “Grigorchuk overgroup”. This group contains Grigorchuk’s example of an infinite torsion group [13] as a subgroup.

1.1. Notation. The notation we use is quite standard, with groups acting on the right. We therefore write $[a, b] = a^{-1}b^{-1}ab$, and $a^b = b^{-1}ab$. More generally, we write $g^{k_1h_1+\dots+k_nh_n} = (g^{k_1})^{h_1} \dots (g^{k_n})^{h_n}$ for $k_i \in \mathbb{Z}$ and g, h_i group elements. The derived subgroup $[G, G]$ of G is written G' . The cyclic group of order n is written $\frac{\mathbb{Z}}{n\mathbb{Z}}$.

The notation $a = \langle\langle a_0, a_1 \rangle\rangle\sigma$ describes an automorphism a of the binary tree. It first acts as a_0 on the left subtree and as a_1 on the right subtree, and then applies the permutation σ to these two subtrees.

On the other hand, if g is an element of a free group F that acts (in principle non-faithfully) on a tree, this action will in our cases be described by a map $\Psi: F \rightarrow (F \times F) \rtimes \mathfrak{S}_2$. Then the notation $\Psi(g) = \langle\langle g_0, g_1 \rangle\rangle\sigma$ describes the image of g using the same convention as above.

If v is a finite word over an alphabet X then $|v|$ denotes its length, i.e., such n that $v \in X^n$.

2. Iterated monodromy and self-similar groups

2.1. Iterated monodromy groups. We present in this subsection a review of the notions and results related to iterated monodromy groups. More details can be found in [17].

A polynomial $f \in \mathbb{C}[z]$ is said to be *post-critically finite* if the orbit of every critical point is finite. The union $P_f = \bigcup_{n \geq 1} f^{\circ n}(C_f)$ of the orbits of critical points is called the *post-critical set* of f ; here C_f is the set of critical points of f .

Suppose that f is a post-critically finite polynomial. We consider it as a branched self-covering of the complex plane. Moreover, it is a covering map $f: \mathbb{C} \setminus f^{-1}(P_f) \rightarrow \mathbb{C} \setminus P_f$ of the punctured plane $\mathbb{C} \setminus P_f$ by its open subset $\mathbb{C} \setminus f^{-1}(P_f)$.

Consider now the general situation of a covering $f: \mathcal{M}_1 \rightarrow \mathcal{M}$ of a path connected and locally path connected topological space \mathcal{M} by an open subset \mathcal{M}_1 . We can iterate the partially defined map f to obtain coverings $f^n: \mathcal{M}_n \rightarrow \mathcal{M}$ of \mathcal{M} by open subsets.

Take a basepoint $t \in \mathcal{M}$. Then the fundamental group $\pi_1(\mathcal{M}, t)$ acts naturally on the set of preimages $f^{-n}(t)$: the image of a point $z \in f^{-n}(t)$ under the action of a loop $\gamma \in \pi_1(\mathcal{M}, t)$ is equal to the endpoint of the f^n -preimage of γ that starts at z . Such a preimage exists and is unique, since $f^n: \mathcal{M}_n \rightarrow \mathcal{M}$ is a covering.

In this way, we get an action of $\pi_1(\mathcal{M}, t)$ on the formal disjoint union $\bigsqcup_{n \geq 0} f^{-n}(t)$, the *backward orbit* of t . The quotient of $\pi_1(\mathcal{M}, t)$ by the kernel of its action is called the *iterated monodromy group* of f and is denoted $\text{IMG}(f)$.

The backward orbit T has a natural structure of a rooted tree. The root is the unique element t of the set $f^{-0}(t)$ and the vertex $z \in f^{-n}(t)$ of the n th level of the tree is connected to the vertex $f(z) \in f^{-(n-1)}(t)$ of the $(n - 1)$ -st level.

It is easy to see that the actions of the fundamental group and of the iterated monodromy group preserve the structure of the rooted tree.

Suppose that the degree of the covering f (i.e., the number of preimages of a point) is finite and is equal to d . Choose an *alphabet* X of d letters and consider the *tree of words* X^* over this alphabet, i.e., the free monoid generated by X , in which a word v is connected to all words of the form vx for $x \in X$. The root of X^* is the empty word. Since both the tree of words X^* and the tree of preimages T are regular d -trees, they are isomorphic.

A particularly nice class of isomorphisms $\Lambda: X^* \rightarrow T$ are constructed in the following way. Choose an arbitrary bijection $\Lambda: X \rightarrow f^{-1}(t)$ of the first level of the tree X^* with the first level of the tree T . Choose also a collection of *connecting paths* ℓ_x from t to $\Lambda(x)$ in \mathcal{M} .

We define now the isomorphism Λ inductively level by level. It is already defined on the zeroth and first levels. Suppose that it is defined on the n th level. Let $v \in X^n$ be an arbitrary vertex of the n th level and let $x \in X$ be an arbitrary letter. Then the path ℓ_x has a unique f^n -preimage starting at $\Lambda(v)$. We declare $\Lambda(xv)$ to be the endpoint of this preimage.

The map $\Lambda: X^* \rightarrow T$ that we just constructed is an isomorphism of rooted trees, and the following proposition makes it possible to compute the action of elements of the fundamental group on the tree T .

Proposition 2.1. *Let us conjugate the action of $\pi_1(\mathcal{M}, t)$ on the tree T by the isomorphism $\Lambda: X^* \rightarrow T$ constructed above. Then the resulting action of $\pi_1(\mathcal{M}, t)$ on X^* is computed by the following recursive formula:*

$$(xv)^\gamma = y(v)^{\ell_x \gamma_x \ell_x^{-1}},$$

where $x \in X$, $v \in X^*$ and $\gamma \in \pi_1(\mathcal{M}, t)$ are arbitrary, γ_x is the f -preimage of γ starting at $\Lambda(x)$ and y is such that $\Lambda(y)$ is the end of γ_x .

2.2. Automata. Recursive formulæ as in Proposition 2.1 are conveniently interpreted in terms of automata theory. We interpret $\text{IMG}(f)$ (or $\pi_1(\mathcal{M}, t)$) as an automaton which being in a state γ and reading as input a letter x produces as output the letter y and moves to state $\ell_x \gamma_x \ell_x^{-1}$. Then it is ready to read a new input.

We describe such automata (or their subsets) by their *Moore diagram*. It is a graph whose vertices are identified with the states of the automaton, in which we have an arrow from a state q_1 to a state q_2 labeled by a letter $x \in X$ if the automaton, being in state q_1 , after reading the letter x goes to the state q_2 . We also label the vertices of the automaton by the transformations of the letters that they define.

In particular, if $X = \{0, 1\}$ and the automaton describes an action of a group G on the tree X^* , then every state $g \in G$ acts either by the rule $(0w)^g = 0(w)^{g_0}$ and $(1w)^g = 1(w)^{g_1}$ or by the rule $(0w)^g = 1(w)^{g_0}$ and $(1w)^g = 0(w)^{g_1}$, where g_0, g_1 are next states of the automaton and $w \in X^*$ is arbitrary. In the first case we say that g is *inactive* and write $g = \langle\langle g_0, g_1 \rangle\rangle$, in the second case we say that g is *active*

and write $g = \langle\langle g_0, g_1 \rangle\rangle\sigma$. In both cases we have in the Moore diagram an arrow from g to g_i labeled by i . We will mark the inactive states in a Moore diagram by white dots, and the active states by black dots.

The notations $g = \langle\langle g_0, g_1 \rangle\rangle$ and $g = \langle\langle g_0, g_1 \rangle\rangle\sigma$ come from the decomposition of the automorphism group $\text{aut}(X^*)$ of the binary rooted tree into the (permutational) wreath product $\text{aut}(X^*) \wr \mathfrak{S}_2 = \text{aut}(X^*)^X \rtimes \mathfrak{S}_2$. The factor \mathfrak{S}_2 acts on the tree just by its action on the first letter of words, and the factor $\text{aut}(X^*)^X$ acts separately on the each of the subtrees xX^* :

$$(0v)^{\langle\langle g_0, g_1 \rangle\rangle} = 0(v^{g_0}), \quad (1v)^{\langle\langle g_0, g_1 \rangle\rangle} = 1(v^{g_1}).$$

Then $\sigma \in \mathfrak{S}_2$ denotes the transposition and $\langle\langle g_0, g_1 \rangle\rangle\sigma$ is equal to the composition of $\langle\langle g_0, g_1 \rangle\rangle$ and σ .

2.3. Self-similar groups

Definition 1. A group G acting faithfully on the tree X^* is said to be *self-similar* if for every $g \in G$ and every $x \in X$ there exist $h \in G$ and $y \in X$ such that

$$(xw)^g = yw^h$$

for all $w \in X^*$.

In other terms a self-similar group is an automaton such that the set of transformations of X^* defined by its states is a group with respect to composition.

In particular, Proposition 2.1 shows that iterated monodromy groups act on X^* self-similarly (i.e., they are self-similar groups).

If G is a self-similar group acting on the binary tree then each of its elements g is decomposed either as $g = \langle\langle g_0, g_1 \rangle\rangle$ or as $g = \langle\langle g_0, g_1 \rangle\rangle\sigma$ according to its action on $X \subset X^*$ and, by the definition of self-similarity, we have $g_0, g_1 \in G$.

Hence every self-similar group G comes with an associated *wreath recursion* $G \rightarrow G \wr \mathfrak{S}_X$. On the other hand, every such homomorphism defines recursively an action of G on X^* (which is also called self-similar, though it may be non-faithful).

If $\Psi: G \rightarrow G \wr \mathfrak{S}_X$ is a homomorphism, then for $g \in G$ and $v \in X^*$ we define the *restriction* $g|_v$ recursively by the rule

$$g = \langle\langle g|_0, g|_1 \rangle\rangle\pi, \quad \pi \in \mathfrak{S}_X$$

and

$$g|_{vx} = g|_v|x$$

for all $v \in X^*$ and $x \in X$. We also assume that $g|_\emptyset = g$, where \emptyset denotes the empty word.

If the action of G on X^* is faithful, then the restrictions are uniquely defined by the condition

$$(vu)^g = v^g u^{g|v}$$

for $v, u \in X^*$.

Proposition 2.2. *Let G be a self-similar group and let N be a normal subgroup of G which belongs to the stabilizer of the first level. If G/N is torsion free, then so is G .*

Proof. Suppose that $g \in G$ has finite order. Since G/N is torsion free, we have $g \in N$. This implies that g belongs to the stabilizer of the first level, so $(g^n)|_x = (g|_x)^n$ for all $x \in X$. The elements $g|_x$ therefore also have finite order. Repeating the same argument, we obtain inductively that g fixes X^* , so $g = 1$. \square

Definition 2. A self-similar group G is called *recurrent* (or *self-replicating*) if its action is transitive on the first level of the tree X^* and if for some (and thus for all) $x \in X$ the homomorphism $g \mapsto g|_x : G_x \rightarrow G$ is onto, where G_x is the stabilizer of x in G .

It is easy to prove that if a self-similar group is recurrent, then it is transitive on every level of the tree X^* (the group is then called *level-transitive*).

Definition 3. A homomorphism $\Psi : G \rightarrow G \wr \mathfrak{S}_X$ is *contracting* if there exists a finite set $N \subset G$ such that for every $g \in G$ there exists $n \in \mathbb{N}$ such that $g|_v \in N$ for all words v of length $\geq n$. The smallest such set N is called the *nucleus* of Ψ .

We say that a self-similar group is contracting if the associated wreath recursion is contracting. The nucleus of the recursion is called then the nucleus of the group.

It follows directly from the definitions that the nucleus is an automaton, i.e., that for every element g of the nucleus and for every $x \in X$ the restriction $g|_x$ is in the nucleus.

2.4. Bounded automata

Definition 4. An automorphism $g \in \text{aut}(X^*)$ is *finite state* if it is defined by a finite automaton, or, equivalently, if the set $\{g|_v \mid v \in X^*\}$ is finite.

An automorphism $g \in \text{aut}(X^*)$ is called *bounded* if it is finite-state and the sequence

$$q_n = |\{v \in X^n \mid g|_v \neq 1\}|$$

is bounded. Here 1 is the identity tree automorphism.

It is not hard to see that the set of all bounded automorphisms of the tree X^* is a group.

Bounded automata and tree automorphisms were defined and studied for the first time by S. Sidki in [18]. The following description of bounded automata follows from his results.

Definition 5. An automorphism $g \in \text{aut}(X^*)$ is called *finitary* if there exists $n \in \mathbb{N}$ such that $g|_v = 1$ for all $v \in X^*$ of length more than n .

An automorphism $g \in \text{aut}(X^*)$ is called *directed* if there exists $v \in X^*$ such that $g|_v = g$ and $g|_u$ is finitary for every $u \in X^*$ such that $u \neq v$ and $|u| = |v|$. Then the infinite word $v^\omega = vvv\dots$ is called the *kneading sequence* of g .

If g is finitary, then it acts only on the first n letters of every word for some fixed n . If it is directed, then all of its non-trivial action on the tree is concentrated around the path described by the kneading sequence.

It is easy to see that every finitary and every directed automorphism is bounded.

Proposition 2.3. *If $g \in \text{aut}(X^*)$ is bounded, then there exists n such that $g|_v$ for every $v \in X^n$ is either finitary or directed.*

We have the following properties of groups of bounded automata.

Theorem 2.4 ([8]). *If G acts on X^* by bounded automata, is finitely generated and self-similar, then it is contracting.*

Theorem 2.5 ([5]). *The group of all bounded automorphisms of the tree X^* is amenable.*

2.5. Branch groups. We say that a group G acting on a regular tree X^* is *regular weakly branch on $H \leq G$* if H is non-trivial and contains the geometric direct product $H \times \dots \times H$ with $|X|$ factors. Here by *geometric direct product* we mean the group generated by copies of H acting disjointly on all subtrees xX^* with x ranging over X . The group G is *regular branch on H* if it is weakly branch on some subgroup H that has finite index in G .

Proposition 2.6 ([4], Theorem 6.9). *Let G be a regular weakly branch group on its subgroup L , and suppose that a subgroup K of L has a regular orbit on X^n for some n . Then G contains $\lim_{\leftarrow} K = \bigcup_{n \geq 0} K \wr \dots \wr K$.*

In particular, if there exists such a $K \cong \frac{\mathbb{Z}}{p\mathbb{Z}}$, then G contains every finite p -group.

Proof. Since G is weakly branch, it contains the subgroup $K_0 = K$, and for all $i \geq 1$ the subgroup $K_i = 1 \times \dots \times K_{i-1} \times \dots \times 1$ with $|X|^n$ factors, where the K_{i-1} is at any position on a regular orbit of K .

The group generated by $K_0 \cup K_1 \cup \dots$ is isomorphic to $\lim \wr^n K$.

It is known [15] that every extension embeds in a wreath product; since finite p -groups are iterated extensions of $\frac{\mathbb{Z}}{p\mathbb{Z}}$, the second claim follows. \square

3. The groups $\mathfrak{K}(v)$

Let $v = x_1x_2 \dots x_{n-1}$ be a word over the alphabet $X = \{0, 1\}$. We denote by $\mathfrak{K}(v)$ the subgroup of $\text{aut}(X^*)$ generated by the elements a_1, \dots, a_n defined by

$$a_1 = \langle\langle 1, a_n \rangle\rangle\sigma, \quad a_{i+1} = \begin{cases} \langle\langle a_i, 1 \rangle\rangle & \text{if } x_i = 0, \\ \langle\langle 1, a_i \rangle\rangle & \text{if } x_i = 1, \end{cases}$$

when $1 \leq i < n$.

In other words, the group $\mathfrak{K}(v)$ is generated by the automaton whose Moore diagram is shown on Figure 1. In this diagram, only the edges leading to non-trivial states are drawn, and the active state a_1 is labelled by σ .

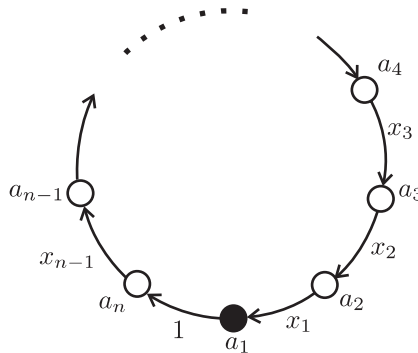


Figure 1. Automaton generating $\mathfrak{K}(v)$.

Note that, in the group $\mathfrak{K}(v)$, if we replace the generators a_i by their inverses a'_i , we obtain the description

$$a'_1 = \langle\langle a'_n, 1 \rangle\rangle\sigma, \quad a'_{i+1} = \begin{cases} \langle\langle a'_i, 1 \rangle\rangle & \text{if } x_i = 0, \\ \langle\langle 1, a'_i \rangle\rangle & \text{if } x_i = 1, \end{cases}$$

when $1 \leq i < n$. Therefore, if we modify the definition of $\mathfrak{K}(v)$ by setting $a_1 = \langle\langle a_n, 1 \rangle\rangle\sigma$, we do not change the group.

Let us denote by v' the word obtained from the word v by changing 0 to 1 and 1 to 0. It is easy to see then that the groups $\mathfrak{K}(v)$ and $\mathfrak{K}(v')$ are conjugate; more precisely, $\mathfrak{K}(v') = \mathfrak{K}(v)^\alpha$, where $\alpha \in \text{aut}(X^*)$ is defined by the recursion $\alpha = \langle\langle \alpha, \alpha \rangle\rangle\sigma$. Note that α also interchanges 0 and 1, i.e., $v^\alpha = v'$.

Lemma 3.1. *The group $\mathfrak{R}(v)$ is recurrent and level-transitive.*

Proof. The projections of the elements a_{i+1} , $a_{i+1}^{a_1}$ and a_1^2 , all fixing the first level of X^* , give all generators; the action is therefore recurrent.

To check that the action of a recurrent group is level-transitive, it suffices to check that it is transitive on the first level; this is achieved by a_1 . □

3.1. Wreath recursion. Let us denote by F_n the free group on $\{a_1, \dots, a_n\}$, and by $\mathfrak{S}_2 = \{1, \sigma\}$ the symmetric group on $X = \{0, 1\}$. Let $\Psi: F_n \rightarrow F_n \wr \mathfrak{S}_2$ be given by the recursive definition of the group $\mathfrak{R}(v)$, i.e.,

$$\Psi(a_1) = \langle\langle 1, a_n \rangle\rangle\sigma, \quad \Psi(a_{i+1}) = \begin{cases} \langle\langle a_i, 1 \rangle\rangle & \text{if } x_i = 0, \\ \langle\langle 1, a_i \rangle\rangle & \text{if } x_i = 1, \end{cases} \quad (1)$$

when $1 \leq i < n$.

We can iterate the map Ψ to obtain a homomorphism $\Psi^k: F_n \rightarrow F_n \wr \mathfrak{S}_2 \wr \dots \wr \mathfrak{S}_2$, where \mathfrak{S}_2 appears k times in the iterated permutational wreath product.

Lemma 3.2. *The wreath recursion Ψ is contracting: if $\Psi^n(g) = \langle\langle g_0, \dots, g_{2^n-1} \rangle\rangle\pi$ and $\|g\| > 2$, then $\|g_i\| < \|g\|$ for all i .*

The nucleus of $\mathfrak{R}(v)$ (and of the wreath recursion Ψ) may be expressed as follows: let d be maximal such that $vx = u^d$ for some $x \in \{0, 1\}$. Then the nucleus of $\mathfrak{R}(v)$ contains $1 + n(d + 1)$ elements. Taking indices modulo n , they are 1 , $a_i^{\pm 1}$, and $a_i^\epsilon a_{i+(n/d)j}^{-\epsilon}$ for any $j \in \{1, \dots, d - 1\}$, with the sign $\epsilon = 1$ if $x_{n/d} = x = 0$ and $\epsilon = -1$ if $x_{n/d} = x = 1$.

Proof. Let us prove that the described set of elements of F_n is the nucleus of the recursion Ψ .

It follows directly from the wreath recursion that if $\Psi(g) = \langle\langle g_0, g_1 \rangle\rangle\sigma^t$, then $\|g_0\| + \|g_1\| \leq \|g\|$. Let us call an element g *stable* if $\|g_i\| = \|g\|$ for some $i \in \{0, 1\}$ and the element g_i is also stable. In other words, an element $g \in F_k$ is stable if for every k there exists a word $v_k \in X^k$ (which is necessarily unique) such that $\|g|_{v_k}\| = \|g\|$. Note then that for every $u \in X^k$ different from v_k we have $g|_u = 1$, and v_k is the prefix of v_{k+1} . Hence there exists an infinite sequence w_g such that v_k is the beginning of length k of w_g . We call the sequence w_g the *kneading sequence* of g .

If we prove that the set of stable elements is finite, we will prove that the recursion is contracting and the nucleus is the set of stable elements, since the length of elements is non-increasing under taking restrictions.

It is easy to see that if g is stable, then g^{-1} is also stable and $w_{g^{-1}} = (w_g)^g$. If g is stable and $g = a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} \dots a_{i_k}^{\epsilon_k}$ for $\epsilon_i \in \{\pm 1\}$ and $k = |g|$, then w_g is also the kneading sequence of $a_{i_1}^{\epsilon_1}$, $(w_g)^{a_{i_1}^{\epsilon_1}}$ is the kneading sequence of $a_{i_2}^{\epsilon_2}$, $(w_g)^{a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2}}$

is the kneading sequence of $a_{i_3}^{\epsilon_3}$, etc. Otherwise the length of g would decrease along w_g . Hence the set of kneading sequences of the stable elements coincides with the set K of kneading sequences of the generators and their inverses. Let Γ be the graph with set of vertices K in which for every a_i there is an edge from the kneading sequence w_{a_i} to the kneading sequence $w_{a_i^{-1}} = (w_{a_i})^{a_i}$ labeled by a_i (if we go in the opposite direction then the edge is labeled by a_i^{-1}). Then for every stable element $g = a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} \dots a_{i_k}^{\epsilon_k}$ there exists a path (without back-tracking) in the graph Γ labeled by $a_{i_1}^{\epsilon_1}, a_{i_2}^{\epsilon_2}, \dots, a_{i_k}^{\epsilon_k}$.

It follows from the definitions that the kneading sequence of the element a_i is $(x_i x_{i-1} \dots x_1 1 x_{n-1} \dots x_{i+1})^\omega$ and the kneading sequence of a_i^{-1} is $(x_i x_{i-1} \dots x_1 0 x_{n-1} \dots x_{i+1})^\omega$. Let, as above, d be the maximal number such that $vx = u^d$ for some $x \in \{0, 1\}$. We denote by $x' \neq x$ the other letter of the alphabet. If $x = 1$, then the kneading sequence of a_i is of period n/d and the kneading sequence of a_i^{-1} is of period n . If $x = 0$, then the period of the kneading sequence of a_i is n and the period of the kneading sequence of a_i^{-1} is n/d . In each case the set K of kneading sequences of stable elements of $\mathfrak{K}(v)$ contains $n + n/d$ elements. The graph Γ consists of n/d disjoint ‘‘stars’’: every sequence of period n is connected to the sequence of period n/d obtained by changing the respective x' to x . Every sequence of period n is hence a vertex of Γ of valence 1 and every sequence of period n/d is a vertex of valence d . Consequently, paths without back-tracking have length at most 2 and the labels read on these paths are the elements described in the lemma. The rest of the lemma easily follows. \square

Proposition 3.3. *Let e_1, \dots, e_n be a free basis of the group \mathbb{Z}^n . The map $\rho: F_n \rightarrow \mathbb{Z}^n$ given by $a_i \mapsto e_i$ can be extended to a homomorphism $\rho: \mathfrak{K}(v) \rightarrow \mathbb{Z}^n$. Consequently, $\mathfrak{K}(v)/\mathfrak{K}(v)' \cong \mathbb{Z}^n$.*

Proof. Write $\pi: F_n \rightarrow \mathfrak{K}(v)$ the natural quotient map. Assume by contradiction that ρ does not factor through π ; then there exists $g \in \ker \pi \setminus \ker \rho$. Let $g = a_{i_1}^{\pm 1} a_{i_2}^{\pm 1} \dots a_{i_k}^{\pm 1}$ be a shortest (i.e., one with smallest k) element of F_n in $\ker \pi \setminus \ker \rho$. Clearly we may assume g is not contained in the nucleus, since every non-trivial word of length ≤ 2 has an active state, and is therefore non-trivial in $\text{aut}(X^*)/\text{aut}(X^*)'$.

Let us write $\Psi(g) = \langle\langle g_0, g_1 \rangle\rangle$. Then, by considering the formulae, we see that the sum of the lengths of g_0 and g_1 is at most equal to k ; it could be shorter, if some cancellation inside g_0 or g_1 occurs. The formulae (1) also show that $\rho(g_0) + \rho(g_1) = T(\rho(g))$, where $T: \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ is the isomorphism $e_i \mapsto e_{i-1 \bmod n}$. Consequently, either $\rho(g_0)$ or $\rho(g_1)$ is non-trivial. Since $\mathfrak{K}(v)$ is contracting, g_0 and g_1 are shorter than g , and we have contradicted the minimality of g . \square

Lemma 3.4. *Define the subgroups $\mathcal{E}_i \leq F_n$ inductively by $\mathcal{E}_0 = 1$ and*

$$\mathcal{E}_{i+1} = \Psi^{-1}(\mathcal{E}_i \times \mathcal{E}_i).$$

Then $\mathcal{E}_\infty = \bigcup_{i=1}^\infty \mathcal{E}_i$ is the kernel of the natural homomorphism $\pi : F_n \rightarrow \mathfrak{R}(v)$.

Proof. By contraction, for every $g \in F_n$ there exists k such that all coordinates of $\Psi^k(g)$ belong to the nucleus. The intersection of the nucleus with $\ker(\pi)$ is trivial. Therefore, if g belongs to $\ker(\pi)$, then there exists k such that all the coordinates of $\Psi^k(g)$ are trivial, i.e., $g \in \ker \Psi^k = \mathcal{E}_k$. \square

3.2. Endomorphisms. Let us define the following endomorphism φ of the free group F_n :

$$\varphi(a_n) = a_1^2, \quad \varphi(a_i) = \begin{cases} a_{i+1} & \text{if } x_i = 0, \\ a_1 a_{i+1} a_1^{-1} & \text{if } x_i = 1, \end{cases} \quad (2)$$

when $1 \leq i < n$.

We will also write φ for its image in $\mathfrak{R}(v)$. It follows directly from the definition that

$$\varphi(a_i) = \langle\langle a_i, 1 \rangle\rangle \text{ when } 1 \leq i < n, \quad \varphi(a_n) = \langle\langle a_n, a_n \rangle\rangle.$$

Proposition 3.5. *The substitution φ induces an endomorphism of the group $\mathfrak{R}(v)$ such that*

$$\varphi(g) = \langle\langle g, \rho_n(g) \rangle\rangle,$$

where ρ_n is the composition of the abelianization $a_i \mapsto e_i \in \mathbb{Z}^n$ and the projection $e_n \mapsto a_n, e_i \mapsto 1$ for $i \neq n$.

Lemma 3.6. *For every $\ell \geq 1$ we have*

$$\mathcal{E}_{\ell+1} \leq \mathcal{E}_1 \cdot \varphi(\mathcal{E}_\ell) \cdot a_1^{-1} \varphi(\mathcal{E}_\ell) a_1.$$

Proof. Consider $g \in \mathcal{E}_{\ell+1}$. Then $\Psi(g) = \langle\langle g_0, g_1 \rangle\rangle$, with $g_0, g_1 \in \mathcal{E}_\ell$. We have $\Psi(\varphi(g_i)) = \langle\langle g_i, \rho_n(g_i) \rangle\rangle$ for all $i \in \{0, 1\}$. Moreover, $\rho_n(g_i) = 1$, since $g_i = 1$ in $\mathfrak{R}(v)$ and the image of ρ is free. Hence,

$$\Psi(g) = \langle\langle g_0, g_1 \rangle\rangle = \Psi(\varphi(g_0) \cdot a_1^{-1} \varphi(g_1) a_1),$$

so $g = h \cdot \varphi(g_0) \cdot a_1^{-1} \varphi(g_1) a_1$ for some $h \in \mathcal{E}_1$. \square

3.3. L-presentation. We give a recursive presentation of $\mathfrak{R}(v)$ by generators and relations.

For $r \geq 0$ define the set of commutators, following [1]

$$\begin{aligned} \mathcal{R}_r = & \{ [a_i, a_j^{a_1^{2k}}] \mid 2 \leq i, j \leq n, 0 \leq 2k \leq r, x_{i-1} \neq x_{j-1} \} \\ & \cup \{ [a_i, a_j^{a_1^{2k+1}}] \mid 2 \leq i, j \leq n, 0 \leq 2k + 1 \leq r, x_{i-1} = x_{j-1} \}. \end{aligned} \quad (3)$$

Lemma 3.7. *The subgroup $\mathcal{E}_1 < F_n$ is the normal closure of $\mathcal{R}_\infty = \bigcup_{r \geq 0} \mathcal{R}_r$.*

Proof. It is easy to see that $\mathcal{R}_r \subset \mathcal{E}_1$ for all r . Let now $g \in \mathcal{E}_1$ be any non-trivial reduced group word. We can write it in the form $a_1^k c_1 c_2 \dots c_m$, where $c_s = a_{i_s}^{\pm a_1^{j_s}}$ for some $i_s \in \{2, \dots, n\}$ and some $j_s \in \mathbb{Z}$. Since $g = 1$ in $\mathfrak{R}(v)$, we have $k = 0$ by Proposition 3.3. Let us write $\Psi(g) = \langle\langle g_0, g_1 \rangle\rangle$. Then each of g_0, g_1 respectively is the product of some $a_{i_s-1}^{\pm a_n^{k_s}}$, where $|2k_s - j_s| \leq 1$ and only those s with j_s congruent modulo 2, respectively not congruent, to x_{i_s-1} , are selected.

Now since $g_0 = g_1 = 1$, there must exist $s < t$ such that $i_s = i_t$ and $k_s = k_t$ and c_s, c_t have opposing signs, both of them occurring in the same g_i , and none of the c_{s+1}, \dots, c_{t-1} contributing to that g_i . Then the relations in \mathcal{R}_∞ allow the commutation of c_s with c_{s+1}, \dots, c_{t-1} , and its eventual cancellation with c_t . One then proceeds by induction on the length of g . \square

Theorem 3.8. *The group $\mathfrak{R}(v)$ has the following presentation:*

$$\mathfrak{R}(v) = \langle a_1, \dots, a_n \mid \varphi^\ell(\mathcal{R}_2) \text{ for all } \ell \geq 0 \rangle,$$

where φ and \mathcal{R}_2 and given respectively in (2) and (3).

Proof. In view of Lemmata 3.4, 3.6 and 3.7, it suffices to show for all $r \geq 0$ that \mathcal{R}_r is a consequence of $\mathcal{R}^* = \bigcup_{k \geq 0} \varphi^k(\mathcal{R}_2)$. Consider therefore a relation $w_{ijk} = [a_i, a_j^{a_1^k}]$ with $k \geq 3$. We write $g \equiv h$ to mean that they are equivalent under relations in \mathcal{R}^* .

We first write $a_j^{a_1^k} = a_j^{a_1^{k-2}} [a_j, a_1^2]^{a_1^{k-2}}$; then by $[a, bc] = [a, c][a, b]^c$ the relation w_{ijk} follows from the relations $w_{ij(k-2)}$ and $[a_i, [a_j, a_1^2]^{a_1^{k-2}}]$.

We then note that $[a_p, [a_p, a_1]] \in \mathcal{R}_2$ for all p , and more generally $[a_p, [a_p, a_q]] \in \mathcal{R}^*$ for all $p \geq q$: if $x_{p-1} = x_{q-1}$, this relation is obtained as $\varphi([a_{p-1}, [a_{p-1}, a_{q-1}]])$, while if $x_{p-1} \neq x_{q-1}$, it follows from $[a_p, a_q] \in \mathcal{R}_2$. In particular $[a_n, [a_n, a_{i-1}]] \in \mathcal{R}^*$, and $[a_1^2, [a_1^2, a_i]] \in \mathcal{R}^*$. This allows to reduce $[a_i, [a_j, a_1^2]^{a_1^{k-2}}]$ to $[a_i, [a_j, a_1^2]^e]$ for some $e \in \{1, a_1^{-1}\}$. Now if $e = 1$ then we have $[a_i, [a_j, a_1^2]] \equiv w_{ij2}[a_i, a_j]$, and if $e = a_1^{-1}$ then we have $[a_i, [a_j, a_1^2]^{a_1^{-1}}] = w_{ij1}^{-a_1^{-1}} w_{ij1}^{-a_1^{-1+2a_j^{-1}}}$. \square

Since the endomorphism φ is injective, we can embed the group $\mathfrak{R}(v)$ into its ascending HNN-extension by φ , i.e., in the group generated by $\mathfrak{R}(v)$ and an element t whose action by conjugation on $\mathfrak{R}(v)$ coincides with φ .

Setting $a = a_1^{-1}$, we have a new generating system $\{a, a^t, a^{t^2}, \dots, a^{t^{n-1}}\}$ of the group $\mathfrak{R}(v)$; the identification is

$$\begin{aligned} a_{i+1} &= a^{x_i} a_i^t a^{-x_i} = a^{x_i} a^{x_{i-1}t} \dots a^{x_1 t^{i-1}} a^{-t^i} a^{-x_1 t^{i-1}} \dots a^{-x_{i-1}t} a^{-x_i}, \\ a^{t^i} &= a_{i+1}^{-a^{x_i + x_{i-1}t + \dots + x_1 t^{i-1}}}. \end{aligned} \quad (4)$$

The last theorem then yields

Theorem 3.9. Write $p(t) = x_{n-1}t + x_{n-2}t^2 + \dots + x_1t^{n-1} \in \mathbb{Z}[t]$. Then the group $\mathfrak{K}(v)$ is isomorphic to the subgroup $\langle a, a^t, a^{t^2}, \dots, a^{t^{n-1}} \rangle$ of the finitely presented group

$$\langle a, t \mid a^{t^n - 2a^{p(t)}}, [a^{t^i}, a^{t^j a}], [a^{t^i}, a^{t^j a^3}] \text{ for all } 1 \leq i, j < n \rangle.$$

Proof. The first relation is $a_n^t = a_1^2$. The others are obtained by rewriting $[a_i, a_j^{a^k}]$ for $0 \leq k \leq 2$ in terms of a and t . Indeed if $x_i = x_j$ then the relation $[a_{i+1}, a_{j+1}^{a^1}]$ yields

$$\begin{aligned} & [a^{x_{i-1}t} \dots a^{x_1 t^{i-1}} a^{-t^i} a^{-x_1 t^{i-1}} \dots a^{-x_{i-1}t}, \\ & (a^{x_{j-1}t} \dots a^{x_1 t^{j-1}} a^{-t^j} a^{-x_1 t^{j-1}} \dots a^{-x_{j-1}t}) a^1], \end{aligned}$$

which is equivalent to $[a^{t^i}, a^{t^j a}]$ for all i, j . If $x_i \neq x_j$, then the relation $[a_{i+1}, a_{j+1}]$ gives the same commutation relation as above, while $[a_{i+1}, a_{j+1}^{a^2}]$ gives the equivalent form $[a^{t^i}, a^{t^j a^3}]$. □

3.4. Torsion. We show that $\mathfrak{K}(v)$ is torsion-free for all v :

Proposition 3.10. Let $\rho: \mathfrak{K}(v) \rightarrow \mathfrak{K}(v)/\mathfrak{K}(v)' \cong \mathbb{Z}^n$ be the canonical epimorphism. An element $g \in \mathfrak{K}(v)$ is level-transitive if and only if all coordinates of $\rho(g)$ are odd.

Proof. We write $\rho(a_i) = e_i$, where (e_i) is a free basis of \mathbb{Z}^n . Recall that $\text{aut}(X^*)/\text{aut}(X^*)'$ is $(\frac{\mathbb{Z}}{2\mathbb{Z}})^\omega$, the identification being $\tau: g \mapsto (i_0, i_1, \dots) \in (\frac{\mathbb{Z}}{2\mathbb{Z}})^\omega$ where i_m is the parity of the number of active restrictions $g|_v$ for all $v \in X^m$. A tree automorphism is *active* if it acts non-trivially on the first level.

It is well known that $g \in \text{aut}(X^*)$ is level-transitive if and only if $\tau(g) = (1, 1, \dots)$. Now if $g \in \mathfrak{K}(v)$, then $\tau(g)_m$ is the parity of the exponent of $e_{m+1 \bmod n}$ in $\rho(g)$. □

Proposition 3.11. The group $\mathfrak{K}(v)$ is torsion free.

Proof. Apply Proposition 2.2 with $N = G'$. □

3.5. Weak branchness. We show that the groups $\mathfrak{K}(v)$ are weakly branch and residually poly- \mathbb{Z} :

Theorem 3.12. Let $v \in X^*$ be a non-empty sequence. Then the group $\mathfrak{K}(v)$ is weakly branch on $\mathfrak{K}(v)'$.

Note that the group $\mathfrak{R}(\emptyset)$ is isomorphic to \mathbb{Z} and so is not weakly branch.

Proof. As a first step, let us check that $\mathfrak{R}(v)' \times \mathfrak{R}(v)' \leq \mathfrak{R}(v)$. It is easy to check that

$$\langle a_1, \dots, a_{n-1}, a_1^{a_n}, \dots, a_{n-1}^{a_n} \rangle \times \langle a_1, \dots, a_{n-1}, a_1^{a_n}, \dots, a_{n-1}^{a_n} \rangle \leq \mathfrak{R}(v).$$

But

$$\mathfrak{R}(v)' \leq \langle a_1, \dots, a_{n-1}, a_1^{a_n}, \dots, a_{n-1}^{a_n} \rangle,$$

since $\mathfrak{R}(v)'$ is contained in the kernel of the homomorphism ρ_n . Hence $\mathfrak{R}(v)' \times \mathfrak{R}(v)' \leq \mathfrak{R}(v)$.

Let now $g = \langle\langle g_0, g_1 \rangle\rangle$ be an arbitrary element of the subgroup $\mathfrak{R}(v)' \times \mathfrak{R}(v)'$. Then $\varphi(g_0) = \langle\langle g_0, \rho_n(g_0) \rangle\rangle \in \mathfrak{R}(v)'$ and $a_1^{-1}\varphi(g_1)a_1 = \langle\langle \rho_n(g_1), g_1 \rangle\rangle \in \mathfrak{R}(v)'$, where φ and ρ_n are as in Proposition 3.5. But $\rho_n(g_0) = \rho_n(g_1) = 1$, since $g_0, g_1 \in \mathfrak{R}(v)'$. Therefore, $g = \varphi(g_0) \cdot a_1^{-1}\varphi(g_1)a_1 \in \mathfrak{R}(v)'$.

It suffices now to check that $\mathfrak{R}(v)'$ is not trivial. For that purpose, note that $[a_1, a_2] = \langle\langle a_1^{\pm a_n}, a_1^{\mp 1} \rangle\rangle$ has non-trivial image in \mathbb{Z}^n . \square

Proposition 3.13. *We have $\mathfrak{R}(v)' / (\mathfrak{R}(v)' \times \mathfrak{R}(v)') \cong \mathbb{Z}^{n-1}$.*

Proof. Since the elements $[a_i, a_j]$ belong to $\mathfrak{R}(v)' \times \mathfrak{R}(v)'$ if $i, j \geq 2$, we see that $\mathfrak{R}(v)' / (\mathfrak{R}(v)' \times \mathfrak{R}(v)')$ is abelian and generated by the $[a_1, a_{i+1}]$ for $i \in \{1, \dots, n-1\}$.

We then have $[a_1, a_{i+1}] = \langle\langle a_1^{\mp a_n}, a_1^{\pm 1} \rangle\rangle$; by Proposition 3.3, these are independent in $(\mathfrak{R}(v) / \mathfrak{R}(v)')^2$. \square

It follows that the groups $\mathfrak{R}(v)$ are not branch; otherwise, they would be branch on a subgroup K with $\mathfrak{R}(v)' \leq K \leq \mathfrak{R}(v)$, the last inclusion of finite index. Then $K / \mathfrak{R}(v)'$ has rank n , so $K / (\mathfrak{R}(v)' \times \mathfrak{R}(v)')$ has Hirsch length $2n - 1$, but it contains $(K \times K) / (\mathfrak{R}(v)' \times \mathfrak{R}(v)')$ of Hirsch length $2n$, a contradiction.

It follows that $\mathfrak{R}(v)$ admits arbitrarily large poly- \mathbb{Z} quotients: the successive quotients along the descending series $\mathfrak{R}(v) > \mathfrak{R}(v)' > \mathfrak{R}(v)' \times \mathfrak{R}(v)' > \dots$ are all free abelian.

Corollary 3.14. *The group $\mathfrak{R}(v)$ is left orderable.*

4. The groups $\mathfrak{R}(w, v)$

Let $w = y_1 \dots y_k \in X^*$ and $v = x_1 \dots x_n \in X^*$ be a pair of non-empty words such that $y_k \neq x_n$. We denote by $\mathfrak{R}(w, v)$ the subgroup of $\text{aut}(X^*)$ generated by the elements $b_1, \dots, b_k, a_1, \dots, a_n$ defined by

$$b_1 = \sigma, \quad b_{j+1} = \begin{cases} \langle\langle b_j, 1 \rangle\rangle & \text{if } y_j = 0, \\ \langle\langle 1, b_j \rangle\rangle & \text{if } y_j = 1, \end{cases}$$

when $1 \leq j < k$, and

$$a_1 = \begin{cases} \langle\langle b_k, a_n \rangle\rangle & \text{if } y_k = 0 \text{ and } x_n = 1, \\ \langle\langle a_n, b_k \rangle\rangle & \text{if } y_k = 1 \text{ and } x_n = 0, \end{cases} \quad a_{i+1} = \begin{cases} \langle\langle a_i, 1 \rangle\rangle & \text{if } x_i = 0, \\ \langle\langle 1, a_i \rangle\rangle & \text{if } x_i = 1, \end{cases}$$

when $1 \leq i < n$.

In other words, the group $\mathfrak{R}(w, v)$ is generated by the automaton whose Moore diagram is shown on Figure 2. In this diagram, only the edges leading to non-trivial states are drawn, and the active state b_1 is labelled by σ .

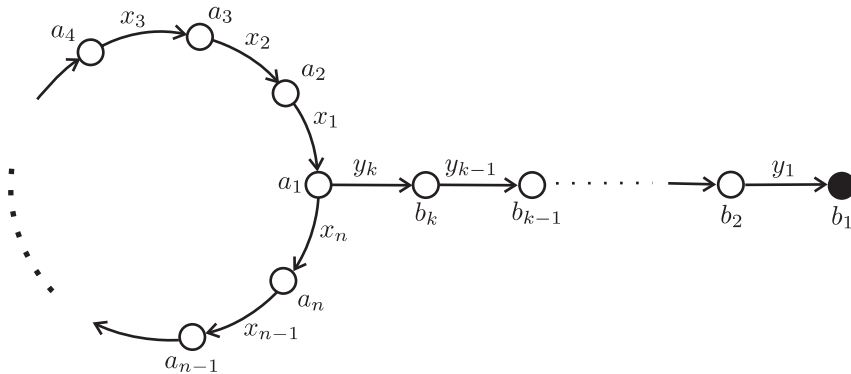


Figure 2. Automaton generating $\mathfrak{R}(w, v)$.

Let us denote by v' the word obtained from the word v by changing 0 to 1 and 1 to 0. Recall from Section 3 that the automorphism $\alpha \in \text{aut}(X^*)$ defined by the recursion $\alpha = \langle\langle \alpha, \alpha \rangle\rangle \sigma$ exchanges v and v' , and conjugates $\mathfrak{R}(v)$ into $\mathfrak{R}(v')$. Similarly, the groups $\mathfrak{R}(w, v)$ and $\mathfrak{R}(w', v')$ are conjugate under α .

By convention, let us accept that $b_{k+1} = a_{n+1} = a_1$.

Lemma 4.1. *The group $\mathfrak{R}(w, v)$ is recurrent and level-transitive.*

Proof. The projections of the $a_{i+1}, a_{i+1}^{b_1}, b_{j+1}$ and $b_{j+1}^{b_1}$, all fixing the first level of X^* , give all generators; the action is therefore recurrent.

To check that the action of a recurrent group is level-transitive, it suffices to check that it is transitive on the first level; this is achieved by b_1 . □

4.1. Wreath recursion. We aim to give a recursive presentation of $\mathfrak{R}(w, v)$ by generators and relations. Until the end of this section, we assume that $(k, n) \neq (1, 1)$; indeed if $k = n = 1$ then $\mathfrak{R}(w, v)$ is infinite dihedral, and has to be taken care of separately.

Proposition 4.2. *The generators a_i, b_j of $\mathfrak{R}(w, v)$ have order 2, and the abelianization $\mathfrak{R}(w, v)/\mathfrak{R}(w, v)'$ is $(\frac{\mathbb{Z}}{2\mathbb{Z}})^{k+n}$, generated by the images of the a_i and b_j .*

Proof. It is clear that the generators are involutions. The image of $\mathfrak{R}(w, v)$ in $\text{aut}(X^*)/\text{aut}(X^*)'$ is generated by the infinite pre-periodic vectors $(0, \dots, 1, \bar{0})$ corresponding to the images of b_j , and by $(0, \dots, 0, \overline{0, \dots, 1, \dots, 0})$ corresponding to the images of a_i ; these $k + n$ vectors are linearly independent. \square

Let d be maximal such that $v = u^d$ for some $u \in X^*$. We have then, considering indices modulo n ,

$$\Psi([a_i, a_{i+j(n/d)}]) = \langle\langle 1, [a_{i-1}, a_{i-1+j(n/d)}] \rangle\rangle \quad \text{or} \quad \langle\langle [a_{i-1}, a_{i-1+j(n/d)}], 1 \rangle\rangle,$$

depending on whether $x_{i-1} = x_{i-1+j(n/d)}$ or not. It follows that a_i and $a_{i+j(n/d)}$ commute for every $i \in \{1, \dots, n\}$ and every $j \in \{1, \dots, d - 1\}$.

Let F be the group given by the presentation

$$F = \langle a_1, \dots, a_n, b_1, \dots, b_k \mid a_i^2, b_j^2, [a_i, a_{i+\ell(n/d)}] \text{ for all } 1 \leq i \leq n, 1 \leq j \leq k, 1 \leq \ell < d \rangle.$$

It is the free product of k copies of $\frac{\mathbb{Z}}{2\mathbb{Z}}$ and n/d copies of $(\frac{\mathbb{Z}}{2\mathbb{Z}})^d$. Let us denote the subgroup $\langle a_i, a_{i+n/d}, \dots, a_{i+(d-1)n/d} \rangle \cong (\frac{\mathbb{Z}}{2\mathbb{Z}})^d$ by A_i .

Let $\Psi: F \rightarrow F \wr \mathfrak{S}_2$ be given by the recursive definition of the group $\mathfrak{R}(w, v)$, i.e.,

$$\Psi(b_1) = \sigma, \quad \Psi(b_{j+1}) = \begin{cases} \langle\langle b_j, 1 \rangle\rangle & \text{if } y_j = 0, \\ \langle\langle 1, b_j \rangle\rangle & \text{if } y_j = 1, \end{cases}$$

when $1 \leq j < k$, and

$$\Psi(a_1) = \begin{cases} \langle\langle b_k, a_n \rangle\rangle & \text{if } x_n = 1, \\ \langle\langle a_n, b_k \rangle\rangle & \text{if } x_n = 0, \end{cases} \quad \Psi(a_{i+1}) = \begin{cases} \langle\langle a_i, 1 \rangle\rangle & \text{if } x_i = 0, \\ \langle\langle 1, a_i \rangle\rangle & \text{if } x_i = 1, \end{cases}$$

when $1 \leq i < n$.

We can iterate the map Ψ to obtain a homomorphism $\Psi^k: F \rightarrow F \wr \mathfrak{S}_2 \wr \dots \wr \mathfrak{S}_2$, where \mathfrak{S}_2 appears k times in the iterated permutational wreath product.

Lemma 4.3. *The wreath recursion Ψ is contracting. Its nucleus is $\{b_1, \dots, b_k\} \cup \bigcup A_i$.*

Proof. Let us denote by $\|g\|$ the minimal number of generators a_i in a representation of g as a product of generators.

It follows from the definition of the recursion Ψ that for every $g = \langle\langle g_0, g_1 \rangle\rangle \sigma^s \in \mathfrak{R}(w, v)$ we have $\|g\| \geq \|g_0\| + \|g_1\|$. As in the proof of Lemma 3.2, we say that

$g \in \mathfrak{K}(w, v)$ is *stable* if there exists an infinite sequence w_g (the *kneading sequence* of g) such that for every l we have $\|g\| = \|g|_{v_l}\|$ where v_l is the beginning of length l of w_g . It also follows that for any word $v \in X^l$ different from v_l the restriction $g|_v$ belongs to $\langle b_i \rangle_{i=1, \dots, k}$, and hence that there exists $m \in \mathbb{N}$ such that $g|_{vw} = 1$ for all $w \in X^m$.

Consequently, if we prove that the set of stable elements is finite, then we show that the recursion is contracting and the nucleus is the set of stable elements and their restrictions.

As in the proof of Lemma 3.2, consider the set K of kneading sequences of the (stable) generators a_i , and the corresponding graph Γ . Here also the labels along the paths of the graph Γ correspond to the stable elements.

The kneading sequence of a_i is equal to $w_{a_i} = (x_{i-1}x_{i-2} \dots x_1x_nx_{n-1} \dots x_i)^\omega$. If d is as above, then the kneading sequence of a_i is of period n/d and the graph Γ consists of n/d vertices with d loops attached to every vertex. The labels of the loops attached to vertex w_{a_i} are $a_i, a_{i+n/d}, \dots, a_{i+(d-1)n/d}$. Consequently, every stable element belongs to the finite group A_i for some $i \in \{1, \dots, n/d\}$. The rest of the lemma follows. □

Lemma 4.4. *Define the subgroups $\mathcal{E}_\ell \leq F$ inductively by $\mathcal{E}_0 = 1$ and*

$$\mathcal{E}_{\ell+1} = \Psi^{-1}(\mathcal{E}_\ell \times \mathcal{E}_\ell).$$

Then $\mathcal{E}_\infty = \bigcup_{\ell=1}^\infty \mathcal{E}_\ell$ is the kernel of the natural homomorphism $F \rightarrow \mathfrak{K}(w, v)$.

Proof. Similar to Lemma 3.4. □

4.2. Endomorphisms. In this subsection we define, for each $\mathfrak{K}(w, v)$, an endomorphism φ of F , and elements $s, t \in F$ and $m \in \{1, 2\}$, with the following meaning: $\Psi(a_1^t) = \langle\langle b_k, a_n^s \rangle\rangle$; in $\mathfrak{K}(w, v)$ the subgroup $\langle a_n^s, b_k \rangle$ is dihedral of order 2^{m+1} ; and the subgroup $\langle a_1^t, b_1 \rangle$ is dihedral of order 2^{m+2} ; and $\varphi(s) \in \{t, b_1t\}$. We use these to construct an endomorphism of $\mathfrak{K}(w, v)$, considering different cases in turn and taking care of small values of k and n .

Assume first that $k, n \geq 2$. Then $\langle a_n^s, b_k \rangle$ is trivial in $\mathfrak{K}(w, v)$ for at least one value of $s \in \langle b_1 \rangle$. Let $t \in \langle b_1, b_2 \rangle$ be such that $\Psi(a_1^t) = \langle\langle b_k, a_n^s \rangle\rangle$. Set $m = 1$.

Assume next that $k \geq 3$ and $n = 1$. Then there exists $r \in \langle b_1 \rangle$ such that $\langle b_k^r, b_{k-1} \rangle = 1$ in $\mathfrak{K}(w, v)$; so there exists $s \in \langle b_1, b_2 \rangle$ such that $\langle a_1^s, b_k \rangle = 1$ in $\mathfrak{K}(w, v)$; so there exists $t \in \langle b_1, b_2, b_3 \rangle$ such that $\Psi(a_1^t) = \langle\langle b_k, a_1^s \rangle\rangle$. Set also $m = 1$.

Assume next that $k = 2$ and $n = 1$. Then there exists $r, s \in \langle b_1 \rangle$ such that $b_2^r = \langle\langle b_1, 1 \rangle\rangle$ in $\mathfrak{K}(w, v)$ and $a_1^{rs} = \langle\langle b_2, a_1 \rangle\rangle$ in $\mathfrak{K}(w, v)$; so there exists $t \in \langle b_1, b_2 \rangle$ such that $a_1^t = \langle\langle b_2, a_1^s \rangle\rangle$. Set $m = 2$.

Finally, consider the case $k = 1$ and $n \geq 2$. Set $s = 1$, and let $t \in \langle b_1 \rangle$ be such that $a_1^t = \langle\langle b_1, a_n \rangle\rangle$. Set $m = 2$.

Now define the endomorphism φ of F by

$$\varphi(b_j) = \begin{cases} b_{j+1} & \text{if } y_j = 0, \\ b_{j+1}^{b_1} & \text{if } y_j = 1, \end{cases} \text{ when } 1 \leq j < k, \quad \varphi(b_k) = a_1^t,$$

$$\varphi(a_i) = \begin{cases} a_{i+1} & \text{if } x_i = 0, \\ a_{i+1}^{b_1} & \text{if } x_i = 1, \end{cases} \text{ when } 1 \leq i < n, \quad \varphi(a_n) = \begin{cases} a_1 & \text{if } x_n = 0, \\ a_1^{b_1} & \text{if } x_n = 1. \end{cases}$$

We then have

$$\Psi\varphi(b_j) = \langle\langle b_j, 1 \rangle\rangle \text{ when } 1 \leq j < k, \quad \Psi\varphi(b_k) = \langle\langle b_k, a_n^s \rangle\rangle,$$

$$\Psi\varphi(a_i) = \langle\langle a_i, 1 \rangle\rangle \text{ when } 1 \leq i < n, \quad \Psi\varphi(a_n) = \langle\langle a_n, b_k \rangle\rangle.$$

Proposition 4.5. *The endomorphism $\varphi: F \rightarrow F$ induces an endomorphism of the group $\mathfrak{K}(w, v)$ such that*

$$\varphi(g) = \langle\langle g, \rho(g) \rangle\rangle,$$

where ρ is an endomorphism with finite image; more precisely, $\rho(b_j) = \rho(a_i) = 1$ for $j < k$ and $i < n$; and $\rho(b_k) = a_n^s$ and $\rho(a_n) = b_k$, so that $\rho(F)$ is dihedral of order 2^{m+1} .

Proof. It is clear that ρ is an endomorphism if $m = 1$, since then it factors through $\mathfrak{K}(w, v) \rightarrow \text{aut}(X^*)/\text{aut}(X^*)'$.

Consider then the case $k = 1$. The map ρ then factors through $\mathfrak{K}(w, v) \rightarrow \mathfrak{K}(w, v) \wr \mathfrak{S}_2 \rightarrow (\text{aut}(X^*)/\text{aut}(X^*)') \wr \mathfrak{S}_2$.

Consider finally the case $k = 2, n = 1$. The map ρ can then be seen, by direct calculation, to factor through $\mathfrak{K}(w, v) \rightarrow \text{aut}(X^4)$. \square

Define now $\bar{F} = F/\langle\langle \varphi^\ell(a_n^s b_k)^{2^m} \mid \ell \in \mathbb{N} \rangle\rangle$, and denote by $\bar{\mathcal{E}}_\ell$ the image of \mathcal{E}_ℓ in \bar{F} . Again φ induces an endomorphism of \bar{F} . Note that $\bar{F} \rightarrow \mathfrak{K}(w, v)$ maps isomorphically $\langle a_n^s, b_k \rangle$ and $\langle a_1^t, b_1 \rangle$.

Lemma 4.6. *For every $\ell \geq 1$ we have*

$$\overline{\mathcal{E}_{\ell+1}} \leq \bar{\mathcal{E}}_1 \cdot \varphi(\bar{\mathcal{E}}_\ell) \cdot b_1 \varphi(\bar{\mathcal{E}}_i) b_1.$$

Proof. The proof is similar to that of Lemma 3.6. Consider $g \in \overline{\mathcal{E}_{\ell+1}}$. Then $\Psi(g) = \langle\langle g_0, g_1 \rangle\rangle$, with $g_0, g_1 \in \bar{\mathcal{E}}_\ell$. We have $\Psi\varphi(g_i) = \langle\langle g_i, \rho(g_i) \rangle\rangle$ for all $i \in \{0, 1\}$, where ρ is the endomorphism with finite image given by Proposition 4.5. Moreover, $\rho(g_i) = 1$ in \bar{F} , since $\rho(g_i)$ is a relation in $\mathfrak{K}(w, v)$. Hence,

$$\Psi(g) = \langle\langle g_0, g_1 \rangle\rangle = \Psi(\varphi(g_0) \cdot b_1 \varphi(g_1) b_1),$$

and $g = h \cdot \varphi(g_0) \cdot b_1 \varphi(g_1) b_1$ for some $h \in \bar{\mathcal{E}}_1$. \square

4.3. L-presentation. Let O denote the elements of $\langle a_1^t, b_1 \rangle < \bar{F}$ that contain an odd number of b_1 's, and let E denote those elements that contain an even number of b_1 's. Both of these sets are finite. Define then

$$\begin{aligned} \mathcal{R} = & \{ [b_i, b_j^w] \mid 2 \leq i, j \leq k, w \in O \text{ if } y_{i-1} = y_{j-1}, w \in E \text{ if } y_{i-1} \neq y_{j-1} \} \\ & \cup \{ [a_i, b_j^w] \mid 2 \leq i \leq n, 2 \leq j \leq k, w \in O \text{ if } x_{i-1} = y_{j-1}, \\ & \hspace{15em} w \in E \text{ if } x_{i-1} \neq y_{j-1} \} \\ & \cup \{ [a_i, a_j^w] \mid 2 \leq i, j \leq n, w \in O \text{ if } x_{i-1} = x_{j-1}, w \in E \text{ if } x_{i-1} \neq x_{j-1} \}. \end{aligned}$$

Lemma 4.7. *We have $\bar{\mathcal{E}}_1 \leq \mathcal{R}^{\bar{F}} \leq \bar{\mathcal{E}}_\infty \leq \bar{F}$.*

Proof. It is easy to see that $\mathcal{R} \subset \bar{\mathcal{E}}_\infty$. Let now $g \in \bar{\mathcal{E}}_1$ be any non-trivial reduced group word. We can write it in the form $hc_1c_2 \dots c_m$, where $c_s = a_{i_s}^{h_s}$ or $= b_{j_s}^{h_s}$ for some $i_s, j_s \geq 2$ and some $h, h_s \in \langle a_1^t, b_1 \rangle$. Since $g = 1$ in $\mathfrak{R}(v)$, we have $h = 1$; this is clear if a_1^t and b_1 commute, because then $\langle a_1^t, b_1 \rangle$ embeds in $\mathfrak{R}(w, v)/\mathfrak{R}(w, v)'$, by Proposition 4.2. On the other hand, if $h = (a_1^t b_1)^2 \neq 1$, then the first projection of g contains precisely two a_n 's that do not cancel, and there will exist two vertices such that each of the a_n 's projects on one; then these states will be non-trivial, again using Proposition 4.2, so $g \neq 1$, a contradiction.

Let us write $\Psi(g) = \langle\langle g_0, g_1 \rangle\rangle$. Then each of g_0, g_1 respectively is the product of some $a_{i_s}^{k_s}$ and $b_{j_s}^{k_s}$, where $k_s \in \langle a_n, b_k \rangle$ and only those s with $h_s \in O$ respectively in E , are selected.

Now since $g_0 = g_1 = 1$, there must exist $s < t$ such that c_s, c_t are both b_*^* or both a_*^* and $h_s = h_t$, both of them occurring in the same g_i , and none of the c_{s+1}, \dots, c_{t-1} contributing to that g_i . The relations in \mathcal{R} and \bar{F} allow the commutation of c_s with c_{s+1}, \dots, c_{t-1} , and its eventual cancellation with c_t . One then proceeds by induction on the length of g . □

Theorem 4.8. *The group $\mathfrak{R}(w, v)$ has the following presentation:*

$$\mathfrak{R}(w, v) = \langle a_1, \dots, a_n, b_1, \dots, b_k \mid \{ \text{relations of } \bar{F} \} \cup \varphi^\ell(\mathcal{R}) \text{ for all } \ell \geq 0 \rangle,$$

where φ and \mathcal{R} are given respectively before Proposition (4.5) and in (3).

Proof. This follows from Lemmata 4.4, 4.6 and 4.7. □

Since the endomorphism φ is injective, we can embed the group $\mathfrak{R}(w, v)$ into its ascending HNN-extension by φ , i.e., in the group generated by $\mathfrak{R}(w, v)$ and an element t whose action by conjugation on $\mathfrak{R}(w, v)$ coincides with φ .

There is an $u \in \langle b_2, b_2^{b_1}, b_3, b_3^{b_1} \rangle$ such that $\varphi(b_k) = a_1^{ub_1^{yk}}$; the group $\langle a_1^u, b_1 \rangle$ is dihedral of order 2^{m+2} , and we have $(a_1^u b_1)^{2^{m+1}} = 1$. We set $a = a_1^u$ and $b = b_1$

to obtain a new generating system $\{b, b^t, \dots, b^{t^{k-1}}, a, a^t, \dots, a^{t^{n-1}}\}$ of the group $\mathfrak{R}(w, v)$; the identification with $\{a_i, b_j\}$ is similar to that in (4). The last theorem then yields

Theorem 4.9. *Write $p(t) = x_n + x_{n-1}t + \dots + x_1t^{n-1} \in \mathbb{Z}[t]$ and $q(t) = y_k + y_{k-1}t + \dots + y_1t^{k-1} \in \mathbb{Z}[t]$. Let $m \in \{1, 2\}$ and $u \in \langle b^t, b^{tb}, b^{t^2}, b^{t^2b} \rangle$ be as above. Then the group $\mathfrak{R}(w, v)$ is isomorphic to the subgroup $\langle a, a^t, \dots, a^{t^{n-1}}, b, b^t, \dots, b^{t^{k-1}} \rangle$ of the finitely presented group (where the generator a is redundant)*

$$\begin{aligned} \langle a, b, t \mid a^2, b^2, (ab)^{2^{m+1}}, b^{t^k} a^{b^q(t)}, a^{u^{-1}(t^n - b^{p(t)})}, \\ [a, a^{t^{jn/d}}] \text{ for all } j \in \{1, \dots, d-1\}, \\ [b^{t^i}, b^{t^j b(ab)^{2^\ell}}] \text{ for all } 1 \leq i, j < k \text{ and } 0 \leq \ell \leq 2^m, \\ [a^{t^i}, b^{t^j b(ab)^{2^\ell}}] \text{ for all } 1 \leq i < n, 1 \leq j < k \text{ and } 0 \leq \ell \leq 2^m, \\ [a^{t^i}, a^{t^j b(ab)^{2^\ell}}] \text{ for all } 1 \leq i, j < n \text{ and } 0 \leq \ell \leq 2^m \rangle. \end{aligned}$$

Proof. The first three relations express $\langle a_1^u, b_1 \rangle$ as a dihedral group. The fourth one is $b_k^t = a_1^{ub_1^{y_k}}$, and the fifth one is $a_n^t = a_1^{b_1^{x_n}}$. The next ones are relations in F , and the last three rows are the commutation relations of the form $[a_i, a_j^w]$ and $[a_i, b_j^w]$ for $w \in E$ or $w \in O$. □

4.4. Branchness. We see in this subsection that $\mathfrak{R}(w, v)$ is branch as soon as $k > 1$ or $n > 1$:

Theorem 4.10. (1) $\mathfrak{R}(w, v)$ is branch on $\mathfrak{R}(w, v)'$ if $k \geq 2$ and $n \geq 2$, or if $k \geq 3$ and $n = 1$; we have $\mathfrak{R}(w, v)/\mathfrak{R}(w, v)' \cong (\frac{\mathbb{Z}}{2\mathbb{Z}})^{k+n}$ generated by $\{a_i, b_j\}$ and $\mathfrak{R}(w, v)' / (\mathfrak{R}(w, v)' \times \mathfrak{R}(w, v)') = (\frac{\mathbb{Z}}{2\mathbb{Z}})^{k+n-1}$ generated by $\{[b_1, b_j]_{2 \leq j \leq k}, [b_1, a_i]_{1 \leq i \leq n}\}$.

(2) $\mathfrak{R}(w, v)$ is branch on $L = \langle [b_1, b_2 a_1] \rangle^{\mathfrak{R}(w, v)}$ if $k = 2$ and $n = 1$; we have $\mathfrak{R}(w, v)' / L = (\frac{\mathbb{Z}}{2\mathbb{Z}})^2$ generated by $\{[b_1, b_2], [b_2, a_1]\}$. Set $x = [b_1, b_2 a_1]$; then $L / (L \times L) = \frac{\mathbb{Z}}{4\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$, generated by $\{x, x^{a_1}\}$ with the relations $x^4 = (x^{a_1})^4 = x^2 x^{2a_1} = 1$.

(3) $\mathfrak{R}(w, v)$ is branch on $L = \langle [a_i, a_j]_{1 \leq i < j \leq n}, [a_i, b_1]_{1 \leq i < n} \rangle^{\mathfrak{R}(w, v)}$ if $k = 1$ and $n \geq 2$; we have $\mathfrak{R}(w, v)' / L = 2$, generated by $[a_n, b_1]$, and $L / (L \times L) = \frac{\mathbb{Z}}{4\mathbb{Z}} \times (\frac{\mathbb{Z}}{2\mathbb{Z}})^{n-2}$, generated by $\{[a_i, b_1]\}$ where $[a_1, b_1]$ has order 4 and the other generators have order 2. *smallskip*

(4) If $k = n = 1$, then $\mathfrak{R}(w, v)$ is infinite dihedral, and is not even weakly branch.

Proof. We use the endomorphism from Proposition 4.5. In all cases, it is easy to check that L has finite index as claimed, and that $\varphi(L) = L \times 1 \subset \mathfrak{K}(w, v) \times 1$.

It then remains to check that $\varphi(L) \subset L$. This is obvious if $L = \mathfrak{K}(w, v)'$ is characteristic. If $k = 2$ and $n = 1$, we obviously have $\varphi(\gamma_3(\mathfrak{K}(w, v))) \subset \gamma_3(\mathfrak{K}(w, v))$; but we also have $\varphi(x) = [b_2^r, a_1^t a_1^u] \in \gamma_3(\mathfrak{K}(w, v))$, so $\varphi(L) \subset L$ with $L = \langle \gamma_3(\mathfrak{K}(w, v)), x \rangle$ as claimed.

If $k = 1$ and $n \geq 2$, then $\varphi[a_i, a_j] = [a_{i+1}^s, a_{j+1}^t]$ for some $s, t \in \langle b_1 \rangle$; now $i + 1 < n$ or $j + 1 < n$, so $[a_{i+1}, a_{j+1}] \in L$ and either $[a_{i+1}, b_1] \in L$ and $[a_{j+1}, b_1] \in L$, so $[a_{i+1}^s, a_{j+1}^t] \in L$. Similarly $\varphi[a_i, b_1] = [a_{i+1}^s, a_1^t] \in L$. \square

4.5. Torsion. First, we note that the group $\mathfrak{K}(w, v)$ always contains elements of infinite order:

Proposition 4.11. *Every element x which is a product, in any order, of all the generators a_i, b_j of $\mathfrak{K}(w, v)$ has infinite order.*

Proof. Consider such an x . Its image in $\text{aut}(X^*)/\text{aut}(X^*)'$ is $(1, 1, \dots)$ so this element acts level-transitively; in particular, it has infinite order. \square

Proposition 4.12. *If $k \neq 1$ or $n \neq 1$, then $\mathfrak{K}(w, v)$ contains every finite 2-group as a subgroup. In particular, it contains torsion elements of arbitrarily large order.*

Proof. This follows directly from Proposition 2.6, since the groups $\mathfrak{K}(w, v)$ are regular branch on a subgroup L containing a torsion element.

If $k \geq 2$ and $n \geq 2$, this torsion element may be chosen as $[b_1, b_2]$. If $k = 2$ and $n = 1$, we may check that $(b_1 a_1)^4$ belongs to L and has order 2. If $k = 1$ and $n \geq 2$ then $[b_1, a_1]$ belongs to L and has order 4. \square

5. Kneading sequences and quadratic polynomials

5.1. Review of results in holomorphic dynamics. Let $f(z) = z^2 + c$ be a quadratic polynomial. Suppose that the orbit of the critical point 0 under the iterations of f is finite. Such polynomials are called *post-critically finite*. We distinguish two cases: when 0 belongs to a finite cycle (*periodic case*) and when it does not, but its orbit is still finite (*pre-periodic case*).

Recall that if $M \subset \mathbb{C}$ is a connected and closed set with connected complement, then there exists a unique biholomorphic isomorphism Φ_M of the complement $\bar{\mathbb{C}} \setminus M$ with $\{z \in \bar{\mathbb{C}} \mid |z| > R\}$ such that $\Phi_M(\infty) = \infty$ and $\Phi'_M(\infty) = 1$. The *external ray* \mathbf{R}_α is the image of the ray $\{r \cdot e^{2\pi i \alpha} \mid r \in (R, \infty)\}$ under Φ_M^{-1} . One says that an external ray \mathbf{R}_α *lands*, if the limit $\lim_{r \searrow R} \Phi_M^{-1}(r \cdot e^{2\pi i \alpha})$ exists. Here and below the

angle α is considered to be an element of the group \mathbb{R}/\mathbb{Z} , i.e., the angles are counted in full turns.

We use known facts about the dynamics of iterations of quadratic maps (see [9], [10], [11]). The filled-in Julia set K_c of $z^2 + c$ is the set of points which do not escape to infinity under iteration, and the Fatou set is the open set $\bar{\mathbb{C}} \setminus \partial K_c$. We assume that $z^2 + c$ is post-critically finite, hence K_c is connected. External rays to the Mandelbrot set are called *parameter rays*, and external rays to K_c are called *dynamical rays*.

It is easy to see that the image of the dynamical ray \mathbf{R}_α under the action of $z^2 + c$ is equal to the ray $\mathbf{R}_{2\alpha}$.

Suppose first that 0 belongs to a cycle of length n under iteration of $z^2 + c$. Then c belongs to a *hyperbolic component* M_c of the interior of the Mandelbrot set. For any other point c_1 of that component, the quadratic polynomial $z^2 + c_1$ also has a unique attracting cycle of length n . If $\Phi(c_1)$ denotes the multiplier of this cycle (i.e., the product of derivatives in all points of the cycle), then Φ is a conformal isomorphism of M_c with the open unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. We obviously have $\Phi(c) = 0$, hence c is called the *center* of the hyperbolic component M_c . The isomorphism $\Phi: M_c \rightarrow \mathbb{D}$ extends to a homeomorphism of the boundary of M_c with the unit circle. The preimage of 1 under this homeomorphism is called the *root* of the component M_c . There exist exactly two angles θ such that the parameter ray \mathbf{R}_θ lands on the root of M_c , except for $n = 1$, when such ray is unique.

In the dynamical plane, the point c belongs to a Fatou component U_c , which is periodic with period n under f . There is a unique point r on the boundary of U_c , fixed under the map $f^n: U_c \rightarrow U_c$ (since $f^n|_{U_c}$ is topologically conjugate (via the *Böttcher map*) to the restriction of z^2 to \mathbb{D}). This point and its forward images are called the *roots* of their corresponding Fatou components.

A parameter ray \mathbf{R}_θ lands on the root of the hyperbolic component M_c if and only if the dynamical ray \mathbf{R}_θ lands on the root of the Fatou component U_c . Moreover, the number $\theta \in \mathbb{R}/\mathbb{Z}$ belongs to a cycle of length n under the doubling map $\alpha \mapsto 2\alpha: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$. In particular the angle θ is equal to $p/(2^n - 1)$ for some integers p, n , and the ray $\mathbf{R}_{2^k\theta}$ lands at the root of the Fatou component to which $f^k(c)$ belongs.

Conversely, for every rational number $\theta \in \mathbb{R}/\mathbb{Z}$ with odd denominator, the parameter ray \mathbf{R}_θ lands on the root of a hyperbolic component M_c , and if c is the center of the component (i.e., the preimage of 0 under the multiplier map), then 0 has the same period under $z^2 + c$ as θ has under the doubling map, and the dynamical ray \mathbf{R}_θ lands on the root of the Fatou component of $z^2 + c$ containing c .

Suppose now that 0 is pre-periodic. Then c belongs to the boundary of the Mandelbrot set (it is a *Misiurewicz point*) and there exists a finite set of angles θ such that the parameter rays \mathbf{R}_θ land on c . For each such θ the external ray \mathbf{R}_θ in the dynamical plane of $z^2 + c$ lands on c . The pre-period of θ under the doubling map is the same as the pre-period of c under $z^2 + c$, but the period of θ may be a multiple of the period of c . Here *pre-period* and *period* of a point x under a map f are the minimal positive

integers k and n such that $f^{k+n}(x) = f^k(x)$.

For example, the point $c \approx -0.1011 + 0.9563i$ is the landing point of the parameter rays \mathbf{R}_α for $\alpha = \frac{9}{56}, \frac{11}{56}$ and $\frac{15}{56}$. The point c has pre-period of length 3 and period of length 1 (i.e., it lands on a fixed point). But the angles have period 3, namely

$$9/56 \mapsto 9/28 \mapsto 9/14 \mapsto 2/7 \mapsto 4/7 \mapsto 1/7 \mapsto 2/7.$$

The period of c is determined by θ as the period of the *kneading sequence* of θ . Let S_0 be the image in \mathbb{R}/\mathbb{Z} of the interval $[\theta/2, (1 + \theta)/2]$, and let S_1 be the image in \mathbb{R}/\mathbb{Z} of $[(1 + \theta)/2, (2 + \theta)/2]$. For every $\alpha \in \mathbb{R}/\mathbb{Z}$, denote by $I_\theta(\alpha)$ its θ -itinerary, defined as the sequence $a_0 a_1 \dots$, where

$$a_k = \begin{cases} 0 & \text{if } 2^k \alpha \in S_0, \\ 1 & \text{if } 2^k \alpha \in S_1, \\ * & \text{if } 2^k \alpha \in \{\theta/2, (1 + \theta)/2\}. \end{cases}$$

The itinerary $I_\theta(\theta)$ is called the *kneading sequence* of the point $\theta \in \mathbb{R}/\mathbb{Z}$ and is denoted $\hat{\theta}$.

If θ is periodic under the doubling map with period of length n , then its kneading sequence is of the form $vv\dots$, where

$$v = 1x_2 \dots x_{n-1}*,$$

for some $x_i \in \{0, 1\}$.

If θ is strictly pre-periodic with a pre-period of length k and period of length n , then its kneading sequence is of the form $wvv\dots$, where $w = 1x_2 \dots x_k$ and v are some words over $\{0, 1\}$ of length k and n respectively with different last letters.

It may happen that v is a proper power. Then the period of the kneading sequence is a factor of the period of θ under the doubling map. In any case if θ is pre-periodic (i.e., if its smallest denominator is even), and the parameter ray \mathbf{R}_θ lands on c , then the period of c under iteration of $z^2 + c$ is equal to the period of the kneading sequence $\hat{\theta}$.

5.2. Iterated monodromy groups of quadratic polynomials. Suppose that w is either a kneading sequence of the form $(x_1 \dots x_{n-1}*)^\omega$, or a kneading sequence of the form $y_1 \dots y_k(x_1 \dots x_n)^\omega$, where $y_k \neq x_n$ and $x_1 \dots x_n$ is not periodic, i.e., is not a proper power (note that every pre-periodic sequence can be uniquely represented in that form). Then we denote by $\mathfrak{K}(w)$ the group $\mathfrak{K}(x_1 \dots x_{n-1})$ in the first case and $\mathfrak{K}(y_1 \dots y_k, x_1 \dots x_n)$ in the second.

Theorem 5.1. *Let $f(z) = z^2 + c$ be a post-critically finite quadratic polynomial. Let $\theta \in \mathbb{R}/\mathbb{Z}$ be an angle such that the parameter ray \mathbf{R}_θ lands either on the root of the hyperbolic component M_c (if c is periodic) or on c (if c is pre-periodic).*

Then $\text{IMG}(z^2 + c)$ is isomorphic to $\mathfrak{K}(\hat{\theta})$. Moreover, the action of $\text{IMG}(z^2 + c)$ on the tree of preimages is conjugate with the action of $\mathfrak{K}(\hat{\theta})$ on the binary tree.

We consider independently the periodic and pre-periodic cases.

Proof in the periodic case. We use the *invariant spiders*, described in [14], to make cuts in \mathbb{C} . A *spider* is a collection of disjoint closed paths γ_z , called *legs*, connecting every point $z \in P_f$ to infinity. A spider \mathcal{S} is f -invariant if $f^{-1}(\mathcal{S}) \supset \mathcal{S}$, up to an isotopy relative to P_f .

The dynamical ray \mathbf{R}_θ has two preimages under f : the ray $\mathbf{R}_{\theta/2}$ and $\mathbf{R}_{(1+\theta)/2}$. These rays land on two points belonging to the boundary of the Fatou component containing 0. These two points are the preimages of the root of the component of c . Let us connect zero to these two points by *internal rays* of the Fatou component, i.e., by images of rays under the holomorphic map fixing 0 and mapping the component to the disc \mathbb{D} .

The union of the rays $\mathbf{R}_{\theta/2}$, $\mathbf{R}_{(1+\theta)/2}$ and the internal rays just constructed also divide the plane into two connected components. Let us denote the component to which c belongs by S_1 , and the other component (to which the ray \mathbf{R}_0 belongs) by S_0 .

For each $i \in \{0, 1\}$, the restriction of $z^2 + c$ to the component S_i is a homeomorphism of S_i onto the set $\mathbb{C} \setminus \gamma_c$, where γ_c is a curve connecting c to infinity. The curve γ_c is the union of the dynamical ray \mathbf{R}_θ with an internal ray.

Let $t \notin \bigcup_{n \geq 0} f^n(\gamma_c)$ be a basepoint and let $T = \bigsqcup_{n \geq 0} f^{-n}(t)$ be the corresponding tree of preimages. We will construct an isomorphism between the tree T and the binary tree $\{0, 1\}^*$ using *itineraries*.

If $z \in f^{-n}(t)$ is a vertex of the tree T , then the corresponding vertex of the binary tree is given by the word $\Lambda(z) = x_{n-1}x_{n-2} \dots x_0$, where $x_k \in \{0, 1\}$ is such that

$$f^k(z) \in S_{x_k}.$$

It follows directly from the definition that Λ is a level-preserving bijection and that if $\Lambda(z) = x_{n-1} \dots x_0$, then $\Lambda(f(z)) = x_{n-1} \dots x_1$. Hence $\Lambda : T \rightarrow \{0, 1\}^*$ is an isomorphism of rooted trees. From now on, we identify the trees X^* and T by this isomorphism.

Let $P_f = \bigcup_{i \geq 1} f^i(c)$ be the post-critical orbit of f , and set $\mathcal{M} = \mathbb{C} \setminus P_f$, so that f is a covering map from $f^{-1}(\mathcal{M})$ to \mathcal{M} . If n is the period of the point c , then n is also the period of the curve γ_c under iteration of f . Moreover, the curves $f^k(\gamma_c)$ are pairwise disjoint for $k = 0, 1, \dots, n - 1$. More precisely, they can have common points, but they do not intersect transversally, so that they become pairwise disjoint after small homotopies in \mathcal{M} , see [14]. Then the set $\{f^k(\gamma_c) \mid 0 \leq k \leq n - 1\}$ is an invariant spider. For every $z \in P_f$ we denote by γ_z the unique leg $f^k(\gamma_c)$ of the spider which connects the point z to infinity. We have then $f(\gamma_z) = \gamma_{f(z)}$. The right-hand side of Figure 3 shows the preimage of a spider.

An example of spider $\bigcup_{z \in P_f} \gamma_z$ is shown on the left hand side of Figure 3, for $c \approx -0.1225 + 0.7448i$ (the ‘‘Douady rabbit’’). The corresponding θ is equal either to $1/7$ or to $2/7$ (it is $2/7$ in our picture), the critical point belongs to a cycle of length 3 and the root of all three components is a common fixed point.

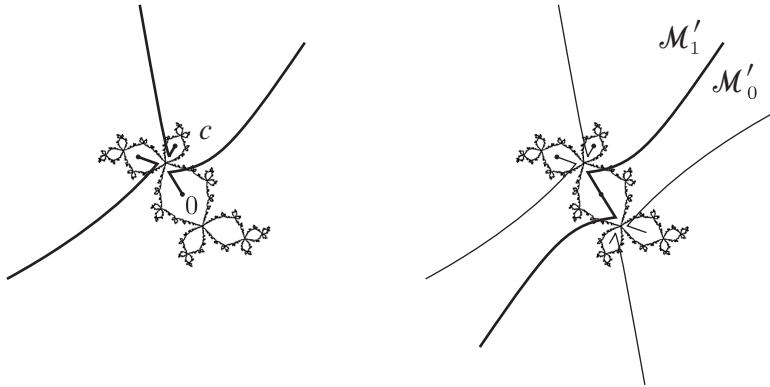


Figure 3. Invariant spider.

Write $\mathcal{M}' = \mathcal{M} \setminus \bigcup_{z \in P_f} \gamma_z$. It follows that \mathcal{M}' is simply connected and $f^{-1}(\mathcal{M}') \subset \mathcal{M}'$, up to isotopy. The set $f^{-1}(\mathcal{M}')$ has two connected components $\mathcal{M}'_0 = S_0 \cap \mathcal{M}'$ and $\mathcal{M}'_1 = S_1 \cap \mathcal{M}'$.

Define for $t, u \in \mathcal{M}'$ and $z \in P_f$ the path $g_z(t, u)$ starting in t , ending in u , disjoint from all the paths γ_y of the spider for $y \neq z$, and intersecting γ_z only once in such a direction that the part of γ_z containing z is to the left of the path $g_z(t, u)$. The path $g_z(t, u)$ is obviously defined uniquely up to homotopy in \mathcal{M} . We also denote by $g(t, u)$ the path starting in t , ending in u and disjoint from the legs of the spider. The path $g(t, u)$ is also uniquely defined up to homotopy in \mathcal{M} . If $y \notin P_f$ then $g_y(t, u)$ is, by definition, the path $g(t, u)$.

It follows from the definitions that, for $z \neq c$,

$$f^{-1}(g_z(t, u)) = \{g_{z_0}(t_0, u_0), g_{z_1}(t_1, u_1)\},$$

where $f^{-1}(z) = \{z_0, z_1\}$ and $f^{-1}(t) = \{t_0, t_1\}$ and $f^{-1}(u) = \{u_0, u_1\}$. We may assume, up to changing our notation, that $z_0 \in S_0$ and $z_1 \in S_1$. Then $t_0, u_0 \in S_0$ and $t_1, u_1 \in S_1$.

If $z = c$, then the only preimage of z is 0 and only one of the components of $f^{-1}(g_c(t, u))$ intersects the path γ_0 . Then this component is homotopic to $g_0(t_0, u_0)$, and the other component is homotopic to $g(t_1, u_1)$. If γ_0 is the extension of the ray $\mathbf{R}_{\theta/2}$, then $t_0, u_1 \in S_0$ and $t_1, u_0 \in S_1$. If γ_0 is the extension of the ray $\mathbf{R}_{(1+\theta)/2}$, then $t_0, u_1 \in S_1$ and $t_1, u_0 \in S_0$.

We also have that the f -preimages of the path $g(t, u)$ are the paths $g(t_0, u_0)$ and $g(t_1, u_1)$, where $t_0, u_0 \in S_0$ and $t_1, u_1 \in S_1$.

It is not hard to see now that if γ_0 is the extension of the ray $\mathbf{R}_{(1+\theta)/2}$, then the generators $h_k = g_{f^k(0)}(t, t)$ of $\pi_1(\mathcal{M}, t)$ act on the tree X^* in the same way as the

generators a_k of the group $\mathfrak{R}(\hat{\theta})$. If γ_0 is the extension of the ray $\mathbf{R}_{\theta/2}$, then the generators h_k act in the same way as the generators a_k^{-1} .

We conclude that the actions of $\text{IMG}(f)$ on T and of $\mathfrak{R}(\hat{f})$ on X^* are conjugate. Note that the proof does not depend on the choice of θ for a given c (though the curve γ_0 does depend on θ). □

The wreath recursion defining $\mathfrak{R}(\hat{\theta})$ is given by the paths disjoint with the legs of the spider connecting the basepoint to its preimages. A different choice of the connecting paths (and hence of the wreath recursion) are convenient in some other situations (see, for instance [6]).

Proof in the pre-periodic case. As before, we consider dynamical rays R_α . In the pre-periodic case, both preimages of R_θ land on 0, and they divide the plane into two connected components. The problem here is that there is no invariant spider with disjoint legs, when the period of the angle is greater than that of the kneading sequence.

However, we can find a sequence of spiders $\mathfrak{S}_0, \mathfrak{S}_1, \dots$, such that $\mathfrak{S}_{k+1} \subset f^{-1}(\mathfrak{S}_k)$. Take any spider \mathfrak{S}_0 consisting of dynamical rays landing on P_f and define inductively \mathfrak{S}_k to be the set of paths belonging to $f^{-1}(\mathfrak{S}_{k-1})$ and landing on the points of P_f . The points of P_f are not critical, therefore for every $z \in P_f$ there exists a unique path $\gamma_{z,k+1} \in f^{-1}(\mathfrak{S}_{k-1})$ landing on z . Hence the spider \mathfrak{S}_k is well defined.

We can then define the paths $g_{z,k}(t, u)$ for every spider \mathfrak{S}_k in the same way as in the periodic case. If $\gamma_{c,k}$ is the element of \mathfrak{S}_k landing on c , then we denote by $S_{0,k+1}$ and $S_{1,k+1}$ the components of $\mathbb{C} \setminus f^{-1}(\gamma_{c,k})$, where $S_{1,k+1}$ is the component containing c . The isomorphism $\Lambda: T \rightarrow X^*$ is then defined using the itineraries of points with respect to these partitions of the plane. Namely, if $z \in f^{-k}(t)$ is a vertex of the tree T , then $\Lambda(z) = x_{n-1}x_{n-2} \dots x_0$, where $x_k \in \{0, 1\}$ is such that

$$f^k(z) \in S_{x_k, n-k}.$$

It is also easy to prove that Λ is an isomorphism of rooted trees.

The same formulae for the preimages of the paths $g_{z,k}(t, u)$ hold as in the periodic case. The only differences will be that the index $k + 1$ appears at the preimages of paths and at names of the components $S_{i,k}$, and that the point 0 does not belong to the post-critical set. Therefore the preimages of $g_c(t, u)$ are the paths $g(t_0, u_0)$ and $g(t_1, u_1)$, with $t_0, u_1 \in S_{0,k+1}$ and $t_1, u_0 \in S_{1,k+1}$. The partitions of the plane into components $S_{0,k+1}$ and $S_{1,k+1}$ agree with the kneading sequence (i.e., every $z \in P_f$ belongs either only to the sectors $S_{0,k}$ or only to the sectors $S_{1,k}$), since the kneading sequences of all rays landing on c are equal to $\hat{\theta}$. These considerations prove that the generators $g_{z,0}(t, t)$ of $\text{IMG}(f)$ act in the same way as the generators a_i, b_i of the group $\mathfrak{R}(\hat{\theta})$. □

The following result follows from the classical results in symbolic dynamics of quadratic polynomials (see [9], [16]). It follows also from general results on iterated monodromy groups of expanding maps (see [17], Theorem 6.4.4).

Theorem 5.2. *Under the conditions of Theorem 5.1, the limit dynamical system of the group $\mathfrak{K}(\hat{\theta})$ is topologically conjugate to the action of the polynomial $z^2 + c$ on its Julia set.*

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