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Profinite completions of orientable Poincaré duality groups of dimension four and Euler characteristic zero

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Abstract. Let p be a prime number, $\mathcal{T}$ a class of finite groups closed under extensions, subgroups and quotients, and suppose that the cyclic group of order p is in $\mathcal{T}$.

We find some sufficient and necessary conditions for the pro-$\mathcal{T}$ completion of an abstract orientable Poincaré duality group G of dimension 4 and Euler characteristic 0 to be a profinite orientable Poincaré duality group of dimension 4 at the prime p with Euler p-characteristic 0. In particular we show that the pro-p completion $\widehat{G}_p$ of G is an orientable Poincaré duality pro-p group of dimension 4 and Euler characteristic 0 if and only if G is p-good.

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Introduction

In this paper we study pro-$\mathcal{T}$ completions of abstract Poincaré duality groups of dimension 4 with Euler characteristic 0, where $\mathcal{T}$ is a class of finite groups that is subgroup, extension and quotient closed and the cyclic group of order p is in $\mathcal{T}$ for a fixed prime p. This paper can be considered as a natural continuation of an earlier paper where profinite and pro-p completions of an abstract orientable Poincaré duality group G of dimension 3 were studied [6].

One of the results obtained in [6] is an algebraic proof of the Reznikov's claim that the pro-p completion of the fundamental group of a closed orientable hyperbolic 3-manifold that violates the Thurston Conjecture is an orientable pro-p Poincaré duality group provided the pro-p completion is infinite [11]. A quite different proof of the same claim was independently discovered by T. Weigel [16].

We call a profinite group a strong PD_n group at p if it is a profinite Poincaré duality group of dimension n at p according to the definition of [15] and keep the name of profinite PD_n group at p for groups satisfying the original Tate's definition [10], [14]. We discuss in details both definitions in the preliminaries.

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In the case of pro-p groups both definitions are equivalent, but it is not known whether they are equivalent in general.

Theorem 1. *Let p be a prime number and G be an abstract orientable Poincaré duality group of dimension 4 and Euler characteristic $\chi(G) = 0$.*

Let $\mathcal{T}$ be a class of finite groups closed under subgroups, extensions and quotients, let the cyclic group of order p be in $\mathcal{T}$ and let $\mathcal{C}$ be a directed set of normal subgroups of finite index in G such that $\mathcal{C}$ induces the pro-$\mathcal{T}$ topology of G.

Then

$$\widehat{G}_{\mathcal{C}} = \varprojlim_{U \in \mathcal{C}} G/U$$

is an orientable profinite Poincaré duality group of dimension 4 at the prime p with Euler p-characteristic $\chi_p(\widehat{G}_{\mathcal{C}}) = 0$ if and only if all of the following conditions hold:

a) $\mathrm{cd}_p(\widehat{G}_{\mathcal{C}})$ *is finite and the Sylow p-subgroups of $\widehat{G}_{\mathcal{C}}$ are not free or trivial, i.e.,* $2 \leq \mathrm{cd}_p(\widehat{G}_{\mathcal{C}}) < \infty$;

b) *for every $U \in \mathcal{C}$ we have* $\sum_{0 \leq i \leq 4} (-1)^i \dim_{\mathbb{F}_p} H_i(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) = 0$;

c) *for every $U \in \mathcal{C}$ we have* $2 \dim_{\mathbb{F}_p} H_1(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H_2(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) = 2$.

Furthermore, if the conditions a), b) *and* c) *hold, then $\widehat{G}_{\mathcal{C}}$ is a strong profinite orientable Poincaré duality group of dimension* 4 *at p.*

Remarks. 1. Since $\infty > \mathrm{cd}_p(\widehat{G}_{\mathcal{C}}) \geq 1$ every Sylow p-subgroup of $\widehat{G}_{\mathcal{C}}$ is infinite.

2. If condition a) is substituted with $2 \leq \mathrm{cd}_p(\widehat{G}_{\mathcal{C}}) \leq 4$ condition b) can be substituted with $\chi_p(\widehat{G}_{\mathcal{C}}) = 0$, since $0 = \chi_p(\widehat{U}_{\mathcal{C}}) = \sum_{0 \leq i \leq 4} (-1)^i \dim_{\mathbb{F}_p} H_i(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p)$.

3. Theorem 1 implies that if conditions a), b) and c) hold then the only possibility for $\mathrm{cd}_p(\widehat{G}_p)$ is 4.

An abstract group G is said to be good if the natural map between continuous and abstract cohomology $H^i(\widehat{G}, M) \to H^i(G, M)$ is an isomorphism for every finite discrete G-module M, where $\widehat{G}$ is the profinite completion of G. The group G is p-good if $H^i(\widehat{G}_p, M) \to H^i(G, M)$ is an isomorphism for every p-primary finite discrete $\widehat{G}_p$- module M, where $\widehat{G}_p$ is the pro-p completion of G.

Theorem 1 easily implies that the pro-p completion $\widehat{G}_p$ of an abstract orientable PD_4 group G of Euler characteristic 0 is an orientable pro-p PD_4 group of Euler characteristic 0 if and only if G is p-good (see Corollary 2 c)).

It would be interesting to find out whether this generalizes to any dimension, i.e., whether for G an abstract orientable PD_n group of Euler characteristic 0 the pro-p completion $\widehat{G}_p$ is an orientable pro-p PD_n group of Euler characteristic 0 if and only if G is p-good.

In section 4 we show that when pro-p completions are considered the first of the conditions of Theorem 1 can be substituted with $\widehat{G}_p$ is not virtually procyclic. The new ingredient in the proofs of the following theorems is the application of some results about virtually Poincaré duality pro-p groups and the number of higher dimensional ends of a pro-p group [7], [8].

Theorem 2. *Let p be a prime number and G be an abstract orientable Poincaré duality group of dimension 4 and Euler characteristic $\chi(G) = 0$ with pro-p completion $\widehat{G}_p$. Let $\mathcal{C}$ be a directed set of normal subgroups of p-power index in G such that $\mathcal{C}$ induces the pro-p topology of G.*

Then $\widehat{G}_p$ is an orientable pro-p Poincaré duality group of dimension 4 with Euler characteristic $\chi(\widehat{G}_p) = 0$ if and only if all of the following conditions hold:

a) *$\widehat{G}_p$ is not virtually procyclic;*

b) *for every $U \in \mathcal{C}$ we have $\sum_{0\le i\le 4}(-1)^i \dim_{\mathbb{F}_p} H_i(\widehat{U}_p, \mathbb{F}_p) = 0$;*

c) *for every $U \in \mathcal{C}$ we have $2\dim_{\mathbb{F}_p} H_1(\widehat{U}_p, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H_2(\widehat{U}_p, \mathbb{F}_p) = 2$.*

Finally we show that if condition c) from Theorem 1 is slightly modified then the only possibility for the pro-p completion of G that is not an orientable PD_4 pro-p group is to be virtually $\mathbb{Z}_p$-by-$\mathbb{Z}_p$.

Theorem 3. *Let p be a prime number and G be an abstract orientable Poincaré duality group of dimension 4 and Euler characteristic $\chi(G) = 0$ with pro-p completion $\widehat{G}_p$. Let $\mathcal{C}$ be a directed set of normal subgroups of p-power index in G such that $\mathcal{C}$ induces the pro-p topology of G. Suppose that:*

a) *$\widehat{G}_p$ is not virtually procyclic and is not an orientable pro-p Poincaré duality group of dimension 4;*

b) *for every $U \in \mathcal{C}$ we have $\sum_{0\le i\le 4}(-1)^i \dim_{\mathbb{F}_p} H_i(\widehat{U}_p, \mathbb{F}_p) = 0$;*

c) $\sup_{U\in\mathcal{C}}(2\dim_{\mathbb{F}_p} H_1(\widehat{U}_p, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H_2(\widehat{U}_p, \mathbb{F}_p)) = m < \infty$.

Then $\widehat{G}_p$ is virtually $\mathbb{Z}_p$-by-$\mathbb{Z}_p$.

In [5], examples of orientable PD_3 groups M with pro-p completion $\widehat{M}_p$ procyclic (both cases of finite or infinite occur) were constructed. Then the group $G = \mathbb{Z} \times M$ is an orientable PD_4 group with $\chi(G) = 0$ and the pro-p completion $\widehat{G}_p$ is either virtually $\mathbb{Z}_p$ or $\mathbb{Z}_p$-by-$\mathbb{Z}_p$. The group M is a double of a knot group and so M and G are not soluble, though in both cases $\widehat{G}_p$ is soluble.

1. Preliminaries on abstract and profinite Poincaré duality groups

1.1. Basic definitions and properties. Let G be an abstract group and S be a commutative ring. A $S[G]$-module V is of type FP_m for some $0 \le m \le \infty$ if there exists a projective $S[G]$-resolution of V

$$\mathcal{R}\colon\ \cdots \to R_i \to R_{i-1} \to \cdots \to R_0 \to V \to 0,$$

with R_i finitely generated for $i \leq m$. The group G is said to be of type FP_m if the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ is of type FP_m.

For a profinite group H, a profinite ring R and a profinite $R[[H]]$-module W we say that W is of type FP_m over R if there is a profinite projective $R[[H]]$- resolution of W

$$\mathcal{Q}\colon \cdots \to Q_i \to Q_{i-1} \to \cdots \to Q_0 \to W \to 0,$$

with Q_i finitely generated for $i \leq m$. The profinite group H is of homological type FP_m over R if the trivial $R[[H]]$-module R is of type FP_m.

An abstract group G is a Poincaré duality group of dimension n, provided that G is a group of cohomological dimension $\mathrm{cd}(G) = n$ of type FP_∞ and $H^*(G, \mathbb{Z}[G]) = \mathrm{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}[G])$ is concentrated in dimension n, where it is $\mathbb{Z}$. If the G-action on $H^n(G, \mathbb{Z}[G])$ is the trivial one, G is orientable; otherwise G is called non-orientable. There is an equivalent definition of abstract Poincaré duality group of dimension n, i.e., there is an isomorphism $H^i(G, M) \simeq H_{n-i}(G, D \otimes_{\mathbb{Z}} M)$ for all G-modules M and all i, where the dualizing module D is $H^n(G, \mathbb{Z}[G]) \simeq \mathbb{Z}$ [2], Ch. 8, Prop. 10.1.

There are two definitions of a profinite Poincaré duality group H of dimension n at a prime p [10], 3.4.6, [15]. The definitions differ at the point whether H should be of type FP_∞ over $\mathbb{Z}_p$. As mentioned in the introduction, we call the groups satisfying the definition of [15] strong profinite PD_n groups at p and the groups satisfying the original Tate's definition [10], 3.4.6, [14] we call profinite PD_n groups at p . A strong PD_n group at p has cohomological p-dimension $\mathrm{cd}_p(H) = n$, has type FP_∞ over $\mathbb{Z}_p$ and $H^k(H, \mathbb{Z}_p[[H]]) = \mathrm{Ext}^k_{\mathbb{Z}_p[[H]]}(\mathbb{Z}_p, \mathbb{Z}_p[[H]])$ is 0 for $k \neq n$ and is $\mathbb{Z}_p$ for $k = n$. If the action of H on $H^n(H, \mathbb{Z}_p[[H]])$ is trivial H is called orientable.

By [15], strong profinite PD_n groups at p are profinite PD_n groups at p. For a profinite PD_n group H at p and A an arbitrary p-primary finite discrete H-module the groups $H^i(H, A)$ are finite for all i [10], 3.4.6, [14]. The precise definition of a profinite PD_n group H at p can be found in [10], Chapter 3. Some important properties of such a group H are $\mathrm{cd}_p(H) = n$ and $\dim_{\mathbb{F}_p} H^i(H, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^{n-i}(H, \mathbb{F}_p)$ for all $0 \leq i \leq n$. A profinite PD_n group H at p is a strong profinite PD_n group at p if it is of type FP_∞ over $\mathbb{Z}_p$. In [6] the definition of strong profinite PD_n groups at p was adopted (though the name strong was not used). Note that pro-p PD_n groups are always of type FP_∞ over $\mathbb{Z}_p$ and over $\mathbb{F}_p$, hence are strong pro-p PD_n groups.

Let G be an abstract group of finite cohomological dimension and of type FP_∞. The Euler characteristic $\chi(G)$ as defined in [2], Ch. IX, Sec. 6, is

$$\begin{aligned}\sum_i (-1)^i \operatorname{rk}_{\mathbb{Z}} H_i(G, \mathbb{Z}) &= \sum_i (-1)^i \dim_{\mathbb{F}_p} H_i(G, \mathbb{F}_p) \\ &= \sum_i (-1)^i \dim_{\mathbb{F}_p} H^i(G, \mathbb{F}_p).\end{aligned}$$

If U is a subgroup of finite index in G by [2], Ch. 9, Thm. 6.3, $\chi(U) = (G : U)\chi(G)$.

For a profinite group H of finite cohomological p-dimension $\mathrm{cd}_p(H)$ and type

FP_∞ over $\mathbb{Z}_p$ the Euler p-characteristic $\chi_p(H)$ of H is

$$\begin{aligned}\sum_i (-1)^i \operatorname{rk}_{\mathbb{Z}_p} H_i(H,\mathbb{Z}_p) &= \sum_i (-1)^i \dim_{\mathbb{F}_p} H_i(H,\mathbb{F}_p)\\ &= \sum_i (-1)^i \dim_{\mathbb{F}_p} H^i(H,\mathbb{F}_p),\end{aligned}$$

where $H_i(H,\cdot)$ and $H^i(H,\cdot)$ are the continuous homology and cohomology.

1.2. Korenev's results. Recently more homological properties of pro-p PD_n groups were discovered in [7] and [8]. As shown in [8], if a pro-p group H of type FP_n over $\mathbb{F}_p$ has the property that $H^i(H,\mathbb{F}_p[[H]]) = 0$ for all $0 \le i < n$ and $0 < \dim_{\mathbb{F}_p} H^n(H,\mathbb{F}_p[[H]]) < \infty$, then H is virtually a pro-p PD_n group. In particular, $\dim_{\mathbb{F}_p} H^n(H,\mathbb{F}_p[[H]]) = 1$ and H is of type FP_∞. An earlier version of the above result was proved in [7], where the case $n = 1$ was considered.

Note that for pro-p groups it is still not known whether Stalling's type theorem holds, i.e., if H is a pro-p group with $\dim_{\mathbb{F}_p} H^1(H,\mathbb{F}_p[[H]]) > 0$, then H splits as a free product with amalgamation or an HNN extension over a finite subgroup.

2. Profinite completions of abstract Poincaré duality groups

Let G be an abstract group of type FP_∞ and of finite cohomological dimension and let

$$\mathcal{R}\colon 0 \to R_m \xrightarrow{\partial_m} R_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} R_1 \xrightarrow{\partial_1} R_0 \xrightarrow{\partial_0} \mathbb{Z} \to 0 \tag{1}$$

be a projective resolution of the trivial (right) $\mathbb{Z}[G]$-module $\mathbb{Z}$ with all projectives finitely generated. Let $\mathcal{C}$ be a directed set of normal subgroups of finite index in G, i.e., for $U_1, U_2 \in \mathcal{C}$ there is $U \in \mathcal{C}$ such that $U \subseteq U_1 \cap U_2$. Define

$$\widehat{G}_{\mathcal{C}} = \varprojlim_{U\in\mathcal{C}} G/U$$

and for $U \in \mathcal{C}$ we define $\widehat{U}_{\mathcal{C}}$ as the inverse limit U/M over those $M \in \mathcal{C}$ such that $M \subseteq U$.

Consider the complex $\mathcal{R}_U = \mathcal{R} \otimes_{\mathbb{Z}[U]} \mathbb{F}_p$ for $U \in \mathcal{C}$. Let

$$\widehat{\mathcal{R}}\colon 0 \to \widehat{R}_m \xrightarrow{\hat\partial_m} \widehat{R}_{m-1} \xrightarrow{\hat\partial_{m-1}} \cdots \xrightarrow{\hat\partial_2} \widehat{R}_1 \xrightarrow{\hat\partial_1} \widehat{R}_0 \xrightarrow{\hat\partial_0} \mathbb{Z} \to 0 \tag{2}$$

be the inverse limit of the complexes $\mathcal{R}_U$ over $U \in \mathcal{C}$. Thus by [6], (1),

$$\widehat{\mathcal{R}} \simeq \mathcal{R} \otimes_{\mathbb{Z}[G]} \mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$$

and by [6], Lemma 2.1,

$$H_i(\widehat{\mathcal{R}}) \simeq \varprojlim_{U\in\mathcal{C}} H_i(\mathcal{R}_U) \simeq \varprojlim_{U\in\mathcal{C}} H_i(U,\mathbb{F}_p). \tag{3}$$

In the following lemma Tor denotes the left derived functor of $\otimes$ in the category of abstract modules.

Lemma 1 ([6], Thm. 2.5). *Suppose that G is an abstract group of type* FP_∞ *and finite cohomological dimension, $\mathcal{C}$ a directed set of normal subgroups U of finite index in G. Suppose further that for a fixed prime p and for all $i \geq 1$,*

$$\varprojlim_{U \in \mathcal{C}} H_i(U, \mathbb{F}_p) = 0.$$

Then

$$\mathrm{Tor}_i^{\mathbb{Z}[G]}(\mathbb{Z}, (\mathbb{Z}/p^m\mathbb{Z})[[\widehat{G}_{\mathcal{C}}]]) = 0 \quad \textit{and} \quad \mathrm{Tor}_i^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}_p[[\widehat{G}_{\mathcal{C}}]]) = 0$$

for all $m \geq 1$ and $i \geq 1$. In particular $\widehat{G}_{\mathcal{C}}$ is of type FP_∞ *over $\mathbb{Z}_p$.*

Lemma 2 ([6], Cor. 2.7). *Suppose that G is an abstract group of finite cohomological dimension* $\mathrm{cd}(G) = m$ *and type* FP_∞. *Let $\mathcal{C}$ be a directed set of normal subgroups U of finite index in G. Suppose further that*

$$\varprojlim_{U \in \mathcal{C}} H_i(U, \mathbb{F}_p) = 0$$

for a fixed prime p and for all $1 \leq i \leq m$.

Then the profinite group $\widehat{G}_{\mathcal{C}}$ is of finite cohomological p-dimension $\mathrm{cd}_p(\widehat{G}_{\mathcal{C}}) \leq m$. *Further, it is of type* FP_∞ *over $\mathbb{F}_p$ and over $\mathbb{Z}_p$, and its Euler p-characteristic $\chi_p(\widehat{G}_{\mathcal{C}}) = \chi(G)$.*

Theorem 4. *Let G be an abstract Poincaré duality group of dimension m and let $\mathcal{C}$ be a directed set of normal subgroups of finite index in G. Suppose further that there is a subgroup G_0 of finite index in G such that G_0 is orientable, that there is some $U_0 \in \mathcal{C}$ with $U_0 \subseteq G_0$ and that, for all $i \geq 1$,*

$$\varprojlim_{U \in \mathcal{C}} H_i(U, \mathbb{F}_p) = 0.$$

Then $\widehat{G}_{\mathcal{C}}$ is a strong profinite Poincaré duality group of dimension m at p, $(\widehat{G_0})_{\mathcal{C}}$ is a strong orientable profinite Poincaré duality group of dimension m at p and $\chi_p(\widehat{G}_{\mathcal{C}}) = \chi(G)$.

Proof. Let

$$\mathcal{R}\colon 0 \to R_m \xrightarrow{\partial_m} R_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} R_1 \xrightarrow{\partial_1} R_0 \xrightarrow{\partial_0} \mathbb{Z} \to 0 \tag{4}$$

be a projective resolution of the trivial $\mathbb{Z}[G_0]$-module $\mathbb{Z}$ with all projectives finitely generated.

Then $H^i(\mathcal{S}) = H^i(G_0, \mathbb{Z}[G_0])$ is 0 for $i \neq m$ and $\mathbb{Z}$ for $i = m$, where $\mathcal{S} = \mathrm{Hom}_{\mathbb{Z}[G_0]}(\mathcal{R}^{\mathrm{del}}, \mathbb{Z}[G_0])$ is the dual complex. Thus $\mathcal{S}$ is a complex of left $\mathbb{Z}[G_0]$-modules. Define $\mathcal{T}$ the complex obtained from $\mathcal{S}$ by adding its unique non-trivial cohomology

$$\mathcal{T}: 0 \to S^0 \to S^1 \to S^2 \to \cdots \to S^m \to H^m(\mathcal{S}) = \mathbb{Z} \to 0.$$

In particular, the complex $\mathcal{T}$ is a projective resolution of the trivial left $\mathbb{Z}[G_0]$-module $\mathbb{Z}$. By Lemma 1, $\mathrm{Tor}_i^{\mathbb{Z}[G_0]}(\mathbb{Z}, \mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]) = 0$ and similarly we get that $\mathrm{Tor}_i^{\mathbb{Z}[G_0]}(\mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]], \mathbb{Z}) = 0$ for $i \geq 1$. Thus

$$\widehat{\mathcal{T}} = \mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]] \otimes_{\mathbb{Z}[G_0]} \mathcal{T} : 0 \to T^0 \to T^1 \to T^2 \to \cdots \to T^m \to \mathbb{Z}_p \to 0$$

is a projective resolution of the trivial abstract left $\mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]$-module $\mathbb{Z}_p$ with all projectives finitely generated, and hence is a profinite projective resolution of $\mathbb{Z}_p$ over $\mathbb{Z}_p[[(\widehat{G}_0)_{\mathcal{C}}]]$.

Let $\widehat{\mathcal{T}}^{\mathrm{del}}$ be the complex obtained from $\widehat{\mathcal{T}}$ by deleting the term $\mathbb{Z}_p$.

Note that $\widehat{\mathcal{T}}^{\mathrm{del}}$ is obtained from the complex $\mathcal{R}^{\mathrm{del}}$ of projective finitely generated $\mathbb{Z}[G_0]$-modules by applying first the functor $\mathrm{Hom}_{\mathbb{Z}[G_0]}(\cdot, \mathbb{Z}[G_0])$ and then the functor $\mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]] \otimes_{\mathbb{Z}[G_0]}$. The composition of these functors is the same as the composition of the functor $\otimes_{\mathbb{Z}[G_0]} \mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]$ and the functor $\mathrm{Hom}_{\mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]}(\,\cdot\,, \mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]])$ if applied on a complex of finitely generated, projective $\mathbb{Z}[G_0]$-modules. Thus

$$\widehat{\mathcal{T}}^{\mathrm{del}} \simeq \mathrm{Hom}_{\mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]}(\mathcal{P}^{\mathrm{del}}, \mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]),$$

where $\mathcal{P} = \mathcal{R} \otimes_{\mathbb{Z}[G_0]} \mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]] \simeq \widehat{\mathcal{R}}$ is an exact complex of right $\mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]$-modules by Lemma 1. Then

$$\begin{aligned} H^i((\widehat{G_0})_{\mathcal{C}}, \mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]) &= \mathrm{Ext}^i_{\mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]}(\mathbb{Z}_p, \mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]) \\ &\simeq H^i(\mathrm{Hom}_{\mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]}(\mathcal{P}^{\mathrm{del}}, \mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]])) \simeq H^i(\widehat{\mathcal{T}}^{\mathrm{del}}) \end{aligned}$$

is 0 for $i \neq m$ and is $\mathbb{Z}_p$ otherwise. Thus $(\widehat{G_0})_{\mathcal{C}}$ is a strong profinite PD_m group at p and is orientable since in the complex $\widehat{\mathcal{T}}$ the module $\mathbb{Z}_p$ is the trivial one, i.e., $(\widehat{G_0})_{\mathcal{C}}$ acts trivially on $\mathbb{Z}_p$.

Note that $(\widehat{G_0})_{\mathcal{C}}$ is a subgroup of finite index in $\widehat{G}_{\mathcal{C}}$ and, by Lemma 2, $\widehat{G}_{\mathcal{C}}$ is FP_∞ over $\mathbb{Z}_p$ and $\mathrm{cd}_p(\widehat{G}_{\mathcal{C}}) \leq m$. By [15], 4.2.9,

$$H^i((\widehat{G_0})_{\mathcal{C}}, \mathbb{Z}_p[[(\widehat{G_0})_{\mathcal{C}}]]) \simeq H^i(\widehat{G}_{\mathcal{C}}, \mathbb{Z}_p[[\widehat{G}_{\mathcal{C}}]]).$$

Then $H^*(\widehat{G}_{\mathcal{C}}, \mathbb{Z}_p[[\widehat{G}_{\mathcal{C}}]])$ is concentrated in dimension m, where it is $\mathbb{Z}_p$, so $\widehat{G}_{\mathcal{C}}$ is a strong profinite PD_m group at p. □

3. Profinite completions of Poincaré duality groups of dimension 4 and Euler characteristic 0

Lemma 3. *Let G be an abstract orientable Poincaré duality group of dimension* 4 *and Euler characteristic $\chi(G) = 0$. Then*

$$\begin{aligned} &2 \dim_{\mathbb{F}_p} H_1(G, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H_2(G, \mathbb{F}_p) \\ &\qquad = 2 = 2 \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p). \end{aligned}$$

Proof. Indeed $\chi(G) = 0$ together with $H_{4-i}(G, \mathbb{F}_p) \simeq H^i(G, \mathbb{F}_p) \simeq H_i(G, \mathbb{F}_p)$ for $i = 0$ and $i = 1$ gives

$$\begin{aligned} 0 = \chi(G) &= \sum_{0 \le i \le 4} (-1)^i \dim_{\mathbb{F}_p} H_i(G, \mathbb{F}_p) \\ &= 1 - \dim_{\mathbb{F}_p} H_1(G, \mathbb{F}_p) + \dim_{\mathbb{F}_p} H_2(G, \mathbb{F}_p) \\ &\quad - \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) + \dim_{\mathbb{F}_p} H^0(G, \mathbb{F}_p) \\ &= 2 - 2\dim_{\mathbb{F}_p} H_1(G, \mathbb{F}_p) + \dim_{\mathbb{F}_p} H_2(G, \mathbb{F}_p). \end{aligned}$$

The proof is completed by the isomorphisms $H^1(G, \mathbb{F}_p) \simeq G/[G, G]G^p \simeq H_1(G, \mathbb{F}_p)$ and $H^2(G, \mathbb{F}_p) \simeq H_2(G, \mathbb{F}_p)$. □

Lemma 4. *Let H be a profinite orientable Poincaré duality group of dimension* 4 *at p with Euler p-characteristic $\chi_p(H) = 0$.*

Then

$$\begin{aligned} &2\dim_{\mathbb{F}_p} H_1(H, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H_2(H, \mathbb{F}_p) \\ &\quad = 2 = 2\dim_{\mathbb{F}_p} H^1(H, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H^2(H, \mathbb{F}_p). \end{aligned}$$

Proof. Note that for $0 \le i \le 4$ we have $\dim_{\mathbb{F}_p} H_i(H, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^i(H, \mathbb{F}_p)$ by Pontryagin duality and $\dim_{\mathbb{F}_p} H^i(H, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H^{4-i}(H, \mathbb{F}_p)$ by Tate's definition of Poincaré duality. Then the proof is completed as the proof of Lemma 3. □

Proof of Theorem 1. Suppose now that the conditions a), b) and c) hold.

Since G is an abstract orientable PD_4 group every subgroup of finite index in G is an abstract orientable PD_4 group. In particular this holds for any $U \in \mathcal{C}$ and we have $H_4(U, \mathbb{F}_p) \simeq H^0(U, \mathbb{F}_p) \simeq \mathbb{F}_p$. Then the inverse limit of $H_4(U, \mathbb{F}_p)$ over $U \in \mathcal{C}$ is either $\mathbb{F}_p$ or 0. It cannot be $\mathbb{F}_p$ otherwise there exists an ideal in $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$ isomorphic to $\mathbb{F}_p$ and this easily contradicts the fact that $\widehat{G}_{\mathcal{C}}$ has an infinite Sylow p-subgroup (note that $\mathrm{cd}_p(H) = \mathrm{cd}_p(\widehat{G}_{\mathcal{C}}) \le 4 < \infty$ for H a Sylow p-subgroup of $\widehat{G}_{\mathcal{C}}$). Indeed if the inverse limit is $\mathbb{F}_p$ by (3) $H_4(\widehat{\mathcal{R}}) \simeq \mathbb{F}_p$ and by going down to a subgroup of finite index if necessary, we can assume that $\widehat{G}_{\mathcal{C}}$ acts trivially on $H_4(\widehat{\mathcal{R}}) \subseteq \widehat{R}_4$. Note that $\widehat{R}_4$ is a finite rank projective $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$-module, hence a direct summand of the finite rank free $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$-module F. Thus the trivial $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$-module $\mathbb{F}_p$ is a submodule of F and projecting to one of the free factors $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$ of F, we see that $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$ contains the trivial $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$-module $\mathbb{F}_p$ as a submodule, a contradiction. A different argument using restriction and corestriction can be used as in the proof of [6], Prop. 3.1.

Note that we have shown that

$$H_4(\widehat{\mathcal{R}}) \simeq \varprojlim_{U \in \mathcal{C}} H_4(U, \mathbb{F}_p) = 0.$$

As tensor product is a right exact functor $H_0(\widehat{\mathcal{R}}) = 0$. The condition that $\mathcal{T}$ is subgroup, extension and quotient closed and contains the cyclic group with p

elements implies that $\mathcal{T}$ contains all finite p-groups. Then, since $\mathcal{C}$ induces the pro-$\mathcal{T}$ topology of G, we obtain that for every $U \in \mathcal{C}$ there is a subgroup $U_1 \in \mathcal{C}$ with $U_1 \subseteq [U,U]U^p$. Hence the canonical map

$$\varphi_{1,U} : H_1(U,\mathbb{F}_p) \to H_1(\widehat{U}_{\mathcal{C}},\mathbb{F}_p)$$

is an isomorphism and

$$\begin{aligned} H_1(\widehat{\mathcal{R}}) &\simeq \varprojlim_{U\in\mathcal{C}} H_1(U,\mathbb{F}_p) \simeq \varprojlim_{U\in\mathcal{C}} H_1(\widehat{U}_{\mathcal{C}},\mathbb{F}_p) \\ &= H_1(\varprojlim_{U\in\mathcal{C}} \widehat{U}_{\mathcal{C}},\mathbb{F}_p) = H_1(1,\mathbb{F}_p) = 0. \end{aligned}$$

We claim that the canonical map

$$\varphi_{2,U} : H_2(U,\mathbb{F}_p) \to H_2(\widehat{U}_{\mathcal{C}},\mathbb{F}_p) \tag{5}$$

is an isomorphism.

Indeed $H_2(U,\mathbb{F}_p) \simeq H_2(\mathcal{R} \otimes_{\mathbb{Z}[U]} \mathbb{F}_p) \simeq H_2(\widehat{\mathcal{R}} \otimes_{\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]} \mathbb{F}_p)$. The partial profinite projective $\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]$-resolution $\widehat{R}_2 \xrightarrow{\widehat{\partial}_2} \widehat{R}_1 \xrightarrow{\widehat{\partial}_1} \widehat{R}_0 \to \mathbb{F}_p \to 0$ of $\mathbb{F}_p$ can be extended to a partial profinite projective $\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]$-resolution

$$\mathcal{S} : S \xrightarrow{\nu} \widehat{R}_2 \xrightarrow{\widehat{\partial}_2} \widehat{R}_1 \xrightarrow{\widehat{\partial}_1} \widehat{R}_0 \to \mathbb{F}_p \to 0,$$

where S contains $\widehat{R}_3$ as a closed submodule and ν is an extension of $\hat{\partial}_3$ from (2). Thus $H_2(\widehat{U}_{\mathcal{C}},\mathbb{F}_p) \simeq H_2(\mathcal{S} \otimes_{\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]} \mathbb{F}_p)$ is a quotient of $H_2(\widehat{\mathcal{R}} \otimes_{\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]} \mathbb{F}_p)$, and $\varphi_{2,U}$ is surjective.

Then by Lemma 3, Lemma 4 and condition c) of the theorem, it follows that

$$\begin{aligned} \dim_{\mathbb{F}_p} \ker(\varphi_{2,U}) &= \dim_{\mathbb{F}_p} H_2(U,\mathbb{F}_p) - \dim_{\mathbb{F}_p} H_2(\widehat{U}_{\mathcal{C}},\mathbb{F}_p) \\ &= (2\dim_{\mathbb{F}_p} H_1(U,\mathbb{F}_p) - 2) - (2\dim_{\mathbb{F}_p} H_1(\widehat{U}_{\mathcal{C}},\mathbb{F}_p) - 2) = 0. \end{aligned}$$

In particular (5) holds and we have

$$\begin{aligned} H_2(\widehat{\mathcal{R}}) &\simeq \varprojlim_{U\in\mathcal{C}} H_2(U,\mathbb{F}_p) \simeq \varprojlim_{U\in\mathcal{C}} H_2(\widehat{U}_{\mathcal{C}},\mathbb{F}_p) \\ &= H_2(\varprojlim_{U\in\mathcal{C}} \widehat{U}_{\mathcal{C}},\mathbb{F}_p) = H_2(1,\mathbb{F}_p) = 0. \end{aligned}$$

Note that we have proved by now that

$$H_i(\widehat{\mathcal{R}}) = 0 \quad \text{for } i \neq 3. \tag{6}$$

Let

$$\mathcal{P} : P \xrightarrow{\mu} \widehat{R}_3 \xrightarrow{\hat{\partial}_3} \widehat{R}_2 \xrightarrow{\hat{\partial}_2} \widehat{R}_1 \xrightarrow{\hat{\partial}_1} \widehat{R}_0 \xrightarrow{\hat{\partial}_0} \mathbb{F}_p \to 0$$

be a partial profinite projective resolution of the trivial $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$-module $\mathbb{F}_p$ with $\widehat{R}_4$ a closed submodule of P and μ an extension of $\hat{\partial}_4$. Hence $\mathcal{P}$ is a partial profinite projective resolution of the trivial $\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]$-module $\mathbb{F}_p$ for every $U \in \mathcal{C}$. Then the natural embedding of the 4-skeleton $\widehat{\mathcal{R}}^{(4)}$ in $\mathcal{P}$ induces an epimorphism

$$H_3(U, \mathbb{F}_p) \simeq H_3(\widehat{\mathcal{R}} \,\widehat{\otimes}_{\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]} \mathbb{F}_p) \to H_3(\mathcal{P} \,\widehat{\otimes}_{\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]} \mathbb{F}_p) \simeq H_3(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p).$$

Consequently the canonical map

$$\varphi_{3,U} : H_3(U, \mathbb{F}_p) \to H_3(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p)$$

is an epimorphism, hence

$$\dim_{\mathbb{F}_p} H_3(U, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H_3(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) \geq 0. \tag{7}$$

Note that by condition b),

$$\begin{aligned} &\sum_{0 \leq i \leq 4} (-1)^i \dim_{\mathbb{F}_p} H_i(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) \\ &\qquad = 0 = \chi(G) = \chi(U) = \sum_{0 \leq i \leq 4} (-1)^i \dim_{\mathbb{F}_p} H_i(U, \mathbb{F}_p) \end{aligned} \tag{8}$$

and since $\varphi_{i,U}$ is an isomorphism for $i = 1, 2$ it follows that $\dim_{\mathbb{F}_p} H_i(U, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H_i(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p)$ for $i = 1, 2$. Then by $H_4(U, \mathbb{F}_p) \simeq H^0(U, \mathbb{F}_p) \simeq \mathbb{F}_p$, (7) and (8) we get

$$\begin{aligned} 0 &\leq \dim_{\mathbb{F}_p} H_3(U, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H_3(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) \\ &= \sum_{0 \leq i \neq 3 \leq 4} (-1)^i (\dim_{\mathbb{F}_p} H_i(U, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H_i(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p)) \\ &= \dim_{\mathbb{F}_p} H_4(U, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H_4(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) \\ &\leq \dim_{\mathbb{F}_p} H_4(U, \mathbb{F}_p) = 1 \end{aligned}$$

and so

$$\dim_{\mathbb{F}_p} \ker(\varphi_{3,U}) = \dim_{\mathbb{F}_p} H_3(U, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H_3(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) \leq 1. \tag{9}$$

Consider the short exact sequence

$$0 \to \ker(\varphi_{3,U}) \to H_3(U, \mathbb{F}_p) \to H_3(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) \to 0$$

and the corresponding exact sequence

$$\begin{aligned} 0 &\to \varprojlim_{U \in \mathcal{C}} \ker(\varphi_{3,U}) \to \varprojlim_{U \in \mathcal{C}} H_3(U, \mathbb{F}_p) \\ &\to \varprojlim_{U \in \mathcal{C}} H_3(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) \simeq H_3(\varprojlim_{U \in \mathcal{C}} \widehat{U}_{\mathcal{C}}, \mathbb{F}_p) = 0 \to \cdots . \end{aligned}$$

Then by (9),

$$H_3(\widehat{\mathcal{R}}) \simeq \varprojlim_{U \in \mathcal{C}} H_3(U, \mathbb{F}_p) \simeq \varprojlim_{U \in \mathcal{C}} \ker(\varphi_{3,U})$$

is either zero or $\mathbb{F}_p$.

Define $V = H_3(\widehat{\mathcal{R}})$ and suppose that $V \neq 0$, consequently $V \simeq \mathbb{F}_p$. Let $U \in \mathcal{C}$ be such that U acts trivially on V. We claim that since $2 \leq \mathrm{cd}_p(\widehat{G}_{\mathcal{C}}) = t < \infty$, the projective dimension of V as a profinite $\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]$-module is $\max\{t-4, 0\}$.

Indeed if $0 \to W_1 \to W \to W_2 \to 0$ is a short exact sequence of profinite modules with W projective, then either the projective dimension of W_1 is the projective dimension of W_2 minus 1 or W_1 and W_2 are projective, i.e., both have projective dimension 0 (this follows from the fact that the projective dimension k of a profinite $\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]$-module M is the minimal non-negative integer k such that $\widehat{\mathrm{Ext}}^{k+1}_{\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]}(M, S) = 0$ for every discrete finite p-primary $\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]$-module S, where $\widehat{\mathrm{Ext}}$ is the derived functor of continuous Hom). Since the trivial $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$-module $\mathbb{F}_p$ has profinite projective dimension t over $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$, by (6) we get that $\ker(\widehat{\partial_3})$ has projective dimension $s = \max\{t-4, 0\}$ as a profinite $\mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]$-module.

Hence $\ker(\widehat{\partial_3})$ has projective dimension $s = \max\{t-4, 0\}$ as a profinite $\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]$-module.

Consider the short exact sequence of profinite $\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]$-modules

$$\mathcal{A}\colon 0 \to A_1 = \widehat{R}_4 \xrightarrow{\hat{\partial}_4} A_0 = \ker(\hat{\partial}_3) \to V \to 0, \tag{10}$$

where $V \simeq \mathbb{F}_p$ is the trivial module. Since A_1 is projective, for every discrete finite p-primary $\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]$-module S and $i \geq 2$, there is an isomorphism

$$\widehat{\mathrm{Ext}}^i_{\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]}(V, S) \simeq \widehat{\mathrm{Ext}}^i_{\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]}(A_0, S).$$

In particular if $\widehat{\mathrm{Ext}}^i_{\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]}(V, S) \neq 0$ for some $i \geq 2$ (i.e., $\mathrm{pd}_{\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]}(V) \geq 2$) we get that $\mathrm{pd}_{\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]}(V) = \mathrm{pd}_{\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]}(A_0)$. Finally since

$$\mathrm{pd}_{\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]}(V) = \mathrm{pd}_{\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]}(\mathbb{F}_p) = \mathrm{cd}_p(\widehat{U}_{\mathcal{C}}) = \mathrm{cd}_p(\widehat{G}_{\mathcal{C}}) = t \geq 2,$$

we obtain that

$$t = \mathrm{pd}_{\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]}(V) = \mathrm{pd}_{\mathbb{F}_p[[\widehat{U}_{\mathcal{C}}]]}(A_0) = s = \max\{t-4, 0\} < t,$$

a contradiction.

Thus

$$H_3(\widehat{\mathcal{R}}) = 0$$

and we have shown that

$$H_i(\widehat{R}) = 0 \quad \text{for all } i \geq 1.$$

Then by (3) we can apply Theorem 4 to deduce that $\widehat{G}_{\mathcal{C}}$ is a strong profinite orientable PD_4 group at p.

Finally we observe that if $\widehat{G}_{\mathcal{C}}$ is a profinite orientable PD_4 group at p, then obviously all conditions a), b) and c) hold. □

Corollary 1. *Let p be a prime number and G be an abstract orientable Poincaré duality group of dimension 4 and Euler characteristic $\chi(G) = 0$. Let $\mathcal{T}$ be a class of finite groups closed under subgroups, extensions and quotients, let the cyclic group of order p be in $\mathcal{T}$ and let $\mathcal{C}$ be a directed set of normal subgroups of finite index in G such that $\mathcal{C}$ induces the pro-$\mathcal{T}$ topology of G.*

Then the following conditions are equivalent:

a) $\widehat{G}_{\mathcal{C}}$ *is an orientable profinite Poincaré duality group of dimension 4 at the prime p with Euler p-characteristic $\chi_p(\widehat{G}_{\mathcal{C}}) = 0$;*

b) $\widehat{G}_{\mathcal{C}}$ *is a strong orientable profinite Poincaré duality group of dimension 4 at the prime p with Euler p-characteristic $\chi_p(\widehat{G}_{\mathcal{C}}) = 0$;*

c) $\mathrm{Tor}_i^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]) = 0$ *for every $i \geq 1$;*

d) $\mathrm{Tor}_i^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}_p[[\widehat{G}_{\mathcal{C}}]]) = 0$ *for every $i \geq 1$.*

Proof. By Theorem 1 item a) is equivalent with item b). Using again Theorem 1, $\widehat{G}_{\mathcal{C}}$ is an orientable profinite PD_4 group at p with $\chi_p(\widehat{G}_{\mathcal{C}}) = 0$ if and only if the conditions a), b) and c) from Theorem 1 hold. The proof of Theorem 1 shows that if these three conditions hold, then $\widehat{\mathcal{R}}$ is an exact complex.

Conversely, if $\widehat{\mathcal{R}}$ is an exact complex, that is,

$$0 = H_i(\widehat{\mathcal{R}}) \simeq \varprojlim_{U \in \mathcal{C}} H_i(U, \mathbb{F}_p) \tag{11}$$

for $i \geq 1$, we get by Theorem 4 that $\widehat{G}_{\mathcal{C}}$ is a strong orientable profinite PD_4 group at p with $\chi_p(\widehat{G}_{\mathcal{C}}) = 0$, hence is a profinite orientable PD_4 group at p.

Thus item a) is equivalent with $H_i(\widehat{\mathcal{R}}) = 0$ for all $i \geq 1$.

Since $H_i(\widehat{\mathcal{R}}) \simeq \mathrm{Tor}_i^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]])$ for $i \geq 1$ we see that a) and c) are equivalent. Furthermore, by Lemma 1, if (11) holds then d) holds, i.e., a) implies d).

If item d) holds then $\mathcal{S} = \mathcal{R} \otimes_{\mathbb{Z}[G]} \mathbb{Z}_p[[\widehat{G}_{\mathcal{C}}]]$ is an abstract projective resolution of $\mathbb{Z}_p$ over $\mathbb{Z}_p[[\widehat{G}_{\mathcal{C}}]]$ of finite length and finitely generated projectives in any dimension, so $\mathcal{S}$ is a profinite projective resolution of $\mathbb{Z}_p$ as a profinite $\mathbb{Z}_p[[\widehat{G}_{\mathcal{C}}]]$-module, hence as a profinite $\mathbb{Z}_p$-module.

Since $\mathcal{S} \otimes_{\mathbb{Z}_p} \mathbb{F}_p \simeq \mathcal{S} \, \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{F}_p$ we have

$$\begin{aligned} \mathrm{Tor}_i^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]) &= H_i(\mathcal{R} \otimes_{\mathbb{Z}[G]} \mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]) \\ &\simeq H_i(\mathcal{S} \otimes_{\mathbb{Z}_p} \mathbb{F}_p) \\ &\simeq H_i(\mathcal{S} \, \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{F}_p) = \widehat{\mathrm{Tor}}_i^{\mathbb{Z}_p}(\mathbb{Z}_p, \mathbb{F}_p) = 0 \quad \text{for } i \geq 1, \end{aligned}$$

where $\widehat{\mathrm{Tor}}$ denotes the left derived functor of $\widehat{\otimes}$ in the category of profinite modules, i.e., d) implies c). □

Corollary 2. *Let p be a prime number and G be an abstract orientable Poincaré duality group of dimension 4 and Euler characteristic $\chi(G) = 0$. Let $\mathcal{T}$ be a class of* *finite groups closed under subgroups, extensions and quotients, let the cyclic group of order p be in $\mathcal{T}$ and let $\mathcal{C}$ be a directed set of normal subgroups U of finite index in G such that $\mathcal{C}$ induces the pro-$\mathcal{T}$ topology of G.*

Then for the pro-$\mathcal{T}$ completion $\widehat{G}_{\mathcal{C}}$ of G the following results hold:

a) *$\widehat{G}_{\mathcal{C}}$ is an orientable profinite Poincaré duality group of dimension 4 at p with Euler p-characteristic $\chi_p(\widehat{G}_{\mathcal{C}}) = 0$ if and only if, for every $U \in \mathcal{C}$, the canonical maps between abstract and continuous homology*

$$\varphi_{i,U}\colon H_i(U, \mathbb{F}_p) \to H_i(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p)$$

are isomorphisms for all i;

b) *$\widehat{G}_{\mathcal{C}}$ is an orientable Poincaré duality group of dimension 4 at p with Euler p-characteristic $\chi_p(\widehat{G}_{\mathcal{C}}) = 0$ if and only if, for every $U \in \mathcal{C}$, the canonical maps between continuous and abstract cohomology*

$$\mu_{i,U}\colon H^i(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) \to H^i(U, \mathbb{F}_p)$$

are isomorphisms for all i;

c) *the pro-p completion of G is an orientable Poincaré duality pro-p group of dimension 4 and Euler characteristic 0 if and only if G is p-good.*

Proof. 1. If $\varphi_{i,U}$ is an isomorphism for every $U \in \mathcal{C}$

$$\begin{aligned}\varprojlim_{U\in\mathcal{C}} H_i(U, \mathbb{F}_p) &\simeq \varprojlim_{U\in\mathcal{C}} H_i(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p)\\ &= H_i(\varprojlim_{\widehat{U}\in\mathcal{C}} \widehat{U}_{\mathcal{C}}, \mathbb{F}_p) = H_i(1, \mathbb{F}_p) = 0 \quad \text{for } i \geq 1,\end{aligned}$$

and by Theorem 4, $\widehat{G}_{\mathcal{C}}$ is an orientable profinite PD_4 group at p.

2. Suppose now that $\widehat{G}_{\mathcal{C}}$ is an orientable profinite PD_4 group at p with $\chi_p(\widehat{G}_{\mathcal{C}}) = 0$ and $\mathcal{R}$ is the complex (1) for $m = 4$.

By Corollary 1, $\mathcal{R} \otimes_{\mathbb{Z}[G]} \mathbb{Z}_p[[\widehat{G}_{\mathcal{C}}]]$ is exact and the same holds for G substituted with any $U \in \mathcal{C}$ and any projective resolution of finite type and length at most 4 of the trivial $\mathbb{Z}[U]$-module $\mathbb{Z}$. In particular, $\mathcal{Q} = \mathcal{R} \otimes_{\mathbb{Z}[U]} \mathbb{Z}_p[[\widehat{U}_{\mathcal{C}}]]$ is exact. We can use the exactness of $\mathcal{Q}$ to show that the natural maps $H_i(U, M) \to H_i(\widehat{U}_{\mathcal{C}}, M)$ and $H^i(\widehat{U}_{\mathcal{C}}, M) \to H^i(U, M)$ are isomorphisms for every p-primary finite discrete $\widehat{G}_p$-module M. In particular, $\varphi_{i,U}$ and $\mu_{i,U}$ are isomorphisms. Indeed

$$H_i(\widehat{U}_{\mathcal{C}}, M) \simeq H_i(\mathcal{Q} \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p[[\widehat{U}_{\mathcal{C}}]]} M) \simeq H_i(\mathcal{R} \otimes_{\mathbb{Z}[U]} M) \simeq H_i(U, M)$$

and

$$H^i(\widehat{U}_{\mathcal{C}}, M) \simeq H^i(\widehat{\mathrm{Hom}}_{\mathbb{Z}_p[[\widehat{U}_{\mathcal{C}}]]}(\mathcal{Q}, M)) \simeq H^i(\mathrm{Hom}_{\mathbb{Z}[U]}(\mathcal{R}, M)) \simeq H^i(U, M), \tag{12}$$

where $\widehat{\text{Hom}}$ denotes continuous homomorphisms. In particular, if $\mathcal{T}$ is the class of all finite p-groups and $U = G$, then (12) implies that G is p-good.

3. Now suppose that $\mu_{i,U}$ is an isomorphism for all $i \geq 1$ and $U \in \mathcal{C}$.

We show that all three conditions a), b) and c) of Theorem 1 hold. Indeed, $H^5(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) \simeq H^5(U, \mathbb{F}_p) = 0$ for all $U \in \mathcal{C}$ and consequently by [14], Prop. 21′, $\text{cd}_p(\widehat{G}_{\mathcal{C}}) \leq 4$. Furthermore $H^4(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) \simeq H^4(U, \mathbb{F}_p) \simeq \mathbb{F}_p \neq 0$, in particular $\text{cd}_p(\widehat{U}_{\mathcal{C}}) \geq 4$ and so $4 \leq \text{cd}_p(\widehat{U}_{\mathcal{C}}) \leq \text{cd}_p(\widehat{G}_{\mathcal{C}}) \leq 4$. Finally $\dim_{\mathbb{F}_p} H^i(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) = \dim_{\mathbb{F}_p} H_i(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p)$ for all i by Pontryagin duality. Thus

$$\begin{aligned}\sum_{0 \leq i \leq 4} (-1)^i \dim_{\mathbb{F}_p} H_i(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) &= \sum_{0 \leq i \leq 4} (-1)^i \dim_{\mathbb{F}_p} H^i(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) \\ &= \sum_{0 \leq i \leq 4} (-1)^i \dim_{\mathbb{F}_p} H^i(U, \mathbb{F}_p) = \chi(U) = 0\end{aligned}$$

and

$$\begin{aligned}&2 \dim_{\mathbb{F}_p} H_1(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H_2(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) \\ &\quad = 2 \dim_{\mathbb{F}_p} H^1(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H^2(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) \\ &\quad = 2 \dim_{\mathbb{F}_p} H^1(U, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H^2(U, \mathbb{F}_p) = 2.\end{aligned}$$

4. Finally, if G is p-good, then $\mu_{i,U}$ is the composition of the maps

$$H^i(\widehat{U}_{\mathcal{C}}, \mathbb{F}_p) \to H^i(\widehat{G}_{\mathcal{C}}, \mathbb{F}_p[G/U]) \to H^i(G, \mathbb{F}_p[G/U]) \to H^i(U, \mathbb{F}_p),$$

where $\mathcal{T}$ is the class of all finite p-groups, the first and the last map are Shapiro's isomorphisms and the middle one is an isomorphism since G is p-good. Therefore, $\mu_{i,U}$ is an isomorphism. □

4. More on pro-p completions

Our first result is a more general version of Theorem 1 in the case of pro-p completions. The new ingredient is the use of cohomology with coefficients in $\mathbb{F}_p[[\widehat{G}_p]]$ together with some results from [7] and [8].

Proof of Theorem 2. The conditions of Theorem 2 include the last two of the conditions of Theorem 1 but not the first one, i.e., we are not assuming that $2 \leq \text{cd}(\widehat{G}_p)$. Note that the proof of Theorem 2 needed $2 \leq \text{cd}(\widehat{G}_p)$ in order to show $H_3(\widehat{\mathcal{R}}) \not\simeq \mathbb{F}_p$ (the only other possibility for $H_3(\widehat{\mathcal{R}})$ is 0), where $\widehat{\mathcal{R}}$ is the complex (2) for $m = 4$ and $\widehat{G}_p$ is infinite (the last holds since $\widehat{G}_p$ is not virtually procyclic, hence is not virtually trivial). Then $H_i(\widehat{\mathcal{R}}) = 0$ for $i \neq 3$ and $H_i(\widehat{\mathcal{R}})$ is either 0 or $\mathbb{F}_p$.

Let $\mathcal{R}^{\text{op}}$ be a resolution as in (1) for $m = 4$ but of the trivial left $\mathbb{Z}[G]$-module $\mathbb{Z}$ (recall that in (1) all modules are right $\mathbb{Z}[G]$- modules). Then exchanging left with right modules we get similar results for the complex $\widehat{\mathcal{R}^{\text{op}}} \simeq \mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]] \otimes_{\mathbb{Z}[G]} \mathbb{Z}$, i.e., $H_i(\widehat{\mathcal{R}^{\text{op}}}) = 0$ for $i \neq 3$ and $H_i(\widehat{\mathcal{R}^{\text{op}}}) = 0$ is either 0 or $\mathbb{F}_p$.

We claim that

$$H_3(\widehat{\mathcal{R}^{\mathrm{op}}}) \simeq H^1(\widehat{G}_p, \mathbb{F}_p[[\widehat{G}_p]]). \tag{13}$$

Suppose that (13) holds and that $H_3(\widehat{\mathcal{R}^{\mathrm{op}}}) \simeq \mathbb{F}_p$. Then $\dim_{\mathbb{F}_p} H^1(\widehat{G}_p, \mathbb{F}_p[[\widehat{G}_p]]) = 1$ and by [7], Thm. 3, $\widehat{G}_p$ is virtually $\mathbb{Z}_p$, a contradiction to condition a). Thus $\widehat{\mathcal{R}^{\mathrm{op}}}$ is an exact complex and the proof of the dual version of Theorem 4 (exchanging left with right modules) completes the proof of Theorem 2.

Finally we prove (13). Let

$$\mathcal{R}\colon 0 \to R_4 \xrightarrow{\partial_4} R_3 \xrightarrow{\partial_3} R_2 \xrightarrow{\partial_2} R_1 \xrightarrow{\partial_1} R_0 \xrightarrow{\partial_0} \mathbb{Z} \to 0 \tag{14}$$

be the complex (1) for $m = 4$.

Then $H^i(\mathcal{S}) = H^i(G, \mathbb{Z}[G])$ is 0 for $i \neq 4$ and $\mathbb{Z}$ for $i = 4$, where $\mathcal{S} = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathcal{R}^{\mathrm{del}}, \mathbb{Z}[G])$ is the dual complex, i.e., $\mathcal{S}$ is a complex of left $\mathbb{Z}[G]$-modules. Define $\mathcal{T}$ the complex obtained from $\mathcal{S}$ by adding its unique non-trivial cohomology:

$$\mathcal{T}: 0 \to S^0 \to S^1 \to S^2 \to S^3 \to S^4 \to H^4(\mathcal{S}) = \mathbb{Z} \to 0.$$

In particular the complex $\mathcal{T}$ is a projective resolution of the trivial left $\mathbb{Z}[G]$-module $\mathbb{Z}$. Consequently for

$$\widehat{\mathcal{T}} = \mathbb{F}_p[[\widehat{G}_p]] \otimes_{\mathbb{Z}[G]} \mathcal{T}: 0 \to T^0 \to T^1 \to T^2 \to T^3 \to T^4 \to \mathbb{F}_p \to 0$$

we have

$$H^i(\widehat{\mathcal{T}}) = \mathrm{Tor}_{4-i}^{\mathbb{Z}[G]}(\mathbb{F}_p[[\widehat{G}_p]], \mathbb{Z}) \ \text{ for } i \neq 4 \quad \text{and} \quad H^4(\widehat{\mathcal{T}}) = 0. \tag{15}$$

By the proof of Theorem 1,

$$H_i(\widehat{\mathcal{R}}) = 0 \quad \text{for } i \neq 3, \tag{16}$$

so $\widehat{R}_3 \to \widehat{R}_2 \to \widehat{R}_1 \to \widehat{R}_0 \to \mathbb{F}_p \to 0$ is exact, i.e., a partial projective resolution of the trivial $\mathbb{F}_p[[\widehat{G}_p]]$-module $\mathbb{F}_p$.

The deleted complex $\widehat{\mathcal{T}}^{\mathrm{del}}$ is the complex obtained from $\mathcal{T}$ by deleting the term $\mathbb{F}_p$. As in the proof of Theorem 4, we have

$$\widehat{\mathcal{T}}^{\mathrm{del}} \simeq \mathrm{Hom}_{\mathbb{F}_p[[\widehat{G}_p]]}(\widehat{\mathcal{R}}^{\mathrm{del}}, \mathbb{F}_p[[\widehat{G}_p]]).$$

Then by (16),

$$\begin{aligned} H^1(\widehat{G}_p, \mathbb{F}_p[[\widehat{G}_p]]) &= \mathrm{Ext}^1_{\mathbb{F}_p[[\widehat{G}_p]]}(\mathbb{F}_p, \mathbb{F}_p[[\widehat{G}_p]]) \\ &\simeq H^1(\mathrm{Hom}_{\mathbb{F}_p[[\widehat{G}_p]]}(\widehat{\mathcal{R}}^{\mathrm{del}}, \mathbb{F}_p[[\widehat{G}_p]])) \\ &\simeq H^1(\widehat{\mathcal{T}}^{\mathrm{del}}) \simeq \mathrm{Tor}_3^{\mathbb{Z}[G]}(\mathbb{F}_p[[\widehat{G}_p]], \mathbb{Z}) \simeq H_3(\widehat{\mathcal{R}^{\mathrm{op}}}), \end{aligned}$$

as required. □

Proof of Theorem 3. As in the proof of Theorem 1, we have

$$H_i(\widehat{\mathcal{R}}) = 0 \quad \text{for } i = 0, 1, 4, \tag{17}$$

where $\widehat{\mathcal{R}} \simeq \mathcal{R} \otimes_{\mathbb{Z}[G]} \mathbb{F}_p[[\widehat{G}_p]]$, $\mathcal{R}$ is the complex (1) for $m = 4$ and again as in the proof of Theorem 1 for $U \in \mathcal{C}$ the map

$$\varphi_{2,U} : H_2(U, \mathbb{F}_p) \to H_2(\widehat{U}_p, \mathbb{F}_p)$$

is surjective.

Then by Lemma 3,

$$\begin{aligned} 0 &\le \dim_{\mathbb{F}_p} \ker(\varphi_{2,U}) = \dim_{\mathbb{F}_p} H_2(U, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H_2(\widehat{U}_p, \mathbb{F}_p) \\ &\le 2 \dim_{\mathbb{F}_p} H_1(U, \mathbb{F}_p) - 2 + m - 2 \dim_{\mathbb{F}_p} H_1(\widehat{U}_p, \mathbb{F}_p) \\ &= m - 2, \end{aligned} \tag{18}$$

and hence

$$\dim_{\mathbb{F}_p} \varprojlim_{U \in \mathcal{C}} \ker(\varphi_{2,U}) \le m - 2. \tag{19}$$

Using the exact sequence

$$0 \to \varprojlim_{U \in \mathcal{C}} \ker(\varphi_{2,U}) \to \varprojlim_{U \in \mathcal{C}} H_2(U, \mathbb{F}_p) \to (\varprojlim_{U \in \mathcal{C}} H_2(\widehat{U}_p, \mathbb{F}_p)) = 0 \to \cdots,$$

(3) and (19) we obtain that

$$\begin{aligned} \dim_{\mathbb{F}_p} H_2(\widehat{\mathcal{R}}) &= \dim_{\mathbb{F}_p} \operatorname{Tor}_2^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{F}_p[[\widehat{G}_{\mathcal{C}}]]) \\ &= \dim_{\mathbb{F}_p} H_2(\widehat{\mathcal{R}}) \\ &= \dim_{\mathbb{F}_p} \varprojlim_{U \in \mathcal{C}} H_2(U, \mathbb{F}_p) \\ &= \dim_{\mathbb{F}_p} \varprojlim_{U \in \mathcal{C}} \ker(\varphi_{2,U}) \le m - 2 < \infty. \end{aligned} \tag{20}$$

By (18), $\sum_{0 \le i \le 4} (-1)^i \dim_{\mathbb{F}_p} H_i(U, \mathbb{F}_p) = 0 = \sum_{0 \le i \le 4} (-1)^i \dim_{\mathbb{F}_p} H_i(\widehat{U}_p, \mathbb{F}_p)$ and $H_1(U, \mathbb{F}_p) \simeq H_1(\widehat{U}_p, \mathbb{F}_p)$ we obtain that $\dim_{\mathbb{F}_p} H_3(U, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H_3(\widehat{U}_p, \mathbb{F}_p)$ equals

$$\begin{aligned} &\sum_{0 \le i \le 4, i \ne 3} (-1)^i (\dim_{\mathbb{F}_p} H_i(U, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H_i(\widehat{U}_p, \mathbb{F}_p)) \\ &\quad = \sum_{i=2,4} (\dim_{\mathbb{F}_p} H_i(U, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H_i(\widehat{U}_p, \mathbb{F}_p)) \\ &\quad \le \dim_{\mathbb{F}_p} H_4(U, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H_4(\widehat{U}_p, \mathbb{F}_p) + m - 2 \\ &\quad = m - 1 - \dim_{\mathbb{F}_p} H_4(\widehat{U}_p, \mathbb{F}_p) \\ &\quad \le m - 1 < \infty. \end{aligned} \tag{21}$$

Lemma 5. *For $U \in \mathcal{C}$ and for the canonical map*

$$\varphi_{3,U}\colon H_3(U, \mathbb{F}_p) \to H_3(\widehat{U}_p, \mathbb{F}_p)$$

we have

$$\dim_{\mathbb{F}_p} \operatorname{coker}(\varphi_{3,U}) = \dim_{\mathbb{F}_p} H_3(\widehat{U}_p, \mathbb{F}_p) - \dim_{\mathbb{F}_p} \operatorname{im}(\varphi_{3,U}) \leq \dim_{\mathbb{F}_p} H_2(\widehat{\mathcal{R}}) \tag{22}$$

Proof. In order to prove (22) consider a short exact sequence of complexes of $\mathbb{F}_p[[\widehat{U}_p]]$-modules

$$0 \to \widehat{\mathcal{R}} \to \mathcal{Q} \to \mathcal{S} \to 0, \tag{23}$$

where all modules in $\mathcal{S}$ positioned in dimension ≤ 2 are 0, $\mathcal{S}$ is a shifted profinite deleted projective resolution of the $\mathbb{Z}_p[[\widehat{U}_p]]$-module $H_2(\widehat{\mathcal{R}})$, i.e., the first non-zero projective in $\mathcal{S}$ is in dimension 3 and

$$H_i(\mathcal{Q}) = 0 \quad \text{for } i \leq 2.$$

Furthermore there is a short exact sequence of profinite $\mathbb{F}_p[[\widehat{U}_p]]$- complexes

$$0 \to \mathcal{Q} \to \mathcal{V} \to \mathcal{W} \to 0, \tag{24}$$

where all modules in $\mathcal{W}$ positioned in dimension ≤ 3 are zero, $\mathcal{W}$ is a shifted profinite deleted projective resolution of $H_3(\mathcal{Q})$, i.e., the first non-zero projective is in dimension 4 and

$$H_i(\mathcal{V}) = 0 \quad \text{for } i \leq 3.$$

Since $\widehat{\mathcal{R}} \mathbin{\widehat{\otimes}}_{\mathbb{F}_p[[\widehat{U}_p]]} \mathbb{F}_p = \mathcal{R} \otimes_{\mathbb{Z}[G]} \mathbb{F}_p[[\widehat{G}_p]] \otimes_{\mathbb{F}_p[[\widehat{U}_p]]} \mathbb{F}_p \simeq \mathcal{R} \otimes_{\mathbb{Z}[G]} \mathbb{F}_p[G/U]$ we have $H_3(\widehat{\mathcal{R}} \mathbin{\widehat{\otimes}}_{\mathbb{F}_p[[\widehat{U}_p]]} \mathbb{F}_p) \simeq H_3(G, \mathbb{F}_p[G/U]) \simeq H_3(U, \mathbb{F}_p)$, and since $\mathcal{V}^{(4)}$ is a partial profinite projective resolution of $\mathbb{F}_p$ over $\mathbb{F}_p[[\widehat{U}_p]]$ there is an isomorphism $H_3(\mathcal{V} \mathbin{\widehat{\otimes}}_{\mathbb{F}_p[[\widehat{U}_p]]} \mathbb{F}_p) \simeq H_3(\widehat{U}_p, \mathbb{F}_p)$. Under these isomorphisms the map $\varphi_{3,U}\colon H_3(U, \mathbb{F}_p) \to H_3(\widehat{U}_p, \mathbb{F}_p)$ is the map

$$f_U\colon H_3(\widehat{\mathcal{R}} \mathbin{\widehat{\otimes}}_{\mathbb{F}_p[[\widehat{U}_p]]} \mathbb{F}_p) \to H_3(\mathcal{V} \mathbin{\widehat{\otimes}}_{\mathbb{F}_p[[\widehat{U}_p]]} \mathbb{F}_p),$$

induced by the inclusion of $\widehat{\mathcal{R}}$ in $\mathcal{V}$.

Since the complexes $\mathcal{S}$ and $\mathcal{W}$ from (23) and (24) contain only projectives, we get exact sequences of complexes

$$0 \to \widehat{\mathcal{R}} \mathbin{\widehat{\otimes}}_{\mathbb{F}_p[[\widehat{U}_p]]} \mathbb{F}_p \to \mathcal{Q} \mathbin{\widehat{\otimes}}_{\mathbb{F}_p[[\widehat{U}_p]]} \mathbb{F}_p \to \mathcal{S} \mathbin{\widehat{\otimes}}_{\mathbb{F}_p[[\widehat{U}_p]]} \mathbb{F}_p \to 0$$

and

$$0 \to \mathcal{Q} \mathbin{\widehat{\otimes}}_{\mathbb{F}_p[[\widehat{U}_p]]} \mathbb{F}_p \to \mathcal{V} \mathbin{\widehat{\otimes}}_{\mathbb{F}_p[[\widehat{U}_p]]} \mathbb{F}_p \to \mathcal{W} \mathbin{\widehat{\otimes}}_{\mathbb{F}_p[[\widehat{U}_p]]} \mathbb{F}_p \to 0$$

and the associated exact sequences in homology

$$\cdots \to H_3(\widehat{\mathcal{R}} \mathbin{\widehat{\otimes}}_{\mathbb{F}_p[[\widehat{U}_p]]} \mathbb{F}_p) \xrightarrow{f_{1,U}} H_3(\mathcal{Q} \mathbin{\widehat{\otimes}}_{\mathbb{F}_p[[\widehat{U}_p]]} \mathbb{F}_p)$$
$$\to H_3(\mathcal{S} \mathbin{\widehat{\otimes}}_{\mathbb{F}_p[[\widehat{U}_p]]} \mathbb{F}_p) = \operatorname{Tor}_0^{\mathbb{F}_p[[\widehat{U}_p]]}(H_2(\widehat{\mathcal{R}}), \mathbb{F}_p) \simeq H_2(\widehat{\mathcal{R}}) \otimes_{\mathbb{F}_p[[\widehat{U}_p]]} \mathbb{F}_p \to \cdots$$

and

$$\cdots \to H_3(\mathcal{Q} \mathbin{\widehat{\otimes}}_{\mathbb{F}_p[[\widehat{U}_p]]} \mathbb{F}_p) \xrightarrow{f_{2,U}} H_3(\mathcal{V} \mathbin{\widehat{\otimes}}_{\mathbb{F}_p[[\widehat{U}_p]]} \mathbb{F}_p)$$
$$\longrightarrow H_3(\mathcal{W} \mathbin{\widehat{\otimes}}_{\mathbb{F}_p[[\widehat{U}_p]]} \mathbb{F}_p) = 0 \to \cdots$$

Finally (22) follows from $f_U = f_{2,U} f_{1,U}$, $f_{2,U}$ is surjective and so

$$\begin{aligned} \dim_{\mathbb{F}_p} \operatorname{coker}(f_U) &\le \dim_{\mathbb{F}_p} \operatorname{coker}(f_{1,U}) \\ &\le \dim_{\mathbb{F}_p} (H_2(\widehat{\mathcal{R}}) \otimes_{\mathbb{F}_p[[\widehat{U}_p]]} \mathbb{F}_p) \\ &\le \dim_{\mathbb{F}_p} H_2(\widehat{\mathcal{R}}). \end{aligned}$$ □

Lemma 6. *For all* $i \ge 1$,

$$\operatorname{Tor}_i^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{F}_p[[\widehat{G}_p]]) \simeq H_i(\widehat{\mathcal{R}}) = H_i(\mathcal{R} \otimes_{\mathbb{Z}[G]} \mathbb{F}_p[[\widehat{G}_p]]) \tag{25}$$

is finite.

Proof. By (20), (21) and (22)

$$\begin{aligned} \dim_{\mathbb{F}_p} \ker(\varphi_{3,U}) &= \dim_{\mathbb{F}_p} H_3(U, \mathbb{F}_p) - \dim_{\mathbb{F}_p} \operatorname{im}(\varphi_{3,U}) \\ &\le \dim_{\mathbb{F}_p} H_3(\widehat{U}_p, \mathbb{F}_p) + (m-1) - \dim_{\mathbb{F}_p} \operatorname{im}(\varphi_{3,U}) \\ &\le \dim_{\mathbb{F}_p} H_2(\widehat{\mathcal{R}}) + (m-1) < \infty. \end{aligned} \tag{26}$$

Then using the exact sequences

$$0 \to \varprojlim_{U \in \mathcal{C}} \operatorname{im}(\varphi_{3,U}) \to (\varprojlim_{U \in \mathcal{C}} H_3(\widehat{U}_p, \mathbb{F}_p)) = 0 \to \cdots$$

and

$$0 \to \varprojlim_{U \in \mathcal{C}} \ker(\varphi_{3,U}) \to \varprojlim_{U \in \mathcal{C}} H_3(U, \mathbb{F}_p) \to (\varprojlim_{U \in \mathcal{C}} \operatorname{im}(\varphi_{3,U})) = 0 \to \cdots,$$

and by (26) we deduce that

$$\begin{aligned} \dim_{\mathbb{F}_p} H_3(\widehat{\mathcal{R}}) &= \dim_{\mathbb{F}_p} \operatorname{Tor}_3^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{F}_p[[\widehat{G}_p]]) \\ &= \dim_{\mathbb{F}_p} H_3(\widehat{\mathcal{R}}) \\ &= \dim_{\mathbb{F}_p} \varprojlim_{U \in \mathcal{C}} H_3(U, \mathbb{F}_p) \\ &= \dim_{\mathbb{F}_p} \varprojlim_{U \in \mathcal{C}} \ker(\varphi_{3,U}) \\ &\le \dim_{\mathbb{F}_p} H_2(\widehat{\mathcal{R}}) + (m-1) < \infty. \end{aligned} \tag{27}$$

Finally (17), (20) and (27) complete the proof. □

Consider the dual complex $\mathcal{M} = \operatorname{Hom}_{\mathbb{Z}[G]}(\mathcal{R}^{\mathrm{del}}, \mathbb{Z}[G])$. Define $\mathcal{T}$ the complex obtained from $\mathcal{M}$ by adding its unique non-trivial cohomology:

$$\mathcal{T}: 0 \to M^0 \to M^1 \to M^2 \to M^3 \to M^4 \to H^4(\mathcal{M}) = \mathbb{Z} \to 0.$$

In particular the complex $\mathcal{T}$ is a projective resolution of the trivial left $\mathbb{Z}[G]$-module $\mathbb{Z}$ and as before we define $\widehat{\mathcal{T}} = \mathbb{F}_p[[\widehat{G}_p]] \otimes_{\mathbb{Z}[G]} \mathcal{T}$. Then

$$\widehat{\mathcal{T}}^{\mathrm{del}} \simeq \operatorname{Hom}_{\mathbb{F}_p[[\widehat{G}_p]]}(\widehat{\mathcal{R}}^{\mathrm{del}}, \mathbb{F}_p[[\widehat{G}_p]]), \tag{28}$$

$$H^i(\operatorname{Hom}_{\mathbb{F}_p[[\widehat{G}_p]]}(\widehat{\mathcal{R}}^{\mathrm{del}}, \mathbb{F}_p[[\widehat{G}_p]])) \simeq H^i(\widehat{\mathcal{T}}^{\mathrm{del}}). \tag{29}$$

As in the proof of Theorem 2, let $\widehat{\mathcal{R}^{\mathrm{op}}}$ be the version of $\widehat{\mathcal{R}}$ exchanging right with left modules. Then by the dual version of (25) (i.e., exchanging left with right modules)

$$H^i(\widehat{\mathcal{T}}^{\mathrm{del}}) \simeq \operatorname{Tor}_{4-i}^{\mathbb{Z}[G]}(\mathbb{F}_p[[\widehat{G}_p]], \mathbb{Z}) \simeq H_{4-i}(\widehat{\mathcal{R}^{\mathrm{op}}}^{\mathrm{del}})$$

is finite for all $i \neq 4$ and

$$H^4(\widehat{\mathcal{T}}^{\mathrm{del}}) = 0. \tag{30}$$

Since the complex $\mathcal{S}$ in (23), considered for $U = G$, contains only projectives, we get a short exact sequence of complexes

$$\begin{aligned}\operatorname{Hom}_{\mathbb{F}_p[[\widehat{G}_p]]}(\mathcal{S}, \mathbb{F}_p[[\widehat{G}_p]]) \to \operatorname{Hom}_{\mathbb{F}_p[[\widehat{G}_p]]}(\mathcal{Q}, \mathbb{F}_p[[\widehat{G}_p]])\\ \to \operatorname{Hom}_{\mathbb{F}_p[[\widehat{G}_p]]}(\widehat{\mathcal{R}}, \mathbb{F}_p[[\widehat{G}_p]])\end{aligned}$$

and the corresponding long exact sequence in cohomology

$$\begin{aligned}\cdots \to H^1(\operatorname{Hom}_{\mathbb{F}_p[[\widehat{G}_p]]}(\mathcal{S}, \mathbb{F}_p[[\widehat{G}_p]])) = 0 \to H^1(\operatorname{Hom}_{\mathbb{F}_p[[\widehat{G}_p]]}(\mathcal{Q}, \mathbb{F}_p[[\widehat{G}_p]]))\\ \to H^1(\operatorname{Hom}_{\mathbb{F}_p[[\widehat{G}_p]]}(\widehat{\mathcal{R}}, \mathbb{F}_p[[\widehat{G}_p]])) \to H^2(\operatorname{Hom}_{\mathbb{F}_p[[\widehat{G}_p]]}(\mathcal{S}, \mathbb{F}_p[[\widehat{G}_p]])) = 0\\ \to H^2(\operatorname{Hom}_{\mathbb{F}_p[[\widehat{G}_p]]}(\mathcal{Q}, \mathbb{F}_p[[\widehat{G}_p]])) \to H^2(\operatorname{Hom}_{\mathbb{F}_p[[\widehat{G}_p]]}(\widehat{\mathcal{R}}, \mathbb{F}_p[[\widehat{G}_p]]))\\ \to H^3(\operatorname{Hom}_{\mathbb{F}_p[[\widehat{G}_p]]}(\mathcal{S}, \mathbb{F}_p[[\widehat{G}_p]])) \simeq \operatorname{Ext}^0_{\mathbb{F}_p[[\widehat{G}_p]]}(H_2(\widehat{\mathcal{R}}), \mathbb{F}_p[[\widehat{G}_p]]) \to \cdots\end{aligned}$$

Note that $\operatorname{Ext}^0_{\mathbb{F}_p[[\widehat{G}_p]]}(\mathbb{F}_p, \mathbb{F}_p[[\widehat{G}_p]]) \simeq H^0(\widehat{G}_p, \mathbb{F}_p[[\widehat{G}_p]]) = 0$ since $\widehat{G}_p$ is infinite (remember that $\widehat{G}_p$ is not virtually procyclic, hence is not virtually trivial), where $\mathbb{F}_p$ is the trivial $\mathbb{F}_p[[\widehat{G}_p]]$-module. Then since $H_2(\widehat{\mathcal{R}})$ is finite, it has a filtration of $\mathbb{F}_p[[\widehat{G}_p]]$-modules with simple quotients, and up to isomorphism there is a unique simple $\mathbb{F}_p[[\widehat{G}_p]]$-module that is the trivial $\mathbb{F}_p[[\widehat{G}_p]]$-module $\mathbb{F}_p$, we obtain that $\operatorname{Ext}^0_{\mathbb{F}_p[[\widehat{G}_p]]}(H_2(\widehat{\mathcal{R}}), \mathbb{F}_p[[\widehat{G}_p]]) = 0$.

The inclusion map $\widehat{\mathcal{R}} \to \mathcal{Q}$ induces isomorphisms

$$H^i(\operatorname{Hom}_{\mathbb{F}_p[[\widehat{G}_p]]}(\mathcal{Q}, \mathbb{F}_p[[\widehat{G}_p]])) \to H^i(\operatorname{Hom}_{\mathbb{F}_p[[\widehat{G}_p]]}(\widehat{\mathcal{R}}, \mathbb{F}_p[[\widehat{G}_p]])) \quad \text{for } i = 1, 2, \tag{31}$$

and by (29), (30), (31) and the fact that the 3-skeleton $\mathcal{Q}^{(3)}$ is a partial profinite projective resolution of $\mathbb{F}_p$ over $\mathbb{F}_p[[\widehat{G}_p]]$ it follows that

$$\begin{aligned} H^i(\widehat{G}_p, \mathbb{F}_p[[\widehat{G}_p]]) &\simeq H^i(\mathrm{Hom}_{\mathbb{F}_p[[\widehat{G}_p]]}(\mathcal{Q}, \mathbb{F}_p[[\widehat{G}_p]])) \\ &\simeq H^i(\mathrm{Hom}_{\mathbb{F}_p[[\widehat{G}_p]]}(\widehat{\mathcal{R}}, \mathbb{F}_p[[\widehat{G}_p]])) \end{aligned} \tag{32}$$

is finite for $i = 1, 2$.

Furthermore by [7], Thm. 3, and (32) either $H^1(\widehat{G}_p, \mathbb{F}_p[[\widehat{G}_p]]) = 0$ or $\widehat{G}_p$ is virtually $\mathbb{Z}_p$; the latter cannot hold by assumption. Thus $H^1(\widehat{G}_p, \mathbb{F}_p[[\widehat{G}_p]]) = 0$, and since $\widehat{G}_p$ has type FP_2 over $\mathbb{F}_p$ (remember G is FP_∞) by [8], Thm. 1, Cor. 1, and (32) it follows that

$$\text{either } H^2(\widehat{G}_p, \mathbb{F}_p[[\widehat{G}_p]]) = 0 \quad \text{or} \quad \widehat{G}_p \text{ is virtually a pro-}p \text{ PD}_2 \text{ group.} \tag{33}$$

In the first case we obtain by (29), (32) and (30) that

$$\begin{aligned} H_i(\widehat{\mathcal{R}^{\mathrm{op}}}) &\simeq H^{4-i}(\widehat{\mathcal{T}}) \\ &\simeq H^{4-i}(\mathrm{Hom}_{\mathbb{F}_p[[\widehat{G}_p]]}(\widehat{\mathcal{R}}, \mathbb{F}_p[[\widehat{G}_p]])) \\ &\simeq H^{4-i}(\widehat{G}_p, \mathbb{F}_p[[\widehat{G}_p]]) = 0 \end{aligned} \tag{34}$$

for $i = 2, 3$.

By the dual version of (17) obtained after exchanging left with right modules we have $H_i(\widehat{\mathcal{R}^{\mathrm{op}}}) = 0$ for $i = 0, 1, 4$. This combined with (34) implies that $\widehat{\mathcal{R}^{\mathrm{op}}}$ is exact, i.e., $\mathrm{Tor}_i^{\mathbb{Z}[G]}(\mathbb{F}_p[[\widehat{G}_p]], \mathbb{Z}) = 0$ for all $i \geq 1$. After exchanging left with right modules in the proof of Corollary 1 we get that condition c) of Corollary 1 can be substituted with $\mathrm{Tor}_i^{\mathbb{Z}[G]}(\mathbb{F}_p[[\widehat{G}_p]], \mathbb{Z}) = 0$ for all $i \geq 1$. Thus $\widehat{G}_p$ is an orientable pro-p PD_4 group, a contradiction, and by (33), $\widehat{G}_p$ is virtually a pro-p PD_2 group.

Finally for some $V \in \mathcal{C}$ the pro-p group $\widehat{V}_p$ is a pro-p PD_2 group, hence a Demushkin group. For such a group, we have that $\dim_{\mathbb{F}_p} H_2(\widehat{V}_p, \mathbb{F}_p) = 1$. Since $2\dim_{\mathbb{F}_p} H_1(\widehat{V}_p, \mathbb{F}_p) - \dim_{\mathbb{F}_p} H_2(\widehat{V}_p, \mathbb{F}_p) \leq m$ there is an upper bound on $\dim_{\mathbb{F}_p} H_1(\widehat{V}_p, \mathbb{F}_p)$, i.e., $\widehat{V}_p$ is a finite rank Demushkin group. The classification of all infinite Demushkin groups can be found in [3], [4], [9] and [13] and this classification implies that $\widehat{V}_p$ has infinite abelianization. In particular there is a normal closed subgroup N of $\widehat{V}_p$ such that $\widehat{V}_p/N \simeq \mathbb{Z}_p$. Because every subgroup of infinite index in a Demushkin group is a free pro-p group, N is a free pro-p group and a pro-p group of finite rank, so $N = \mathbb{Z}_p$. Thus $\widehat{V}_p$ is $\mathbb{Z}_p$-by-$\mathbb{Z}_p$. □

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