On rigidity and the isomorphism problem for tree braid groups

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Abstract. We solve the isomorphism problem for braid groups on trees with $n = 4$ or 5 strands. We do so in three main steps, each of which is interesting in its own right. First, we establish some tools and terminology for dealing with computations using the cohomology of tree braid groups, couching our discussion in the language of differential forms. Second, we show that, given a tree braid group $B_n T$ on $n = 4$ or 5 strands, $H^*(B_n T)$ is an exterior face algebra. Finally, we prove that one may reconstruct the tree T from a tree braid group $B_n T$ for $n = 4$ or 5. Among other corollaries, this third step shows that, when $n = 4$ or 5, tree braid groups $B_n T$ and trees T (up to homeomorphism) are in bijective correspondence. That such a bijection exists is not true for higher dimensional spaces, and is an artifact of the 1-dimensionality of trees. We end by stating the results for right-angled Artin groups corresponding to the main theorems, some of which do not yet appear in the literature.

1. Introduction

Given a graph Γ, the *unlabelled configuration space* $U\mathcal{C}^n\Gamma$ of n points on Γ is the space of n-element subsets of distinct points in Γ. The *n-strand braid group of Γ*, denoted $B_n\Gamma$, is the fundamental group of $U\mathcal{C}^n\Gamma$. If Γ is a tree, $B_n\Gamma$ is a *tree braid group*.

Graph braid groups are of interest because of their connections with classical braid groups (see, for instance, [23]) and right-angled Artin groups [8], [24], [16], as well as connections in robotics and mechanical engineering. Graph braid groups can, for instance, model the motions of robots moving about a factory floor [17], [11], [12], or the motions of microscopic balls of liquid on a nano-scale electronic circuit [18].

Ghrist [17] showed that the complexes $U\mathcal{C}^n\Gamma$ are $K(B_n\Gamma, 1)$ spaces. Abrams [1] showed that graph braid groups are fundamental groups of locally CAT(0) cubical complexes, and so for instance have solvable word and conjugacy problem [3]. Crisp and Wiest [8] showed that any graph braid group embeds in some right-angled Artin

group, so graph braid groups are linear, bi-orderable, and residually finite. For more information on what is known about graph braid groups, see for instance [23].

For any class of groups $\mathscr{G}$, it is interesting to ask whether or not one can algorithmically decide if two members G and G' in $\mathscr{G}$ are isomorphic as groups. We call this question the *isomorphism problem* for $\mathscr{G}$.

Isomorphism problems are one of the fundamental topics of study for combinatorial and geometric group theory. Isomorphism problems are the hardest of the three classes of algorithmic problems in group theory formulated by Max Dehn [9]. It is known that the isomorphism problem for finitely presented groups is undecidable in general [2], [22]. However, there are solutions to the isomorphism problem for certain classes of groups. A short list of such classes of groups includes: polycyclic-by-finite groups [25], finitely generated nilpotent groups [19], torsion-free word hyperbolic groups which do not split over the trivial or infinite cyclic group [26], and finitely generated fully residually free groups [4].

The purpose of this paper is to implement an algorithm to solve the isomorphism problem for tree braid groups in some cases. We prove:

Theorem A (The isomorphism problem, cf. Theorem 7.1). *Let G and G' be two groups be given by finite presentations, and assume that $G \cong B_n T$ and $G' \cong B_n T'$ for some positive integer n and finite trees T and T'. If either:*

- *$n = 4$ or 5 or*
- *at least one of G or G' is free,*

then there exists an algorithm which decides whether G and G' are isomorphic. The trees T and T' need not be specified. If one of T and T' has at least 3 essential vertices, then n need not be specified.

To prove Theorem A, the main ingredient is a bijection between trees and tree braid groups. This bijection allows us to algorithmically reconstruct the defining tree T from the tree braid group $B_n T$. This reduces the isomorphism problem for tree braid groups to the isomorphism problem for trees, which has a brute force algorithmic solution.

The bijection between trees and tree braid groups is the strongest and most difficult result of this paper, and is interesting in its own right:

Theorem B (Rigidity for 4 and 5 strand tree braid groups, cf. Theorem 6.9). *Let T and T' be two finite trees, and let $n = 4$ or 5. The tree braid groups $B_n T$ and $B_n T'$ are isomorphic as groups if and only if the trees T and T' are homeomorphic as trees.*

The idea of the proof of Theorem B is to use cohomology to reconstruct the tree T from the tree braid group $B_n T$. This reconstruction involves careful combinatorial bookkeeping in a finite simplicial complex Δ associated to $B_n T$. The complex Δ is the defining complex for an exterior face algebra structure on the cohomology ring

of B_nT (see Section 2.2 for definitions). The main technical used to prove Theorem B is that Δ exists and is unique:

Theorem C (Exterior face algebra structure on cohomology, cf. Theorem 5.2). *Let T be a finite tree. For $n = 4$ or 5, $H^*(B_nT; \mathbb{Z}/2\mathbb{Z})$ is an exterior face algebra. The simplicial complex Δ defining the exterior face algebra structure is unique and at most 1-dimensional.*

Theorem C and other related results lead to an almost complete characterization of when the cohomology of a tree braid group is an exterior face algebra (see Conjecture 5.17).

Throughout the proofs of Theorems B and C, we rely extensively on results due to discrete Morse theory, many of which were presented in previous papers: [15], [13], [16], [14]. We also develop tools to discuss many properties of cohomology rings of tree braid groups, using the language of differential forms.

The remainder of this paper is organized as follows. In Section 2, we introduce terminology about trees that we will need to prove the main results, and define exterior face algebras. In Section 3, we survey results from other sources needed in our proofs. We pay particular attention to the structure of the cohomology ring for tree braid groups, as detailed in [16]. In Section 4, we develop the notation and terminology to talk about the cohomology rings of graph braid groups in the language of differential forms. In Sections 5 and 6, we prove a number of results which lead to Theorems 5.2 and 6.9, respectively. The solution to the isomorphism problem for tree braid groups on 4 or 5 strands then follows in Section 7. Finally, in Section 8, we end with theorems for right-angled Artin groups in a similar vein to Theorems A, B, and C.

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2. Terminology

2.1. Trees. We begin with terminology for trees. Throughout this subsection, see Figure 1 for explicit examples of some of the many of the concepts we define.

Let T be a tree. We call a vertex v of T *essential* if v has degree 3 or more. Two essential vertices are considered *adjacent* if they are connected by a path which crosses no other essential vertices. An essential vertex v is *extremal* it is adjacent to exactly one other essential vertex. A tree is *linear* if there exists an embedded line segment which contains every essential vertex; equivalently, if it has at most two extremal vertices.
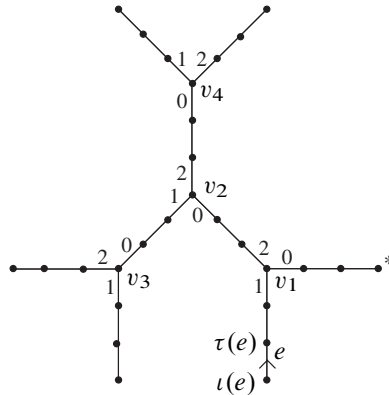
Figure 1. This figure shows an embedding of a tree in the plane. The tree is $T_{\min}$, the minimal nonlinear tree – i.e. the (unique up to homeomorphism) nonlinear tree with the fewest number of essential vertices and the smallest degrees of essential vertices. With the choice of basepoint $*$, we have a Morse $T_{\min}$-embedding. The essential vertices of $T_{\min}$ are labelled v_1, v_2, v_3, and v_4. The vertices v_1, v_3, and v_4 are extremal, and each are adjacent only to v_2. Directions from each essential vertex are labelled. For instance, v_2 is in direction 2 from v_1, and v_4 is in direction 0 from v_3. The edge e has its endpoints $\iota(e)$ and $\tau(e)$ labelled. Here, $T_{\min}$ is sufficiently subdivided for $n = 4$.

Definition 2.1 (Morse T-embedding). Let T be a tree. Embed T into the plane. Let $*$ denote a degree 1 vertex of T, called the *basepoint* of T. The information of the tree T, the embedding of T into the plane, and the choice of $*$ is called a *Morse T-embedding*.

Let T be a tree with a Morse T-embedding. Let e be an edge in T. We call the endpoint of e closer to $*$ in T the *terminal* vertex of e, denoted $\tau(e)$. This convention gives us an orientation on edges in T. Similarly, the endpoint of e further from $*$ in T is the *initial* vertex of e, denoted $\iota(e)$. This convention gives us an orientation on edges in T.

Let S be a collection of vertices and edges of T. We denote by $T - S$ the largest closed subgraph of T which does not contain an element of S. In other words, $T - S$ is formed by deleting every edge in S, as well as any vertex in S and any edge with an endpoint in S.

Let v be a vertex in T. Given a Morse T-embedding, the edges adjacent to v may be numbered $0, \ldots, \deg(v) - 1$ in the order encountered by a clockwise traversal of T from $*$, and where $\deg(v)$ denotes the degree of v. If $v = *$, the unique edge adjacent to $*$ is numbered 1. A *direction* from v is a choice of one of these edge labels. A vertex is said to *lie in* direction d from v if d labels the first edge of the unique simple path (that is, a path with no self intersections) from v to the vertex. By convention, v *lies in* direction 0 from itself. An edge *lies in* direction d from v if, for

one of the endpoints of e, d labels the first edge of the unique simple path from v to that endpoint.

An extremal vertex has the property that every other essential vertex lies in a single direction from it.

Let Δ' denote the union of those open cells of $\prod^n \Gamma$ whose closures intersect the fat diagonal $\Delta = \{(x_1, \ldots, x_n) \mid x_i = x_j \text{ for some } i \neq j\}$. Let $U\mathcal{D}^n\Gamma$ denote the quotient of the space $\prod^n \Gamma - \Delta'$ by the action of the symmetric group given by permuting coordinates. Note that $U\mathcal{D}^n\Gamma$ inherits a CW complex structure from the Cartesian product: an open cell in $U\mathcal{D}^n\Gamma$ has the form $\{y_1, \ldots, y_n\}$ such that each y_i is either a vertex or an edge and the closures of the y_i are mutually disjoint. The set notation is used to indicate that order does not matter. We call $U\mathcal{D}^n\Gamma$ the *unlabelled discretized configuration space* of Γ. Under most circumstances, the $U\mathcal{C}^n\Gamma$ is homotopy equivalent to $U\mathcal{D}^n\Gamma$. Specifically:

Theorem 2.2 (Sufficient subdivision; [21] for $n = 2$, [1] for $n > 2$). *For any $n > 1$ and any graph Γ with at least n vertices, $U\mathcal{C}^n\Gamma$ strong deformation retracts onto $U\mathcal{D}^n\Gamma$ if*

(1) *each path between distinct vertices of degree not equal to 2 passes through at least $n - 1$ edges; and*

(2) *each path from a vertex to itself which is not null-homotopic in Γ passes through at least $n + 1$ edges.*

A graph Γ satisfying the conditions of this theorem for a given n is called *sufficiently subdivided* for n. It is clear that, for any n, every graph is homeomorphic to a sufficiently subdivided graph for n.

Throughout the rest of this paper, we assume that we are dealing with graphs which are sufficiently subdivided for at least $n + 2$ strands.

We mention here that Abrams [1] proved that the universal cover of the space $U\mathcal{D}^n\Gamma$ is a CAT(0) cubical complex for any graph Γ. This implies that graph braid groups have solvable word and conjugacy problems [3].

2.2. Exterior face algebras. We now consider exterior face algebras. Let K be a finite simplicial complex with vertices $\{v_1, \ldots, v_k\}$. For any field F with identity, the *exterior face algebra* $\Lambda_F(K)$ of K over F is the quotient of the exterior algebra $\Lambda_F[v_1, \ldots, v_{k+1}]$ by the ideal generated by products of vertices of non-faces of K. In other words, $\Lambda_F(K)$ is the F-vector space having the products $v_{i_1} v_{i_2} \ldots v_{i_j}$ $(0 \leq j \leq k, i_1 < i_2 < \cdots < i_j)$ as a basis, and subject to the following multiplicative relations:

- $v_i v_j = -v_j v_i$ for $0 \leq i, j \leq n$,
- $v_i^2 = 0$ for $0 \leq i \leq n$, and
- $v_{i_1} \ldots v_{i_k} = 0$ if $i_1 < \cdots < i_k$ and $\{v_{i_1}, \ldots, v_{i_k}\}$ is not a face of K.

An exterior algebra corresponds to the exterior face algebra of a standard simplex. Note the shift of indices: an i-cell in K corresponds to an element of degree $i + 1$ in $\Lambda(K)$.

For our purposes, F will always be the field $\mathbb{Z}/2\mathbb{Z}$, so we suppress the subscript F from now on. For this field, $\Lambda(K)$ is a quotient of a polynomial ring:

$$\Lambda(K) = (\mathbb{Z}/2\mathbb{Z})\,[v_1, \ldots, v_n]\,/I(K),$$

where $I(K)$ is the ideal of $(\mathbb{Z}/2\mathbb{Z})\,[v_1, \ldots, v_n]$ generated by the set

$$\{v_1^2, \ldots, v_n^2\} \cup \{v_{i_1} \ldots v_{i_k} \mid i_1 < \cdots < i_k; \{v_{i_1}, \ldots, v_{i_k}\} \text{ is not a face of } K\}.$$

In the $\mathbb{Z}/2\mathbb{Z}$ case, Gubeladze [20] has shown that exterior face algebras are in bijective correspondence with their defining simplicial complexes:

Theorem 2.3 (Gubeladze's theorem on exterior face algebra rigidity; [20]). *Let K and K' be two finite simplicial complexes. Then $\Lambda(K)$ and $\Lambda(K')$ are isomorphic as algebras if and only if K and K' are isomorphic as simplicial complexes.*

This rigidity allows us to speak of 'the' simplicial complex defining an exterior face algebra.

As an example application of Gubeladze's theorem, see Theorem 8.3 in Section 8, on right-angled Artin groups.

3. Previous results

Now that we have introduced notation and terminology concerning trees and exterior face algebras, we turn our attention to recalling previous results – in particular important results from [15], [13], and [16]. We begin with the first two references, on fundamental group and homology.

3.1. Morse theory, the fundamental group, and homology. For a tree T, consider a Morse T-embedding (Definition 2.1). By [15], this embedding induces a 'Morse matching' on $U\mathcal{D}^n T$. For the sake of brevity, we do not define or detail the Morse matching here, but instead define the cells of $U\mathcal{D}^n T$ which are *critical* with respect to this matching. More detailed expositions on the Morse matching and the classification of cells of $U\mathcal{D}^n T$ can be found in [23], where a Morse matching is referred to as a discrete gradient vector field (the exposition in [23] is based on the original work in [15]).

Definition 3.1 (Blocked, respectful, and critical). Begin with a Morse T-embedding. Let c be an open cell in $U\mathcal{D}^n T$. Consider a vertex $v \in c$. If $v = *$, then v is *blocked* by $*$ in c. If $v \neq *$, let e be the unique edge in T with $\iota(e) = v$. If $e \cap x \neq \emptyset$ for

some edge or vertex $x \in c$, $x \neq v$, then again v is *blocked* by x in c. If v is not blocked in c, v is *unblocked*.

Now consider an edge $e \in c$. The edge e is *disrespectful* in c if: there exists a vertex $v \in c$ blocked by e and which, in a clockwise traversal of the tree T from $*$, is traversed after one endpoint of e but before the other. Otherwise, the edge e is *respectful* in c.

A cell c is *critical* if there are neither unblocked vertices nor respectful edges in c. See Figure 2 for an example.

Critical cells were seen to be very useful in describing graph braid groups, as evidenced in [15], [16], [13], etc. In particular, as we will see, critical cells lend themselves well to describing generating sets and even relations for fundamental group, homology, and cohomology.

A k-cell in $U \mathcal{D}^n T$ corresponds exactly to $n-k$ strands sitting on T and k strands simultaneously crossing k disjoint edges. To see the connection between critical cells and braids, consider a critical 1-cell $c = \{v_1, \ldots, v_{n-1}, e\}$ with unique edge $e \in c$. Then c corresponds to the following braid: start with n strands sitting next to $*$ in T. Move each strand out to a distinct vertex in $\{v_1, \ldots, v_{n-1}, \iota(e)\}$ in order so that the strand furthest away from $*$ goes to the highest-numbered vertex, the second-furthest goes to the second-highest, etc. Next, move the strand at $\iota(e)$ to $\tau(e)$. Finally, move each of the strands back to $*$, in order so that the strand on the lowest-numbered vertex in $\{v_1, \ldots, v_{n-1}, \tau(e)\}$ moves first and ends nearest $*$, the second-lowest ends second-nearest, etc. See Figure 2.
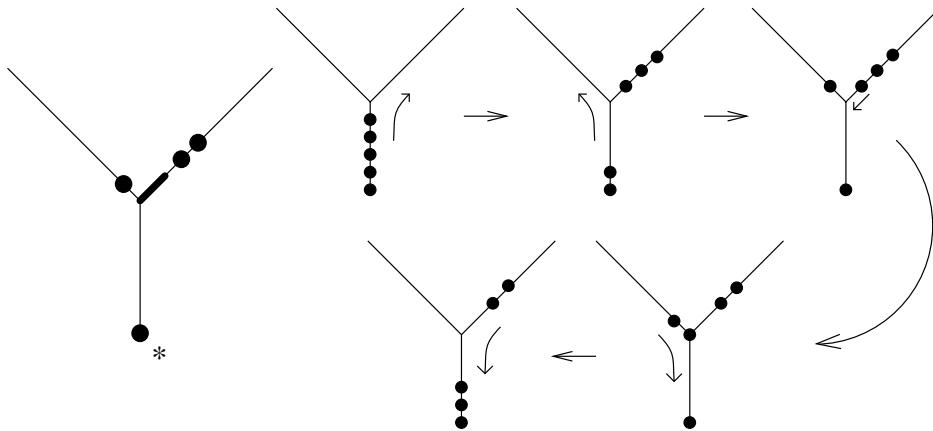

Figure 2. On the left is a critical 1-cell c. The remaining figures are a diagrammatic illustration of the braid that c represents.

The power of discrete Morse theory and the classification of critical cells is evident in the following useful theorems:

Theorem 3.2 (Morse presentation for $B_n T$; [15], corollary of Theorem 2.5). *Let T be a tree. Fix a Morse T-embedding. Then $B_n T$ has a presentation for which the generators may be identified with the set of critical 1-cells, and the relations are determined by the set of critical 2-cells.* □

A presentation derived from a Morse T-embedding as in Theorem 3.2 is called a *Morse* presentation, and its generators are Morse generators.

Theorem 3.3 (Homology; [13]). *Let T be a tree. Fix a Morse T-embedding. Then $H_i B_n T$ is free abelian of rank equal to the cardinality of the set of critical i-cells of T. In particular, $H_i B_n T$ has a generating set which may be identified with the set of critical i-cells of T.* □

Such a distinguished basis for homology is called a *Morse basis*, composed of elements called *Morse generators*.

We end this subsection with the following useful calculation about certain tree braid groups, to which we will repeatedly refer:

Theorem 3.4 (Radial rank; [15], Corollary 4.2). *If Γ is a* radial *tree – i.e. a tree with exactly one essential vertex v – then the tree braid group $B_n \Gamma$ is free of rank*

$$Y_n(x) := \sum_{i=2}^{x-1} \left[\binom{n+x-2}{n-1} - \binom{n+x-i-1}{n-1} \right],$$

where $x = \deg(v)$. □

Note that the function $Y_n(x)$ is a monotonically increasing function of x for $x \geq 3$ and of n for $n \geq 2$. For instance, $Y_2(x) = x^2/2 - 3x/2 + 1$, and $Y_3(x) = x^3/3 - x^2/2 - 5x/6 + 1$.

3.2. Cohomology. To describe the structure of cohomology for tree braid groups, we first need to introduce some definitions. We begin by defining a partial ordering and an equivalence relation on cells of $U \mathcal{D}^n T$. Using this partial order and equivalence relation, we are able to state two of the main results from [16], which are computational and structural statements about cohomology for tree braid groups, and will be referred to often. We end the section with some notation, suggested by one of these results, for what will be called reduced 1-cells.

Let $E(c)$ denote the set of edges of the i-cell c. We abuse the notation by also letting $E(c)$ denote the subset $\bigcup_{e \in E(c)} e$ of T. For two cells c and c', write $c \sim c'$ if

(1) $E(c) = E(c')$, and

(2) for any connected component C of $T - E(c)$,

$$|C \cap (c - E(c))| = |C \cap (c' - E(c'))|.$$

It is straightforward to see that $\sim$ is an equivalence relation on the set of open cells in $U\mathcal{D}^n T$. Let $[c]$ denote the equivalence class of a cell c.

If every vertex in c is blocked, we say c is *reduced*. If c is reduced but there exists an edge of c such that $\tau(e)$ is not essential, c is *extraneous*. If c is reduced, c is not a 0-cell, and for every edge e of c no vertex of c is blocked by $\tau(e)$, then again c is *extraneous*.

We define a partial order $\leq$ on the equivalence classes based on the face relation $\leq$ on cells, writing $[c_0] \leq [c_1]$ if there exist representatives $\hat{c}_0 \in [c_0]$, $\hat{c}_1 \in [c_1]$ such that $\hat{c}_0 \leq \hat{c}_1$ – i.e. $\hat{c}_0$ is a face of $\hat{c}_1$. We record some properties of $\sim$ and $\leq$ here:

Theorem 3.5 (Properties of $\leq$ and $\sim$; c.f. [16]).

(1) *The relation $\leq$ is indeed a partial order.*

(2) *Suppose that $c_1, \ldots, c_k$ are 1-cells from distinct equivalence classes. If the set $\{[c_1], \ldots, [c_k]\}$ has an upper bound $[s]$ with respect to $\leq$, then the collection has a least upper bound. Furthermore, if $e_1, \ldots, e_k$ are the edges of T satisfying $e_i \in c_i$, then the edges $e_1, \ldots, e_k$ are the edges of s and are pairwise disjoint.*

(3) *For any k-cell s, there exists a unique collection $\{[c_1], \ldots, [c_k]\}$ of equivalence classes of 1-cells such that $[s]$ is the least upper bound of $\{[c_1], \ldots, [c_k]\}$ with respect to $\leq$.*

(4) *If c is a critical cell in $U\mathcal{D}^n\Gamma$ and $[c'] \leq [c]$, then $c' \sim \hat{c}$ for some critical cell $\hat{c}$.*

(5) *There exists exactly one reduced cell in each $\sim$-equivalence class. Every critical cell is reduced, but not every reduced cell is critical. In particular, a critical cell is the unique critical cell in its equivalence class.*

Proof. Most of this theorem is from [16], Lemma 4.1 and the preceding discussion. The only part not proven in [16] is Part (5).

For Part (5), that every equivalence class contains a reduced cell is clear. That every critical cell is reduced follows from the definitions. That there exist reduced cells which are not critical is clear. That a critical cell is the unique critical cell in its equivalence class follows from the first two statements of Part (5), and was also proven in [16].

It remains to prove that two reduced cells c_1 and c_2 in the same equivalence class must be equal. By [16], Lemma 4.1(1), $E(c_1) = E(c_2)$, so let $E := E(c_1) = E(c_2)$. Consider a connected component C of $T - E$. Let $*_C$ be the smallest vertex of C. The idea of the proof is that T is sufficiently subdivided so that any collection of at most $n-1$ blocked vertices inside of C must be 'stacked up' at $*_C$, and is in particular uniquely determined. The uniqueness will force c_1 and c_2 to coincide, for each such connected component. We formalize this idea.

Since $*_C$ is the smallest vertex of C, either $*_C = *$ or $e(*_C)$ intersects an edge in E. Define a vertex v_C of T as follows. If $*_C = *$, let $v_C := *$. If $*_C \neq *$, then since $*_C$ is the smallest vertex of C, there exists an edge $e \in E$ with $e(*_C) \cap e \neq \emptyset$.

In this case, define $*_C$ to be $\tau(e)$. In the former case, v_C has degree 1 in T, and in the latter case, v_C has degree at least 3 in T. Since T is sufficiently subdivided for $n+2$ strands, v_C is at least $n+1$ edges away from any other vertex of T which does not have degree 2 in T. Since $*_C$ is at most 2 edges away from v_C, $*_C$ is at least $n-1$ edges away from any other vertex of T which does not have degree 2 in T. Since each edge of E has an essential terminal endpoint, $*_C$ is at least $n-2$ edges away from any other vertex of C which does not have degree 2 in C. Thus, since C is a (connected subset of a) tree, for any $i = 0, \ldots, n-1$, there is a unique vertex $v_{C,i}$ in C such that the unique path from $v_{C,i}$ to $*_C$ contains exactly k edges.

By the definition of $\sim$, c_1 and c_2 have the same number k of vertices in C. We claim that both c_1 and c_2 contain the k vertices $v_{C,1}, \ldots, v_{C,k}$ of C. For, assume otherwise. Without loss of generality, assume c_1 does not contain the vertex $v_{C,i}$. Let v be the smallest vertex of $c_1 \cap C$ greater than $v_{C,i}$. Then $\tau(e(v))$ is in C, since $*_C$ is in the same direction from $v_{C,i}$ in T as $*$ is. But $\tau(e(v))$ does not intersect c_1, since C contains only vertices of c_1 and v was chosen to be as small as possible. Thus, v is unblocked in c_1. This contradicts the hypothesis that c_1 is reduced. Thus, c_1 and c_2 have the same edge set E and the same vertices in each connected component of $T - E$, $c_1 = c_2$. $\square$

Let $C_*(U \mathcal{D}^n T)$ denote the cellular chain complex of chains of cells in $U \mathcal{D}^n T$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Let $\phi_{[c]} \colon C_*(U \mathcal{D}^n T) \to \mathbb{Z}/2\mathbb{Z}$ denote the characteristic function of the $\sim$-equivalence class of c: $\phi_{[c]}(c') = 1$ if and only if $c' \sim c$. For c a critical cell, let $c^* \colon H_*(B_n T) \to \mathbb{Z}/2\mathbb{Z}$ denote the dual of c viewed as a basis element of cohomology, by the Universal Coefficient Theorem. Note the boundary maps for chains are all 0 [13], so $H^*(B_n T) \cong \mathrm{Hom}(H_*(B_n, T), \mathbb{Z}/2\mathbb{Z})$. Then, using the ordering $\leq$ on equivalence classes of 1-cells, we may state the following theorem:

Theorem 3.6 (Cohomology; [16], Proposition 4.5 and preceding discussion). *Let T be a tree. Fix a Morse T-embedding (see Definition 2.1). Then under the induced Morse matching the following holds:*

(1) *If c is a critical cell in $U \mathcal{D}^n T$, then $c^* = [\phi_{[c]}]$. A distinguished basis for i-dimensional cohomology is*

$$\{c^* \mid c \text{ a critical } i\text{-cell}\}.$$

(2) *Let s be a critical i-cell in $U \mathcal{D}^n T$. Let $[s]$ be the least upper bound of $\{[c_1], \ldots, [c_i]\}$, where the $[c_1], \ldots, [c_i]$ are distinct equivalence classes of 1-cells. Then, without loss of generality, $c_1, \ldots, c_i$ are critical, and*

$$c_1^* \cup \cdots \cup c_i^* = s^*.$$

In particular, $H^*(B_n T)$ is generated as a ring by duals of critical 1-cells.

(3) *If $[c_1], \ldots, [c_i]$ are distinct equivalence classes of critical 1-cells having the least upper bound $[s]$, then*

$$\left[\phi_{[c_1]}\right] \cup \cdots \cup \left[\phi_{[c_i]}\right] = \left[\phi_{[s]}\right].$$

If $[c_1], \ldots, [c_i]$ are not all pairwise distinct or have no upper bound, then $\left[\phi_{[c_1]}\right] \cup \cdots \cup \left[\phi_{[c_i]}\right] = 0.$

Note a critical cell has dimension at most $(\lfloor \frac{n}{2} \rfloor)$ [15]. It follows from the first part of Theorem 3.6 that cohomology is trivial in all dimensions greater than $\lfloor \frac{n}{2} \rfloor$.

As with homology, such a distinguished basis for cohomology is called a *Morse basis*, composed of elements called *Morse generators*. The power of this theorem comes from the characterization not only of a Morse basis, but of the cup product structure.

3.3. Reduced cells. Throughout the remainder of this paper, we will be extensively using reduced cells and computations involving reduced cells. For future reference, we establish here notation for many reduced 1-cells, and record some properties of reduced cells.

For a vector $\vec{x} \in \mathbb{N}^\infty$ of nonnegative integers, indexed from 0 to ∞, we let x_i denote the ith entry of $\vec{x}$. Given a tree T, a Morse T-embedding, and an essential vertex v of T, the vector $\vec{x}$ is a v-*vector* if $x_i = 0$ for all $i \geq \deg(v)$. When writing v-vectors, we will omit the entries 0 for indices $i \geq \deg(v)$. The *length* of $\vec{x}$, denoted $|\vec{x}|$, is the sum $\sum_{i=0}^\infty x_i$ (all of our lengths will be finite).

Let c be a reduced 1-cell of $U\mathcal{D}^n T$ with unique edge e such that e has an essential terminal endpoint $a = \tau(e)$. By Theorem 3.5, c is unique in its equivalence class $[c]$. Thus, c is uniquely determined by the number of vertices in each connected component of $T - e$. This means that c is uniquely determined by specifying the endpoint $a = \tau(e)$, the direction d from a along e, and the number of vertices in c in each direction from a. Let $\vec{x} \in \mathbb{N}^\infty$ be the a-vector such that, for $i \in \{0, \ldots, \deg(a) - 1\}$ x_i is the number of vertices in c in direction i from a. Note $n = |\vec{x}|$.

Definition 3.7 (Reduced 1-cell notation). To encode the reduced 1-cell c with edge e such that $\tau(e)$ is the essential vertex a, we write

$$(a, d, \vec{x}) \text{ or equivalently } (a, e, \vec{x}),$$

where d and $\vec{x}$ are as above. We say that $\vec{x}$ is *the a-vector for c*, and that c *lies over* the vertex a.

This notation is a slightly modified version of the notation developed in the paper [15], and appears in [16].

Using this notation, a fairly immediate observation is:

Lemma 3.8. *Let $c = (a, e, \vec{x})$ be a non-extraneous reduced 1-cell. Then $x_0 \leq n - 2$.*

Proof. Let d be the direction from a to $\iota(e)$. Since e is not extraneous, there exists some vertex v of c such that the direction d' from a to v is not 0 or d. Thus, $x_d, x_{d'} \geq 1$. But $|\vec{x}| = n$, so $x_0 \leq n - 2$. $\qquad\square$

As Theorem 3.6 suggests, upper bounds of equivalence classes of reduced cells play an important role. We record many properties of upper bounds in the following lemma. Note that, in the proof of the last statement, we explicitly construct the reduced representative of upper bound of the equivalence classes of two non-extraneous reduced 1-cells.

Lemma 3.9 (Upper Bound Lemma). *Let $c_1 = (a, d, \vec{x})$ and $c_2 = (b, f, \vec{y})$ be non-extraneous reduced 1-cells where $a \leq b$ in the order on vertices. Let α be the direction from a to b. Then:*

(1) *the direction from b to a is 0.*

(2) *$\{[c_1], [c_2]\}$ has an upper bound if and only if*

    (a) *$a \neq b$, and*

    (b) *$x_\alpha + y_0 \geq n + \epsilon$, where $\epsilon = \epsilon(c_1, \alpha)$ is 1 if $d = \alpha$ and 0 otherwise.*

(1) *If $\{[c_1], [c_2]\}$ has an upper bound, then $x_\alpha \geq 2 + \epsilon$.*

(2) *If $\{[c_1], [c_2]\}$ has an upper bound $[s]$, let s be the reduced representative of $[s]$. We may explicitly describe s. The edge f of c_2 is disrespectful in the reduced representative s of $[s]$ if and only if f is disrespectful in c_2. The edge e of c_1 is disrespectful in s if and only if either*

    (a) *$0 < \alpha < d$ and $x_\alpha + y_0 > n$, or*

    (b) *there exists some $i \neq \alpha$ such that $0 < i < d$ and $x_i > 0$.*

The $\epsilon(c_1, \alpha)$ in the lemma is called the *upper bound constant* in direction α for c_1.

Proof. (1) If $a = b$, then the direction from $b = a$ to itself is by definition 0. If $a < b$, that the direction from b to a is 0 follows directly from the definition of $a < b$.

(2) If $\{[c_1], [c_2]\}$ has an upper bound, let $[s]$ be a least upper bound. By Theorem 3.5, $[s]$ exists, is unique, and has a unique element – say, s – in which all vertices are blocked. Since $\tau(e)$ and $\tau(f)$ are essential, s is reduced. Also by Theorem 3.5, the edges e and f are disjoint – in particular, $a \neq b$. Finally, note that x_α is the number of elements of not just c_1 but also s in direction α from a, and similarly y_β is the number of elements of s in direction β from b. Since $\beta = 0$, $n - y_\beta$ is the number of elements of s in all directions not equal to β from b. Since T is a tree, $n - y_\beta$ must be at most the number of elements in s in direction α from a – that is, x_α. But if $d = \alpha$, then the edge of c_1 which is also an edge of s is in direction α from a but direction β from b. Thus if $d = \alpha$, $n - y_\beta$ must be at most $x_\alpha - 1$. The desired inequality then follows.

Now assume that c_1 and c_2 satisfy the desired properties. Let c_1' be the 1-cell with $(n - x_\alpha + \epsilon)$ strands on T consisting of the edge of c_1 and all vertices of c_1 not in direction α from a. Similarly, let c_2' be the 1-cell with $(n - y_\beta)$ strands on T consisting of the edge of c_2 and all vertices of c_2 not in direction α from a. Since $a \neq b$, we may define the 2-cell s' to be $c_1' \cup c_2'$ with k strands on T, where $k = (n - x_\alpha + \epsilon) + (n - y_\beta)$. Since $x_\alpha + y_\beta \geq n + \epsilon$, $k \leq n$. Since T is sufficiently subdivided for $n + 2$, there are at least n vertices of T between a and b not contained in s'. Let s be the 2-cell with n strands on T which is s' plus exactly $n - k$ vertices between a and b. Then by the definitions of $\sim$ and $\leq$, $[s]$ is an upper bound for $\{[c_1], [c_2]\}$.

(3) If $\{[c_1], [c_2]\}$ has an upper bound, then $x_\alpha \geq n + \epsilon - y_0 \geq 2 + \epsilon$, as $y_0 \leq n - 2$ by Lemma 3.8.

(4) If $\{[c_1], [c_2]\}$ has an upper bound $[s]$, then by Theorem 3.5, the reduced representative s of $[s]$ is unique. We explicitly construct s.

Let s_1 be the reduced cell on y_0 strands which is c_1 but with only $x_\alpha - (n - y_0)$ strands in direction α from a. As $x_\alpha \geq n + \epsilon - y_0$, even if e is in direction α from a, s_1 still contains the edge e. The cell s_1 corresponds to deleting the $n - y_0$ largest strands with respect to the order on vertices in the direction α from a. Let s_2 be the reduced cell on $n - y_0$ strands which is c_2 but with the vertices in direction 0 from b removed. Finally, let $s = s_1 \cup s_2$. Since each of s_1 and s_2 is reduced, so is α. Note f is disrespectful in c_2 if and only if f is disrespectful in s_2, if and only if f is disrespectful in s, by the definition of disrespectful. Since e is disrespectful in c_1, e will remain disrespectful in s_1 and therefore s unless there was only one direction i such that $0 < i < d$ and $x_i > 0$ – namely, $i = \alpha$ – and there are no vertices in direction $i = \alpha$ from a in s_1. There are no vertices in direction α from a in s_1 if and only if $0 < \alpha < d$ and $x_\alpha + y_0 = n$. This finishes the proof of the lemma. $\square$

4. Cohomology in terms of differential forms

We will couch our further discussion of cohomology of tree braid groups in the terminology of de Rham cohomology and differential forms. Although we are dealing with CW complexes instead of manifolds, hopefully the similarity of the formulas, particularly in defining differentials, will justify our abuse of notation.

Recall that $C_*(U\mathcal{D}^n T)$ is the cellular chain complex on $U\mathcal{D}^n T$. We denote certain cochains on our space by the term *form*, where a k-form will be a k-cochain. The 0-forms will be functions on cellular chains in $C_*(U\mathcal{D}^n T)$ which take values in $\mathbb{Z}/2\mathbb{Z}$.

For a vertex $v \in T$ and a direction i from v, let $D_{v,i}$ be the function from cells of $U\mathcal{D}^n T$ to $\mathbb{Z}$ which takes a cell c and counts the number of vertices or edges in c in direction i from v. Thus, for instance, for a reduced cell $(a, d, \vec{x})$, $D_{a,i}(a, d, \vec{x}) = x_i$. Let $\bar{D}_{v,i}$ be similarly a function which takes a cell c and counts the number of vertices or edges in c in direction i from v, *but* where each edge of v is considered

in the direction of its terminal endpoint from v instead of its initial endpoint. Thus, for instance, $\bar{D}_{a,i}(a, d, \vec{x}) = x_i$ if $i \neq 0, d$, but $\bar{D}_{a,0}(a, d, \vec{x}) = x_0 + 1$ and $\bar{D}_{a,d}(a, d, \vec{x}) = x_d - 1$.

Example 4.1. Consider the 2-cell s shown in Figure 3. This figure depicts a 2-cell s on a tree with three essential vertices a, b, and c. We have:

$$D_{a,i}(s) = \begin{cases} 0 & \text{if } i = 0, \\ 6 & \text{if } i = 1, \\ 1 & \text{if } i = 2, \end{cases} \quad D_{b,i}(s) = \begin{cases} 1 & \text{if } i = 0, \\ 1 & \text{if } i = 1, \\ 3 & \text{if } i = 2, \\ 2 & \text{if } i = 3, \end{cases} \quad \bar{D}_{b,i}(s) = \begin{cases} 2 & \text{if } i = 0, \\ 1 & \text{if } i = 1, \\ 3 & \text{if } i = 2, \\ 1 & \text{if } i = 3. \end{cases}$$

In terms of differential forms, the following is an incomplete list of forms which map c to 1:

$$f\left(a, \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}\right), \quad f\left(b, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}\right), \quad f\left(c, \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}\right),$$

$$f\left(a, \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}\right) d\left(b, 2, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}\right), \quad f\left(c, \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}\right) d\left(b, 2, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}\right),$$

$$f\left(b, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}\right) d\left(c, 1, \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}\right), \quad d\left(b, 2, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}\right) \wedge d\left(c, 1, \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}\right).$$

Lemma 4.2. *Let $c = (a, e, \vec{x})$ be a reduced 1-cell. For any cell s such that $[c] \leq [s]$, $e \in s$. Furthermore, $D_{a,i}(s) = D_{a,i}(c) = \vec{x}_i$ for each $i \in \{0, \ldots, \deg(a) - 1\}$.*

Proof. By the definition of $\leq$, s must contain e. The lemma then follows from the definition of $\sim$. $\square$

For a vertex a and an a-vector $\vec{x}$ with $|\vec{x}| = n$, define the 0-form $f(a, \vec{x})$: $C_*(U\mathcal{D}^n T) \rightarrow \mathbb{Z}/2\mathbb{Z}$ by:

$$c \mapsto \begin{cases} 1 & \text{if } D_{a,i}(c) = x_i \text{ for all } i \in \{0, \ldots, \deg(a) - 1\}, \text{ or} \\ & \bar{D}_{a,i}(c) = x_i \text{ for all } i \in \{0, \ldots, \deg(a) - 1\}, \\ 0 & \text{otherwise,} \end{cases}$$

extended linearly to all cellular chains. The constant function 1 is a 0-form: for any vertex $a \neq *$ of degree 1 in T, $f(a, 0) = 1$.

Now we define k-forms. Let $c = (a, e, \vec{x})$ be a reduced 1-cell. Define the basic 1-form $dc = d(a, e, \vec{x}) \colon C_*(U\mathcal{D}^n T) \rightarrow \mathbb{Z}/2\mathbb{Z}$ by:

$$c' \mapsto \begin{cases} 1 & \text{if } e \in c' \text{ and } f(a, \vec{x})([c']) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and extend linearly to all cellular chains. In general, a basic k-form is

$$f(a, \vec{x})dc_1 \wedge \cdots \wedge dc_k,$$

where $f(a, \vec{x})$ is a 0-form as above and each of $c_1, \ldots, c_k$ is a distinct reduced 1-cell. Here, the wedge product represents conjunction: for $[c] \in H_*(B_nT)$, the k-form sends $[c]$ to 1 if and only if $[c] \mapsto 1$ under $f(a, \vec{x})$ and under each dc_i. If not all of the c_i are distinct, then the k-form is identically the 0-function.

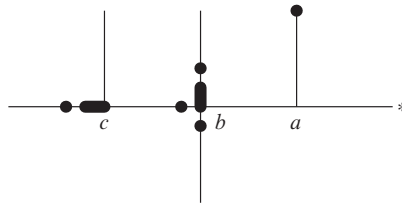As an example of forms, see Figure 3.

Figure 3. This figure depicts the 2-cell s used in Example 4.1.

Thus defined, forms have the following interpretation for cohomology:

Proposition 4.3 (Forms and cohomology). *Let T be a tree. Fix a Morse T-embedding. Then under the induced Morse matching:*

(1) *If c is a reduced 1-cell, then $\phi_{[c]} = dc$. In particular, if c is a critical 1-cell, then $c^* = [dc]$.*

(2) *If $c_1, \ldots, c_k$ are reduced 1-cells, then*

$$[\phi_{[c_1]}] \cup \ldots [\phi_{[c_k]}] = [dc_1 \wedge \cdots \wedge dc_k].$$

   In particular, if $c_1, \ldots, c_k$ are critical 1-cells, then

$$c_1^* \cup \cdots \cup c_k^* = [dc_1 \wedge \cdots \wedge dc_k].$$

(3) *If $c_1, \ldots, c_k$ are reduced 1-cells such that $\{[c_1], \ldots, [c_k]\}$ has no upper bound or contains a repeated element, then*

$$[dc_1 \wedge \cdots \wedge dc_k] = [0].$$

Proof. Part (3) and that $c^* = [\phi_{[c]}]$ follow from Theorem 3.6. The remaining statements follow from chasing definitions. ☐

Now that we have defined forms, we wish to define differentials for these forms. We will see in Proposition 4.6 that coboundaries and differentials coincide. Before defining differentials, though, we give a motivating theorem, Theorem 4.5, for why coboundaries are important. To state the theorem, we need a few more definitions.

Construct a simplicial complex K from $U\mathscr{D}^n T$ as follows. First define a simplicial complex K''. The vertex set of K'' corresponds to distinct equivalence classes of 1-cells $[c]$. Vertices $[c_1], \ldots, [c_k]$ span a $(k-1)$-simplex in K'' if and only if $\{[c_1], \ldots, [c_k]\}$ has an upper bound $[s]$, and is labelled by $[s]$. Note the labels on the faces of K'' induce an injective map from equivalence classes of cells in $U\mathscr{D}^n T$ to $\Lambda(K'')$ (see Theorem 3.5). Let K' denote the $(\lfloor \frac{n}{2} \rfloor - 1)$-skeleton of K''; then $\Lambda(K') = \Lambda(K'')/I''$, where I'' is the ideal of $\Lambda(K'')$ generated by all simplices of dimension greater than $\lfloor \frac{n}{2} \rfloor - 1$. Finally, define the complex K to be the subcomplex of K' where the only vertices left correspond to equivalence classes of 1-cells $[c]$ with non-extraneous reduced representatives. Then $\Lambda(K) = \Lambda(K')/I'$, where I' is the ideal of $\Lambda(K')$ generated by equivalence classes of extraneous reduced 1-cells (see Section 3.2).

Definition 4.4 (Necessary forms and cells). Let $\omega = f(a, \vec{x}) dc_1 \wedge \cdots \wedge dc_k$ be a k-form. The k-form ω is *necessary* if:

(1) $k < \lfloor n/2 \rfloor$ [1],
(2) there exists an edge $e \in T$ such that $(a, e, \vec{x})$ is a non-extraneous reduced 1-cell,
(3) the set $\{[c_1], \ldots, [c_k], [(a, e, \vec{x})]\}$ has an upper bound $[s]$, and
(4) the edge e is the unique respectful edge in the reduced representative s of $[s]$.

The reduced 1-cell $(a, e, \vec{x})$ is the *necessary reduced 1-cell for ω*, and is called *necessary*.

We leave it as an exercise for the reader to verify that there exists a unique necessary reduced 1-cell for a given necessary form.

For an arbitrary k-form ω, the $(k+1)$-chain which is the sum of all equivalence classes in the support of the coboundary of ω will be called the *coboundary support chain* for ω.

Theorem 4.5 (Presentation for cohomology; [14], Theorem 4.5). *We have that*

$$H^*(B_n T) \cong \Lambda(K)/I,$$

where I is the ideal of $\Lambda(K)$ generated by all coboundary support chains for necessary forms, viewed as cochains. The isomorphism is induced by the injective map from equivalence classes of cells in $U\mathscr{D}^n T$ to $\Lambda(K'')$.

Proof. This theorem is almost a rewording of Farley's Theorem 4.5 from [14]. The difference is that Farley does not define the ideals I, I', and I'', but instead views $H^*(B_n T)$ as isomorphic to a quotient of $\Lambda(K'')$. A statement of the result presented here which more closely resembles Farley's theorem is that

$$H^*(B_n T) \cong ((\Lambda(K'')/I'')/I')/I.$$

---

[1]The reason for this dimension restriction is that the simplicial complex K of Theorem 4.5 has no faces in dimension $\lfloor \frac{n}{2} \rfloor$ or larger; see Theorem 3.6.

That we may quotient by the ideal I'' follows from Theorem 3.6: cohomology is trivial in all dimensions greater that $\lfloor \frac{n}{2} \rfloor$.

That we may then quotient by I' is more difficult to see. Clearly I' is an ideal of $\Lambda(K')$. Let $c = (a, e, \vec{x})$ be an extraneous 1-cell. If the edge e of c is such that $\tau(e)$ is essential, then this text and Farley agree: both quotient by the ideal generated by $\partial f(a, \vec{x})$. Now consider if the edge e of c is such that $\tau(e)$ is not essential. We may still talk about directions from $\tau(e)$, but there are at most 2 directions. Let $x_0 := D_{\tau(e),0}(c)$, and let $x_1 := D_{\tau(e),1}(c)$ if a second direction from $\tau(e)$ exists or 0 otherwise. Even though the cell c is not necessarily uniquely determined from the information $(\tau(e), e, \vec{x})$, its equivalence class $[c]$ is. For the purposes of this proof, we will abuse notation and write $[c] = [(\tau(e), e, \vec{x})]$. For example, consider the cell c in Figure 4 with edge e. If exactly one of the non-filled-in vertices is an element of c, then $[c] = [(\tau(e), e, \left[\begin{smallmatrix} 1 \\ 4 \end{smallmatrix} \right])]$. As the tuple $(\tau(e), e, \left[\begin{smallmatrix} 1 \\ 4 \end{smallmatrix} \right])$ does not depend on which of the two non-filled-in vertices is an element of c, c is not uniquely determined by it.
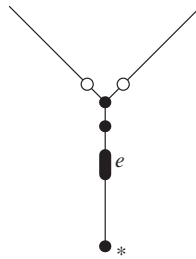

Figure 4. An extraneous 1-cell c whose edge e is such that $\tau(e)$ is not essential.

Consider the $\tau(e)$-vector $\vec{x}'$, where $x_0' := x_0 - 1$ and $x_1' := x_1 + 1$. If $x_0 = 0$, or there are less than x_1' vertices in direction 1 from $\tau(e)$ in all of T, then there are no 1-cells c' satisfying $[c'] = [(\tau(e), e, \vec{x}')]$. In this case, Farley includes $[c]$ in the generating set of his quotient ideal, as we have done here. Otherwise, tracing Farley's definitions yields that the Farley generating set does not include $[c]$, but instead includes the chain $[c] + [(\tau(e), e, \vec{x}')]$. In the quotient, $[c]$ and $[(\tau(e), e, \vec{x}')]$ are equivalent. Inducting on the value x_0, we see that indeed $[c]$ is equivalent to 0 in the quotient. The ideal I' is thus precisely the subideal of Farley's ideal corresponding to these chains.

Carefully tracing Farley's definitions and the definition of necessary show that the remaining generating cochains of Farley's ideal precisely coincide with coboundary support chains for necessary forms. Note that in [14], the convention that $\tau(e) < \iota(e)$ is switched. $\square$

Let $\omega = f(a, \vec{x})dc_1 \wedge \cdots \wedge dc_k$ be a necessary cochain. Necessary cochains were defined so that each term in the coboundary support chain of ω has the form

$dc_0 \wedge dc_1 \wedge \cdots \wedge dc_k$, where each c_i is a non-extraneous critical 1-cell for $i = 0, 1, \ldots, k$. We leave this to the reader to verify. In particular, since each c_i is non-extraneous and $k < \lfloor n/2 \rfloor$, every term in the coboundary support chain of ω represents a nontrivial element of $\Lambda(K)$.

We already know that cohomology is generated by duals of critical cells. Theorem 4.5 tells us that we may effectively ignore all extraneous reduced 1-cells. *From now on, all reduced cells will be assumed to be non-extraneous unless otherwise stated.*

Theorem 4.5 also tells us that coboundaries of necessary forms show us how to rewrite duals of noncritical cells in terms of our Morse basis. The isomorphism in the theorem takes a characteristic function $\phi_{[c]}$ of a cell c to the element of $\Lambda(K)/I$ corresponding to $[c]$.

We wish to understand coboundaries of necessary forms. To do so, we define differentials of forms, and show that our differential coincides with the operation of coboundary.

Define the *differential* of a form as follows. For a basic 0-form $f(a, \vec{x})$, define

$$df(a, \vec{x}) = \sum \big(f(a, \vec{x})(\partial c) \big) \, dc,$$

where the sum runs over all reduced 1-cells c, and ∂c is the CW-boundary of c as a cellular chain. We are left with an expression of $df(a, \vec{x})$ as a sum of basic 1-forms, all of which have 1 as the leading 0-form. Note that $d1 = 0$. For a basic k-form $\omega = f(a, \vec{x})dc_1 \wedge \cdots \wedge dc_k$, define

$$d\omega := d(f(a, \vec{x}) \wedge dc_1 \wedge \cdots \wedge dc_k) = (df(a, \vec{x})) \wedge dc_1 \wedge \cdots \wedge dc_k.$$

Extend linearly to all k-forms. Since $d1 = 0$, it is clear that $d^2 = 0$.

We are now ready to prove the relationship between differentials and coboundaries:

Proposition 4.6 (Coboundary and differential). *The differential operation d precisely coincides with the coboundary operation.*

Proof. To compute the coboundary $\delta\omega$ of an k-cochain $\omega = f(a, \vec{x}) \wedge dc_1 \wedge \cdots \wedge dc_k$, we consider all $\sim$-equivalence classes of $(k + 1)$-cells in $U\mathcal{D}^n T_{\min}$. Let $[s]$ be an equivalence class of $(k + 1)$-cells, and let $s \in [s]$.

Consider a face s' of s. The face s' corresponds to replacing some edge f of s with one of its endpoints. For any $i \in \{1, \ldots, k\}$, we claim that if $dc_i(s) = 0$, then $dc_i(s') = 0$. For, let e_i denote the edge of c_i. If $e_i \notin s$, then $e_i \notin s'$. Consider if $e_i \in s$. If $e_i \notin s'$ (that is, if $f = e_i$), then $dc_i(s') = 0$ and there is nothing to prove. If $e_i \in s'$ (that is, if $f \neq e_i$), then since T is a tree both endpoints of f are in the same direction from $\tau(e)$. In particular, the number of vertices in each direction from $\tau(e)$ is the same for both s and s'. If $e_i \in s$, but $dc_i(s) = 0$, then by definition the number of vertices of s (and thus s') in some direction – say j – from $\tau(e_i)$ is not equal to the number of vertices in c_i in direction j from $\tau(e_i)$; thus, $dc_i(s') = 0$.

We have proven that for any $i \in \{1, \ldots, k\}$, if $dc_i(s) = 0$, then for any face s' of s, we will have $dc_i(s') = 0$. By the definition of coboundary, $\delta\omega(s) = 1$ if and only if the sum of ω evaluated on the 2^{k+1} k-faces of s is 1 in $\mathbb{Z}/2\mathbb{Z}$. Thus, if there exists an $i \in \{1, \ldots, k\}$ such that $dc_i(s) = 0$, then $\delta\omega(s) = 0 = d\omega(s)$.

Assume that $dc_i(s) = 1$ for each $i \in \{1, \ldots, k\}$. Then s contains the edge e_i of the cell c_i for each i. If $e_i = e_j$ for some $i \neq j$, then since $dc_i(s) = dc_j(s) = 1$, it must be that $c_i = c_j$. But then $dc_i \wedge dc_j = dc_i^2 = 0$, and again both $\delta\omega(s) = 0$ and $d\omega(s) = 0$. Now assume the edges e_i are all distinct. Let e denote the edge of s which is not one of the e_i. Consider again a face s' of s corresponding to replacing some edge f of s with one of its endpoints. If $f \neq e$, then $f = e_i$ for some i, so dc_i and therefore ω evaluated on the given face will be 0. Let s' and s'' be the two faces of s corresponding to replacing e with one of its endpoints. Thus, since $dc_i(s) = dc_i(s') = dc_i(s'') = 1$ for each $i \in \{1, \ldots, k\}$,

$$\delta\omega(s) = \omega(s') + \omega(s'') = f(a, \vec{x})(s') + f(a, \vec{x})(s'').$$

Also, by the definition of differential,

$$d\omega(s) = f(a, \vec{x})(\partial s) ds(s) = f(a, \vec{x})(s') + f(a, \vec{x})(s'').$$

In this final case, we have again shown that $\delta\omega(s) = d\omega(s)$. Since $[s]$ was chosen arbitrarily, $\delta\omega = d\omega$ as desired. $\qquad \square$

The proof of Proposition 4.6 actually gives us some computational techniques for considering differentials. In particular, because of the last equation in the proof, we have the following scholium:

Corollary 4.7 (Restricting the differential). *Let $f(a, \vec{x})$ be a basic 0-form and let s be a reduced k-cell.*

(1) *If $f(a, \vec{x})(\partial s) = 1$ then s contains an edge e with $\tau(e) = a$.*
(2) *For each $i \in \{0, \ldots, \deg(a) - 1\}$, let $x_i' := D_{a,i}(s)$, and let d be the direction from a along e. If $f(a, \vec{x})(\partial s) = 1$ then either $\vec{x} = \vec{x}'$ or $\vec{x}'$ differs from $\vec{x}$ by the subtraction of 1 in the 0th entry and the addition of 1 in the dth entry.* $\qquad \square$

More can be said about $d\omega$ when $\omega = f(a, \vec{x})dc_1 \wedge \cdots \wedge dc_k$ is a basic k-form. Many of the 1-forms in the sum used to define $d(f(a, \vec{x}))$ multiply to 0 when wedged with $dc_1 \wedge \cdots \wedge dc_k$. That is, many of the terms of the expansion of $d\omega$ with respect to the definition of $df(a, \vec{x})$ are trivial. We define an annihilator function to get rid of the trivial terms.

Definition 4.8 (Annihilator). Let $c_1, \ldots, c_k$ be reduced 1-cells. Define an *annihilator* function $A_{c_1, \ldots, c_k}$ on 1-forms as follows. Let $d(a, e, \vec{x})$ be a basic 1-form. Then the annihilator acts either as the identity or the 0 function on $d(a, e, \vec{x})$: $A_{c_1, \ldots, c_k}(d(a, e, \vec{x})) = d(a, e, \vec{x})$ if and only if

- the equivalence classes $[c_1], \dots, [c_k]$, and $[(a, e, \vec{x})]$ are all distinct, and
- the set of equivalence classes $\{[c_1], \dots, [c_k], [(a, e, \vec{x})]\}$ has an upper bound;

otherwise, $A_{c_1,\dots,c_k}(d(a, e, \vec{x})) = 0$. Extend $A_{c_1,\dots,c_k}$ linearly to all 1-forms.

Corollary 4.9 (Coboundaries as annihilators). *We have that*

$$\delta\omega = d\omega = A_{c_1,\dots,c_k}(df(a, \vec{x})) \wedge dc_1 \wedge \cdots \wedge dc_k.$$

Proof. This follows from the definition of the annihilator $A_{c_1,\dots,c_k}$, Proposition 4.6, and part (3) of Proposition 4.3.   □

We refer the reader to the recent paper [14] for more information on coboundaries and cohomology presentations.

5. Exterior face algebra structures on cohomology

Recall that a linear tree is one in which there exists an embedded line segment containing every essential vertex. Consider the following theorem:

Theorem 5.1. *Let T be a tree. If T is linear or $n \le 3$, then $H^*(B_n T; \mathbb{Z}/2\mathbb{Z})$ is an exterior face algebra.*

Proof. A tree braid group $B_n T$ is a right-angled Artin group (see Section 8 for a definition) if and only if T is linear or $n \le 3$ ([7] proves linear trees are right-angled Artin; [17] and [15] prove $B_n T$ is free if $n \le 3$; [16] proves the only if direction). For any right-angled Artin group, the cohomology ring is an exterior face algebra [6].   □

Let $T_{\min}$ be the 'minimal' nonlinear tree of Figure 1: $T_{\min}$ has exactly 4 essential vertices, each of degree 3, such that not all 4 essential vertices lie on an embedded line segment. In [16], it was shown that $H^*(B_4 T_{\min}; \mathbb{Z}/2\mathbb{Z})$ is an exterior face algebra $\Lambda(\Delta)$; Figure 5 shows the complex Δ. In this section, our goal is to expand the easy observations in Theorem 5.1 to show that the cohomology of a four, or even five, strand tree braid group is an exterior face algebra holds in general:

Theorem 5.2 (Exterior face algebra structure on 4 and 5 strand cohomology). *Let T be a finite tree. For $n = 4$ or 5, $H^*(B_n T; \mathbb{Z}/2\mathbb{Z})$ is an exterior face algebra. The simplicial complex Δ defining the exterior face algebra structure is unique and at most 1-dimensional.*

We will comment on more general tree braid groups at the end of the section. As our coefficients for cohomology are always in $\mathbb{Z}/2\mathbb{Z}$, we suppress this in the notation from now on. Even though a 1-dimensional simplicial complex (for instance, Δ) is a graph, we will continue to refer to them as simplicial complexes to maintain
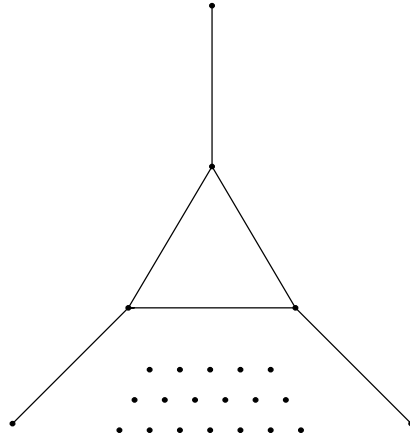
Figure 5.   The simplicial complex Δ giving the exterior face algebra structure on $H^*(B_4 T_{\min}; \mathbb{Z}/2\mathbb{Z})$.

the distinction between defining complexes for exterior face algebras and graphs underlying graph braid groups.

Some of the statements of Theorem 5.2 are immediate, and we prove them now.

Lemma 5.3 (Uniqueness and the dimension bound). *Let T be a finite tree. For $n = 4$ or 5, if $H^*(B_n T)$ is an exterior face algebra then the simplicial complex Δ defining the exterior face algebra structure is unique and at most 1-dimensional.*

Proof. By Theorem 3.6 and a dimension computation for $U\mathcal{D}^n T$ in [15], cohomology is trivial in all dimensions greater than $\lfloor n/2 \rfloor = 2$. Thus, if $H^*(B_n T)$ is an exterior face algebra corresponding to some simplicial complex Δ, Δ must be at most $(2-1)$-dimensional, by the shift of indices in the definition of exterior face algebras. By Theorem 2.3, such a complex Δ is unique. $\square$

To prove Theorem 5.2, it remains to show that $H^*(B_n T)$ is an exterior face algebra. We will spend much of the remainder of this section proving this, via a series of lemmas. In essence, we will define a 1-dimensional simplicial complex Δ and a homomorphism $\Psi \colon \Lambda(\Delta) \to H^*(B_n T)$, and then show the homomorphism is an isomorphism. The bulk of our effort will be in defining Δ, via a change of basis for $H^*(B_n T)$.

Until the end of the proof of Theorem 5.2, we will assume that $n \in \{4, 5\}$.

5.1. Two computational lemmas. In this subsection we establish two computational lemmas to be used throughout the remainder of this section.

Lemma 5.4 (Necessary 0-forms). *Let $c = (a, e, \vec{x})$ be a reduced noncritical 1-cell. Then $f(a, \vec{x})$ is necessary, c is necessary for the 0-form $f(a, \vec{x})$, and $f(a, \vec{x})$ is the unique 0-form for which c is necessary.*

Proof. This is a direct consequence of the definitions of necessary and $f(a, \vec{x})$. □

Lemma 5.5 (Necessary 1-forms). *Let $c = (a, e, \vec{x})$ and $c_1 = (b, e_1, \vec{y})$ be critical 1-cells. If $\omega := f(a, \vec{x}) dc_1$ is necessary and c is the necessary reduced 1-cell for ω, then:*

(1) *the direction from b to a is 0,*

(2) *the direction α from a to b satisfies $0 < \alpha < d$, and*

(3) *α is the unique direction between 0 and d for which $x_\alpha \neq 0$.*

Proof. Let $[s]$ be the least upper bound of $\{[c], [c_1]\}$. By the definition of $\leq$, s must contain the edges e and e_1. But the only edges of s are e and e_1, for if s contains any other edges, replacing each with one of its endpoints yields a smaller upper bound.

Since e is disrespectful in c, there is a vertex of c between $\tau(e)$ and $\iota(e)$ in the order on vertices, i.e. in direction d_0 from a, where $0 < d_0 < d$. By Lemma 4.2, s still has a vertex or edge in direction d_0 from a.

We finish the proof by contradiction. If the direction from b to a is nonzero, then the direction from a to b must be 0 (since T is a tree) – in particular, e_1 is not in direction d_0 from a. If α does not satisfy $0 < \alpha < d$, then $\alpha \neq d_0$, and again e_1 is not in direction d_0 from a. If c has two nonzero directions less than d such that c has vertices in both directions from a, then without loss of generality we may assume d_0 is such that $d_0 \neq \alpha$, and again e_1 is not in direction d_0 from a.

Thus, if any of the conclusions of this lemma do not hold, then s has a vertex in direction d_0 from a. Assume without loss of generality that s is reduced, and let v be the least vertex in direction d_0 from a. Then v must be blocked in s by either $*$, e_1, or e. Since $d_0 \neq 0$, v cannot be blocked by $*$. Since $d_0 \neq \alpha$, v cannot be blocked by e_1. So, v must be blocked by e. This makes e disrespectful in s, since $0 < d_0 < d$. This contradicts the definition of necessary, and proves the lemma. □

5.2. Towards changing bases. In this subsection, we describe why we need to change bases to find the exterior face algebra structure on cohomology. We also define many matrices, some associated to necessary forms and some to critical 1-cells, which we will use in the next section to define our change of basis matrix.

Fix a Morse T-embedding, so that we have a classification of critical cells in T. Consider the finite simplicial complex Δ', defined as follows. Let the vertex set of Δ' be identified with the set $\{c^* \mid c$ a critical 1-cell$\}$. A set of vertices $\{c_1^*, c_2^*\}$ span a 1-simplex, labelled $c_1^* \cup c_2^*$, if and only if $c_1^* \cup c_2^*$ is nontrivial.

Let $\Psi' : \Lambda(\Delta') \to H^*(B_n T)$ be the map which takes an element of $\Lambda(\Delta')$ corresponding to a vertex of Δ' labelled c^* and maps it to the i-cohomology class c^*. Since

critical 1-cells form a free basis for $H^1(B_nT)$, the map extends to all of $H^1(B_nT)$. By the definition of Δ', Ψ' is surjective onto $H^1(B_nT)$ and moreover extends linearly to a surjective homomorphism, since $H^1(B_nT)$ generates all of $H^*(B_nT)$ (see Theorem 3.6).

If Ψ' were injective, then Ψ' would be the desired isomorphism. Recall that a critical 2-cell s uniquely determines the pair $\{c_1, c_2\}$ of critical 1-cells for which it is an upper bound (Theorem 3.6). As critical 2-cells form a basis for 2-dimensional cohomology, Ψ' is injective if and only if, for every pair $\{c_1, c_2\}$ of critical 1-cells which has a least upper bound $[s]$, we may find a representative $s \in [s]$ such that s is a critical 2-cell. Unfortunately, though, this is not the case, and Ψ' is not injective.

Our goal is to modify the Morse generating set of $H^*(B_nT)$ to address these cases. We will define a new simplicial complex Δ using this new generating set so that the corresponding map $\Psi \colon \Lambda(\Delta) \to H^*(B_nT)$ will be an isomorphism.

To modify the generating set of duals of critical 1-cells, we need to specify a change of basis. Since duals of critical 1-cells freely generate $H^1(B_nT)$, we may introduce a vector-theoretic interpretation of $H^1(B_nT)$. Then, using this vector-theoretic interpretation, we specify an invertible change of basis matrix M. The matrix M will be a product of invertible matrices, one for each critical cell, in a particular order, as we will see.

Define $<_r$ to be a (usually non-unique) total order on reduced 1-cells $(a, d, \vec{x})$ induced by lexicographically ordering the triple $(a, -x_0, d)$. Note we are ordering all reduced 1-cells here, not just critical ones. Let rm be the number of reduced 1-cells, sm the number of critical 1-cells, and tm the number of non-critical reduced 1-cells, so $rm = sm + tm$. Define a map ri on reduced 1-cells, so that for a reduced 1-cell c, its image $ri(c) \in \{1, \dots, M\}$ is its index in the total order $<_r$, so that the $<_r$-smallest reduced 1-cell has index 1, the second smallest has index 2, etc. Also define maps $si(c)$ and $ti(c)$ on critical and non-critical reduced 1-cells respectively, so that for a critical (respectively, noncritical) 1-cell c, $si(c)$ is its index among critical (respectively, noncritical) cells in the total order $<_r$. Ergo, the $<_r$-smallest critical 1-cell maps to 1 under ci, the second smallest to 2, etc.

There is a bijection between reduced 1-cells and the standard basis vectors for $\mathbb{F}_2^r m$, where a reduced 1-cell c corresponds to the vector $\vec{v}_c$ consisting of all 0s except a 1 in the $ri(c)^{\text{th}}$ row. By Theorems 3.3 and 3.6, this bijection induces a surjective homomorphism from $\mathbb{F}_2^{rm}$ to each of the rings $H_1(B_nT)$ and $H^1(B_nT)$. The surjection is an isomorphism on the sm-dimensional subspace corresponding to the critical 1-cells, where for a critical cell c, $\vec{v}_c$ is mapped to $[c]$ or c^*, respectively.

Theorem 4.5 tells us that, if we want to rewrite our basis for cohomology to reduce the number of relations in our presentation, then we need to focus on necessary k-forms. So, we use these vector representatives of critical 1-cells to associate to each necessary k-form ω a matrix M_ω.

Let $\omega = f(a, \vec{x})dc_1 \wedge \cdots \wedge dc_k$ be a necessary k-form with c the necessary reduced 1-cell for ω. Consider the annihilator portion $A_{c_1, \dots, c_k}(df(a, \vec{x}))$ of $d\omega$. Let $\vec{u}_\omega \in \mathbb{F}_2^{rm}$ be the vector whose nonzero entries exactly correspond to nonzero terms

of $A_{c_1,\ldots,c_k}(df(a,\vec{x}))$. That is, $(u_\omega)_{ri(c')} = 1$ if and only if dc' is a nonzero term of $A_{c_1,\ldots,c_k}(df(a,\vec{x}))$. Define the $rm \times rm$ matrix M_ω to be the identity matrix I_m, but with the $ri(c)^{\text{th}}$ column replaced by $\vec{u}_\omega$.

Lemma 5.6. *For any necessary k-form $\omega = f(a,\vec{x})dc_1 \wedge \cdots \wedge dc_k$, the necessary reduced 1-cell c for ω is the $<_r$-smallest reduced 1-cell c' such that dc' appears as a nonzero term in $A_{c_1,\ldots,c_k}(df(a,\vec{x}))$.*

Note this lemma holds for arbitrary n, not just $n = 4$ and 5.

In Convention 5.8, we will slightly modify the definition of $<_r$ in the cases $n = 4$ or 5. One consequence will be that, for a necessary 1-form whose associated necessary critical 1-cell c is exceptional of Type I (to be defined soon), this lemma and its corollary will not hold – in fact, M_ω will be upper triangular. The modification will not affect our applications of these results, though.

Proof of Lemma 5.6. Let $c = (a, e, \vec{x})$ be the reduced 1-cell for the ω. Let $c' \neq c$ be any other reduced 1-cell such that dc' appears nontrivially in $df(a,\vec{x})$. By restricting the differential (Corollary 4.7), c' must have an edge e' with $\tau(e') = a$. By the definition of differential, we know that $f(a,\vec{x})(\partial c') = 1$. Let c'_ι and c'_τ denote the faces of c', corresponding to replacing e' with $\iota(e')$ and $\tau(e')$, respectively. Then exactly one of $f(a,\vec{x})(c'_\iota)$ or $f(a,\vec{x})(c'_\tau)$ is 1. For each direction i from a,

$$D_{a,i}(c'_\iota) = \bar{D}_{a,i}(c'_\iota) = D_{a,i}(c'),$$

and

$$D_{a,i}(c'_\tau) = \bar{D}_{a,i}(c'_\tau) = \bar{D}_{a,i}(c').$$

We need to show that $c <_r c'$. If $f(a,\vec{x})(c'_\tau) = 1$ then for $i = 0$ we have $D_{a,0}(c') = \bar{D}_{a,0}(c') = x_0 - 1$, so $c <_r c'$. It remains to consider the case when $f(a,\vec{x})(c'_\iota) = 1$. Then $D_{a,i}(c') = x_i = D_{a,i}(c)$, and in particular, $c' = (a, e', \vec{x})$.

By the definition of necessary, c is such that:

(1) the set $\{[c_1], \ldots, [c_k], [c]\}$ has an upper bound $[s]$, and

(2) the edge e is respectful in the reduced representative s of $[s]$.

Let $d > 0$ be the direction from a along e and let $d' > 0$ be the direction from a along e'. Since $D_{a,i}(c') = x_i$ for each i and $e' \in c'$, $D_{a,d'}(c) = D_{a,d'}(c') \geq 1$. Unless $d = d'$ so that $c = c'$, s must have a vertex in direction d' from a. Since s is reduced, that vertex must be blocked by e in s. Since e is respectful in s, $d' > d$. Thus, $c <_r c'$. $\qquad\qquad\square$

Corollary 5.7 (M_ω is lower triangular). *For any necessary k-form ω, the matrix M_ω is lower triangular and invertible.*

Proof. That $c <_r c'$ for any other c' for which dc' appears nontrivially in the annihilator $A_{c_1,\ldots,c_k}(df(a,\vec{x}))$ makes M_ω a lower triangular matrix. By the definition
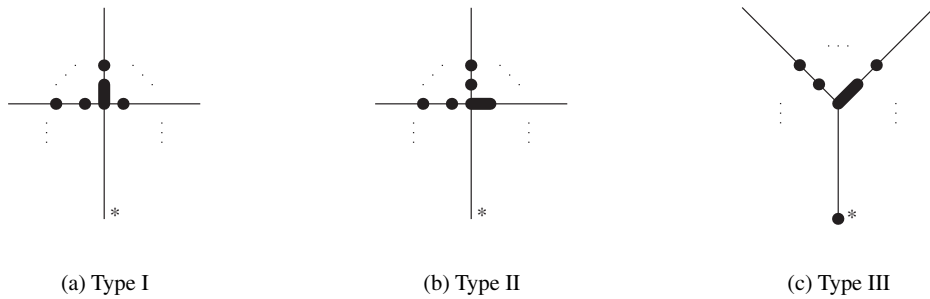
(a) Type I   (b) Type II   (c) Type III

Figure 6. The three types of exceptional critical 1-cells. For each type, all that is drawn are the relevant edges of the tree T: T may have other essential vertices, and the essential vertex may have degree greater than 4 (for Types I and II) or 3 (for Type III), as suggested by the ellipses. As exceptional cells are critical, the diagrams depict reduced 1-cells, even though a subdivision of T is not shown.

of necessary 1-cell, dc also appears nontrivially in $A_{c_1,\ldots,c_k}(df(a,\vec{x}))$. Thus, M_ω is a lower triangular matrix, all of whose diagonal entries are 1. Therefore, M_ω is invertible. $\square$

At this point, we have associated to each necessary form ω an invertible matrix M_ω. We now use these matrices associated to forms to define a matrix M_c for each reduced 1-cell c.

Let c be a reduced 1-cell.

If $c = (a, d, \vec{x})$ is not critical, then c is necessary for the necessary 0-form $\omega = f(a, \vec{x})$ (Lemma 5.4). Define M_c to be the matrix M_ω, but with a 0 on the diagonal in the $ri(c)^{\text{th}}$ row. In terms of multiplication by M_c, this will correspond to *replacing dc with a cohomologically equivalent cochain determined by the coboundary of ω.*

If c is critical, there are three types of exceptions we must make when defining M_c. Before stating the exceptions, we state the general cases. If c is critical, not exceptional, and not necessary, define the matrix M_c to be the identity matrix. If c is critical, not exceptional, and necessary, define M_c to be the matrix M_ω for any 1-form ω for which c is necessary. This corresponds to *rewriting dc with a 1-cochain that will eliminate the relation corresponding to ω.* We will prove momentarily that M_c is well defined.

We now state the three types of exceptions. The motivation for the definitions of M_c for the exceptional cases is not intuitively apparent, but the lemmas in the remainder of this section will justify our choices. All of the exceptional types have $n = 5$. Let $c = (a, d, \vec{x})$ be critical. Assume that there exist two directions dir1 and dir2 from a for which $x_{\text{dir1}}, x_{\text{dir2}} \geq 2$. Without loss of generality, assume dir1 < dir2. Let dir3 be: dir1 if $x_{\text{dir1}} = 3$, or dir2 if $x_{\text{dir2}} = 3$, or the unique index such that $x_{\text{dir3}} = 1$. As $|\vec{x}| = 5$, dir3 is uniquely determined.

(1) (Type I) $0 < \mathrm{dir}1 < \mathrm{dir}2 < \mathrm{dir}3$, and $d = \mathrm{dir}2$: M_c is the identity matrix.

(2) (Type II) $0 < \mathrm{dir}1 < \mathrm{dir}2 < \mathrm{dir}3$, and $d = \mathrm{dir}3$: Let c' be the critical cell $(a, \mathrm{dir}2, \vec{x})$ – that is, replace the edge of e in direction dir3 from a with an edge in direction dir2 from a. Then M_c is the $rm \times rm$ identity matrix, but with an extra 1 in the $ri(c)^{\mathrm{th}}$ row and $ri(c')^{\mathrm{th}}$ column. Note as defined $c' <_r c$, so M_c is *upper* triangular. We want M_c to be lower triangular, so we will slightly modify the order $<_r$ in Convention 5.8, below. The Type I cell c' and the Type II cell c are said to *correspond* to each other. Note there is a bijection between cells of Type I and cells of Type II given by this correspondence.

(3) (Type III) $0 = \mathrm{dir}3 < \mathrm{dir}1 < \mathrm{dir}2$, and $d = \mathrm{dir}2$: For $i = 1, \ldots, \deg(a) - 1$, define $\vec{y}_i$ to be the a-vector which is $\vec{x}$, but with $(\vec{y}_i)_0 = x_0 - 1 = 0$ and $(\vec{y}_i)_i = x_i + 1$. Let $\vec{u}' \in \mathbb{F}_2^{rm}$ be the vector whose only nonzero entries are exactly a 1 in the $ri(c')^{\mathrm{th}}$ row for every reduced cell c' such that $c' = c$ or $c' = (a, \mathrm{dir}2, \vec{y}_i)$ for some $i \in \{1, \ldots, \deg(a) - 1\}, i \neq d$. Define M_c to be the $rm \times rm$ identity matrix I_m, but with the $ri(c)^{\mathrm{th}}$ row replaced by the transpose $(\vec{u}')^{\top}$. Note that $c <_r (a, \mathrm{dir}2, \vec{y}_i)$ for each $i \in \{1, \ldots, \deg(a) - 1\}$, so M_c is lower triangular.

Convention 5.8 (A modification to $<_r$). We slightly modify the total order $<_r$ on reduced 1-cells by having corresponding exceptional critical cells of Types I and II switch places. By convention, the matrices M_ω and M_c defined above are defined using this modified total order. All further references to $<_r$, and the associated functions ri and ci, will be to the modified total order.

Note that with this new convention, Lemma 5.6 no longer holds for necessary forms ω where the associated necessary 1-cell is exceptional of Type I – in fact, instead of being lower triangular, M_ω will be upper triangular. This is fine, as M_ω was not used to define any matrix M_c. It is a small exercise to verify that, for all other cases, Lemma 5.6 still holds.

Lemma 5.9 (M_c is well defined). *Let c be a reduced 1-cell. The matrix M_c is well defined and lower triangular.*

Proof. For the three exceptional cases, the matrix M_c was uniquely determined. If c is exceptional of Type I, M_c is the identity and is lower triangular. If c is exceptional of Type II, then Convention 5.8 makes M_c lower triangular. If c is exceptional of Type III, the argument that M_c is lower triangular given in the definition of M_c still applies, even with Convention 5.8.

Aside from the three exceptional cases, if c is not necessary or not critical, the matrix M_c is uniquely specified and lower triangular by Corollary 5.7. So, assume c is necessary and critical. Corollary 5.7 shows that M_c will be lower triangular if it is well defined. To claim that M_c is well defined is to claim that, if ω_1 and ω_2 are any two 1-forms for which c is necessary, $M_{\omega_1} = M_{\omega_2}$. Let $c = (a, d, \vec{x})$. By the definition

of necessary, there exist critical 1-cells $c_1 = (b_1, d_1, \vec{y})$ and $c_2 = (b_2, d_2, \vec{z})$ such that $\omega = f(a, \vec{x}) \wedge dc_1$ and $\omega = f(a, \vec{x}) \wedge dc_2$. By the Upper Bound Lemma (Lemma 3.9), the directions α_1 and α_2 from a to b_1 and b_2, respectively, are such that $\vec{x}_{\alpha_1} = n - y_0 \geq 2$ and $\vec{x}_{\alpha_2} = n - z_0 \geq 2$. By Lemma 5.5, $0 < \alpha_1 < d$ and $0 < \alpha_2 < d$, so if $\alpha_1 \neq \alpha_2$, c is an exceptional case of Type II. Since we are not addressing the exceptional cases, $\alpha_1 = \alpha_2$.

The formulas $\vec{x}_{\alpha_1} = n - y_0$ and $\vec{x}_{\alpha_2} = n - z_0$ imply $y_0 = z_0$. By the restricting the differential (Corollary 4.7), a nonzero term of $df(a, \vec{x})$ is of the form ds, where s is a reduced 1-cell which lies over the vertex a. Since $y_0 = z_0$ and a is in direction 0 from both of b_1 and b_2, by the Upper Bound Lemma (3.9), $[s]$ has an upper bound with $[c_1]$ if and only if $[s]$ has an upper bound with $[c_2]$. In particular, by the definition of annihilator, $A_{c_1}(ds) = A_{c_2}(ds)$ for each such s – i.e. $A_{c_1}(df(a, \vec{x})) = A_{c_2}(df(a, \vec{x}))$. The equality of the matrices M_{ω_1} and M_{ω_2} follows. $\square$

5.3. Changing bases and finishing Theorem 5.2.

We now have lower triangular matrices M_c for each reduced 1-cell c. We want to use the matrices M_c to define a change of basis for $H^1(B_n T)$.

From now on, we think of the matrices M_c as acting on reduced cocycles, where we identify a reduced cocycle dc' and the vector in $\mathbb{F}_2^{rm}$ whose only nonzero entry is in the $ri(c')^{\text{th}}$ row.

To define our change of basis, we need to specify how to rewrite duals of critical cells. We do so on the cochain level, using 1-forms associated to both critical and noncritical reduced 1-cells. For critical reduced 1-cells, we define a matrix Ms. For noncritical reduced 1-cells, we define a matrix Mt. We multiply the matrices Ms and Mt to define the desired change of basis matrix M, as follows.

Definition 5.10 (The matrices Ms, Mt, and M). Define the matrix Ms as:

$$Ms := \prod_{\substack{i=1 \\ si(c)=i}}^{sm} M_c,$$

where the product is over all critical cells c and is written so that the $<_r$-largest cell c is such that M_c is on the right, applied first to any target vector. Define the matrix Mt as:

$$Mt := \prod_{\substack{i=1 \\ ti(c)=tm-1+i}}^{tm} M_c,$$

where the product is over all non-critical reduced cells c and is written so that the $<_r$-smallest cell c is such that M_c is on the right, applied first to any target vector. Define the matrix M, which will be the desired change of basis matrix, as:

$$M := MtMs.$$

The matrices Ms, Mt, and M have many nice properties, some of which we describe now.

Let c_0 be a reduced 1-cell. By definition, $M_c dc_0 = dc_0$ for every $c \neq c_0$. Also by definition, $M_{c_0} dc_0$ consists of terms of the form dc_0', where c_0 and c_0' lie over the same vertex. Since each M_c is lower triangular (Lemma 5.9), $c_0 \leq_r c_0'$, even when c is exceptional.

If c_0 is a critical 1-cell, then $Mt dc_0 = dc_0$ and

$$Ms dc_0 = \prod_{\substack{i=1 \\ si(c)=i}}^{sm} M_c dc_0 = M_{c_0} dc_0.$$

This means that Ms agrees with the matrix M_{c_0} in the $ri(c_0)^{\text{th}}$ column. Since M_c is lower triangular and invertible for any critical 1-cell c, the matrix Ms is also lower triangular and invertible.

If c_0 is a noncritical 1-cell, $M_{c_0} dc_0$ is by definition cohomologous to dc_0. As M_{c_0} is the identity outside of column $ri(c_0)$, the matrix Mt preserves cohomology classes. As the matrix M_c is lower triangular for each noncritical 1-cell c, the matrix Mt is lower triangular. Moreover, as the matrix M_{c_0} contains a 0 on the diagonal in the $ri(c_0)^{\text{th}}$ place, the matrix Mt has no nonzero entries in the $ri(c_0)^{\text{th}}$ row. Finally, since c_0 is noncritical, $Ms dc_0 = dc_0$.

These observations imply that $M = MtMs$, restricted to basic 1-forms corresponding to critical 1-cells, is a change of basis matrix on cohomology classes, where for a critical 1-cell c, $Mc^* = [Mdc]$. We now have our change of basis matrix M on $H^1(B_n T)$ and therefore on $H^*(B_n T)$. We may thus write M as acting on cohomology classes (like c^*) instead of cochains (like dc), and will do so freely from now on. For future reference, we record some of our observations in a theorem:

Theorem 5.11 (Change of Basis Theorem).

(1) *The matrix M gives a change of basis isomorphism for $H^1(B_n T)$ and therefore $H^*(B_n T)$. For a critical 1-cell c, $Mc^* := [Mdc]$.*

(2) *The matrices M, Ms, and Mt are lower triangular. The matrix Ms is invertible.*

(3) *For a critical 1-cell c, $Ms dc = M_c dc$. The cochain $M_c dc$ consists of terms of the form dc_0', where c and c_0' lie over the same vertex and $c \leq_r c_0'$.*

(4) *The matrix Mt preserves cohomology classes.*   □

We need to analyze the effect of M on all of cohomology, not just $H^1(B_n T)$. In particular, we need to know how M affects cup products.

Lemma 5.12 (Cup products after changing basis). *Let c and c' be critical 1-cells, where $c \leq_r c'$. If $Mc^* \cup M(c')^* \neq [0]$, then $[c]$ and $[c']$ have a least upper bound $[s]$ with reduced representative s and either:*

   (1) *s is critical, or*

   (2) *s is not critical and c is exceptional of Type* I.

Proof. We will repeatedly use the properties of M, Ms, and Mt stated in Theorem 5.11 to prove this lemma, and do so without further reference.

Since $M = MtMs$ and Mt preserves cohomology classes, $Mc^* \cup M(c')^* \neq [0]$ if and only if $[Msdc] \cup [Msdc'] \neq [0]$. We have that $Msdc = M_cdc$ and $Msdc' = M_{c'}dc'$. Thus, $Mc^* \cup M(c')^* \neq [0]$ if and only if $[M_cdc] \cup [M_{c'}dc'] \neq [0]$.

Express c as $(a, e, \vec{x})$ and c' as $(a', e', \vec{x'})$.

If $a = a'$, then every summand in both M_cdc and $M_{c'}dc'$ will lie over a. By the Upper Bound Lemma (Lemma 3.9), no two reduced 1-cells which lie over the same vertex have an upper bound. By the definition of $\wedge$, it follows that $M_cdc \wedge M_{c'}dc'$ is the 0 function, so $[Msdc] \cup [Msdc'] = [0]$. Thus, we may assume that $a \neq a'$.

Consider summands dc_0 and dc_0' of M_cdc and $M_{c'}dc'$, respectively. Express c_0 as $(a, d_0, \vec{y})$ and c_0' as $(a', d_0', \vec{y'})$. By the definition of M_c and restricting the differential (Corollary 4.7), $\vec{y}$ differs from $\vec{x}$ for at most two indices, namely 0 and some index dir_0. If $\vec{y} \neq \vec{x}$ then $y_0 = x_0 - 1$ and $y_{\mathrm{dir}_0} = x_{\mathrm{dir}_0} + 1$. In particular, $x_0 \geq y_0$. Furthermore, by definition, if c is not exceptional of Type III, then $d_0' = d_0$. Similar statements hold for dc_0', but we will only need to use that $x_0' \geq y_0'$.

We claim that $[M_cdc] \cup [M_{c'}dc'] \neq [0]$ if and only if $[M_cdc] \cup [dc'] \neq [0]$, as follows.

Consider if $[M_cdc] \cup [dc'] = [0]$ but $[M_cdc] \cup [M_{c'}dc'] \neq [0]$. Then there exists some summand $dc_0' \neq dc'$ of $M_{c'}dc'$ such that $[M_cdc] \cup [dc_0'] \neq [0]$. Thus, there must exist a summand dc_0 of M_cdc such that $dc_0 \wedge dc_0'$ is not cohomologous to 0. In particular, $[c_0]$ and $[c_0']$ have an upper bound. By the Upper Bound Lemma, since $x_0' \geq y_0'$, $[c_0]$ and $[c']$ have an upper bound. Since $[M_cdc] \cup [dc'] = [0]$, there exists a sequence of Tietze transformations which equates $dc_0 \wedge dc'$ with $(M_cdc \wedge dc' - dc_0 \wedge dc')$. Each Tietze transformation involving a relation will involve the coboundary of a necessary 1-form. By restricting the differential, such a 1-form has either the form $f(a, \vec{z})dc'$ or $f(a', \vec{z'})dc_0$ for some a-vector $\vec{z}$ or a'-vector $\vec{z'}$. By Lemma 5.5, such a 1-form must be of the form $f(a, \vec{z})dc'$, since a is in direction 0 from a'. By the definition of necessary, $f(a, \vec{z})dc_0'$ will also be a necessary 1-form. It follows that a corresponding sequence of Tietze transformations equates $dc_0 \wedge dc_0'$ with $(M_cdc \wedge dc_0' - dc_0 \wedge dc_0')$. But then $[M_cdc \wedge dc_0'] = [0]$, a contradiction.

To finish the claim, it remains to prove that if $[M_cdc] \cup [dc'] \neq [0]$, then $[M_cdc] \cup [M_{c'}dc'] \neq [0]$. If $[M_cdc] \cup [dc'] \neq [0]$ but $[M_cdc] \cup [M_{c'}dc'] = [0]$, then there exists some summand – say dc_0' – of $M_{c'}dc'$ such that $dc_0 \wedge dc'$ and $dc_0 \wedge dc_0'$ are both in a single relation – that is, in the differential of a single necessary 1-form ω'. By restricting the differential, it must be that $\omega' = f(a', \vec{z})dc_0$ for some a'-vector $\vec{z}$. This contradicts Lemma 5.5, as the direction from a' to a is 0. This proves the claim.

To prove the lemma, we show that if $[M_cdc] \cup [dc'] \neq [0]$, then $[c]$ and $[c']$ have a least upper bound $[s]$ with reduced representative s, and either s is critical or s is

not critical but c is exceptional of Type I.

By Theorem 3.6, $[dc_0] \cup [dc'_0] \neq [0]$ only if $[c_0]$ and $[c'_0]$ have an upper bound. Let $\alpha > 0$ be the direction from a to a'. By the Upper Bound Lemma, $[c_0]$ and $[c'_0]$ have a common upper bound if and only if

$$y_\alpha + y'_0 \geq n + \epsilon_0, \tag{1}$$

where $\epsilon_0 := \epsilon(c_0, \alpha)$ is the upper bound constant. We record the interpretation of equation 1 for the case that $c_0 = c$, as these two equations are the heart of what remains in the proof. We have that $[c]$ and $[c']$ have a common upper bound if and only if

$$x_\alpha + x'_0 \geq n + \epsilon, \tag{2}$$

where $\epsilon := \epsilon(c, \alpha)$.

If $[c]$ and $[c']$ have a common upper bound, by the Upper Bound Lemma, then $x_\alpha \geq 2$. Also, by definition, $x_d \geq 1$.

Consider the number $y_\alpha - \epsilon_0$. By definition, $\vec{y}$ differs from $\vec{x}$ only in the 0th and $(d'_0)^{\text{th}}$ entries. Since $\alpha > 0$, $y_\alpha = x_\alpha$ unless $\alpha = d'_0$. But unless c is exceptional of Type III, $d'_0 = d_0$, and when $\alpha = d'_0$, $\epsilon_0 = 1$. Therefore, unless $\alpha = d'_0$ and c is exceptional of Type III, $y_\alpha - \epsilon_0 = x_\alpha$. Unless $\alpha = d$, $\epsilon = 0$, so $x_\alpha = x_\alpha - \epsilon$. Thus, unless $\alpha = d$, or $\alpha = d'_0$ and c is exceptional of Type III, if $[c_0]$ and $[c']$ then $[c]$ and $[c']$ have a common upper bound $[s]$.

Consider the case that c is exceptional of Type III. Let $\omega = f(a, \vec{x}) dc_1$ be a necessary 1-form for which c is necessary. Let α_1 be the direction from a towards the vertex over which c_1 lies. By the Upper Bound Lemma, if $[c_0]$ and $[c']$ have an upper bound, then $y_\alpha - \epsilon_0 \geq 2$. By the definition of M_c, if $\alpha \neq \alpha_1$, then $y_\alpha - \epsilon_0 \leq 1$. Thus, if $[c_0]$ and $[c']$ have an upper bound, then $\alpha = \alpha_1$. If $[c]$ and $[c']$ also have an upper bound, then $x'_0 = 3$. By definition, any term dc'_0 of $M_c dc$ is such that $[c'_0]$ and $[c_1]$ have an upper bound. By the Upper Bound Lemma, we also have that $[c'_0]$ and $[c']$ have an upper bound. Moreover, consider the 1-form $\omega' = f(a, \vec{x}) dc'$. By restricting the differential (Corollary 4.7), we have that $d\omega' = M_c dc \wedge dc'$. Thus, $[M_j dc] \cup [M_j dc'] = [0]$. If $[c]$ and $[c']$ do not also have an upper bound, then $x'_0 = 2$ and $y_\alpha = 3$. Thus there is only one such c_0: when $d'_0 = \alpha$. By restricting the differential, we have that $d(f(a, \vec{y}) dc') = dc_0 \wedge dc'$, so

$$[M_j dc] \cup [M_j dc'] = [M_c dc \wedge dc'] = [dc_0 \wedge dc'] = [0].$$

Now consider the case when c is not exceptional of Type III. If $\alpha = d$, we have a number of subcases.

- If c is exceptional of Type II, then $y_d = 1$. This implies $x'_d \geq n - 1$, so $[c_0]$ and $[c']$ cannot have an upper bound.
- If c is not necessary and c is not exceptional of Type II, then M_c is the identity, and $c_0 = c$. If $[c_0]$ and $[c']$ have an upper bound $[s]$, then $[c] = [c_0]$ and $[c']$ have the upper bound $[s]$. Let s be the reduced representative of $[s]$. If e is not

disrespectful in s, c would be necessary for the necessary 1-form $f(a, \vec{x})dc'$, contradicting that c is not necessary. Thus, if $[c_0]$ and $[c']$ have an upper bound, e must be disrespectful in s, and by the Upper Bound Lemma, s is critical.

- If c is necessary and not exceptional of Type II, then let $\omega = f(a, \vec{x})dc_1$ be a 1-form for which c is necessary. Let α_1 be the direction from a towards the vertex over which c_1 lies. By Lemma 5.5, $0 < \alpha_1 < d$, and $x_{\alpha_1} \geq 2$. Since c is not exceptional of Type III, $d_0' = d_0$. By equations 1 and 2, $y_\alpha - \epsilon_0 = x_\alpha$. Thus, since $x_\alpha' \leq n - 2$, if $[c_0]$ and $[c']$ have an upper bound then $x_\alpha \geq 2 + \epsilon_0$.

  - If $n = 4$, since $|\vec{x}| = n$, $x_\alpha = x_{\alpha_1} = 2$ and c is completely determined. In this case, by restricting the differential, we have $A_{c_1}(df(a, \vec{x})) = dc$, so $c_0 = c$. But then $\epsilon_0 = 1$, and $2 = x_\alpha \geq 2 + 1 = 3$. Thus, if $n = 4$, then $[c_0]$ and $[c']$ cannot have an upper bound.

  - If $n = 5$, since $|\vec{x}| = 5$ we know there is one vertex of c unaccounted for: there is some index β such that $x_\beta = 1$ when $\beta \notin \{\alpha, \alpha_1\}$ or $x_\beta = 3$ when $\beta \in \{\alpha, \alpha_1\}$. Since c is not exceptional of Type III, $\beta \neq 0$.

    * If $\beta > d$, then c is of Type I, and so $c_0 = c$ by the definition of M_c and the lemma holds.

    * If $\beta = d$, then the only nonzero entries of $\vec{x}$ are $x_{\alpha_1} = 2$ and $x_d = 3$. Since $\alpha \neq \alpha_1$ and $x_\alpha \geq 2 + \epsilon_0$, $\alpha = d$. By restricting the differential we have that $d\omega = dc \wedge dc_1$, so $M_c dc = dc$ and $c_0 = c$. Since c' lies over a vertex in direction $\alpha = d$ from a, the lemma then follows from the Upper Bound Lemma.

    * If $\beta < d$, then c is not necessary: the cell $(a, \beta, \vec{x})$ is the necessary 1-cell for $f(a, \vec{x})dc_1$. This contradicts the assumption that c is necessary, so the lemma holds.

It remains to prove the lemma when c is not exceptional of Type III and $\alpha \neq d$. In this case, by equations (1) and (2), if $[c_0]$ and $[c']$ have a common upper bound then $[c]$ and $[c']$ have a common upper bound $[s]$. Let s be the reduced representative of $[s]$. If e is disrespectful in s, we're done. Note that if c is exceptional of Type II, then e is disrespectful in s, by the Upper Bound Lemma. If e is respectful in s, then by the Upper Bound Lemma, $0 < \alpha < d$, $x_\alpha + y_0 = n$, and $x_i = 0$ for all $0 < i < d$, $i \neq \alpha$. The lemma allows for c to be exceptional of Type I in this case. Consider if c is not exceptional (as we have addressed each of the three exceptional cases). By definition, any term dc_0' of $M_c dc$ is such that $[c_0']$ and $[c_1]$ have an upper bound. By the Upper Bound Lemma, we also have that $[c_0']$ and $[c']$ have an upper bound. Moreover, consider the 1-form $\omega' = f(a, \vec{x})dc'$. By restricting the differential, we have that $d\omega' = M_c dc \wedge dc'$. Thus, $[M_j dc] \cup [M_j dc'] = [0]$. This finishes the proof. $\square$

Lemma 5.12 tells us that multiplication by M is very nice in ensuring that when Mc^* and $M(c')^*$ cup nontrivially, $[c]$ and $[c']$ have an upper bound with critical

representative, with only one type of exception. If c is exceptional of Type I, then Lemma 5.12 does not say much about Mc^*. There is something to say, though:

Lemma 5.13. *Let $c_1 = (a, d_1, \vec{x})$ be a critical cell of Type* I *and let $c_2 = (a, d_2, \vec{x})$ be the corresponding critical cell of Type* II. *Let d_0 be the index of the remaining nonzero coordinate of $\vec{x}$. For any critical* 1-*cell $c' = (a', e', \vec{x'})$,*

  (1) *if $Mc_1^* \cup M(c')^* \neq [0]$ then $[c_2]$ and $[c']$ have a least upper bound $[s]$, $c_2^* \cup (c')^* \neq [0]$, and a' is in direction d_0 from a.*
  (2) *if $Mc_2^* \cup M(c')^* \neq [0]$ then $[c_2]$ and $[c']$ have a least upper bound $[s]$, $c_2^* \cup (c')^* \neq [0]$ and a' is in direction d_1 from a.*

The way to think of this is that Mc_1^* is the d_0 'portion' of c_2^* (and c_1^*), while Mc_2^* is the d_1 'portion' of c_2^* (and trivially c_1^*).

Proof. By the Upper Bound Lemma (Lemma 3.9), $[c_1]$ and $[c']$ have an upper bound if and only if a' is in direction d_0 from a and $[c_2]$ and $[c']$ have an upper bound. Also, that $[c_2]$ and $[c']$ have an upper bound implies a' is in either of the directions d_0 or d_1 from a. By the Upper Bound Lemma, if $[c_2]$ and $[c']$ have an upper bound $[s]$, then the reduced representative s of $[s]$ is critical, so $c_2^* \cup [dc'] \neq [0]$.

If $Mc_1^* \cup M[dc'] \neq [0]$, then by Lemma 5.12, $[c_1]$ and $[c']$ have an upper bound. Thus, by above, if $Mc_1^* \cup M[dc'] \neq [0]$, then $c_2^* \cup [dc'] \neq [0]$.

If $Mc_2^* \cup M[dc'] \neq [0]$, then by Lemma 5.12, $[c_2]$ and $[c']$ have an upper bound, so by above, $c_2^* \cup [dc'] = [ds] \neq [0]$. By above, a' is in one of the directions d_0 or d_1 from a. We prove, assuming $Mc_2^* \cup M[dc'] \neq [0]$, that a' cannot be in direction d_0 from a. By the definition of M and the matrices M_{c_1} and M_{c_2}, $Mdc_2 = dc_1 + dc_2$. So, $Mdc_2 \wedge dc' = dc_1 \wedge dc' + dc_2 \wedge dc'$. But this is precisely the coboundary of the 1-form $f(a, \vec{x})dc'$ by restricting the differential (Corollary 4.7). Thus,

$$[0] = [Mdc_2 \wedge dc'] = [Mdc_2] \cup [Mdc'] = Mc_2^* \cup M[dc'].$$

This contradicts the assumption that $Mc_2^* \cup M[dc'] \neq [0]$. We have proven that if $Mc_2^* \cup M[dc'] \neq [0]$ then a' is in direction d_1 from a. This finishes the proof. $\square$

We are now ready to state and prove the final step of Theorem 5.2.

Definition 5.14 (The simplicial complex Δ). Define the finite simplicial complex Δ as follows. The vertex set of Δ is identified with the set $\{Mc^* \mid c \text{ a critical 1-cell}\}$. A vertex Mc^* is said to *lie over* the vertex of T over which c lies. A set of vertices $\{Mc_1^*, Mc_2^*\}$ span a 1-simplex, labelled $Mc_1^* \cup Mc_2^*$, if and only if $Mc_1^* \cup Mc_2^*$ is nontrivial.

Theorem 5.15 (The exterior face algebra structure). *We have that*

$$H^*(B_n T) \cong \Lambda(\Delta).$$

Proof. Consider the map $\Psi\colon \Lambda(\Delta) \to H^*(B_nT)$ given by, for c a critical 1-cell, mapping the element in $\Lambda(\Delta)$ corresponding to the 0-cell of Δ labelled Mc^* to the 1-cohomology class Mc^*. Since critical 1-cells form a free basis for $H^1(B_nT)$ and M is a change of basis matrix, the map extends to all of $H^1(B_nT)$. By the definition of Δ, Ψ is surjective onto $H^1(B_nT)$ and moreover extends linearly to a surjective homomorphism, since $H^1(B_nT)$ generates all of $H^*(B_nT)$ (see Theorem 3.6).

We claim that Ψ is also injective. Since $n \in \{4, 5\}$, it suffices to consider 1-cells of Δ. A simple counting argument will finish the result. For any 1-cell $\{Mc_1^*, Mc_2^*\}$ of Δ, by Lemma 5.12 there is a critical 2-cell s such that $[s]$ is the least upper bound of $[c_1]$ and $[c_2]$, *unless* possibly one of c_1 or c_2 is exceptional of type I. By Theorem 3.6, s uniquely determines and is uniquely determined by c_1 and c_2. By Lemma 5.13, even when one of c_1 or c_2 is exceptional of Type I, there is still a uniquely determined critical 2-cell s corresponding to c_1 and c_2. Thus, there are at most as many edges of Δ as there are critical 2-cells. Since Ψ is surjective and by Theorem 3.6, $H^2(B_nT)$ has rank equal to the number of critical 2-cells, Ψ must also be injective. Thus, Ψ is the desired isomorphism. $\square$

Theorem 5.15 gives us an important strengthening of Lemmas 5.12 and 5.13:

Corollary 5.16 (Multiplication by M). *Let $c = (a, d, \vec{x})$ and $c' = (a', d', \vec{x'})$ be critical 1-cells with $c \leq_r c'$. Then*

$$Mc^* \cup M(c')^* \neq [0]$$

if and only if $[c]$ and $[c']$ have a least upper bound $[s]$ with reduced representative s and either

(1) *c is not exceptional of Types I or II and s is critical,*

(2) *c is exceptional of Type I and s is not critical, or*

(3) *c is exceptional of Type II, s is critical, and the direction α from a to a' is not the smallest direction for which $x_\alpha \neq 0$.*

The results of this section show that exterior face algebra structures do crop up for cohomology rings of tree braid groups, but restrictive assumptions were made on the tree braid groups in question. We have shown that if $n < 6$ or T is linear, then $H^*(B_nT)$ is an exterior face algebra. We conjecture the converse holds:

Conjecture 5.17 (Arbitrary number of strands). *Let T be a tree. Then $H^*(B_nT)$ is an exterior face algebra if and only if $n < 6$ or T is linear.*

Our conjecture is based on the reasoning that, for T a nonlinear tree and $n \geq 6$, we may find a particular structure in $H^*(B_nT)$ which we believe prevents $H^*(B_nT)$ from being an exterior face algebra. We do not go into the details here, but we claim if we apply the change of basis techniques of this section to $H^*(B_nT)$ for $n \geq 6$,

the result is that $H^*(B_nT)$ is isomorphic to some particular quotient of an exterior face algebra. That quotient has as a subalgebra the quotient pictured in Figure 7. We believe that this subalgebra is 'poisonous': any algebra containing the given subalgebra in a 'nice' way will not be an exterior face algebra. Of course, we do not know the appropriate definition of 'nice', and showing that an algebra is *not* an exterior face algebra is very difficult, even in this restricted setting.
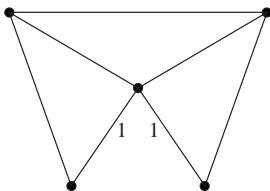

Figure 7. Shown is a simplicial complex. The 1s labelling two 1-cells of the simplicial complex indicate that in the corresponding exterior face algebra, we quotient by identifying the two edges. We believe that, if this quotient is a subalgebra of a given algebra in a nice way, then the given algebra cannot be an exterior face algebra. We also believe that the cohomology algebra of tree braid groups for nonlinear trees and at least 6 strands contain this quotient as a subalgebra in a nice way.

6. Rigidity

In this section, our goal is to prove Theorem 6.9, the rigidity of 4 and 5 strand tree braid groups. We do so by explicitly reconstructing the tree T from the corresponding tree braid group B_nT. The tree T_Δ, defined in the first subsection, will be the reconstruction desired, as proven in the second subsection.

6.1. The tree T_Δ. In this subsection we try to construct a tree T_Δ from an arbitrary 1-dimensional simplicial complex Δ. The tree T_Δ will be used in Section 6.2 to prove Theorem 6.9. In Theorem 6.9, we will show that, if Δ is the simplicial complex giving the exterior face structure for $H^*(B_nT)$ for some tree T and $n = 4$ or 5, then T_Δ is homeomorphic to T. By Theorem 5.2, for $n = 4$ or 5 there exists a unique 1-dimensional simplicial complex associated to $H^*(B_nT)$ – this is why we restrict Δ to be 1-dimensional. Even with the dimension restriction, for many Δ, T_Δ will not exist; it will turn out that T_Δ will exist exactly when we want it to (see Corollary 6.13).

Throughout this section, refer to Example 6.1 on page 505 for an example of the concepts defined. Also see Example 6.11 on page 513 for further reference.

Let Δ be an arbitrary 1-dimensional simplicial complex, and fix a constant $n \in \{4, 5\}$. For each vertex $v \in \Delta$, let N_v denote the *vertex neighborhood* of v in Δ: the set of all vertices $v' \in \Delta$ adjacent to v. Let $\leq_N$ denote the preorder on vertices of

Δ induced by vertex neighborhood inclusions, so $v \leq_N v'$ if and only if $N_v \subseteq N_{v'}$. Most pairs of vertices will not be comparable, but enough will be.

The preorder $\leq_N$ induces an equivalence relation $=_N$ on vertices of Δ: if $v \leq_N v'$ and $v' \leq_N v$, we write $v =_N v'$. Note that $\leq_N$ induces a partial order on the vertex set $V(\Delta)$ of Δ modulo equivalence classes under $=_N$. We write the $(=_N)$-equivalence class of $v \in V(\Delta)$ as $[v]$. For an $(=_N)$-equivalence class $[v]$, the *descendants* of $[v]$ are those $(=_N)$-equivalence classes $[v']$ such that $[v'] \leq [v]$ (i.e. $N_{v'} \subseteq N_v$) and $N_{v'} \neq \emptyset$. A descendant $[v']$ of $[v]$ is a *child* of $[v]$ and $[v]$ is a *parent* of $[v']$ if there are no descendants of $[v]$ between $[v']$ and $[v]$ with respect to $\leq_N$ – i.e. there does not exist a w such that $N_{v'} \subsetneq N_w \subsetneq N_v$.

We now define a tree T_Δ from the simplicial complex Δ. Let $|\Delta|$ denote the number of vertices of Δ, and for any subset S of $V(\Delta)$, let $|S|$ denote its size.

Recall the function $Y_m(x)$, defined in Theorem 3.4. We will use $Y_m(x)$ repeatedly to define T_Δ.

Let v_0 be a vertex of Δ which is maximal under $\leq_N$. Thus, N_{v_0} is maximal under inclusion among all vertex neighborhoods of vertices of Δ.

If N_{v_0} is empty, define the tree T_Δ to be a radial tree (see Theorem 3.4) whose essential vertex a has degree $\deg(a)$ such that

$$Y_n(\deg(a)) = |\Delta|$$

if such a number exists. If such a number does not exist, then T_Δ is undefined.

Now assume that N_{v_0} is nonempty. Define the finite undirected graph H' to be the *neighborhood hierarchy* of $[v_0]$, as follows. Create a vertex p_1 in H', and for each descendant $[v]$ of $[v_0]$ create a vertex $p_{[v]}$ in H', so that the vertex set of H' is

$$\{p_1\} \cup \{p_{[v]} \mid [v] \text{ is a descendant of } [v_0]\}.$$

Every vertex $p_{[v]}$ is connected to $p_{[v']}$ for each child $[v']$ of $[v]$. Also, there is one edge connecting p_1 to $p_{[v_0]}$ (when Δ is as in Theorem 5.15, think of p_1 as corresponding to the cochain 1, so that every 1-cochain is in its 'neighborhood').

If $n = 4$, the neighborhood hierarchy H' is already almost the tree T_Δ, but if $n = 5$, the H' contains too many vertices. To address this issue, we define a subgraph H of the graph H'. If $n = 4$, define the finite undirected graph H to be H'. If $n = 5$, consider if there exists a child $[v_0']$ of $[v_0]$ such that

(1) $[v_0']$ has exactly half as many descendants as $[v_0]$, and
(2) if $[v]$ and $[v']$ are children of $[v_0]$ with a common descendant, then one of $[v]$ or $[v']$ is a descendant of $[v_0']$.

If such a child $[v_0']$ exists, define H to be H' but with the descendants of $[v_0']$ removed from H' – that is, H is H' minus the vertices corresponding to descendants of $[v_0']$ and any edges connecting them to any other vertices. If no such child exists, H is undefined.

We want to construct a tree T_Δ by 'growing' T_Δ from the graph H. To do so, we want to associate to each vertex $p_{[v]}$ of H a number $p \deg_{[v]}$, which will be the

desired degree of $p_{[v]}$. We say $p_{[v]}$ is a *leaf* of H if $[v]$ has no children. If $p_{[v]}$ is a *leaf* of H, then define $p \deg_{[v]}$ to be the positive integer greater than 2 satisfying

$$Y_{(n-2)}(p \deg_{[v]}) = |N_v|$$

for any $v \in [v]$; if such a number does not exist, $p \deg_{[v]}$ is undefined. If $p_{[v]}$ is not a leaf of H, then define $p \deg_{[v]}$ to be the positive integer greater than 2 satisfying

$$Y_2(p \deg_{[v]}) = |[v']|$$

for every child $[v']$ of $[v]$; if such a number does not exist, $p \deg_{[v]}$ is undefined. Also define $p \deg_1$ to be the positive integer greater than 2 satisfying

$$Y_2(p \deg_1) = |[v_0]|;$$

if such a number does not exist, $p \deg_1$ is undefined.

If $p \deg_{[v]}$ exists, it is unique, since Y_2 and $Y_{(n-2)}$ are monotonic increasing on values greater than 2.

We may now define the T_Δ. The tree T_Δ is undefined if any of the following hold:

(1) H is undefined;

(2) H is not a tree;

(3) there exists $p_{[v]}$ in H such that $p \deg_{[v]}$ is undefined, or $p \deg_1$ is undefined;

(4) there exists $p_{[v]}$ in H which is not a leaf and $[v]$ has more than $d_{[v]} - 1$ children.

If we are not in one of the cases where T_Δ is undefined, then T_Δ is the tree H together with some edges added for the purpose of increasing the degrees of the vertices $p_{[v]}$. Each distinct added edge connects a vertex $p_{[v]}$ to a distinct vertex of degree 1 in T_Δ. We add edges so that the degree of $p_{[v]}$ in T_Δ is exactly $p \deg_{[v]}$. The degree of p_1 in T_Δ is $p \deg_1$.

There is one exceptional case when defining T_Δ. If $n = 5$ and H has exactly three vertices, then the graph H' had exactly five vertices. Thus, there were exactly two children $[v_0']$ and $[\hat{v}_0']$ which had exactly half the number of descendants of $[v_0]$. The vertices of H' are then exactly p_1, $p_{[v_0]}$, $p_{[v_0']}$, $p_{[\hat{v}_0']}$, and some vertex $p_{[w]}$. Let a, b, and c be positive integers greater than 2 such that

$$Y_2(a) = |[v_0]|, \quad Y_3(b) - Y_2(b) = |[w]|, \quad Y_2(c) = |N_w|,$$

if such integers exist. If any of a, b, or c is undefined, then T_Δ is undefined. If a, b, and c are all defined, then define T_Δ to be the linear tree having no vertices of degree 2 and exactly three essential vertices A, B, and C, where the essential vertices have degrees a, b, and c, respectively, and A and C are extremal.

We note that, according to the definition presented here, the tree T_Δ may depend on the choice of $\leq_N$-maximal vertex v_0. As a consequence of Theorem 6.9, we will have that, when Δ is the unique simplicial complex associated to $H^*(B_n T)$ for T a tree and $n = 4$ or 5, the tree T_Δ is independent of v_0.

Example 6.1. Consider the simplicial complex Δ shown in Figure 5, and reproduced in Figure 8. We have labelled the vertices of Δ with nontrivial neighborhoods. With the vertices of Δ as labelled in Figure 8, the vertex v_0 is maximal under $\leq_N (N_{v_0} = \{v_1, v_4, v_5\})$. Since $N_{v_1} = \{v_4\}$ and $N_{v_2} = \{v_5\}$, $[v_1]$ and $[v_2]$ are descendants of $[v_0]$. The neighborhood hierarchy of $[v_0]$ is shown on the right of Figure 8 with solid edges. Fixing n to be 4, the tree H constructed above is the neighborhood hierarchy. To grow T_Δ from H, we compute the desired degrees in T_Δ of the vertices of H. The desired degree of p_1 is 3, since $Y_2(3) = 1 = |[v_0]|$. The desired degree of $p_{[v_0]}$ is $p\deg_{[v_0]} = 3$, since $p_{[v_0]}$ is not a leaf of H and $Y_2(3) = 1 = |[v_1]| = |[v_2]|$. The desired degree of $p_{[v_1]}$ is $p\deg_{[v_1]} = 3$, since $p_{[v_1]}$ is a leaf of H and $Y_{(4-2)}(3) = 1 = |N_{v_1}|$. Finally, the desired degree of $p_{[v_2]}$ is $p\deg_{[v_2]} = 3$, since $p_{[v_2]}$ is a leaf of H and $Y_{(4-2)}(3) = 1 = |N_{v_1}|$. Thus, the tree T_Δ is as depicted on the right of Figure 8.
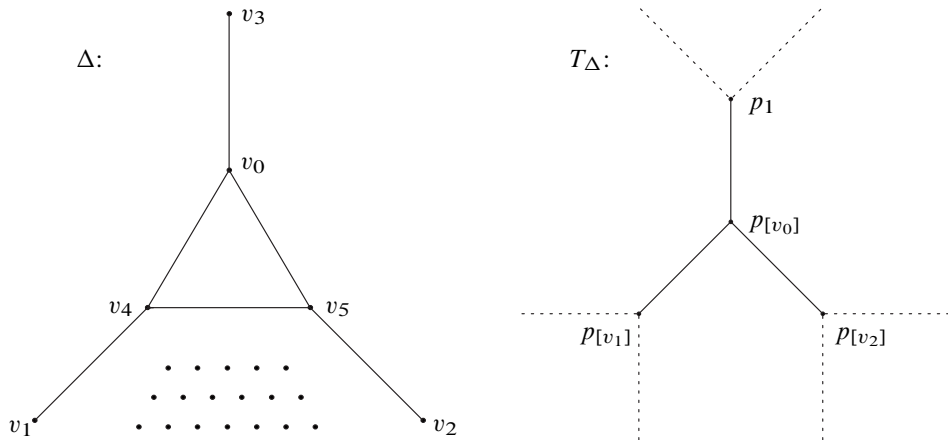

Figure 8. An example calculation of the tree T_Δ.

6.2. The four and five strand case. We now wish to prove the rigidity for tree braid groups in the restricted setting of $n = 4$ or 5. As usual, we need a series of definitions and lemmas to complete the proof. These lemmas will completely describe, for a given critical cell c, which critical cells c' are such that $Mc^* \cup M(c')^* \neq [0]$. It is this information that we use to prove Theorem 6.9. All lemmas in this subsection will assume $n \in \{4, 5\}$.

For any critical 1-cell c, define the *M-cup neighborhood* N_c of c to be the set

$$N_c := \{c_1 \mid Mc^* \cup Mc_1^* \neq [0]\}.$$

When Δ is as in Theorem 5.15 and $v = Mc^*$ is a vertex of Δ, the vertex neighborhood N_v of v in Δ is by definition in bijective correspondence with the M-cup neighborhood of c.

Lemma 6.2 (The direction d_c). *Let $c = (a, e, \vec{x})$ be a critical 1-cell. Either the M-cup neighborhood N_c of c is empty or there exists a unique direction d from a such that, for every $c_1 \in N_c$, c_1 lies in direction d_c from a.*

The direction d_c is called the *CUB* (*Critical Upper Bound*) *direction* for c.

Proof. This lemma follows from repeated application of the Upper Bound Lemma, Lemma 3.9. Throughout this proof, we use the Upper Bound Lemma, usually without reference.

Assume the converse: that N_c is nonempty and there are critical 1-cells $c_1, c_2 \in N_c$ such that the directions d_1 and d_2 from a toward c_1 and c_2, respectively, are distinct. By definition, we know $x_d \geq 1$. We have that $x_{d_1}, x_{d_2} \geq 2$. Also, if $d_1 = d$ then $x_{d_1} \geq 3$, while if $d_2 = d$, $x_{d_2} \geq 3$. If $x_{d_1} \geq 3$ and $d_2 \neq d$, then $x_{d_2} \leq 1$, a contradiction. If $x_{d_2} \geq 3$ while instead $d_2 = d$, then $x_{d_1} \leq 2$, again a contradiction. Thus, $x_{d_1} < 3$. Since $2 \leq x_{d_1} < 3$, $x_{d_1} = 2$, and so $d_1 \neq d$. Similarly, $x_{d_2} = 2$ and $d_2 \neq d$.

Consider if $d_1 = 0$. Then c is not exceptional of Type I. Let $[s_2]$ be the least upper bound of $[c]$ and $[c_2]$, and let s_2 be the reduced representative of $[s_2]$. By Corollary 5.16, s_2 exists and is critical. But by the Upper Bound Lemma, e is respectful in s_2, a contradiction. Thus, $d_1 \neq 0$. Similarly, $d_2 \neq 0$.

If $d_1 > d$, note c is not exceptional of Type I. Let $[s_2]$ be the least upper bound of $[c]$ and $[c_2]$, and let s_2 be the reduced representative of $[s_2]$. By Corollary 5.16, s_2 exists and is critical. but by the Upper Bound Lemma, e is respectful in s_2, a contradiction. Thus, $d_1 < d$. Similarly, $d_2 < d$.

The only possibility left is that c is exceptional of Type II. The existence of both c_1 and c_2 contradicts Lemma 5.13. This finishes the proof. $\square$

Consider a critical 1-cell $c = (a, d, \vec{x})$ with nonempty M-cup neighborhood N_c, and d_c as in Lemma 6.2. Just as the CUB direction d_c helps determine N_c, there is a number, CUB(c), that we may associate to c to help determine N_c. Before defining CUB(c), we need the following definition. Let $\epsilon_c(d')$ be the *cup constant* of c in direction d', associated to c as follows. If $d' = 0$ or $d' > d$, define $\epsilon_c(d') := 0$. If $0 < d' < d$ and there is some index i with $0 < i < d$ and $i \neq d'$ with $x_i > 0$, define $\epsilon_c(d') := 0$. If $n = 5$ and c is exceptional of Type I, define $\epsilon_c(d') := 0$. Otherwise, define $\epsilon_c(d') := 1$. Note that, unless $n = 5$ and c is exceptional of Type I, if the upper bound constant $\epsilon(c, d')$ for c in direction d' is 1, then the cup constant $\epsilon_c(d')$ for c in direction d' is also 1. For convenience, we let $\epsilon_c := \epsilon_c(d_c)$. Then, the number $x_{d_c} - \epsilon_c$ is denoted CUB(c), and is the *CUB* (*Critical Upper Bound*) *number* for c.

Lemma 6.3 (A bound on CUB(c)). *For a critical 1-cell c with nonempty M-cup neighborhood, $2 \leq \text{CUB}(c) \leq n - 2$.*

Proof. If c is exceptional of Type I, then $\mathrm{CUB}(c) = 2$ and there is nothing to prove. So, assume c is not exceptional of Type I. Let $c = (a, d, \vec{x})$. By the Upper Bound Lemma, $x_{d_c} \geq 2 + \epsilon(c, d_c)$. Since c has non-empty M-cup neighborhood, this inequality is strict if there exists no index $i \neq d_c$ such that $0 < i < d$ and $x_i > 0$, by Corollary 5.16. But by the definition of ϵ_c, this means $x_{d_c} \geq 2 + \epsilon_c$. Thus,

$$\mathrm{CUB}(c) = x_{d_c} - \epsilon_c \geq 2.$$

Since c is critical, there exist at least two nonzero coordinates of $\vec{x}$, so $x_{d_c} \leq n - 1$. If $x_{d_c} < n - 1$, there is nothing to prove, so consider if $x_{d_c} = n - 1$. If $d_c = 0$ or $d_c > d$, then c is not critical, so it must be that $0 < d_c \leq d$. If $d_c = d$, then $\epsilon_c = 1$. If $0 < d_c < d$, then there exists no index $j \neq d_c$ with $0 < j < d$ and $x_j > 0$, so again $\epsilon_c = 1$. Either way,

$$\mathrm{CUB}(c) = x_{d_c} - \epsilon_c \leq n - 1 - 1 = n - 2. \qquad \square$$

The CUB direction and number determine N_c in the following sense:

Lemma 6.4 (The structure of neighborhoods). *Let $c = (a, e, \vec{x})$ and $c' = (a', e', \vec{x'})$ be critical 1-cells. Then $c \in N_{c'}$ and $c' \in N_c$ if and only if:*

(1) $a \neq a'$,

(2) *a' lies in direction d_c from a,*

(3) *a lies in direction $d_{c'}$ from a', and*

(4) $\mathrm{CUB}(c) + \mathrm{CUB}(c') \geq n$.

Proof. Note $c \in N_{c'}$ if and only if $c' \in N_c$ by definition. If $c \in N_{c'}$ and $c' \in N_c$, then the first three conditions follow from Lemma 6.2 and the fourth condition from the Upper Bound Lemma, Lemma 3.9.

Now consider if the four conditions hold. That $[c]$ and $[c']$ have an upper bound $[s]$ follows from the Upper Bound Lemma. Let s be the reduced representative of $[s]$. We claim that either s is critical or the $\leq_N$-smaller of c and c' is exceptional of Type I. If the claim holds, then by Corollary 5.16, $c \in N_{c'}$ and $c' \in N_c$. If s is not critical, then one of e or e' is respectful in s. Without loss of generality, assume e is respectful in s. By the Upper Bound Lemma, $d_{c'} = 0$ and d_c satisfies $0 < d_c < \mathrm{dir}$, where dir is the direction from a along e. Furthermore, $x_i = 0$ for every $i \neq d_c$ satisfying $0 < i < \mathrm{dir}$, and $x_{d_c} + x'_0 = n$. By definition, $\epsilon_{c'}(d_{c'}) = 0$. Since

$$n - \epsilon_c = (x_{d_c} - \epsilon_c) + (x'_0 - \epsilon'_c) = \mathrm{CUB}(c) + \mathrm{CUB}(c') \geq n,$$

it follows that $\epsilon_c = 0$. By definition, it must be that c is exceptional of Type I. This proves the claim. $\qquad \square$

Corollary 6.5 (Neighborhood containment). *Let $c = (a, d, \vec{x})$ and $c' = (a', d', \vec{x'})$ be critical 1-cells with nonempty M-cup neighborhoods. Then $N_{c'} \subseteq N_c$ if and only if $\mathrm{CUB}(c') \leq \mathrm{CUB}(c)$ and either:*

(1) $a = a'$ and $d_c = d_{c'}$, or
(2) $a \neq a'$ and

    (a) a' is in direction d_c from a, and
    (b) a is not in direction $d_{c'}$ from a'.

Proof. These conditions are equivalent to the statement "if $c_0 \in N_{c'}$ then $c_0 \in N_c$", by Lemma 6.4.   □

Corollary 6.6 (Neighborhood equality). *Let $c = (a, d, \vec{x})$ and $c' = (a', d', \vec{x}')$ be critical 1-cells with nonempty M-cup neighborhoods. Then $N_c = N_{c'}$ if and only if*

(1) $a = a'$,
(2) $d_c = d_{c'}$, *and*
(3) $\mathrm{CUB}(c) = \mathrm{CUB}(c')$.   □

Proof. That $N_c = N_{c'}$ is equivalent to $N_c \subseteq N_{c'}$ and $N_{c'} \subseteq N_c$. The result then follows from applying Corollary 6.5 twice.   □

This lemma and its two corollaries almost completely describe the structure of the M-cup neighborhoods as well as their behavior under inclusion. The final two ingredients for the proof of Theorem 6.9 are to count exactly how many critical 1-cells have a given neighborhood, and describe what kinds of critical 1-cells have neighborhoods which are maximal under inclusion. We accomplish these tasks in the following two results.

Lemma 6.7 (Counting). *For any essential vertex a, there exist exactly $Y_n(\deg(a))$ critical 1-cells lying over a. Let d be a direction from a such that there exists some essential vertex besides a itself in direction d from a. Let k be an integer satisfying $2 \leq k \leq n - 2$. Then there exists exactly $Y_{n-k}(\deg(a))$ critical 1-cells lying over a with CUB direction d and CUB number at least k.*

Proof. By Definition 3.7 and the preceding discussion, a critical 1-cell c is determined by: an essential vertex a, an edge e with terminal endpoint a, an a-vector $\vec{x}$, and the condition that at least one vertex of c be blocked by e at a. All of these properties are local properties, and do not depend on T or on the existence or nonexistence of any other essential vertices in T. Thus, there is a bijection between critical 1-cells lying over a and critical 1-cells in a radial tree whose essential vertex has degree $\deg(a)$. The first statement of the lemma then follows from the Radial Rank Theorem, Theorem 3.4.

We now prove the second statement of the lemma. If $c = (a, \mathrm{dir}, \vec{x})$ is a critical 1-cell lying over a with $d_c = d$ and $\mathrm{CUB}(c) \geq k$, then $x_d \geq k + \epsilon_c$. Thus, c has at least k vertices (not including the edge e) in direction d from a. Let c' be the reduced 1-cell corresponding to deleting k vertices from c in direction d from a. That is,

$c' := (a, \mathrm{dir}, \vec{x}')$, where $\vec{x}'$ equals $\vec{x}$ in all but the dth coordinate, and $x'_d = x_d - k$. We make one exception: if c is exceptional of Type I, define $c' := (a, \mathrm{dir}', \vec{x}')$, where $\mathrm{dir}' > \mathrm{dir}$ is the direction towards the edge of the exceptional 1-cell of Type II associated to c.

We claim c' is critical. For, if c' is not critical, then it must be that $0 < d < \mathrm{dir}$, $x_d = k$, and $x_i = 0$ for all $0 < i < \mathrm{dir}$, $i \neq d$. Since $x_d \geq k + \epsilon_c$, it must be that $\epsilon_c = 0$. By the definition of ϵ_c, it follows that c is exceptional of Type I. But when c is exceptional of Type I, we changed c' to be critical.

Now consider a critical 1-cell c'_0 lying over a on $n - k$ strands. Let c_0 be the reduced 1-cell on n strands corresponding to adding k vertices to c'_0 in the direction d from a. As c'_0 is critical, so is c_0. It is straightforward to construct a critical 1-cell c_1 lying over a vertex in direction d from a such that $[c_0]$ and $[c_1]$ have a least upper bound $[s]$ whose reduced representative s is critical. We leave this to the reader. By Lemma 6.2 and Corollary 5.16, unless c_0 is exceptional of Type II and d is the smallest direction from a in which c_0 has strands, $d_{c_0} = d$. It then follows that $\mathrm{CUB}(c_0) \geq k$. Furthermore, since c'_0 is critical, c_0 cannot be of Type I. If c_0 is exceptional of Type II and d is the smallest direction from a in which c_0 has strands, we make an exception and replace c_0 with the corresponding exceptional cell of Type I.

These two constructions (constructing c' from c and c_0 from c'_0) are inverses of each other (we leave it to the reader to check this when c is of Type I or c'_0 is of Type II). Thus, they give a set bijection between 'critical 1-cells c with $d_c = d$ and $\mathrm{CUB}(c) \geq k$' and 'critical 1-cells lying over a with $n - k$ strands'. In particular, these two sets have the same number of elements. The latter set has $Y_{n-k}(\deg(a))$ elements, by Theorem 3.4. This proves the lemma. $\qquad\square$

When Δ is as in Theorem 5.15 and $v = Mc^*$ is a vertex of Δ where $[v]$ is maximal under $\leq_N$, then both the vertex neighborhood N_v and the M-cup neighborhood N_c will be maximal under inclusion. Furthermore:

Corollary 6.8 (Maximal neighborhoods). *Let c be a critical 1-cell. If the M-cup neighborhood N_c of c is maximal under inclusion among all M-cup neighborhoods, then c lies over an extremal vertex a of T, and $\mathrm{CUB}(c) = n - 2$.*

Proof. Let a be the essential vertex over which c lies. If T is radial, then the only essential vertex is a, so a is extremal. If T is not radial, then there exist non-empty M-cup neighborhoods, so the empty set is not maximal under inclusion and therefore N_c is nonempty. Assume that a is not extremal, and let d_c be the direction prescribed by Lemma 6.2. Since a is not extremal, there exist extremal vertices in at least 2 directions from a. Thus there exists an extremal vertex a' in a direction which is not d_c from a. Let d' be the direction from a' to a. By Lemma 6.7, there exists a critical 1-cell c' lying over a' with $d_{c'} = d'$ and $\mathrm{CUB}(c') = \mathrm{CUB}(c)$. By Corollaries 6.5 and 6.6, $N_c \subsetneq N_{c'}$, contradicting the maximality of N_c. Thus a is extremal.

That $\text{CUB}(c) = n - 2$ is similar: assume $\text{CUB}(c) < n - 2$. By Lemma 6.7, there exists a critical 1-cell c' lying over a with $d_{c'} = d_c$ and $\text{CUB}(c') = n - 2$. By Corollaries 6.5 and 6.6, $N_c \subsetneq N_{c'}$, contradicting the maximality of N_c. Thus $\text{CUB}(c) = n - 2$. □

We are now ready to prove the following rigidity result for $n = 4$ and 5 strand tree braid groups:

Theorem 6.9 (Rigidity). *Let T and T' be two finite trees, and let $n = 4$ or 5. The tree braid groups $B_n T$ and $B_n T'$ are isomorphic as groups if and only if the trees T and T' are homeomorphic as trees.*

There are many equivalent formulations of Theorem 6.9. Some of these restatements include:

- there is a bijection between finite trees up to homeomorphism and n-strand tree braid groups up to isomorphism, $n = 4$ or 5;
- there is a bijection between n-strand tree braid groups up to isomorphism, $n = 4$ or 5, and the set of cohomology rings of n strand tree braid groups (that this is a reformulation follows from Theorem 5.2);
- Given an n-strand tree braid group G with $n = 4$ or 5, there exists a unique tree T (up to homeomorphism) such that $G = B_n T$.

To prove Theorem 6.9, we actually show the last equivalent statement.

Note that, in each homeomorphism class of trees, there exists exactly one tree that has no vertices of degree two. For this tree, the vertices are exactly all of the nonmanifold points. The following proof involves construction of a tree with no vertices of degree two, and graph braid groups on this tree. As the fundamental group of the associated configuration spaces is invariant under homeomorphism, this does not affect the associated tree braid group. The reader is cautioned, however, that in order to utilize critical cells via discretized configuration spaces and Theorem 2.2, further subdivision of this tree is required.

For applications of the proof of Theorem 6.9, we refer the reader to Examples 6.10 and 6.11, immediately following the proof.

Proof. Let *Trees* denote the set of all trees up to homeomorphism. Let B_n be the map taking a tree T to its n strand tree braid group $B_n T$. By definition, B_n is well defined. We prove $B_n : \textit{Trees} \to B_n(\textit{Trees})$ is bijective for $n = 4, 5$ by constructing an inverse map. Let $\Psi_n : B_n(\textit{Trees}) \to \textit{Trees}$ be given by $\Psi_n(B_n T) = T_\Delta$, where Δ is the simplicial complex uniquely determined by $H^*(B_n T)$, as in Theorem 5.2, and T_Δ is as in Section 6.1. We claim Ψ_n is well defined and that $\Psi_n \circ B_n$ is the identity map for $n = 4, 5$. Our goal is to analyze Ψ_n so that both properties will be proven simultaneously. This would prove the desired result, as B_4 and B_5 would thus be injective. See Figure 9.
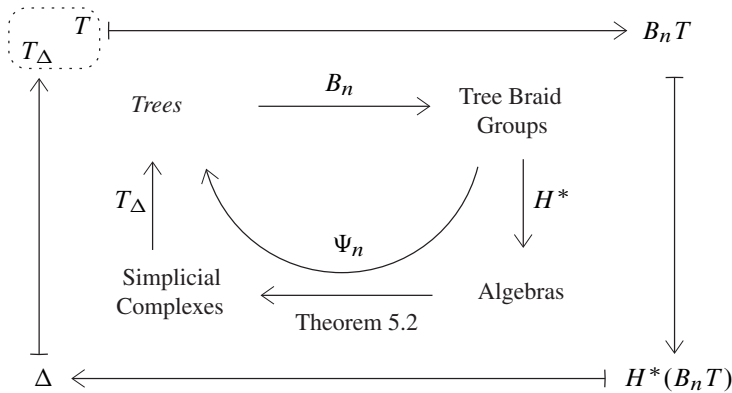
Figure 9. A diagram showing the maps used in the proof of Theorem 6.9. The proof of Theorem 6.9 shows that, when $n = 4$ or 5, T_Δ is homeomorphic to T.

Let T be a tree. If T is radial with central vertex a, then by [16] we know that $H^*(B_nT)$ is free of rank $Y_n(\deg(a))$ (see Theorem 3.4). Then Δ is a collection of $Y_n(\deg(a))$ vertices. By definition, T_Δ is a tree with one essential vertex of degree $\deg(a)$. Thus, T is homeomorphic to T_Δ, as desired.

Assume that T is not radial (and is sufficiently subdivided for n). Let Δ be the unique 1-dimensional simplicial complex giving the exterior face algebra structure on $H^*(B_nT)$ given by Theorem 5.2. Fix a Morse T-embedding, so that the vertices of Δ may be labelled Mc^*, where each c is a critical 1-cell, as in Definition 5.14.

Since T is not radial and $n \geq 4$, there exist critical 2-cells (this is a consequence of Lemmas 6.4 and 6.7, and is an exercise for the reader). By Theorem 3.6, $H^2(B_nT)$ is nontrivial, so Δ contains some vertex with nonempty vertex neighborhood. Let $v_0 = Mc_0^*$ be a vertex of Δ which is maximal under $\leq_N$. We will prove that the tree T_Δ defined by this choice of $\leq_N$-maximal vertex is homeomorphic to T.

By the definition of $\leq_N$, both the vertex neighborhood N_{v_0} of v_0 and the M-cup neighborhood N_{c_0} of c_0 are nonempty and maximal under inclusion. Let q_1 denote the essential vertex in T over which c_0 lies. By Corollary 6.8, q_1 is extremal and $\mathrm{CUB}(c) = n - 2$.

Consider a descendant $[v]$ of $[v_0]$. Pick $v \in [v]$, and let c be the critical 1-cell such that $v = Mc^*$. Let q be the vertex over which c lies. By Lemma 6.3, we have that $2 \leq \mathrm{CUB}(c) \leq n - 2$. By Corollary 6.5, either

(1) $n = 5$, $q = q_1$, $d_c = d_{c_0}$, and $\mathrm{CUB}(c) = \mathrm{CUB}(c_0) - 1 = 2$, or

(2) $q \neq q_1$, q is in direction d_{c_0} from q_1, and q_1 is not in direction d_c from q.

Consider if $\mathrm{CUB}(c) = \mathrm{CUB}(c_0) = n - 2$. Then we are in the second case above. Label the unique essential vertex of T which is adjacent to q and is in direction d_c from q by $q_{[v]}$. By Corollary 6.6, the vertex q, the CUB direction d_c, and the CUB

number CUB(c) are independent of the choice of $v \in [v]$, so $q_{[v]}$ is well defined. By Lemma 6.7, every essential vertex of T except q_1 will have a label of the form $q_{[v]}$. As T is a tree, the labels are unique.

We claim that the map $q_{[Mc^*]} \mapsto p_{[Mc^*]}$ from essential vertices of T to vertices of T_Δ shows that T_Δ is defined and that this map induces the desired homeomorphism Ψ_n between the two trees. To prove our claim, we need to show:

(i) There exists a vertex labelled $q_{[Mc^*]}$ in T if and only if there exists a vertex labelled $p_{[Mc^*]}$ in T_Δ,

(ii) $q_{[M(c')^*]}$ is adjacent to $q_{[Mc^*]}$ if and only if $p_{[M(c')^*]}$ is adjacent to $p_{[Mc^*]}$, and

(iii) $\deg(q_{[Mc^*]}) = \deg(p_{[Mc^*]})$.

(i) If $n = 4$, this is clear. If $n = 5$, let $[v'_0]$ be a child of $[v_0]$, and let c'_0 be a critical 1-cell such that $v'_0 = M^*c'_0$. Consider if CUB(c'_0) = CUB(c_0) $- 1 = 2$. For $[v'_0]$ to be a child of $[v_0]$, it follows from Corollary 6.5 that c'_0 lies over q_1 and $d_{c'_0} = d_{c_0}$. By Corollary 6.6, $[v'_0]$ is thus uniquely determined by the choice of CUB(c'_0) = 2. Either a descendant $[Mc^*]$ of $[v_0]$ is also a descendant of $[v'_0]$ or not. By Corollary 6.5, $[Mc^*]$ is a descendant of $[v'_0]$ if and only if CUB(c) = 2, and is not a descendant of $[v'_0]$ if and only if CUB(c) = 3. Thus, by Lemma 6.7, $[v'_0]$ has exactly half as many descendants as $[v_0]$. If $[Mc^*]$ and $[M(c')^*]$ are distinct children of $[v_0]$ such that neither $[Mc^*]$ nor $[M(c')^*]$ is $[v'_0]$, then CUB(c) = CUB(c') = CUB(c_0) = 3, as $[v'_0]$ is uniquely determined by the choice CUB(c'_0) = 2. By Corollary 6.5, it follows that c and c' both lie over $q_{[v_0]}$. By Corollary 6.6, since $[Mc^*] \neq [M(c')^*]$, $d_c \neq d'_c$. It follows from Corollary 6.5 that $[Mc^*]$ and $[M(c')^*]$ have no common descendants, let alone children.

Thus, $[v'_0]$ is as in the definition of T_Δ. Note this child $[v'_0]$ always exists, by Lemma 6.7. There exists a vertex labelled $q_{[Mc^*]}$ in T if and only if $[Mc^*]$ is a descendant of $[v_0]$ and CUB(c) = 3, if and only if $[Mc^*]$ is a descendant of $[v_0]$ but not $[v'_0]$, if and only if there exists a vertex labelled $p_{[Mc^*]}$ in T_Δ. This proves (i) in the case that CUB(c'_0) = 2.

If CUB(c'_0) = 3, let $[M(c')^*]$ denote the child of $[v_0]$ such that CUB(c') = 2. If $[v_0]$ has any other child $[Mc^*]$ distinct from $[v'_0]$ and $[M(c')^*]$, then $[M(c')^*]$ and $[Mc^*]$ have a common descendant (namely, $[M(c'')^*]$ where CUB(c'') = 2, $d_{c''} = d_c$, and c'' and c lie over the same vertex in T). Thus, unless $[v_0]$ has no other children, $[v'_0]$ is not as in the definition of T_Δ. If $[v_0]$ has no other children, then by Corollary 6.5, every descendant of $[v_0]$ except $[v_0]$ and $[M(c')^*]$ are also descendants of $[v'_0]$. Thus, unless $[v_0]$ has exactly 4 descendants, $[v'_0]$ is again not as in the definition of T_Δ.

If CUB(c'_0) = 3, $[v_0]$ has no other children, and exactly 4 descendants, we are in the exceptional case in the definition of T_Δ. We repeatedly apply Corollary 6.5 to analyze this case. As $[v_0]$ has only one child $[v'_0] = [M(c'_0)^*]$ with CUB(c'_0) = 3, the vertex $q_{[v_0]}$ is adjacent to exactly two essential vertices in T: q_1 and $q_{[v'_0]}$. As $[v'_0]$ has only one descendant $[Mc^*]$, $q_{[v'_0]}$ must be adjacent to only $q_{[v_0]}$. Let x, y, and z

denote the degrees of vertices q_1, $q_{[v_0]}$, and $q_{[v_0']}$, respectively. Then by Lemma 6.7, it follows that:

$$Y_2(x) = |[v_0]|, \quad Y_3(y) - Y_2(y) = |[Mc^*]|, \quad Y_2(z) = |N_c|.$$

Thus, in this exceptional case, we already have that T is homeomorphic to T_Δ. This finishes the proof of (i), as either we must be in this exceptional case or $\mathrm{CUB}(c_0') = 2$.

(ii) Let $v := Mc^*$ and $v' := M(c')^*$ be such that $q_{[v]}$ and $q_{[v']}$ are essential vertices in T. Without loss of generality, assume $q_{[v]}$ is closer to q_1 than $q_{[v']}$. Since $\mathrm{CUB}(c) = \mathrm{CUB}(c') = n-2$, by Corollary 6.5, $q_{[v]}$ and $q_{[v']}$ are adjacent if and only if there are no critical 1-cells c'' such that $N_{c'} \subsetneq N_{c''} \subsetneq N_c$. By definition, this holds if and only if $[v']$ is a child of $[v]$, if and only if $p_{[v]}$ is adjacent to $p_{[v']}$.

(iii) Let c be a critical 1-cell such that $q_{[Mc^*]}$ labels a vertex of T. Let $a' := q_{[Mc^*]}$, and let a denote the vertex over which c lies. Then a' is adjacent to a and in direction d_c from a. If a' is extremal, then by Lemma 6.4, $c' \in N_c$ if and only if c' lies over a' and $d_{c'}$ is the direction from a' to a. By Lemma 6.7, we have

$$Y_{(n-2)}(\deg(a')) = |N_c|.$$

If a' is not extremal, then if follows from Corollary 6.5 that any child $[M(c')^*]$ of $[Mc^*]$ will satisfy either:

(1) $n = 5$, c' lies over a, $d_{c'} = d_c$, and $\mathrm{CUB}(c') = \mathrm{CUB}(c) - 1 = 2$, or

(2) c' lies over a', $\mathrm{CUB}(c') = \mathrm{CUB}(c)$, and a is not in direction $d_{c'}$ from a'.

In the former case, there is no vertex labelled $q_{[M(c')^*]}$ in T. In the latter case, by Lemma 6.7, we have

$$Y_2(\deg(a')) = |N_{c'}|.$$

Thus, regardless of whether a' is extremal or not,

$$\deg(q_{[Mc^*]}) = \deg(p_{[Mc^*]}).$$

This finishes the proof of (iii), and of the lemma. $\qquad\square$

Example 6.10. Consider the tree $T_{\min}$, shown on the right of Figure 10. On the left of Figure 10 is the tree T_Δ generated in Example 6.1. The complex Δ used is in fact the simplicial complex underlying the exterior face algebra structure on $H^*(B_4 T_{\min})$. The proof of Theorem 6.9 gives the isomorphism between T_Δ and T, induced by the map $q_x \mapsto p_x$. Here, $c_0 = \left(q_1, 2, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right)$, $c_1 = \left(q_{[v_0]}, 2, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}\right)$, and $c_1 = \left(q_{[v_0]}, 2, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right)$.

Example 6.11. We wish to show an example of a partial computation for an $n = 5$ strand tree braid group. Consider the tree T shown on the left of Figure 11, depicted
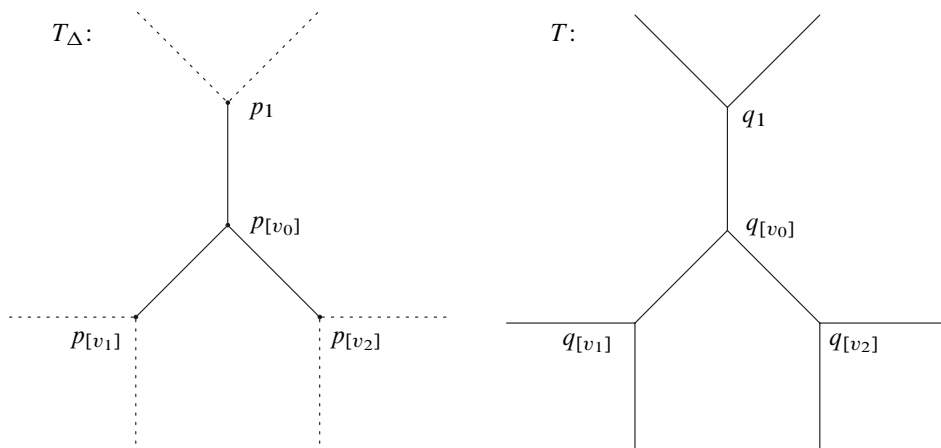
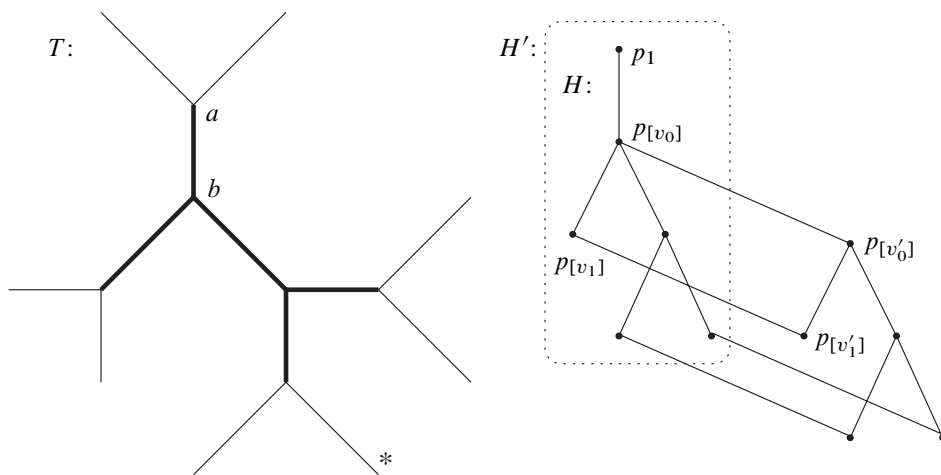Figure 10. The tree T_Δ and the isomorphism of Theorem 6.9 for the tree $T_{\min}$.

Figure 11. An example with $n = 5$.

with a Morse T-embedding. Let Δ be the simplicial complex giving the exterior face algebra structure on $H^*(B_5 T)$. On the right of Figure 11 is the graph H' is the neighborhood hierarchy H' for a vertex v_0 in the simplicial complex Δ. Since we have a Morse T-embedding, we can give v_0 a label: it is $M c_0^*$, where c_0 is the critical 1-cell $\left(a, 2, \left[\begin{smallmatrix} 3 \\ 1 \\ 1 \end{smallmatrix}\right]\right)$. Similarly, $v_0' = M(c_0')^*$, $v_1 = M c_1^*$, and $v_1' = M(c_1')^*$, where $c_0' := \left(a, 2, \left[\begin{smallmatrix} 2 \\ 1 \\ 2 \end{smallmatrix}\right]\right)$, $c_1 := \left(b, 2, \left[\begin{smallmatrix} 0 \\ 4 \\ 1 \end{smallmatrix}\right]\right)$, and $c_1' := \left(b, 2, \left[\begin{smallmatrix} 0 \\ 3 \\ 2 \end{smallmatrix}\right]\right)$. The subtree H of

H', circled in Figure 11, is the subtree from which T_Δ is grown. The subtree H is isomorphic to a subtree of T, shown in bold.

We may expand the proof of Theorem 6.9 to give us the following useful corollary:

Corollary 6.12 (Determining n). *If $G = B_n T$ is a tree braid group on either $n = 4$ or 5 strands for which T has at least 3 essential vertices, then n may be determined from G.*

Proof. Let Δ be the unique simplicial complex associated to $H^*(G)$, by Theorem 5.2. If T has at least 3 essential vertices, T is not radial, so G is not free. If G is not free, there exists a vertex v_0 in the associated complex Δ with nonempty neighborhood and which is maximal under $\leq_N$. If $n = 5$, then $[v_0]$ has a child $[v_0']$ as in the definition of Δ. As v_0 is maximal under $\leq_N$, v_0 lies over an extremal vertex q_1 of T. Let $q_{[v_0]}$ denote the unique essential vertex of T adjacent to q_1. As T has at least 3 essential vertices, there exists a third essential vertex $q \neq q_1$ adjacent to $q_{[v_0]}$. Pick a Morse T-embedding, and let c and c' be critical 1-cells that lie over $q_{[v_0]}$ and such that $d_{c'} = d_c$ is the direction from $q_{[v_0]}$ to q, CUB$(c) = 3$, and CUB$(c') = 2$ (by Lemma 6.7, such c and c' exist). By Corollary 6.5, it follows that $[Mc^*]$ is a child of $[v_0]$ but not of $[v_0']$, and $[M(c')^*]$ is a child of both $[Mc^*]$ and $[v_0']$. Thus, if $n = 5$, there exist two children of $[v_0]$ with a common child. If $n = 4$, then the neighborhood hierarchy of $[v_0]$ must be a tree, by Theorem 6.9. Thus, $n = 5$ if and only if there exist two children of $[v_0]$ with a common child. Note this characterization does not actually depend on fixing a Morse T-embedding.   □

It seems reasonable that some combinatorial argument can address the case when T has exactly two essential vertices. If this is done, the only exception to Corollary 6.12 would be when T has exactly one essential vertex. But if T has only one essential vertex, T is radial, so G is free. Thus, modulo a combinatorial argument about when T has exactly two essential vertices, Corollary 6.12 would read: If G is a tree braid group on either $n = 4$ or 5 strands which is not free, then n may be determined by G.

Theorem 6.9 also proves:

Corollary 6.13. *The tree T_Δ constructed in Section 6.1 is defined if and only if Δ is a simplicial complex giving the exterior face algebra structure of $H^*(B_n T)$ for some tree T and some $n = 4$ or 5.*   □

The fact that one may reconstruct any tree T up to homeomorphism given a (4 or 5 strand) tree braid group $B_n T$ on T is an artifact of the 1-dimensionality of trees. For instance, let $k > 1$ and consider the braid groups on a k-dimensional ball, and on a k-dimensional ball joined at a single point with a line segment. On any number of strands, the two corresponding braid groups are isomorphic.

6.3. More strands. It would be nice to generalize Theorem 6.9 to say that, given a tree braid group on n strands *for any* $n \geq 4$, one may reconstruct the underlying tree. As noted in Conjecture 5.17, we believe that tree braid groups on more than 5 strands do not have an exterior face algebra structure on cohomology in general. Thus the techniques used for $n = 4$ and 5 probably do not apply for $n \geq 6$. However, we still believe a generalization is possible:

Conjecture 6.14 (Rigidity for an arbitrary number of strands). *Let G be an n strand tree braid group where* $n \geq 4$. *Then there exists a unique tree T (up to homeomorphism) such that* $G = B_n T$.

Our intuition for conjecturing this generalization comes from the following situation. Let T be a finite tree and let $n \geq 4$. Fix a Morse T-embedding. As we have a Morse T-embedding, we may talk about critical cells. We may still apply the results of Sections 3 and 4 to T and its critical cells. Consider a critical 1-cell $c_0 = (a_0, d_0, \vec{x})$. Assume that a_0 is an extremal vertex. Let d_{c_0} denote the direction from a_0 towards every other essential vertex of T. Further assume that $x_{d_{c_0}} = n - 2 + \epsilon_{c_0}(d_{c_0})$, where $\epsilon_{c_0}(d_{c_0})$ is the cup constant associated to c_0 in direction d_{c_0}. We call a critical cell of this form *extremal*. Under these assumptions, we claim that:

Lemma 6.15 (Motivation for more strands). *Let* $c_0 = (a_0, d_0, \vec{x})$ *be extremal. Then*

$$H^*(B_{n-2}(T - \{a\})) \cong H^*(B_n T) \cup c_0^*.$$

We will prove this lemma momentarily. Assuming Lemma 6.15 holds, we should be able to reconstruct T by induction: reconstruct $T - \{a\}$ from $B_{n-2}(T - \{a\})$, and then simply 'reattach' a to T.

Of course, in general, when we are given a tree braid group we are not given any information about a Morse T-embedding or a classification of critical cells. The heart of answering Conjecture 6.14 is in finding cohomology classes which behave like c_0^* in the sense of Lemma 6.15.

Proof of Lemma 6.15. We wish to construct the isomorphism on cohomology. To do so, we define functions on several types of objects to build up to the desired map: on b-vectors for essential vertices $b \neq a$; on reduced 1-cells; on k-forms; and finally on cohomology classes.

For any essential vertex b of T which is not a, define a function at_b on b-vectors, which takes a b-vector and adds two (hence the name at) to the $(d_b)^{\text{th}}$ coordinate, where d_b is the direction from b to a. By definition, at_b is injective. Note that at_b is defined regardless of the length of the b-vectors.

Now define a function at on reduced 1-cells: $at(b, f, \vec{y}) := (b, f, at_b(\vec{y}))$. This takes a reduced 1-cell and adds two (again, hence the name) strands in the direction towards a. We must require that $b \neq a$. As with at_b, the map at is injective. Note at is defined regardless of the length of $\vec{y}$. For a reduced k-cell s, by Theorem 3.5

there exists a unique collection $\{[c_1], \ldots, [c_k]\}$ of equivalence classes of 1-cells such that $[s]$ is the least upper bound of $\{[c_1], \ldots, [c_k]\}$. Define $at(s)$ to be the reduced representative of the least upper bound of $\{[at(c_1)], \ldots, [at(c_k)]\}$. We must require that no c_i lies over a. This corresponds to adding two strands to s in the direction towards a from the essential vertex b closest to a over which one of the c_i lies. Think of at as placing two strands at the vertex a in T. Again, as the collection $\{[c_1], \ldots, [c_k]\}$ is unique, at is injective.

We generalize the map at to k-forms, as follows. For a basic k-form $\omega = f(b, \vec{y}) dc_1 \wedge \cdots \wedge dc_k$, define $at(\omega)$ to be

$$f(b, at_b(\vec{y})) d(at(c_1)) \wedge \cdots \wedge d(at(c_k)).$$

Here we require that $b \neq a$ and that each c_i does not lie over a. Again, at is injective and defined regardless of the number of strands.

We claim that the desired isomorphism on cohomology is the map AT, where $AT \colon H^*(B_{n-2}(T - \{a\})) \to H^*(B_n T) \cup c_0^*$ is given by

$$[\omega] \mapsto [at(\omega) \wedge dc_0],$$

where ω is a cochain for $n - 2$ strands on the tree $T - \{a\}$, thought of as a cochain for $n - 2$ strands on the tree T. We prove that AT is well defined with a well-defined inverse map AT^{-1}, where

$$AT^{-1}([\omega \wedge dc_0]) := [at^{-1}(\omega)]$$

if $at^{-1}(\omega)$ is defined; otherwise, $AT^{-1}([\omega \wedge dc_0]) := [0]$. If AT and AT^{-1} are well defined, then AT is a ring homomorphism, by definition and Proposition 4.6. Then, clearly from the definitions of AT and AT^{-1}, they will be indeed inverse homomorphisms whose composition in either order is the identity, giving the desired isomorphism.

First, consider if AT is well defined. Let $\omega = f(b, \vec{y}) dc_1 \wedge \cdots \wedge dc_k \in [0]$ be a necessary k-form, and let $c = (b, f, \vec{y})$ denote the necessary 1-cell for ω. By the definition of necessary, $\{[c_1], \ldots, [c_k], [c]\}$ has an upper bound $[s]$, and f is the unique respectful edge in the reduced representative s of $[s]$. Consider $at(\omega)$. If $at(\omega)$ is necessary, then $at(\omega)$ is cohomologous to 0, and $[at(\omega) \wedge dc_0] = [0]$. Consider if $at(\omega)$ is not necessary. Clearly f is such that $(b, f, at_b(\vec{y})) = at(c)$ is a reduced 1-cell. Moreover, $\{[at(c_1)], \ldots, [at(c_k)], [at(c)]\}$ has an upper bound $[at(s)]$, by the definition of at. Thus, if $at(\omega)$ is not necessary, f must be disrespectful in $at(s)$, by the definition of necessary. As f is respectful in s but disrespectful in $at(s)$, it must be that the two strands added to s in $at(s)$ are the strands which make f disrespectful. A generalization of the Upper Bound Lemma, which we leave as an exercise for the sake of brevity, shows that $\{[at(c_1)], \ldots, [at(c_k)], [at(c)], [c_0]\}$ has an upper bound $[s']$, and that moreover f is the unique respectful edge in the reduced representative s' of $[s']$. Thus, $at(c)$ is necessary for the necessary $(k + 1)$-form $at(\omega) \wedge dc_0$. Thus,

regardless of whether $at(\omega)$ is necessary or not, $[at(\omega) \wedge dc_0] = [0]$. This shows AT is well defined.

Now, consider if AT^{-1} is well defined. Let $\omega = f(b, \vec{y})dc'_1 \wedge \cdots \wedge dc'_k$ be a k-form such that $\omega \wedge dc_0 \in [0]$ is necessary, and let $c' = (b, f, \vec{y})$ denote the necessary 1-cell for $\omega \wedge dc_0$. By the definition of necessary, $\{[c'_1], \ldots, [c'_k], [c_0], [c']\}$ has an upper bound $[s']$, and f is the unique respectful edge in the reduced representative s' of $[s']$. Either $at^{-1}(\omega)$ is defined or not.

If $at^{-1}(\omega)$ is defined and $at^{-1}(\omega)$ is necessary, then $at^{-1}(\omega)$ is cohomologous to 0, and there is nothing to prove. Consider if $at^{-1}(\omega)$ is defined but not necessary. Clearly f is such that $(b, f, at_b^{-1}(\vec{y})) = at^{-1}(c')$ is a reduced 1-cell. Moreover, the set of preimages $\{[at^{-1}(c_1)], \ldots, [at^{-1}(c_k)], [at^{-1}(c)]\}$ has an upper bound $[at^{-1}(s)]$, by the definition of at. Thus, if $at^{-1}(\omega)$ is not necessary, f must be disrespectful in $at^{-1}(s)$, by the definition of necessary. But if f were disrespectful in $at^{-1}(s)$, then f would be disrespectful in $at(s)$, a contradiction. Thus, whether or not $at^{-1}(\omega)$ is necessary, if $at^{-1}(\omega)$ is defined then $AT^{-1}(\omega)$ is well defined.

If $at^{-1}(\omega)$ is not defined, then by definition either $at_b^{-1}(\vec{y})$ is not defined or $at^{-1}(c'_i)$ is not defined for some $i \in \{1, \ldots, k\}$. In the former case, let $c'' := c'$ and in the latter case let $c'' := c'_i$. In either case, express c'' as $(a'', d'', \vec{z})$, and let d''_0 denote the direction from a'' to a_0. Note $[c'']$ and $[c_0]$ have a least upper bound $[s'']$ whose reduced representative s'' is $\sim$-equivalent to a face of s, since c'' and c_0 are both $\sim$-equivalent to faces of s. As $at^{-1}(c'')$ is not defined but $[s'']$ is, by the Upper Bound Lemma it follows that the edge e_0 of c_0 is respectful in s''. But then e_0 is respectful in s. This contradicts the assumption that $\omega \wedge dc_0$ was necessary. Thus, $at^{-1}(\omega)$ must be defined and cohomologous to 0. This proves that AT^{-1} is well defined as a homomorphism.

This finishes the proof.   □

To conclude this section, we state a theorem which clarifies the behavior of free tree braid groups – that is, tree braid groups on radial trees or on fewer than 4 strands:

Theorem 6.16 (The free case). *Let G be a free tree braid group. If the number of strands n for which $G = B_n T$ for some tree T is known and $n \geq 4$, T may be reconstructed up to homeomorphism. Otherwise, T may not be uniquely determined up to homeomorphism.*

Proof. If $n < 4$, then not much can be said about T. If $n \geq 4$, then T is radial. If n is known, Theorem 3.4 gives an explicit equation to solve for the degree of the unique essential vertex. The solution is unique since all of the Y_n functions are monotone increasing. If n is not known, then it is possible for free groups of ranks given by $r = Y_n(x)$ and $r' = Y_{n'}(x')$ to be such that $r = r', n \neq n'$, and $x \neq x'$. For instance: for $n = 4, x = 4, n' = 3$, and $x' = 5$, then $r = r' = 26$; for $n = 5, x = 5, n' = 4$, and $x' = 6$, then $r = r = 155$. In general, for $x' = n + 1$ and $n' = x - 1, r = r'$. We note without proof, though, that empirically it appears this is the *only* situation in which $r = r'$.   □

7. The isomorphism problem and generalizations

Theorems 6.9 and 6.16 allow us to solve the isomorphism problem for tree braid groups when we can reconstruct the defining trees by enumeration:

Theorem 7.1 (The Isomorphism Problem). *Let G and G' be two groups be given by finite presentations, and assume that $G \cong B_n T$ and $G' \cong B_n T'$ for some positive integer n and finite trees T and T'. If either:*

- *$n = 4$ or 5 or*
- *at least one of G or G' is free,*

then there exists an algorithm which decides whether G and G' are isomorphic. The trees T and T' need not be specified. If one of T and T' has at least 3 essential vertices, then n need not be specified.

Proof. If both $B_n T$ and $B_n T'$ are free, then the problem reduces to the isomorphism problem for free groups, which has a solution. If exactly one of $B_n T$ or $B_n T'$ is free, then the groups cannot be isomorphic. If both of $B_n T$ and $B_n T'$ are not free, then we reconstruct the trees T and T' (and the value n when one of T or T' has at least 3 essential vertices) from the corresponding groups, which can be done, according to Theorem 6.9. We need to show that we can reconstruct the trees algorithmically. We begin by extrapolating T from $B_n T$.

Let the *degree* of a tree denote the sum of degrees of all essential vertices of the tree. Note that the number of trees up to homeomorphism of any given degree is finite. Let the *length* of a presentation for a group denote the sum of lengths of the defining relators in the presentation plus the number of generators. Note that the number of distinct presentations for a group of any given length is finite. For any given homomorphism between two groups and any given presentations of both, let the *length* of the homomorphism with respect to the presentations be the sum of the lengths of the images of the generators of the first group under the homomorphism. Note that, for fixed groups and presentations, the number of homomorphisms of a given length is finite.

Enumerate all trees up to homeomorphism in some order from lesser degree to greater. For a given tree and a given n, enumerate all (reduced) presentations of the corresponding n strand tree braid groups in some order from lesser length to greater. For a given tree, a given n, and a given presentation of the corresponding n strand tree braid group, enumerate all homomorphisms to $B_n T$ in some order from lesser length to greater. Finally, enumerate all homomorphisms from n strand tree braid groups to $B_n T$ by diagonalization.

As we know that $B_n T$ is a tree braid group, eventually in this enumeration there will be an isomorphism. Define an algorithm to extrapolate the tree T (up to homeomorphism) and the value n from the group $B_n T$ by running through the above enumeration and finding the first isomorphism, and outputting the corresponding tree T and

value n. Similarly, define an algorithm to extract the tree T' (up to homeomorphism) from $B_n T'$ (which can be shortened, since we now know n).

By Theorems 6.9 and 6.16, the trees T and T' are uniquely determined up to isomorphism for the cases of the theorem. Note Corollary 6.12 implies that n is uniquely determined up to isomorphism if either T or T' contain at least 3 essential vertices.

Now, the isomorphism problem for n strand tree braid groups has been reduced to solving the homeomorphism problem for trees. But there exists an algorithm for solving this problem, so we are done. $\qquad \square$

8. Right-angled Artin groups

Let Δ be a finite graph. For the purposes of this paper, we will always assume Δ is *simple*: it contains no nontrivial embedded edge loops having less than 3 edges. The *right-angled Artin group* $G(\Delta)$ associated to Δ is defined as follows. The group $G(\Delta)$ is generated by the vertices of Δ, and the only relations are that two generators commute if and only if the corresponding vertices are connected by an edge. For more information on right-angled Artin groups, see [5].

The purpose of this section is to state analogues of the main theorems of this paper, but for right-angled Artin groups. Graph braid groups are closely related to right-angled Artin groups, so it is not surprising that similar theorems hold for each. The differences between the two classes of groups is highlighted, however, by the comparative difficulty of proofs for graph braid groups.

For a finite simplicial graph Δ, the *flagification* of Δ, denoted $\overline{\Delta}$, is the simplicial complex whose vertices are the vertices of Δ, and for which k vertices span a k-simplex in $\overline{\Delta}$ if and only if the k vertices span a complete subgraph of Δ. We say Δ is *flag* if $\Delta = \overline{\Delta}$. The analogue of Theorem 5.2 is already known, and was proven by Charney and Davis:

Theorem 8.1 (RAAG cohomology; [6]). *Let Δ be a finite simple graph. Then $H^*(G(\Gamma)) \cong \Lambda(\overline{\Delta})$.*

We remark that in [16] it was proven that a tree braid group $B_n T$ is right-angled Artin if and only if $n \leq 3$ or T is linear. The method of proof essentially involved proving, in the non-right-angled Artin cases, that even if the cohomology ring of a tree braid group is an exterior face algebra, it cannot be an exterior face algebra over a flag simplicial complex.

There are two analogues of Theorem 6.9. One is an important rigidity result and appears in the literature, due to Droms. The other does not yet appear in the literature, but follows from Droms's theorem and Gubeladze's theorem:

Theorem 8.2 (RAAG defining graph rigidity; [10]). *Let Δ and Δ' be finite simple graphs. Then $G(\Delta) \cong G(\Delta')$ if and only if $\Delta \cong \Delta'$.*

Theorem 8.3 (RAAG cohomology rigidity). *There is a setwise bijection between right-angled Artin groups up to isomorphism and their cohomology rings up to isomorphism.*

Proof. By Theorem 8.1, cohomology rings of right-angled Artin groups are always exterior face algebras over flag simplicial complexes. Moreover, every flag simplicial complex corresponds to the cohomology ring of some right-angled Artin group, as there is a bijective correspondence between finite simplicial graphs and flag simplicial complexes. To get from graphs to simplicial complexes, add a k-simplex whenever its 1-skeleton is present; to get from flag simplicial complexes to graphs, take the 1-skeleton. Gubeladze's theorem (Theorem 2.3) shows that there is thus a bijective correspondence between cohomology rings of right-angled Artin groups and flag simplicial complexes. As there are bijections between right-angled Artin groups and finite simplicial graphs, finite simplicial graphs and finite flag simplicial complexes, and finite flag simplicial complexes and cohomology rings of right-angled Artin groups, this proves the theorem.   ☐

We end with the analogue of Theorem 7.1 for right-angled Artin groups. Theorem 8.2 suggests that the isomorphism for right-angled Artin groups may be solved by reconstructing Δ given $G(\Delta)$. One simply needs an algorithm for the reconstruction.

Theorem 8.4 (RAAG isomorphism problem). *Let G and G' be two groups be given by finite presentations, and assume that $G \cong G(\Delta)$ and $G' \cong G(\Delta')$ for some finite simple graphs Δ and Δ'. Then there exists an algorithm which decides whether G and G' are isomorphic. The graphs Δ and Δ' need not be specified.*

The proof of this theorem is similar to the proof of Theorem 7.1, and is a standard diagonalization argument. The theorem follows directly from Theorem 8.2, and is, more or less, generally known.

Proof. Let the *degree* of a graph denote the sum of degrees of all essential vertices. Note that the number of graphs up to homeomorphism of any given degree is finite. Let the *length* of a presentation for a group denote the sum of lengths of the defining relators in the presentation plus the number of generators. Note that the number of distinct presentations for a group of any given length is finite. For any given homomorphism between two groups and any given presentations of both, let the *length* of the homomorphism with respect to the presentations be the sum of the lengths of the images of the generators of the first group under the homomorphism. Note that, for fixed groups and presentations, the number of homomorphisms of a given length is finite.

Enumerate all graphs in some order from lesser degree to greater. For a given graph, enumerate all (reduced) presentations of the corresponding right-angled Artin group in some order from lesser length to greater. For a given graph and a given

presentation of the corresponding right-angled Artin group, enumerate all homomorphisms to G in some order from lesser length to greater. Finally, enumerate all homomorphisms from right-angled Artin groups to G by diagonalization.

As we know that G is a right-angled Artin group, eventually in this enumeration there will be an isomorphism. Define an algorithm to extrapolate the graph Δ from G. Similarly, extrapolate Δ' from G'. By 8.2, Δ and Δ' are uniquely determined. Thus, the isomorphism problem for right-angled Artin groups reduces to the isomorphism problem for graphs. $\qquad\square$

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