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Groups, Geometry, and Dynamics

The free group of rank 2 is a limit of Thompson's group F

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Abstract. We show that the free group of rank 2 is a limit of 2-markings of Thompson's group F in the space of all 2-marked groups. More specifically, we find a sequence of generating pairs for F so that as one goes out the sequence, the length of the shortest relation satisfied by the generating pair goes to infinity.

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1. Introduction

From [14], a k-marked group is a pair (G, S) where S is an ordered k-tuple of generators (the k-marking) of the group G. An isomorphism of marked groups must preserve the markings. With $\mathcal{G}_k$ the set of all isomorphism classes of k-marked groups, one says that two elements of $\mathcal{G}_k$ are no more than e^{-R} apart if they satisfy the same relations of length no longer than R. This gives a metric on $\mathcal{G}_k$, and a group L is a G-limit if it is a limit in $\mathcal{G}_k$ of a sequence of marked groups each of which is isomorphic as an unmarked group to G.

Groups that cannot be obtained as a limit of Thompson's group F are studied in [14].

Recently Akhmedov, Stein and Taback [1] have proven that the free group on k generators is a limit of k-markings of Thompson's group F when $k \geq 3$. The purpose of this paper is to prove the following.

Theorem 1. *The free group on two generators is a limit of* 2*-markings of* F.

Limits in metric spaces whose elements are groups appear in Section 7 of [10]. The space $\mathcal{G}_k$ appears in Section 6 of [8] together with references to prior related notions and a comparison of $\mathcal{G}_k$ with the space in [10]. The constructions in [10] and [8] are used in the study of the growth of groups. See Section 2 of [9] for more recent discussions and questions concerning $\mathcal{G}_k$.

Limits of groups are also used in [12] in the study of equations over groups in connection with the Tarski problem on the first order logic of free groups. See [6] for a recent discussion of this connection.

See [5] for an introduction to Thompson's group F.

There is a closely related notion of free-like. Essentially a group G is k-free-like if both (a) the free group on k generators can be obtained as a G-limit, and (b) the group G is "uniformly non-amenable" with respect to the markings used in the limit. See [11] for the full definition, examples and discussion of the free-like property and related concepts. A recent proof [13] that F is amenable is still being checked at the time of this writing.

The notion of a free group occurring as a G-limit can be easily restated. The free group of rank k is a G-limit if there is a sequence of k-tuples S_n in G so that (a) each S_n generates G, and (b) for each n, no reduced word of length at most n in the elements of S_n and their inverses represents the trivial element of G. Alternatively for (b), one can say that the homomorphism from the free group on k generators taking the generators of the free group to the elements of S_n embeds the n-ball of the free group in G.

Our proof combines two tendencies in F. First, it is not hard to find elements in F that satisfy no short relations. Second it is rather "easy" to generate F. The word "easy" is in quotes because while it might not be hard to pick out sets of elements that generate F, the calculations that show that they generate might be complicated. This is discussed more in the body of the paper.

The technique that we use to embed large balls of the free group in F is taken from [3]. An alternative technique is found in [7]. However it is not clear that the alternative technique lends itself as well to building generators.

In Section 2 we give the outline. In Section 3 we give the details that show that the outline is correct. The outline is the more important part of the paper, and the details should be read only by those that wish to check for correctness.

Background for this paper would include [5] for its general introduction to the group F, for the normal form of an element in terms of the infinite generating set, and for the description in Section 1 of [5] of the "rectangular diagrams" of Thurston which are aids to calculation. Rectangular diagrams will be used extensively in Section 3. We use the representation of F as a group of right acting homeomorphisms of the non-negative real numbers. A discussion close to this view is found in Section 2 of [2] and to some extent in [4].

We would like to thank Mark Sapir for supplying the question answered by Theorem 1, and the referees for a meticulous reading of the paper.

2. Outline

The model of F that we work with is the model on

$$\mathbb{R}_{\geq 0} = \{t \in \mathbb{R} \mid t \geq 0\}$$

in which the generators are the self homeomorphisms x_i, $i \geq 0$, of $\mathbb{R}_{\geq 0}$ operating on the right defined by

$$tx_i = \begin{cases} t, & t \leq i, \\ i + 2(t-i), & i \leq t \leq i+1, \\ t+1, & t \geq i+1. \end{cases}$$

The x_i satisfy the usual relations $x_j x_i = x_i x_{j+1}$ whenever $i < j$, and it is known that x_0 and x_1 generate F.

Our method will be to modify x_0 and x_1 "slightly" so that (I) they still generate, and (II) they do not satisfy short relations. The slightness of the modification will mean that in a region large enough to be useful, the modifications agree with the originals.

2.1. Getting generators. Showing (I), that the modifications still generate, will need the more intricate argument. I learned the following use of the subgroups $F_{[a,b]}$ from Collin Bleak and Bronlyn Wassink.

Let $[a, b]$ be a closed interval, and let $F_{[a,b]}$ be all the elements in F whose support is contained in $[a, b]$. The *support* of an $f \in F$ is the set $\{t \in \mathbb{R}_{\geq 0} \mid tf \neq t\}$.

We will only refer to $F_{[a,b]}$ when $0 \leq a < b$ and both a and b are dyadic (that is, of the form $j/2^k$ with j and k in $\mathbb{Z}$). It is standard that under these restrictions $F_{[a,b]}$ is isomorphic to F, and if $a \leq c < b \leq d$ are all dyadic, then $F_{[a,b]} \cup F_{[c,d]}$ generates all of $F_{[a,d]}$.

When a and b are two consecutive integers, it is very easy to write down generators for $F_{[a,b]}$. For $a = 0$ and $b = 1$, we have the usual model of F on $[0, 1]$ and typical generators for $F_{[0,1]}$ are y_0 and z_0 defined by

$$ty_0 = \begin{cases} 2t, & 0 \leq t \leq \frac{1}{4}, \\ t + \frac{1}{4}, & \frac{1}{4} \leq t \leq \frac{1}{2}, \\ 1 - \frac{1}{2}(1-t), & \frac{1}{2} \leq t \leq 1, \end{cases}$$

$$tz_0 = \begin{cases} 2t, & 0 \leq t \leq \frac{1}{8}, \\ t + \frac{1}{8}, & \frac{1}{8} \leq t \leq \frac{1}{4}, \\ \frac{1}{2} - \frac{1}{2}(\frac{1}{2} - t), & \frac{1}{4} \leq t \leq \frac{1}{2}, \\ t, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

It is standard and an easy exercise that

$$y_0 = x_0^2 x_1^{-1} x_0^{-1},$$
$$z_0 = x_0^3 x_1^{-1} x_0^{-2}.$$

To generate $F_{[i,i+1]}$ for a positive integer i, we use y_i and z_i where

$$\begin{aligned} y_i &= x_i^2 x_{i+1}^{-1} x_i^{-1}, \\ z_i &= x_i^3 x_{i+1}^{-1} x_i^{-2}. \end{aligned} \tag{1}$$

We will also use the elements

$$w_i = x_i x_{i+1}^{-1} \tag{2}$$

and it is also standard and easy exercise that w_i and y_i generate $F_{[i,i+2]}$ in a manner "identical" to the manner in which y_i and z_i generate $F_{[i,i+1]}$.

Lemma 2.1. *Let X_0 and X_1 generate a subgroup G of F and assume the following hold.*

(a) *The support of $X_0 x_0^{-1}$ is contained in a compact subset of* $(0, \infty)$.
(b) *The support of $X_1 x_1^{-1}$ is contained in a compact subset of* $(0, \infty)$.
(c) *The group G generated by X_0 and X_1 contains w_1 and y_1.*
(d) *The translates of the open interval* $(1, 3)$ *under G cover all of* $(0, \infty)$.

Then $G = F$.

Proof. From (c), we know that G contains $F_{[1,3]}$, from (d) we know that G contains all elements of F whose support is contained in a compact subset of $(0, \infty)$, and from (a) and (b) we know that G contains $X_0 x_0^{-1}$ and $X_1 x_1^{-1}$ and thus x_0 and x_1. Thus G contains all of F. □

2.2. Avoiding relations. Showing (II), that our chosen generators do not satisfy short relations, will be done as in [3]. We will take homeomorphisms on the circle that generate a free group and lift these to the real line. The lifts will not be elements of F, but approximations with supports on compact subsets will be elements of F. If the compact subsets are long enough, then as shown in [3] the approximations can only satisfy long relations. Let us refer to these approximations as "almost free" elements.

We will need to know more about these "almost free elements" than just the fact that they only satisfy long relations. We will also need to know some specific relations that they do satisfy. As is typically the case with F, these relations will be commutators of certain words in the almost free elements. Our knowledge of these relations will help us build our generating set.

2.3. Getting generators, revisited. We can now give better descriptions of our generators X_0 and X_1.

We will work with two intervals. On one interval $[0, b-5)$ for some $b > 5$ that we will choose, X_i will agree with x_i for $i = 0, 1$. (The strange way of giving the upper limit of the interval is to make things convenient later.) On a second interval $(b-5, d)$ for a $d > b$ that will depend on n, X_i will agree with an element g_i (also depending on n) for $i = 0, 1$. The g_i will be chosen to so that their restrictions to $(b-5, d)$ satisfy no relations shorter than n, but do satisfy certain specific relations that are longer than n.

We then consider words in the X_i. We will use capital letters to denote such words. For example

$$C = X_0^2 X_1^2 X_0^{-2} X_1^{-2}$$

will be one word. The corresponding lower case letter will denote the corresponding word in the x_i. Thus

$$c = x_0^2 x_1^2 x_0^{-2} x_1^{-2}$$

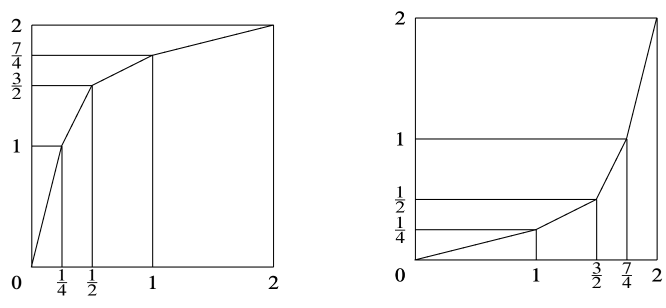
denotes the word corresponding to C.

For certain words (let W denote such a word for this discussion) in the X_i, the corresponding word (w in this example) in the x_i will satisfy $W = w$ in F. A rough idea of the reason is that

(1) W will be trivial on $(b-5, \infty)$,
(2) the X_i agree with the x_i on $(0, b-5)$, and
(3) the word w in the x_i has support in $[0, b-5)$.

See Proposition 2.2 (i) and (ii) below. More discussion can be read in Section 3.2.1 before diving into the details of the proof of the proposition.

2.4. Avoiding relations, revisited. We now describe the X_i. Our first basic building block will be the piece of function below on the left and its inverse below on the right.

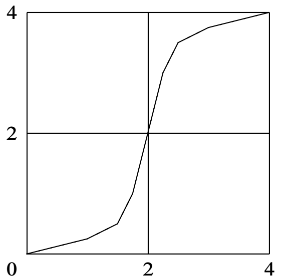


(3)

The reader can verify that the function on the left is w_0^2, a fact that will be convenient for notation but that is not terribly important otherwise. The most important property that we need is that $\frac{1}{2}$ is taken to $\frac{3}{2}$. That is, all points in $[0, 2]$ not within $\frac{1}{2}$

of the left fixed point are carried to within $\frac{1}{2}$ of the right fixed point. Also important is that no point moves more than one.

Our second basic building block will be $w_0^{-2}w_2^2$ that is pictured below.

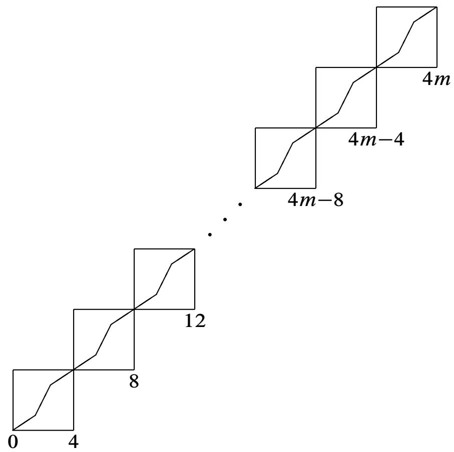


(4)

We consider a composition of m copies of the block from (4) translated by multiples of 4 as follows:

$$(w_0^{-2}w_2^2)(w_4^{-2}w_6^2)\ldots(w_{4(m-1)}^{-2}w_{4(m-1)+2}^2).$$

The graph looks something like the picture below, but the small scale prevents a truly accurate picture.



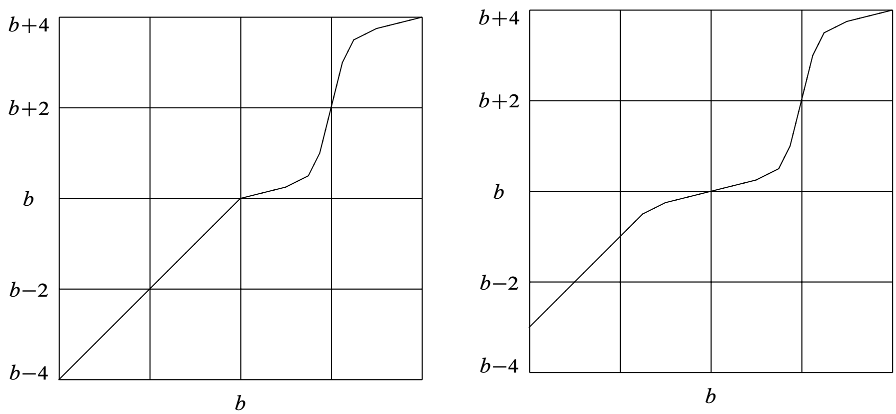
If we conjugate this function by a translation by $b > 0$ (so that it is the product $(w_b^{-2}w_{b+2}^2)(w_{b+4}^{-2}w_{b+6}^2)\ldots$, etc.), then we get a function with graph similar to that above and whose support is on $[b, b+4m]$. Call this function g_0. To make the next discussion easier, we take b to be an integral multiple of 4.

The function g_0 has the property that positive powers of g_0 have the multiples of 4 in $[b, b+4m]$ as attracting fixed points and the odd multiples of 2 in $[b, b+4m]$ as repelling fixed points.

If we conjugate g_0 by translation by 1, then we get a function g_1 whose support is on $J = [b+1, b+4m+1]$, which has the elements from $\mathbb{Z}$ that are equal to 1 modulo 4 in J as attracting fixed points, and which has the elements from $\mathbb{Z}$ that are equal to 3 modulo 4 in J as repelling fixed points.

Let $n = 2m-3$. We give a brief description of the well known argument that given any reduced word in g_0 and g_1 of length less than n, then the function corresponding to that word cannot be the identity, and thus the pair g_0 and g_1 cannot satisfy any relation shorter than n. Associate to each of the four elements $g_0^{\pm 1}$ and $g_1^{\pm 1}$ its set of attracting fixed points in $[b, b+4m]$. Thus g_0 is associated to the integers in $[b, b+4m]$ that are equal to 0 modulo 4, g_0^{-1} is associated to the integers in $[b, b+4m]$ that are equal to 2 modulo 4, and so forth. Let w be a reduced word in $g_i^{\pm 1}$, $i = 0, 1$. Let ζ be an integer within two of $b+2m$ in a set associated with neither the last letter in w, nor the inverse of the first letter. One then shows inductively on the length of w (based on the fact that w is reduced), that the image of ζ under w is within $\frac{1}{2}$ of a point in the set associated to the last letter. Since the image of ζ never moves more than one under the action of each successive letter in w, and since the induction persists as long as the image of ζ stays within $[b+1, b+4m]$, the induction will survive as long as the length of w is not more than n. (We start the interval in the previous sentence at $b+1$ since the behavior of g_1 is under the control of the blocks in (3) only starting at $b+1$.)

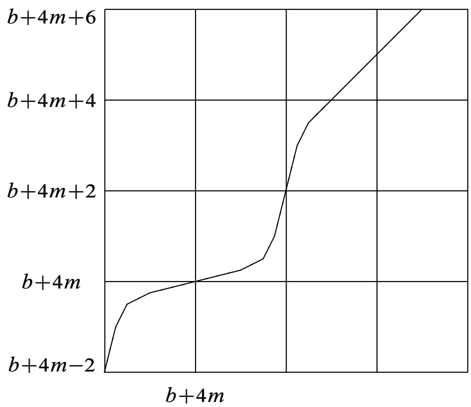
2.5. The generators themselves. We wish to combine the function g_0 with x_0. This cannot be done directly since their behaviors at b do not match. At b, the first is fixed and the second translates by 1. Thus we will alter g_0 in the interval $[b-4, b]$ so as to agree with the picture below on the right instead of its originally defined behavior as pictured below on the left.



(5)

The exact values involved in the new g_0 will be made clear in Section 3. What we need to know now is that the function on the right acts as translation by 1 at $b-2$.

We must similarly change the behavior above $b+4m$ so that its behavior is as shown below. Once again, exact values will be given in Section 3. For now we only need to know that the function acts as translation by 1 on $[b+4m+3, \infty)$.



(6)

Now we can define X_0 to agree with x_0 on $[0, b-2]$, with the modified g_0 on $[b-2, b+4m+3]$ and with x_0 again on $(b+4m+3, \infty)$.

We make similar modifications to g_1 and end up with an X_1 that is X_0 conjugated by a translation by 1. In particular X_1 agrees with x_1 on $[0, b-1]$.

We take m to be a positive integer. We let $i = m+2$.

In the following definitions, we set $X_j = X_0^{1-j} X_1 X_0^{j-1}$ for each $j > 1$ to parallel the relation $x_j = x_0^{1-j} x_1 x_0^{j-1}$ that holds in F. We now define:

$$\begin{aligned} C &= X_0^2 X_1^2 X_0^{-2} X_1^{-2}, \\ S &= X_0 X_2 X_1^{-2}, \\ T &= X_0^2 X_2 X_4 X_3^{-2} X_1^{-1} X_0^{-1}, \\ \Sigma &= C^{-i} S C^i, \\ \Theta &= C^{-i} T C^i, \end{aligned} \qquad \begin{aligned} Z &= [S, \Sigma] = S \Sigma S^{-1} \Sigma^{-1}, \\ W &= [T, \Theta] = T \Theta T^{-1} \Theta^{-1}, \\ P &= Z^{-1} W, \\ Q &= X_1^{-1} P X_1 P^{-1}, \\ H &= X_1^{-2} Q X_1^2, \\ K &= X_1 H X_1^{-1}. \end{aligned} \tag{7}$$

As mentioned above, we define the corresponding lower case symbols, c for C, θ for Θ, etc., as the corresponding words in the x_j.

It is clear that X_0 and X_1 satisfy (a) and (b) of Lemma 2.1.

In the proposition below, the elements y_1 and w_1 are as defined in (1) and (2), and w_1 is not related to the w that corresponds to W defined in (7).

Proposition 2.2. *The following hold for all sufficiently large values of b.*

(i) *The symbols defined in the right hand column of* (7) *represent the same elements of F as the corresponding lower case symbols.*
(ii) *We have the equalities $H = w_1^{-1}$ and $K = y_1^{-1}$.*
(iii) *The elements X_1 and X_2 satisfy no relation of length less than $2m-3$.*
(iv) *Hypothesis* (d) *of Lemma* 2.1 *is satisfied.*

Since (ii) in the above proposition gives (c) of Lemma 2.1, we have that X_0 and X_1 generate F. This and (iii) of the proposition give Theorem 1.

We add a bit more to the outline before surrendering this paper to the details. Items (i) and (ii) are the most technical.

For (i), we will divide $\mathbb{R}_{\geq 0}$ into two regions, $[0, b-5)$ and $(b-5, \infty)$. With b sufficiently large, the lower case symbols corresponding to the words defined in (7) will have supports in $[0, b-5)$. This is the only requirement on b and contains the meaning of "sufficiently large."

We will show that C has two parts to its support. There will be a part in $[0, b-5)$ where C and c agree and a part in a closed interval I_C in $(b-5, \infty)$ for which $\zeta C > \zeta$ for all ζ in the interior of I_C.

The functions defined as S and T will also have two parts to their support. There will be a part in $[0, b-5)$ where S and T will agree with s and t, respectively, and a part with closure in the interior of I_C.

We will show that the conjugates of S and T by C^i will have the parts of their supports in $(b-5, \infty)$ disjoint from the supports of S and T. Thus the commutators Z and W will be trivial on $(b-5, \infty)$. This will verify (i) for Z and W, and the truth of (i) for the rest will follow easily.

The truth of (ii) will follow from a long algebraic calculation.

The argument for (iii) has already been given for the original functions g_0 and g_1. The argument applies to X_0 and X_1 since these agree with the original g_0 and g_1, respectively, on the interval $[b+1, b+4m]$ needed for the argument.

The truth of (iv) will follow from the information gathered in the arguments for (ii) and from the definitions of X_0 and X_1.

The next section gives the details needed to verify the truth of (i), (ii) and (iv) of Proposition 2.2. This will complete the proof of Theorem 1.

3. Details

The elements in (7) were found using a computer program for doing calculations in F that was written by the author almost twenty years ago. It was written partly as an exercise in learning how to use a compiler compiler (sic). The elements in (7) were found by "experiment guided by experience." Once elements were found with the required properties, the problem of how to write a proof of Theorem 1 arose. Listing the definitions in (7) and then instructing the reader to the use the program to check the claimed behaviors was not an option since the program is very large and its inner workings are (after almost 20 years) opaque even to the author. It was then discovered that the algebra behind (ii) of Proposition 2.2 was not that bad, and that the "rectangular diagrams" of Thurston as described in [5] made the verification of all that is needed for (i) of Proposition 2.2 very visual. The result, while manageable, is still not attractive.

A failed attempt was made to find more attractive examples. There may very well be less complicated generators, or ones whose properties are easier to prove. However if such examples exist, they seem hard to find.

3.1. Proposition 2.2 part (ii). We start with the more algebraic calculations. We will not end with a proof of (ii), but a proof that (ii) follows from (i). If (i) holds, then $H = h$ and $K = k$ and proving that $h = w_1^{-1}$ and $k = y_1^{-1}$ will give (ii). Thus we analyze the lower case symbols.

We put the lower case symbols in normal form. Some, such as s and t are already given in normal form.

Recall that m (which does not make an appearance until the next section) is a positive integer, and that i, which appears almost immediately below, is defined as $m + 2$.

First

$$c = x_0^2(x_1^2x_3^{-2})x_0^{-2}.$$

Next

$$\begin{aligned} c^i &= x_0^2(x_1^2x_3^{-2})^i x_0^{-2} \\ &= x_0^2(x_1^2x_3^{-2}x_1^2x_3^{-2} \dots x_1^2x_3^{-2})x_0^{-2} \\ &= x_0^2(x_1^{2i}x_{2i+1}^{-2}x_{2i-1}^{-2} \dots x_5^{-2}x_3^{-2})x_0^{-2}. \end{aligned}$$

We skip σ and θ since they are absorbed into z and w. We start with z.

We have

$$\begin{aligned} z &= s(c^{-i}sc^i)s^{-1}(c^{-i}s^{-1}c^i) \\ &= (x_0x_2x_1^{-2})(x_0^2x_3^2x_5^2 \dots x_{2i-1}^2x_{2i+1}^2x_1^{-2i}x_0^{-2}) \\ &\quad (x_0x_2x_1^{-2})(x_0^2x_1^{2i}x_{2i+1}^{-2}x_{2i-1}^{-2} \dots x_5^{-2}x_3^{-2}x_0^{-2}) \\ &\quad (x_1^2x_2^{-1}x_0^{-1})(x_0^2x_3^2x_5^2 \dots x_{2i-1}^2x_{2i+1}^2x_1^{-2i}x_0^{-2}) \\ &\quad (x_1^2x_2^{-1}x_0^{-1})(x_0^2x_1^{2i}x_{2i+1}^{-2}x_{2i-1}^{-2} \dots x_5^{-2}x_3^{-2}x_0^{-2}). \end{aligned}$$

The long parenthesized expressions on the right of each line contain appearances of $x_0^{\pm 2}$. The appearances of x_0^2 are moved as far left as possible and the appearances of x_0^{-2} are moved as far right as possible.

$$\begin{aligned} z &= (x_0^3x_4x_3^{-2})(x_3^2x_5^2 \dots x_{2i-1}^2x_{2i+1}^2x_1^{-2i}) \\ &\quad (x_0x_4x_3^{-2})(x_1^{2i}x_{2i+1}^{-2}x_{2i-1}^{-2} \dots x_5^{-2}x_3^{-2}) \\ &\quad (x_3^2x_4^{-1}x_0^{-1})(x_3^2x_5^2 \dots x_{2i-1}^2x_{2i+1}^2x_1^{-2i}) \\ &\quad (x_3^2x_4^{-1}x_0^{-1})(x_1^{2i}x_{2i+1}^{-2}x_{2i-1}^{-2} \dots x_5^{-2}x_3^{-2}x_0^{-2}). \end{aligned}$$

We cancel adjacent inverse items.

$$\begin{aligned} z = {} & (x_0^3 x_4)(x_5^2 \dots x_{2i-1}^2 x_{2i+1}^2 x_1^{-2i}) \\ & (x_0 x_4 x_3^{-2})(x_1^{2i} x_{2i+1}^{-2} x_{2i-1}^{-2} \dots x_5^{-2}) \\ & (x_4^{-1} x_0^{-1})(x_3^2 x_5^2 \dots x_{2i-1}^2 x_{2i+1}^2 x_1^{-2i}) \\ & (x_3^2 x_4^{-1} x_0^{-1})(x_1^{2i} x_{2i+1}^{-2} x_{2i-1}^{-2} \dots x_5^{-2} x_3^{-2} x_0^{-2}). \end{aligned}$$

Now we move the remaining appearances of $x_0^{\pm 1}$ from the middle of the expression.

$$\begin{aligned} z = {} & (x_0^4 x_5)(x_6^2 \dots x_{2i}^2 x_{2i+2}^2 x_2^{-2i}) \\ & (x_4 x_3^{-2})(x_1^{2i} x_{2i+1}^{-2} x_{2i-1}^{-2} \dots x_5^{-2}) \\ & (x_4^{-1})(x_4^2 x_6^2 \dots x_{2i}^2 x_{2i+2}^2 x_2^{-2i}) \\ & (x_4^2 x_5^{-1})(x_3^{2i} x_{2i+3}^{-2} x_{2i+1}^{-2} \dots x_7^{-2} x_5^{-2} x_0^{-4}). \end{aligned}$$

We cancel one adjacent pair.

$$\begin{aligned} z = {} & (x_0^4 x_5)(x_6^2 \dots x_{2i}^2 x_{2i+2}^2 x_2^{-2i}) \\ & (x_4 x_3^{-2})(x_1^{2i} x_{2i+1}^{-2} x_{2i-1}^{-2} \dots x_5^{-2}) \\ & (x_4 x_6^2 \dots x_{2i}^2 x_{2i+2}^2 x_2^{-2i}) \\ & (x_4^2 x_5^{-1})(x_3^{2i} x_{2i+3}^{-2} x_{2i+1}^{-2} \dots x_7^{-2} x_5^{-2} x_0^{-4}). \end{aligned}$$

We move the x_1^{2i} from the second line and the x_2^{-2i} from the third line.

$$\begin{aligned} z = {} & (x_0^4 x_1^{2i} x_{2i+5})(x_{2i+6}^2 \dots x_{4i}^2 x_{4i+2}^2 x_{2i+2}^{-2i}) \\ & (x_{2i+4} x_{2i+3}^{-2})(x_{2i+1}^{-2} x_{2i-1}^{-2} \dots x_5^{-2}) \\ & (x_4 x_6^2 \dots x_{2i}^2 x_{2i+2}^2) \\ & (x_{2i+4}^2 x_{2i+5}^{-1})(x_{2i+3}^{2i} x_{4i+3}^{-2} x_{4i+1}^{-2} \dots x_{2i+7}^{-2} x_{2i+5}^{-2} x_2^{-2i} x_0^{-4}). \end{aligned}$$

We move the x_4 from the beginning of the third line.

$$\begin{aligned} z = {} & (x_0^4 x_1^{2i} x_4 x_{2i+6})(x_{2i+7}^2 \dots x_{4i+1}^2 x_{4i+3}^2 x_{2i+3}^{-2i}) \\ & (x_{2i+5} x_{2i+4}^{-2})(x_{2i+2}^{-2} x_{2i}^{-2} \dots x_6^{-2}) \\ & (x_6^2 \dots x_{2i}^2 x_{2i+2}^2) \\ & (x_{2i+4}^2 x_{2i+5}^{-1})(x_{2i+3}^{2i} x_{4i+3}^{-2} x_{4i+1}^{-2} \dots x_{2i+7}^{-2} x_{2i+5}^{-2} x_2^{-2i} x_0^{-4}). \end{aligned}$$

We cancel inverse pairs.

$$z = (x_0^4 x_1^{2i} x_4 x_{2i+6})(x_{2i+5}^{-2} x_2^{-2i} x_0^{-4}).$$

Now we work on w.

$$\begin{aligned}
w &= t(c^{-i}tc^{i})t^{-1}(c^{-i}t^{-1}c^{i}) \\
&= (x_0^2x_2x_4x_3^{-2}x_1^{-1}x_0^{-1})(x_0^2x_3^2x_5^2\dots x_{2i-1}^2x_{2i+1}^2x_1^{-2i}x_0^{-2}) \\
&\quad (x_0^2x_2x_4x_3^{-2}x_1^{-1}x_0^{-1})(x_0^2x_1^{2i}x_{2i+1}^{-2}x_{2i-1}^{-2}\dots x_5^{-2}x_3^{-2}x_0^{-2}) \\
&\quad (x_0x_1x_3^2x_4^{-1}x_2^{-1}x_0^{-2})(x_0^2x_3^2x_5^2\dots x_{2i-1}^2x_{2i+1}^2x_1^{-2i}x_0^{-2}) \\
&\quad (x_0x_1x_3^2x_4^{-1}x_2^{-1}x_0^{-2})(x_0^2x_1^{2i}x_{2i+1}^{-2}x_{2i-1}^{-2}\dots x_5^{-2}x_3^{-2}x_0^{-2}) \\
&= (x_0^2x_2x_4x_3^{-2}x_1^{-1})(x_0x_3^2x_5^2\dots x_{2i-1}^2x_{2i+1}^2x_1^{-2i}) \\
&\quad (x_2x_4x_3^{-2}x_1^{-1})(x_0x_1^{2i}x_{2i+1}^{-2}x_{2i-1}^{-2}\dots x_5^{-2}x_3^{-2}x_0^{-1}) \\
&\quad (x_1x_3^2x_4^{-1}x_2^{-1})(x_3^2x_5^2\dots x_{2i-1}^2x_{2i+1}^2x_1^{-2i}x_0^{-1}) \\
&\quad (x_1x_3^2x_4^{-1}x_2^{-1})(x_1^{2i}x_{2i+1}^{-2}x_{2i-1}^{-2}\dots x_5^{-2}x_3^{-2}x_0^{-2}).
\end{aligned}$$

Moving the internal $x_0^{\pm 1}$ gives the following.

$$\begin{aligned}
w = {} & (x_0^4x_4x_6x_5^{-2}x_3^{-1})(x_4^2x_6^2\dots x_{2i}^2x_{2i+2}^2x_2^{-2i}) \\
& (x_3x_5x_4^{-2}x_2^{-1})(x_1^{2i}x_{2i+1}^{-2}x_{2i-1}^{-2}\dots x_5^{-2}x_3^{-2}) \\
& (x_2x_4^2x_5^{-1}x_3^{-1})(x_4^2x_6^2\dots x_{2i}^2x_{2i+2}^2x_2^{-2i}) \\
& (x_3x_5^2x_6^{-1}x_4^{-1})(x_3^{2i}x_{2i+3}^{-2}x_{2i+1}^{-2}\dots x_7^{-2}x_5^{-2}x_0^{-4}).
\end{aligned}$$

We move the x_1^{2i} from the second line and the x_2^{-2i} from the third line.

$$\begin{aligned}
w = {} & (x_0^4x_1^{2i}x_{2i+4}x_{2i+6}x_{2i+5}^{-2}x_{2i+3}^{-1})(x_{2i+4}^2x_{2i+6}^2\dots x_{4i}^2x_{4i+2}^2x_{2i+2}^{-2i}) \\
& (x_{2i+3}x_{2i+5}x_{2i+4}^{-2}x_{2i+2}^{-1})(x_{2i+1}^{-2}x_{2i-1}^{-2}\dots x_5^{-2}x_3^{-2}) \\
& (x_2x_4^2x_5^{-1}x_3^{-1})(x_4^2x_6^2\dots x_{2i}^2x_{2i+2}^2) \\
& (x_{2i+3}x_{2i+5}^2x_{2i+6}^{-1}x_{2i+4}^{-1})(x_{2i+3}^{2i}x_{4i+3}^{-2}x_{4i+1}^{-2}\dots x_{2i+7}^{-2}x_{2i+5}^{-2}x_2^{-2i}x_0^{-4}).
\end{aligned}$$

We move x_2 and x_3^{-1} from the third line and cancel some adjacent pairs.

$$\begin{aligned}
w = {} & (x_0^4x_1^{2i}x_2x_{2i+5}x_{2i+7}x_{2i+6}^{-2}x_{2i+4}^{-1})(x_{2i+5}^2x_{2i+7}^2\dots x_{4i+1}^2x_{4i+3}^2x_{2i+3}^{-2i}) \\
& (x_{2i+4}x_{2i+6}x_{2i+5}^{-2}x_{2i+3}^{-1})(x_{2i+2}^{-2}x_{2i}^{-2}\dots x_6^{-2}) \\
& (x_5x_7^2\dots x_{2i+1}^2x_{2i+3}^2) \\
& (x_{2i+4}x_{2i+6}^2x_{2i+7}^{-1}x_{2i+5}^{-1})(x_{2i+4}^{2i}x_{4i+4}^{-2}x_{4i+2}^{-2}\dots x_{2i+8}^{-2}x_{2i+6}^{-2}x_3^{-1}x_2^{-2i}x_0^{-4}).
\end{aligned}$$

We move the x_5 from the beginning of the third line.

$$\begin{aligned}
w = {} & (x_0^4x_1^{2i}x_2x_5x_{2i+6}x_{2i+8}x_{2i+7}^{-2}x_{2i+5}^{-1})(x_{2i+6}^2x_{2i+8}^2\dots x_{4i+2}^2x_{4i+4}^2x_{2i+4}^{-2i}) \\
& (x_{2i+5}x_{2i+7}x_{2i+6}^{-2}x_{2i+4}^{-1})(x_{2i+3}^{-2}x_{2i+1}^{-2}\dots x_7^{-2}) \\
& (x_7^2\dots x_{2i+1}^2x_{2i+3}^2) \\
& (x_{2i+4}x_{2i+6}^2x_{2i+7}^{-1}x_{2i+5}^{-1})(x_{2i+4}^{2i}x_{4i+4}^{-2}x_{4i+2}^{-2}\dots x_{2i+8}^{-2}x_{2i+6}^{-2}x_3^{-1}x_2^{-2i}x_0^{-4}).
\end{aligned}$$

We cancel inverse pairs.

$$w = (x_0^4 x_1^{2i} x_2 x_5 x_{2i+6} x_{2i+8} x_{2i+7}^{-2} x_{2i+5}^{-1})(x_3^{-1} x_2^{-2i} x_0^{-4}).$$

We continue with p, q, h, and k.

$$\begin{aligned}
p &= z^{-1} w \\
&= (x_0^4 x_1^{2i} x_4 x_{2i+6} x_{2i+5}^{-2} x_2^{-2i} x_0^{-4})^{-1} \\
&\quad (x_0^4 x_1^{2i} x_2 x_5 x_{2i+6} x_{2i+8} x_{2i+7}^{-2} x_{2i+5}^{-1})(x_3^{-1} x_2^{-2i} x_0^{-4}) \\
&= (x_0^4 x_2^{2i} x_{2i+5}^2 x_{2i+6}^{-1} x_4^{-1} x_1^{-2i} x_0^{-4}) \\
&\quad (x_0^4 x_1^{2i} x_2 x_5 x_{2i+6} x_{2i+8} x_{2i+7}^{-2} x_{2i+5}^{-1} x_3^{-1} x_2^{-2i} x_0^{-4}) \\
&= (x_0^4 x_2^{2i} x_{2i+5}^2 x_{2i+6}^{-1} x_4^{-1})(x_2 x_5 x_{2i+6} x_{2i+8} x_{2i+7}^{-2} x_{2i+5}^{-1} x_3^{-1} x_2^{-2i} x_0^{-4}) \\
&= (x_0^4 x_2^{2i+1} x_{2i+6}^2 x_{2i+7}^{-1} x_5^{-1})(x_5 x_{2i+6} x_{2i+8} x_{2i+7}^{-2} x_{2i+5}^{-1} x_3^{-1} x_2^{-2i} x_0^{-4}) \\
&= (x_0^4 x_2^{2i+1} x_{2i+6}^2 x_{2i+7}^{-1})(x_{2i+6} x_{2i+8} x_{2i+7}^{-2} x_{2i+5}^{-1} x_3^{-1} x_2^{-2i} x_0^{-4}) \\
&= (x_0^4 x_2^{2i+1} x_{2i+6}^3 x_{2i+7}^{-2} x_{2i+5}^{-1} x_3^{-1} x_2^{-2i} x_0^{-4}) \\
&= (x_0^3 x_1^{2i+1} x_{2i+5}^3 x_{2i+6}^{-2} x_{2i+4}^{-1} x_2^{-1} x_1^{-2i} x_0^{-3}).
\end{aligned}$$

A factor of q is $x_1^{-1} p x_1$ which we compute.

$$\begin{aligned}
x_1^{-1} p x_1 &= x_1^{-1}(x_0^3 x_1^{2i+1} x_{2i+5}^3 x_{2i+6}^{-2} x_{2i+4}^{-1} x_2^{-1} x_1^{-2i} x_0^{-3}) x_1 \\
&= x_0^3 x_4^{-1} x_1^{2i+1} x_{2i+5}^3 x_{2i+6}^{-2} x_{2i+4}^{-1} x_2^{-1} x_1^{-2i} x_4 x_0^{-3} \\
&= x_0^3 x_1^{2i+1} x_{2i+5}^{-1} x_{2i+5}^3 x_{2i+6}^{-2} x_{2i+4}^{-1} x_{2i+5} x_2^{-1} x_1^{-2i} x_0^{-3} \\
&= x_0^3 x_1^{2i+1} x_{2i+5}^2 x_{2i+6}^{-2} x_{2i+4}^{-1} x_{2i+5} x_2^{-1} x_1^{-2i} x_0^{-3} \\
&= x_0^3 x_1^{2i+1} x_{2i+5}^2 x_{2i+6}^{-2} x_{2i+6} x_{2i+4}^{-1} x_2^{-1} x_1^{-2i} x_0^{-3} \\
&= x_0^3 x_1^{2i+1} x_{2i+5}^2 x_{2i+6}^{-1} x_{2i+4}^{-1} x_2^{-1} x_1^{-2i} x_0^{-3}.
\end{aligned}$$

And we can now compute q.

$$\begin{aligned}
q &= x_1^{-1} p x_1 p^{-1} \\
&= (x_0^3 x_1^{2i+1} x_{2i+5}^2 x_{2i+6}^{-1} x_{2i+4}^{-1} x_2^{-1} x_1^{-2i} x_0^{-3}) \\
&\quad (x_0^3 x_1^{2i+1} x_{2i+5}^3 x_{2i+6}^{-2} x_{2i+4}^{-1} x_2^{-1} x_1^{-2i} x_0^{-3})^{-1} \\
&= (x_0^3 x_1^{2i+1} x_{2i+5}^2 x_{2i+6}^{-1} x_{2i+4}^{-1} x_2^{-1} x_1^{-2i} x_0^{-3}) \\
&\quad (x_0^3 x_1^{2i} x_2 x_{2i+4} x_{2i+6}^2 x_{2i+5}^{-3} x_1^{-(2i+1)} x_0^{-3}) \\
&= x_0^3 x_1^{2i+1} x_{2i+5}^2 x_{2i+6} x_{2i+5}^{-3} x_1^{-(2i+1)} x_0^{-3} \\
&= x_0^3 x_4^2 x_5 x_4^{-3} x_0^{-3} \\
&= x_1^2 x_2 x_1^{-3}.
\end{aligned}$$

We now get our desired elements.

$$h = x_1^{-2} q x_1^2 = x_1^{-2}(x_1^2 x_2 x_1^{-3}) x_1^2 = x_2 x_1^{-1} = w_1^{-1},$$
$$k = x_1 h x_1^{-1} = x_1 x_2 x_1^{-2} = y_1^{-1}.$$

This completes the proof of (ii) from (i). The calculations above might give the impression that generating F (or more to the point, generating some $F_{[a,b]}$) is a rare phenomenon. This is not so. While certain obvious obstructions make the generation of some $F_{[a,b]}$ a "probability zero" event, it is still surprisingly easy to arrange that it happens.

3.2. Proposition 2.2 Part (i). Almost all of the effort here will go to understanding the supports of the elements in (7). What we do here will also supply the missing details about the defined behavior of g_0 and g_1 that were only vaguely described in (5) and (6).

3.2.1. First look at the supports. We start with some preliminary estimates that relate to b. These will be sharpened later.

Each of the elements defined in (7) acts on $\mathbb{R}_{\geq 0}$ as the identity on certain intervals. It will be necessary to know something about what these intervals are. We will look at both the upper and lower case symbols.

We note that the symbols in (7) occur in three groups. The symbols C, S and T are defined in terms of the X_j, the symbols Σ, Θ, Z, W and P are defined in terms of C, S and T, and the last three symbols are defined in terms of P and X_1.

We consider C, S and T first. When written in terms of X_0 and X_1 using $X_j = X_0^{1-j} X_1 X_0^{j-1}$, we see that

$$T = X_0^2 X_2 X_4 X_3^{-2} X_1^{-1} X_0^{-1} = X_0 X_1 X_0^{-1} X_0^{-1} X_1 X_0 X_1^{-1} X_1^{-1} X_0 X_0 X_1^{-1} X_0^{-1}.$$

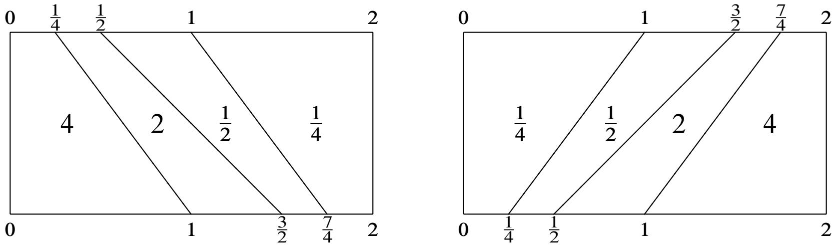
is the longest at 12 letters. We also note that the total exponent sum over all the generators in each of C, S and T is zero. Thus the sum of the positive exponents is never more than 6. Since $X_0^{\pm 1}$ and $X_1^{\pm 1}$ are translations by ± 1 on at least $[3, b-2]$, it follows that if the length of $[3, b-2]$ is 12 or more, then each of C, S and T has a fixed point at 9. From this point we can take b to be at least 17. We will see later that this is overly cautious.

With b as above, it follows similarly from the fact that each of $x_0^{\pm 1}$ and $x_1^{\pm 1}$ is translation by ± 1 on at least $[3, \infty)$ that each of c, s and t has support in $[0, 9]$, and that each of C, S and T agrees with c, s and t, respectively, on $[0, 9]$.

It now follows that each of the symbols in the second group Σ, Θ Z, W, and P has a fixed point at 9 and that it agrees with the corresponding lower case symbol on $[0, 9]$.

Discussion of the third group of symbols can wait until it is shown that $W = w$ and $Z = z$.

3.2.2. Describing the elements. We make use of the rectangular diagrams of [5]. In (8) below are the diagrams for the basic building blocks shown in (3).

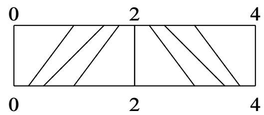


(8)

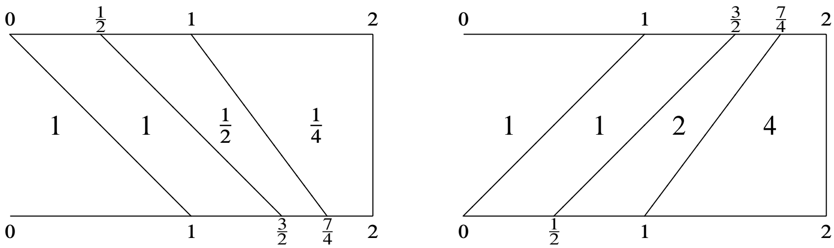
The numbers at the top give coordinates in the domain, and the numbers at the bottom give coordinates for the range. The function is viewed as going from the top of the rectangle to the bottom. The numbers in the middle give the slopes. The slopes will be useful in calculating the effects of some compositions.

The two figures in (8) are mutual inverses. Note that, with the exceptions of the numbers across the middle, the two figures in (8) are reflections of each other across a horizontal line through the center. The slopes in one figure are the reciprocals of the slopes in the other.

To describe more complicated functions, we will put together smaller versions of the pictures above, with less information about coordinates and no information about slopes. For example, the function in (4) would be described by the following.

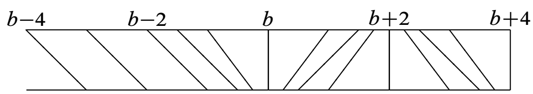


To describe the function in the right figure of (5) we will make use of the diagram in the left of (9) below. The right figure in (9) is the inverse of the left figure.

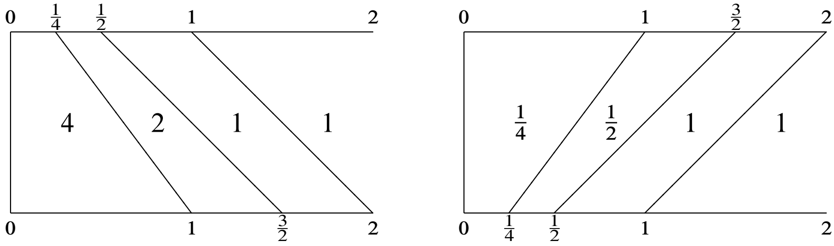


(9)

The coordinates in (9) have been arbitrarily chosen to start at 0, and the left edge is missing since 0 is not a fixed point. A diagram for the function in the right part of (5) is as follows. We do not bother with the coordinates on the bottom.

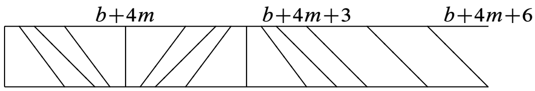


To describe the function in (6), we will use the left figure in (10) below whose inverse is in the right part of (10).



(10)

A diagram for the function in (6) is as follows where we show only a few coordinates.

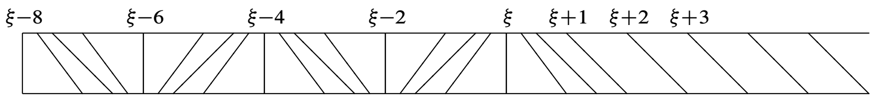


As is seen, coordinates such as $b + 4m$ and $b + 4m + 6$ are cumbersome. From now on ξ will represent $b + 4m + 2$, the rightmost fixed point (which happens to be repelling) of X_0.

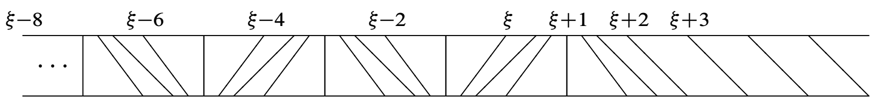
The diagrams are too bulky to show all parts of a given element from (7). We will restrict ourselves to diagrams in the neighborhood of b and diagrams in the neighborhood of $\xi = b + 4m + 2$.

3.2.3. The analysis of supports near ξ. Let us first tackle diagrams at the right end, in the neighborhood of ξ. We start with the generators.

First we have X_0.



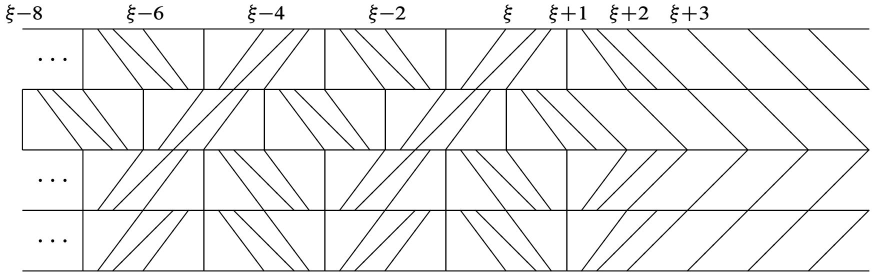
Next we have X_1.



Their inverses are obtained by reflecting across a central horizontal line.

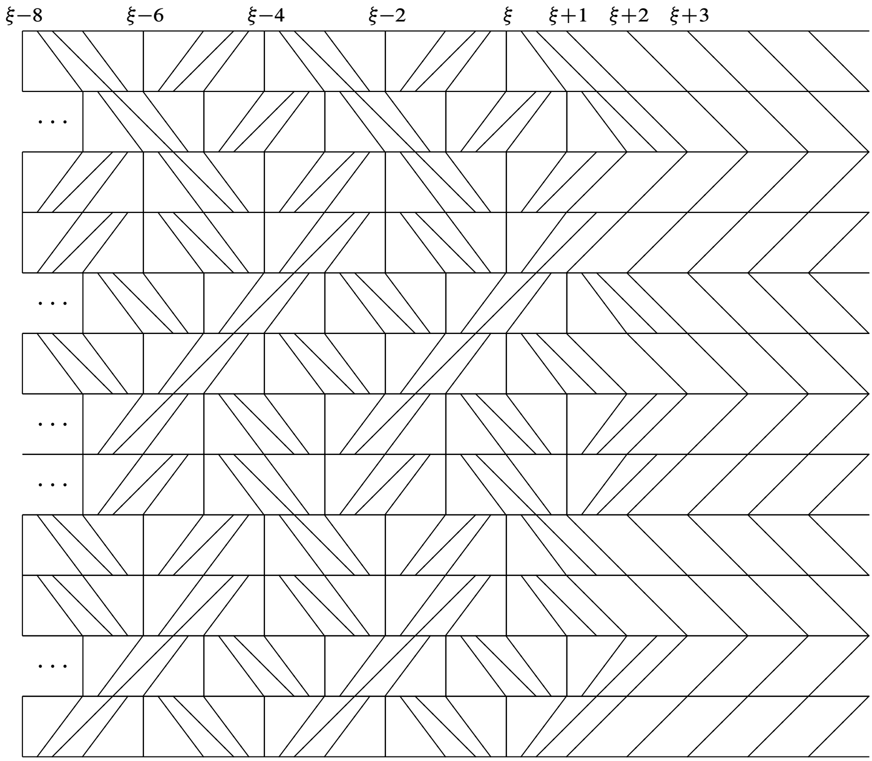
Compositions are shown by stacking the diagrams vertically. We start with the simpler of S and T.

Since $S = X_1 X_0 X_1^{-1} X_1^{-1}$, we get the following diagram for S.



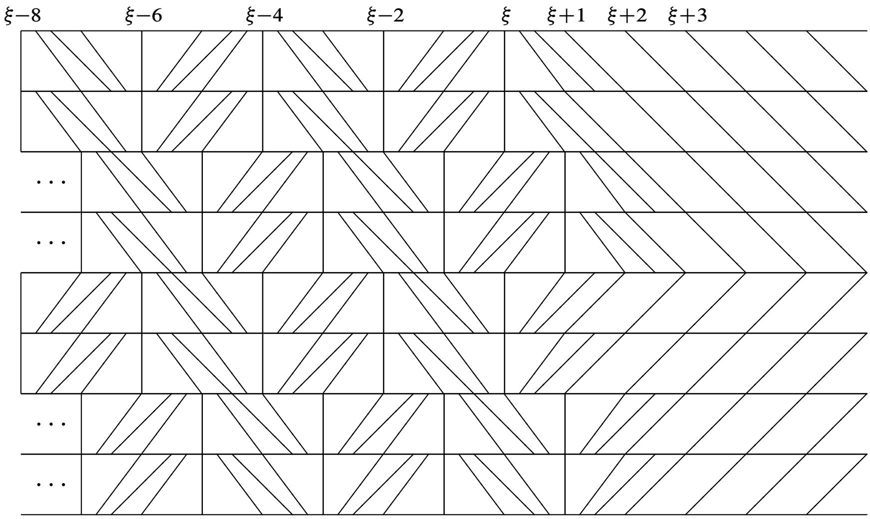
The picture above shows that $\xi + 1\frac{1}{4}$ is an upper bound for the support of S. The actual right endpoint for the support needs more careful inspection. By tracing the slopes from top to bottom, using the figures in (8) and (10), it is seen that between $\xi + 1\frac{1}{8}$ and $\xi + 1\frac{1}{4}$ the slopes encountered are 4, 1, 1 and $\frac{1}{4}$ in that order from top to bottom. However, the slopes between $\xi + 1$ and $\xi + 1\frac{1}{8}$ are 4, 1, $\frac{1}{2}$ and $\frac{1}{4}$. Thus $\xi + 1\frac{1}{8}$ is the right endpoint of the support of S.

Our next task is to tackle $T = X_0 X_1 X_0^{-1} X_0^{-1} X_1 X_0 X_1^{-1} X_1^{-1} X_0 X_0 X_1^{-1} X_0^{-1}$.



In an analysis almost identical to that of S, we get that $\xi + 1\frac{1}{8}$ is the right endpoint of the support of T.

Now we look at $C = X_0X_0X_1X_1X_0^{-1}X_0^{-1}X_1^{-1}X_1^{-1}$.



We get the following from the picture above. For an integer $j > 0$ for which $\xi - 4j$ is in the pattern above, we have $\xi - 4j + \frac{1}{4}$ is carried by C to at least $\xi - 4j + 4\frac{3}{4}$. In particular $\xi - 4 + \frac{1}{4}$ is carried to greater than $\xi + \frac{3}{4}$. Using the information in (10) with the figure above, we get that $\xi + \frac{3}{4}$ is carried to $\xi + 1\frac{3}{16}$. Further, the interval from $\xi + 1$ to $\xi + 1\frac{1}{2}$ is carried affinely with slope $\frac{1}{2}$ to the interval from $\xi + 1\frac{1}{4}$ to $\xi + 1\frac{1}{2}$. Thus $\xi + 1\frac{1}{2}$ is the right endpoint of the support of C.

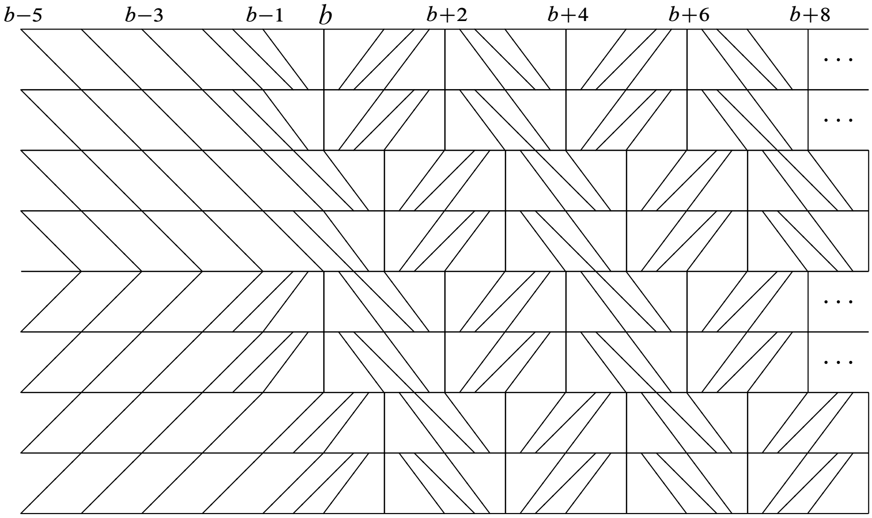
It follows that for an integer $j > 0$ for which $\xi - 4j$ is in the pattern above, we have that any η in $(\xi - 4j + \frac{1}{4}, \xi + 1\frac{1}{2})$ has $\eta C > \eta$. Further, any such η has

$$\begin{aligned} \eta C^{j} &> \xi + \tfrac{3}{4}, \\ \eta C^{j+1} &> \xi + 1\tfrac{3}{16}. \end{aligned} \tag{11}$$

The point is that $\xi + 1\frac{3}{16}$ is greater than $\xi + 1\frac{1}{8}$, the right endpoint of the support of S and of T.

3.2.4. The analysis of supports near *b*. We create pictures for the generators near b in much the same way as we do near ξ. The reader can verify that the following is an accurate combination of diagrams for $X_0^{\pm 1}$ and $X_1^{\pm 1}$ that gives the behavior of

$C = X_0 X_0 X_1 X_1 X_0^{-1} X_0^{-1} X_1^{-1} X_1^{-1}$ near b.

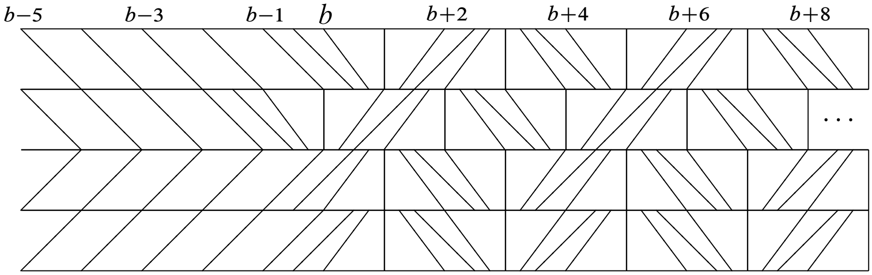


From the diagram above, we get the following information. First, the left endpoint of the support of C near b is $b - 4\frac{1}{2}$. Second, a value of j for which (11) is valid is that j for which $\xi - 4j = b + 2$. This value of j satisfies $b + 4m + 2 - 4j = b + 2$, giving $j = m$. Third, we note that $(b - 3\frac{1}{2})C > b + 2\frac{3}{4}$. Combining this information with (11), we get that for any $\eta \geq (b - 3\frac{1}{2})$ we have

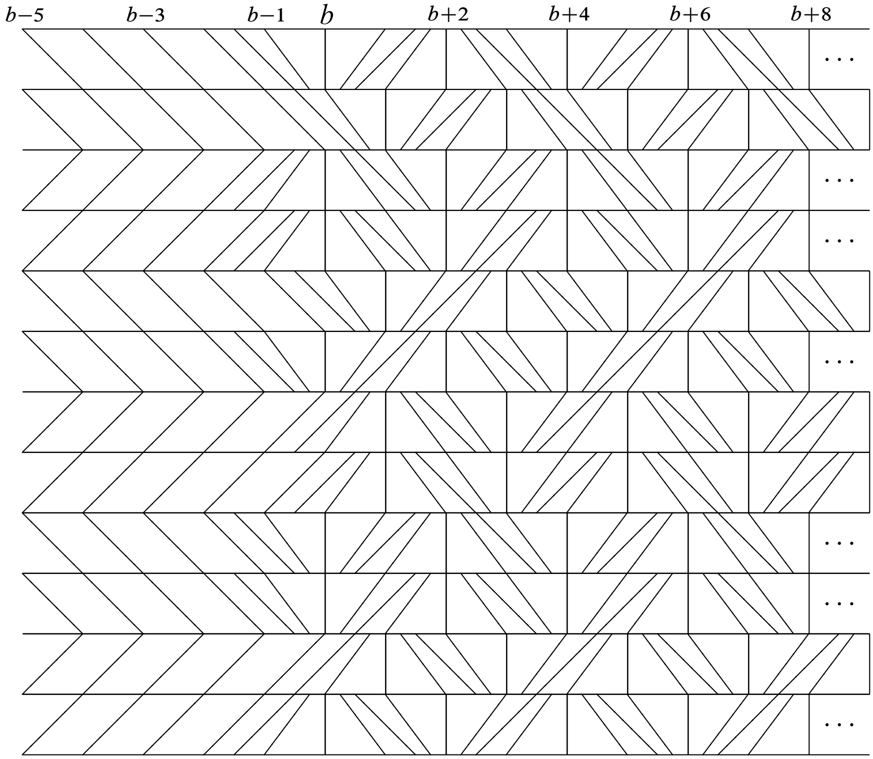
$$\eta C^{m+2} > \xi + 1\tfrac{3}{16}. \qquad (12)$$

We next look at S and T.

The diagram for $S = X_1 X_0 X_1^{-1} X_1^{-1}$ near b follows.



The diagram for $T = X_0 X_1 X_0^{-1} X_0^{-1} X_1 X_0 X_1^{-1} X_1^{-1} X_0 X_0 X_1^{-1} X_0^{-1}$ near b is below.



A trace through the two diagrams above, using the information in (9) about the slopes, shows that the left endpoint of the supports of S and T near b is $b - 2\frac{1}{2}$.

The fact that the left endpoint of the support of C near b is $(b - 4\frac{1}{2})$ and that the left endpoint of the supports of both S and T near b is $(b - 2\frac{1}{2})$ explains why our claims in Section 2 refer to $[0, b - 5)$ and $(b - 5, \infty)$.

3.2.5. End of the proof of Part (i). From Sections 3.2.3 and 3.2.4, we have the following information. Using the fact that $\xi = b + 4m + 2$, we have that the supports of C, S and T in $(b - 5, \infty)$ are given by

$$C: \ (b - 4\tfrac{1}{2},\ b + 4m + 3\tfrac{1}{2}),$$
$$S, T: \ (b - 2\tfrac{1}{2},\ b + 4m + 3\tfrac{1}{8}).$$

From (12) and the fact that $i = m + 2$, we know that for any $\eta \geq (b - 3\frac{1}{2})$ we have

$$\eta C^i > b + 4m + 3\tfrac{3}{16}.$$

From the facts above, we know that the supports of $\Sigma = C^{-i}SC^i$ and $\Theta = C^{-i}TC^i$ in $(b-5,\infty)$ are both contained in

$$(b+4m+3\tfrac{3}{16},\ b+4m+3\tfrac{1}{2})$$

which is disjoint from the supports of S and T in $(b-5,\infty)$. Thus the restrictions of $Z = [S,\Sigma]$ and $W = [T,\Theta]$ to $(b-5,\infty)$ are trivial.

From our discussion in Section 3.2.1, we see that $W = w$ and $Z = z$. From the definition $P = Z^{-1}W$, we get $P = p$. Lastly, with Q, H and K defined in terms of P and X_1, we get the rest of (i) of Proposition 2.2.

3.3. Proposition 2.2 Part (iv). We must show that the translates of the interval $(1,3)$ under words in X_0 and X_1 cover all of $(0,\infty)$.

Since the orbit of $(b-1)$ under X_0 includes all the integers below $b-1$ as well as all fractions of the form $1/2^n$ for a positive integer n, we get that the translates cover at least $(0,b-1)$.

Since $(1,b-1)$ is covered by finitely many translates of $(1,3)$, we may work from now on with $(1,b-1)$ instead of $(1,3)$.

Since C has a fixed point at $(b-5)$ which we take to be bigger than 1, and since $(b-1)$ is carried by powers of C to at least $(b+4m+3\frac{3}{16})$, we can make another replacement and work from now on with the interval $(1, b+4m+3\frac{3}{16})$.

From the discussion above (6), we know that X_0 acts as translation by 1 on $[b+4m+3,\infty)$. Since X_0 has a fixed point in $(b+4m+2)$, we can stretch any open interval containing both $(b+4m+2)$ and $(b+4m+3)$ to any length we want by powers of X_0. Thus we get all points in $(1,\infty)$ covered.

Combining the information in the four paragraphs above completes the argument for (iv).

References

[1] A. Akhmedov, M. Stein, and J. Taback, Free limits of Thompson's group F. Preprint 2009. arXiv:0908.1268

[2] M. G. Brin and F. Guzmán, Automorphisms of generalized Thompson groups. *J. Algebra* **203** (1998), 285–348. Zbl 0930.20039 MR 1620674

[3] M. G. Brin and C. C. Squier, Groups of piecewise linear homeomorphisms of the real line. *Invent. Math.* **79** (1985), 485–498. Zbl 0563.57022 MR 782231

[4] K. S. Brown, Finiteness properties of groups. *J. Pure Appl. Algebra* **44** (1987), 45–75. Zbl 0613.20033 MR 0885095

[5] J. W. Cannon, W. J. Floyd, and W. R. Parry, Introductory notes on Richard Thompson's groups. *Enseign. Math.* (2) **42** (1996), 215–256. Zbl 0880.20027 MR 1426438

[6] C. Champetier and V. Guirardel, Limit groups as limits of free groups. *Israel J. Math.* **146** (2005), 1–75. Zbl 1103.20026 MR 2151593

[7] E. S. Esyp, On identities in Thompson's group. Preprint 2009. arXiv:0902.0199

[8] R. I. Grigorchuk, Degrees of growth of finitely generated groups, and the theory of invariant means. *Izv. Akad. Nauk SSSR Ser. Mat.* **48** (1984), 939–985; English transl. *Math. USSR-Izv.* **25** (1985), 259–300. Zbl 0583.20023 MR 0764305

[9] R. Grigorchuk, Solved and unsolved problems around one group. In *Infinite groups: geometric, combinatorial and dynamical aspects*, Progr. Math. 248, Birkhäuser, Basel 2005, 117–218. Zbl 1165.20021 MR 2195454

[10] M. Gromov, Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.* **53** (1981), 53–73. Zbl 0474.20018 MR 623534

[11] A. Yu. Olshanskii and M. V. Sapir On k-free-like groups. *Algebra i Logika* **48** (2009), 245–257; English transl. *Algebra and Logic* **48** (2009), 140–146.

[12] Z. Sela, Diophantine geometry over groups I: Makanin-Razborov diagrams. *Publ. Math. Inst. Hautes Études Sci.* **93** (2001), 31–105. Zbl 1018.20034 MR 1863735

[13] E. T. Shavgulidze, About amenability of subgroups of the group of diffeomorphisms of the interval. Preprint 2009. arXiv:0906.0107

[14] R. Zarzycki, Limits of (Thompson's) group F. Preprint 2007. arXiv:math/0701601

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