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Abelian state-closed subgroups of automorphisms of m-ary trees

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Abstract. The group $\mathcal{A}_m$ of automorphisms of a one-rooted m-ary tree admits a diagonal monomorphism which we denote by x. Let A be an abelian state-closed (or self-similar) subgroup of $\mathcal{A}_m$. We prove that the combined diagonal and tree-topological closure A^* of A is additively a finitely presented $\mathbb{Z}_m[[x]]$-module, where $\mathbb{Z}_m$ is the ring of m-adic integers. Moreover, if A^* is torsion-free then it is a finitely generated pro-m group. Furthermore, the group A splits over its torsion subgroup. We study in detail the case where A^* is additively a cyclic $\mathbb{Z}_m[[x]]$-module, and we show that when m is a prime number then A^* is conjugate by a tree automorphism to one of two specific types of groups.

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1. Introduction

Automorphisms of one-rooted regular trees $\mathcal{T}(Y)$ indexed by finite sequences from a finite set Y of size $m \geq 2$ have a natural interpretation as automata on the alphabet Y, with states which are again automorphisms of the tree. A subgroup of the group of automorphisms $\mathcal{A}(Y)$ of the tree is said to be *state-closed* in the language of automata (or *self-similar* in the language of dynamics) of degree m, provided that the states of its elements are themselves elements of the same group. If the group is not state-closed then we may consider its state-closure. The prime example of a state-closed group is the group generated by the binary adding machine $\tau = (e, \tau)\sigma$, where σ is the transposition $(0, 1)$.

We study in this paper representations of general abelian groups as state-closed groups of degree m. For this purpose we use topological and diagonal closure oper-

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ations in the automorphism group of the tree. Representations of free abelian groups of finite rank as state-closed groups of degree 2 were characterized in [4].

An automorphism group G of the tree group is said to be *transitive*, provided that the permutation group $P(G)$ induced by G on the set Y is transitive; actions of groups on sets will be applied on the right. It will be shown that the structure of state-closed groups can in a certain sense be reduced to those which are transitive.

The automorphism group $\mathcal{A}(Y)$ of the tree is a topological group with respect to the topology inherited from the tree. This topology allows us to exponentiate elements of $\mathcal{A}(Y)$ by m-ary integers from $\mathbb{Z}_m$. Given a subgroup G of $\mathcal{A}(Y)$, its topological closure $\overline{G}$ with respect to the tree topology belongs to the same variety as G. Also, if G is state-closed then so is $\overline{G}$.

The diagonal map $\alpha \to \alpha^{(1)} = (\alpha, \alpha, \ldots, \alpha)$ is a monomorphism of $\mathcal{A}_m$. Define inductively $\alpha^{(0)} = \alpha$, $\alpha^{(i+1)} = (\alpha^{(i)})^{(1)}$ for $i \geq 0$. It is convenient to introduce a symbol x and write $\alpha^{(i)}$ as α^{x^i} for $i \geq 0$. This will permit more general exponentiation, by formal power series $p(x) \in \mathbb{Z}_m[[x]]$. Given a subgroup G of $\mathcal{A}(Y)$, its *diagonal closure* is the group $\widetilde{G} = \langle G^{(i)} \mid i \geq 0\rangle$. Observe that the diagonal closure operation preserves the state-closed property.

We will show that given an abelian transitive state-closed group A, its diagonal closure $\tilde{A}$ is again abelian. The composition of the diagonal and topological closures when applied to A produces an abelian group denoted by A^*, which can be viewed additively as a finitely generated $\mathbb{Z}_m[[x]]$-module. This approach was first used in [2].

The prime decomposition $m = \prod_{1 \leq i \leq s} p_i^{k_i}$ provides us with the decomposition $\mathbb{Z}_m = \bigoplus_{1 \leq i \leq s} \varepsilon_i \mathbb{Z}_{p_i^{k_i}}$, where ε_i are orthogonal idempotents such that $1 = \sum_{1 \leq i \leq s} \varepsilon_i$, and also gives us the decomposition $\mathbb{Z}_m[[x]] = \bigoplus_{1 \leq i \leq s} \varepsilon_i \mathbb{Z}_{p_i^{k_i}}[[x]]$. When $m = p^k$ and p a prime number, the rings $\mathbb{Z}_m[[x]]$ and $\mathbb{Z}_p[[x]]$ are isomorphic, yet when $k > 1$ they are different representations of the same object and for this reason we distinguish between them.

In Sections 3 and 4 we prove

Theorem 1. *Let A be an abelian transitive state-closed group of degree m. Then*

(1) *the group A^* is isomorphic to a finitely presented $\mathbb{Z}_m[[x]]$-module;*

(2) *if A^* is torsion-free then it is a finitely generated $\mathbb{Z}_m$-module which is also a pro-m group.*

Item (1) is part of Theorem 5 and item (2) is Corollary 1 of Theorem 6.

We consider in Section 5 torsion subgroups of state-closed abelian groups and use methods from virtual endomorphisms of groups (see [3], [1]; reviewed in Section 5.1) to prove the following structural result.

Theorem 2. *Let A be an abelian transitive state-closed group of degree m and* $\mathrm{tor}(A)$ *its torsion subgroup. Then*

(i) $\operatorname{tor}(A)$ *is a direct summand of* A *and has exponent a divisor of the exponent of* $P(A)$;

(ii) *the action of* A *on the* m*-ary tree induces transitive state-closed representations of* $\operatorname{tor}(A)$ *on the* m_1*-tree and of* $\frac{A}{\operatorname{tor}(A)}$ *on the* m_2*-tree, where* $m_1 = |P(\operatorname{tor}(A))|$ *and* $m_2 = |\frac{P(A)}{P(\operatorname{tor}(A))}|$;

(iii) *if* $A = \operatorname{tor}(A)$ *and* $P(A) \cong \bigoplus_{1 \le i \le k} \frac{\mathbb{Z}}{m_i \mathbb{Z}}$, *then* $A^* \cong \bigoplus_{1 \le i \le k} \frac{\mathbb{Z}}{m_i \mathbb{Z}}[[x]]$.

The above results are analogous to Theorem 4.3.4 of [5] on the structure of finitely generated pro-p groups. By item (i) of the theorem, an abelian torsion group G of infinite exponent cannot have a faithful representation as a transitive state-closed group for any finite degree. Put differently, the group G does not admit any simple virtual endomorphism. On the other hand, the group of automorphisms of the p-adic tree is replete with abelian p-subgroups of infinite exponent. Item (iii) follows from Theorem 7, which is a conjugacy result and therefore more general than isomorphism.

We focus our attention in Section 6 on transitive state-closed abelian groups A, for which A^* is additively a cyclic $\mathbb{Z}_m[[x]]$-module. We show

Theorem 3. (1) *Let* $q_{\mathbf{1}}, \dots, q_{\boldsymbol{m}} \in \mathbb{Z}_m[[x]]$ *and let* σ *be the cycle* $(1, 2, \dots, m)$. *Then the expression*

$$\alpha = (\alpha^{q_1}, \dots, \alpha^{q_m})\sigma$$

is a well-defined automorphism of the m*-ary tree and the state-closure* A *of* $\langle \alpha \rangle$ *is an abelian transitive group. The group* A^* *is additively isomorphic to the quotient ring* $\frac{\mathbb{Z}_m[[x]]}{(r)}$, *where*

$$r = m - xq \text{ and } q = q_{\mathbf{1}} + \cdots + q_{\boldsymbol{m}}.$$

(2) *Let* A *be a transitive state-closed abelian group of degree* m *such that* A^* *is additively a cyclic* $\mathbb{Z}_m[[x]]$*-module. Then* $P(A)$ *is cyclic, say generated by* σ, *and* A^* *is the state-diagonal-topological closure of an element of the form* $\alpha = (\alpha^{q_1}, \dots, \alpha^{q_m})\sigma$ *for some* $q_{\mathbf{1}}, \dots, q_{\boldsymbol{m}} \in \mathbb{Z}_m[[x]]$.

Finally we provide a complete description of the group A^* for state-closed groups of prime degree. Let $j \ge 1$ and let $D_m(j)$ be the group generated by the set of states of the generalized adding machine $\alpha = (e, \dots, e, \alpha^{x^{j-1}})\sigma$ acting on the m-ary tree with $\sigma = (1, 2, \dots, m)$. The topological closure of $D_m(j)$ seen as $\mathbb{Z}_m$-module is isomorphic to the ring $\frac{\mathbb{Z}_m[[x]]}{(r)}$, $r = m - x^j$.

Theorem 4. *Let* A *be an abelian transitive state-closed group of prime degree* m *and let* σ *be the* m*-cycle automorphism. If* $\operatorname{tor}(A)$ *is nontrivial then* A^* *is a torsion group conjugate to* $\langle \sigma \rangle^* (\cong \frac{\mathbb{Z}}{m\mathbb{Z}}[[x]])$. *If* A *is torsion-free then* A^* *is a torsion-free group conjugate to the topological closure of* $D_m(j)$ *for some* j.

One of the questions that has remained unanswered is whether a free abelian group of infinite rank admits a faithful transitive state-closed representation, even of prime degree.

2. Preliminaries

We fix the notation $Y = \{1, 2, \ldots, m\}$, $\mathcal{T}_m = \mathcal{T}(Y)$, $\mathcal{A}_m = \mathcal{A}(Y)$ and we let $\mathrm{Perm}(Y)$ be the group of permutations of Y. A permutation $\gamma \in \mathrm{Perm}(Y)$ is extended to an automorphism of the tree by $\gamma\colon\ yu \to y^{\gamma}u$, fixing the non-initial letters of every sequence. An automorphism $\alpha \in \mathcal{A}_m$ is represented as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)\sigma(\alpha)$, where $\alpha_i \in \mathcal{A}_m$ and $\sigma(\alpha) \in \mathrm{Perm}(Y)$. Successive developments of α_i produce for us α_u (a state of α) for every finite string u over Y.

The product of $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)\sigma(\alpha)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)\sigma(\beta)$ in $\mathcal{A}_m$ is

$$\alpha\beta = (\alpha_1\beta_{(1)\sigma(\alpha)}, \ldots, \alpha_m\beta_{(m)\sigma(\alpha)})\sigma(\alpha)\sigma(\beta).$$

Let G be a subgroup of $\mathcal{A}_m$. Denote the subgroup of G which fixes the vertices of the i-th level of the tree by $\mathrm{Stab}_G(i)$. Given $y \in Y$, denote by $\mathrm{Fix}_G(y)$ the subgroup of G consisting of the elements of G, which fix y. The group G is said to be *recurrent* provided it is transitive and $\mathrm{Fix}_G(1)$ projects in the 1st coordinate onto G.

The group $\mathcal{A}_m$ is the inverse limit of its quotients by the i -th level stabilizers $\mathrm{Stab}_{\mathcal{A}_m}(i)$ of the tree and is as such a topological group where each $\mathrm{Stab}_{\mathcal{A}_m}(i)$ is an open and closed subgroup. For a subgroup G of automorphisms of the tree, its topological closure $\overline{G}$ coincides with the set of all infinite products $\ldots g_{\boldsymbol{i}} \ldots g_{\boldsymbol{1}}g_{\boldsymbol{0}}$, or alternately, $g_{\boldsymbol{0}}g_{\boldsymbol{1}} \ldots g_{\boldsymbol{i}} \ldots$, where $g_{\boldsymbol{i}} \in \mathrm{Stab}_G(\boldsymbol{i})$. The group $\overline{G}$ satisfies the same group identities as G. We note that the property of being state-closed is also preserved by the topological closure operation.

Let α be an automorphism of the tree. Then $\overline{\langle\alpha\rangle} = \{\alpha^p \mid p \in \mathbb{Z}_m\}$. More generally, for $q = \sum_{i\geq 0} q_i x^i \in \mathbb{Z}_m[[x]]$ with $q_i \in \mathbb{Z}_m$, we write the expression

$$\alpha^q = \alpha^{q_0}\alpha^{q_1x} \ldots \alpha^{q_ix^i} \ldots,$$

which can be verified to be a well-defined automorphism of the tree.

We recall the reduction of group actions to transitive ones, with a view to a similar reduction for state-closed groups of automorphisms of trees. Let G be a subgroup of $\mathrm{Perm}(Y)$, let $\{Y_i \mid i = 1, \ldots, s\}$ be the set of orbits of G on Y and let $\{\rho_i\colon G \to \mathrm{Perm}(Y_i) \mid i = 1, \ldots, s\}$ be the set of induced representations. Then, each ρ_i is transitive and $\rho\colon G \to \prod_{1\leq i\leq s}\mathrm{Perm}(Y_i) \leq \mathrm{Perm}(Y)$ defined by $g \to (g^{\rho_1}, \ldots, g^{\rho_s})$ is a monomorphism. The reduction for tree actions follows from

Lemma 1. *Let G be a state-closed group of automorphisms of the tree $\mathcal{T}(Y)$ and let X be a $P(G)$-invariant subset of Y. Then $\mathcal{T}(X)$ is G-invariant and for the resulting representation $\mu\colon G \to \mathcal{A}(X)$ the group G^{μ} is state-closed. If G is diagonally closed or is topologically closed then so is G^{μ}.*

Proof. Let xu be a sequence from X and let $\alpha \in G$. Then $(xu)^{\alpha} = x^{\sigma(\alpha)}u^{\alpha_x}$. As $x^{\sigma(\alpha)} \in X$ and $\alpha_x \in G$, it follows that $(xu)^{\alpha}$ is a sequence from X. Also, for any sequence u from X, we have $(\alpha^{\mu})_u = (\alpha_u)^{\mu}$. Thus, G^{μ} is state-closed. The last assertion is clear. □

We note the following important properties of transitive state-closed abelian groups A.

Proposition 1. *Let A be an abelian transitive state-closed group of degree m. Then* $\mathrm{Stab}_A(i) \leq A^{(i)}$ *for all $i \geq 0$. The group $\tilde{A}$ is an abelian transitive state-closed group and is a minimal recurrent group containing A. Moreover, the topological closure and diagonal closure operations commute when applied to A. The diagonal-topological closure A^* of A is an abelian transitive state-closed group.*

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_m)\sigma, \beta = (\beta_1, \ldots, \beta_m) \in A$. Then the conjugate of β by α is

$$\beta^{\alpha} = (\beta_1^{\alpha_1}, \ldots, \beta_m^{\alpha_m})^{\sigma}.$$

As $\alpha_i, \beta_i \in A$ and A is abelian, it follows that $\beta = (\beta_1, \ldots, \beta_m)^{\sigma}$. Furthermore, since A is transitive, $\beta = (\beta_1, \ldots, \beta_1) = (\beta_1)^{(1)}$. Thus, $\mathrm{Stab}_A(i) \leq A^{(i)}$ for all i. A similar verification shows that $\tilde{A} = \langle A^{(i)} \mid i \geq 0 \rangle$ is abelian.

Let G be a recurrent group such that $A \leq G \leq \tilde{A}$. Given $\alpha \in G$, as G is recurrent, there exists $\beta \in \mathrm{Stab}_G(1)$ such that $\beta = (\beta_1, \ldots, \beta_m)$ with $\beta_1 = \alpha$. Since G is transitive and abelian, we have $\beta_1 = \cdots = \beta_m = \alpha$; that is, $\beta = \alpha^{(1)}$. Hence, $A^{(i)} \leq G$ and $G = \tilde{A}$ follows.

The last two assertions of the proposition are clear. □

The following result indicates the smallness of recurrent transitive abelian groups from the point of view of centralizers.

Proposition 2 (Theorem 7 [1]). (1) *Let A be a recurrent abelian group of degree m and let $C_{\mathcal{A}_m}(A)$ be the centralizer of A in $\mathcal{A}_m$. Then $C_{\mathcal{A}_m}(A) = \bar{A}$.*

(2) *Let m be a prime number and A be an infinite transitive state-closed abelian group. Then $C_{\mathcal{A}_m}(A) = \bar{A}$.*

This result will be used in the proofs of Lemma 3 and step 4 of Theorem 9.

3. A presentation for A^*

Let A be a transitive abelian state-closed group of degree m and let A^* be its diagonal-topological closure. Then A^* is additively a $\mathbb{Z}_m[[x]]$-module having the following properties. Given $\alpha \in A^*$, then

(i) $x\alpha = 0$ implies $\alpha = 0$;
(ii) $m\alpha = x\gamma$ for some $\gamma \in A^*$.

Let $P(A)$ be given by its presentation

$$\langle \sigma_i \ (1 \leq i \leq k) \mid \sigma_i^{m_i} = e, \ \text{abelian} \rangle.$$

Choose for each σ_i an element β_i in A, which induces σ_i on Y; denote β_i by $\beta(\sigma_i)$. Then, for any $n \geq 0$, the automorphism of the tree $\beta(\sigma_i)^{(n)}$ is an element of $\tilde{A}$ which induces $(\sigma_i)^{(n)}$ on the $(n+1)$-th level of the tree. Although the notation β_i has been used to indicate the ith entry in an automorphism β, we hope this new usage will not cause confusion.

Theorem 5. *Let A be a transitive abelian state-closed group of degree m. Then A^* is additively a $\mathbb{Z}_m[[x]]$-module generated by*

$$\{\beta_i \mid 1 \leq i \leq k\}$$

subject to the set of defining relations

$$\left\{r_i = \textstyle\sum_{1\leq j\leq k} m_i\beta_i - p_{ij}\beta_j x = 0 \mid 1 \leq i \leq k\right\} \quad \textit{for some } p_{ij} \in \mathbb{Z}_m[[x]].$$

Moreover, there exist $r, q \in \mathbb{Z}_m[[x]]$ such that $r = m - xq$ and $rA^* = (0)$. The elements of A^* can be represented additively as $\sum_{1\leq i\leq k} p_i\beta_i$, where $p_i = \sum_{j\geq 0} p_{ij}x^j$ and each $p_{ij} \in \mathbb{Z}$ with $0 \leq p_{ij} < m$.

Proof. Let $\alpha \in A^*$ and $\sigma(\alpha) = \prod_{1\leq i\leq k} \sigma_i^{r_{i1}}$, $0 \leq r_{i1} < m_i$. Then either $\alpha\left(\prod_{1\leq i\leq k} \beta_i^{r_{i1}}\right)^{-1}$ is the identity element or there exists $l_2 \geq 1$ such that

$$\alpha\left(\textstyle\prod_{1\leq i\leq k} \beta_i^{r_{i1}}\right)^{-1} \in \mathrm{Stab}(l_2)\backslash \mathrm{Stab}(l_2+1)$$

and so, $\alpha\left(\prod_{1\leq i\leq k} \beta_i^{r_{i1}}\right)^{-1} = (\gamma)^{(l_2)}$ for some $\gamma \in A^*$. We treat γ in the same manner as α. In the limit, we obtain

$$\alpha = \prod_{1\leq i\leq k} (\beta_i^{r_{i1}}(\beta_i^{r_{i2}})^{(l_2)} \dots (\beta_i^{r_{ij}})^{(l_j)} \dots) = \prod_{1\leq i\leq k} \beta_i^{q_i},$$

where $0 \leq r_{ij} < m_i$, $1 \leq l_2 < l_3 < \cdots < l_j < \cdots$, and $q_i = r_{i1} + \sum_{j\geq 2} r_{ij}x^{l_j}$ are formal power series in x. Additively we then have

$$\alpha = \sum_{1\leq i\leq k} q_i\beta_i \in \sum_{1\leq i\leq k} \mathbb{Z}_{m_i}[[x]]\beta_i.$$

Each relation $\sigma_i^{m_i} = e$ in P produces in A^* a relation of the form

$$\beta_i^{m_i} = \prod_{1\leq j\leq k} \beta_j^{xp_{ij}},$$

where p_{ij} are elements in the power series, as above; when written additively $\beta_i^{m_i}$ has the form

$$m_i\beta_i = x\left(\textstyle\sum_{1\leq j\leq k} p_{ij}\beta_j\right).$$

Let $F = \bigoplus_{1 \le i \le k} \mathbb{Z}_m[[x]]\dot{\beta}_i$ be a free $\mathbb{Z}_m[[x]]$-module of rank k. Define the $\mathbb{Z}_m[[x]]$-homomorphism

$$\phi\colon \sum_{1 \le i \le k} \mathbb{Z}_m[[x]]\dot{\beta}_i \to A^*, \qquad \sum_{1 \le i \le k} p_i \dot{\beta}_i \mapsto \prod_{1 \le i \le k} \beta_i^{p_i},$$

and let R be the kernel ϕ. Define J to be the $\mathbb{Z}_m[[x]]$-submodule of R generated by

$$\dot{r}_i = m_i \dot{\beta}_i - x\big(\textstyle\sum_{1 \le j \le k} p_{ij} \dot{\beta}_j\big) \quad (1 \le i \le k).$$

We will show that $J = R$. So let $\nu \in R$ and write $\nu = \sum_{1 \le i \le k} \nu_i \dot{\beta}_i$, where

$$\nu_i = \sum_{j \ge 0} \nu_{ij} x^j, \quad \nu_{ij} = \nu_{ij,0} + m w_{ij} \in \mathbb{Z}_m.$$

Then $m_i | \nu_{i0,0}$, $\nu_{i0,0} = m_i \nu'_{i0,0}$; factor $m = m_i m'_i$. Therefore,

$$\begin{aligned}
\nu_i &= \nu_{i0} + \big(\textstyle\sum_{j \ge 1} \nu_{ij} x^{j-1}\big)x,\\
\nu_{i0} &= m_i \nu'_{i0,0} + m w_{i0} = (\nu'_{i0,0} + m'_i w_{i0}) m_i,\\
\nu_i \dot{\beta}_i &= (\nu'_{i0,0} + m'_i w_{i0})(m_i \dot{\beta}_i) + \big(\textstyle\sum_{j \ge 1} \nu_{ij} x^{j-1}\big)x\dot{\beta}_i,\\
&\equiv (\nu'_{i0,0} + m'_i w_{i0})\big(x \textstyle\sum_{1 \le j \le k} p_{ij} \dot{\beta}_j\big) + \big(\textstyle\sum_{j \ge 1} \nu_{ij} x^{j-1}\big)x\dot{\beta}_i \mod J.
\end{aligned}$$

Hence

$$\nu = \sum_{1 \le i \le k} \nu_i \dot{\beta}_i \in x\mu + J, \quad \mu = \sum_{1 \le i \le k} \mu_i \dot{\beta}_i \in R.$$

Hence, by repeating the argument, we obtain

$$\nu \in \big(\textstyle\bigcap_{i \ge 1} x^i R\big) + J = J, \quad J = R.$$

On re-writing the relations $m_i \beta_i = \sum_{1 \le j \le k} p_{ij} x \beta_j$ in the form

$$p_{i1} x \beta_1 + \cdots + (p_{ii} x - m_i)\beta_i + \cdots + p_{kk} x \beta_k = 0$$

we see that the $k \times k$ matrix of coefficients of these equations has determinant $r = m - xq$ for some $q \in \mathbb{Z}_m[[x]]$ and thus r annuls A^*.

The last assertion of the theorem follows by using $r = m - qx \in R$ to reduce the coefficients modulo m. $\square$

4. The *m*-congruence property

A group G of automorphisms of the m-ary tree is said to satisfy the *m-congruence property*, provided that given m^i there exists $l(i) \ge 1$ such that $\mathrm{Stab}_G(l(i)) \le G^{m^i}$ for all i, in which case the topology on G inherited from $\mathcal{A}(Y)$ is equal to the pro-m topology. Since when A^* is written additively, we have $\mathrm{Stab}_G(l(i)) = x^{l(i)} A^*$, the m-congruence property reads $x^{l(i)} A^* \le m^i A^*$.

Theorem 6. *Let $r = m - qx^j \in \mathbb{Z}_m[[x]]$ with $q \in \mathbb{Z}_m[[x]]$ and $j \geq 1$. Let S be quotient ring $\frac{\mathbb{Z}_m[[x]]}{(r)}$. Suppose that S is torsion-free. Then S is a finitely generated pro-m group.*

Proof. From the decomposition $\mathbb{Z}_m[[x]] = \bigoplus_{1\leq i\leq s} \varepsilon_i \mathbb{Z}_{p_i^{k_i}}[[x]]$ corresponding to the prime decomposition $m = \prod_{1\leq i\leq s} p_i^{k_i}$, we obtain

$$r = \sum_{1\leq i\leq s} r_i,$$
$$r_i = \varepsilon_i r = p_i^{k_i} - q_i(x)x^j,$$
$$S = \sum_{1\leq i\leq s} S_i, \quad S_i = \frac{\mathbb{Z}_{p_i^{k_i}}[[x]]}{(r_i)},$$

where each S_i is torsion-free. Thus, it is sufficient to address the case where m is a prime power p^k.

(1) First, we show that S is a pro-m group.

So let $r = p^k - qx^j$ and decompose $q = q(x) = s(x) + p \cdot t(x)$, where each non-zero coefficient of $s(x)$ is an integer relatively prime to p. If $s(x) = 0$ then $q(x) = p \cdot t(x)$ and

$$r = p^k - q(x)x^j = p^k - p.t(x)x^j = p(p^{k-1} - t(x)x^j) \in (r);$$

but as by hypothesis S is torsion free, we have $p^{k-1} - t(x)x^j \in (r)$, which is not possible.

Write $s(x) = x^l u(x)$, where $l \geq 0$ and $u(x)$ is invertible in $\mathbb{Z}_m[[x]]$ with inverse $u'(x)$. Then $q(x) = x^l u(x) + p \cdot t(x)$ and

$$r = p^k - (x^l u(x)x^j + p \cdot t(x)x^j) = p(p^{k-1} - t(x)x^j) - x^{j+l}u(x).$$

Therefore, on multiplying by $u'(x)$, the inverse of $u(x)$, we obtain

$$p(p^{k-1} - t(x)x^j)u'(x) \equiv x^{j+l} \mod r.$$

It follows that

$$x^{j+l}S \leq pS, \quad x^{n(j+l)}S \leq p^n S.$$

(2) Now we show that S is finitely generated as a $\mathbb{Z}_m$-module.

By the previous step there exist $l \geq 1$ and $v(x) \in \mathbb{Z}[[x]]$ such that

$$x^l \equiv mv(x) \mod r.$$

Decompose $v(x) = v_1(x) + v_2(x)x^l$ where the degree of $v_1(x)$ is less that l. Then we deduce modulo r:

$$\begin{aligned}
v(x) &\equiv v_1(x) + v_2(x)mv(x), \\
v_2(x)v(x) &\equiv w(x) \in \mathbb{Z}[[x]], \\
w(x) &= w_1(x) + w_2(x)x^l, \\
v(x) &\equiv v_1(x) + mw(x) \\
&\equiv v_1(x) + mw_1(x) + mw_2(x)x^l \\
&\vdots \\
v(x) &\equiv a_0 + a_1x + \cdots + a_{(l-1)}x^{l-1}, \quad a_i \in \mathbb{Z}_m.
\end{aligned}$$

We have shown that S is generated by $1, x, \ldots, x^{l-1}$ as a pro-m group. □

Corollary 1. *Let A be an abelian transitive state-closed group of degree m. Suppose that the group A^* is torsion-free. Then A^* is a finitely generated pro-m group.*

Proof. With previous notation, the group A^* is a $\mathbb{Z}_m[[x]]$-module generated by

$$\{\beta_i = \beta(\sigma_i) \mid 1 \leq i \leq k\}$$

and is annihilated by $r = m - qx^j \in \mathbb{Z}_m[[x]]$ for some $q \in \mathbb{Z}_m[[x]]$ and $j \geq 1$.

It follows that A^* is an S-module, where $S = \frac{\mathbb{Z}_m[[x]]}{(r)}$. Since S satisfies the m-congruence property, it follows that A^* is a pro-m group.

That A^* is a finitely generated $\mathbb{Z}_m$-module, is a consequence of S being a finitely generated $\mathbb{Z}_m$-module. □

5. Torsion in state-closed abelian groups

5.1. Preliminaries on virtual endomorphisms of groups. Let G be a transitive state-closed subgroup of $\mathcal{A}(Y)$, where $Y = \{1, 2, \ldots, m\}$. Then $[G : \mathrm{Fix}_G(1)] = m$ and the projection on the 1st coordinate of $\mathrm{Fix}_G(1)$ produces a subgroup of G; that is, $\pi_1\colon \mathrm{Fix}_G(1) \to G$ is a virtual endomorphism of G. This notion has proven to be effective in studying state-closed groups. We give a quick review below.

Let G be a group with a subgroup H of finite index m and a homomorphism $f\colon H \to G$. A subgroup U of G is *semi-invariant* under the action of f, provided that $(U \cap H)^f \leq U$. If $U \leq H$ and $U^f \leq U$ then U is *f-invariant.*

The largest subgroup K of H which is normal in G and f-invariant is called the f-core(H). If the f-core(H) is trivial then f and the triple (G, H, f) are said to be a *simple.*

Given a triple (G, H, f) and a right transversal $L = \{x_1, x_2, \ldots, x_m\}$ of H in G, the permutational representation $\pi\colon G \to \mathrm{Perm}(1, 2, \ldots, m)$ is $g^\pi\colon i \to j$, which is induced from the right multiplication $Hx_i g = Hx_j$. We produce recursively a representation $\varphi\colon G \to \mathcal{A}(m)$ as follows:

$$g^{\varphi} = ((x_i g \cdot (x_{(i)g^{\pi}})^{-1})^{f\varphi})_{1\le i\le m} g^{\pi}.$$

One further expansion of g^{φ} is

$$\begin{aligned} g^{\varphi} &= ((((x_j g_i \cdot x^{-1}_{(j)g_i^{\pi}})^{f\varphi})_{1\le j\le m} g_i^{\pi})_{1\le i\le m} g^{\pi}, \\ &= ((((x_j g_i . x^{-1}_{(j)g_i^{\pi}})^{f\varphi})_{1\le j\le m})_{1\le i\le m} (g_i^{\pi})_{1\le i\le m}\ g^{\pi}, \end{aligned}$$

where $g_i = (x_i g . x^{-1}_{(i)g^{\pi}})^{f}$.

The kernel of φ is precisely the f-core(H), G^{φ} is state-closed and $H^{\varphi} = \mathrm{Fix}_{G^{\varphi}}(1)$.

5.1.1. Changing transversals. We will show below that changing the transversal of H in G produces another representation of G, conjugate to the original one by an explicit automorphism of the m-ary tree.

Proposition 3. *Let (G, H, f) be a triple and*

$$L = \{x_1, x_2, \dots, x_m\}, \quad L' = \{x'_1 = h_1 x_1, x'_2 = h_2 x_2, \dots, x'_m = h_m x_m\}$$

right transversals of H in G where $h_i \in H$. Let $\varphi = \varphi_{x_i}$, $\varphi' = \varphi_{h_i x_i}\colon G \to \mathcal{A}(m)$ be the corresponding tree representations and define the following elements of $\mathcal{A}(m)$,

$$\begin{aligned} \gamma &= \gamma_{h_i,\varphi'} = ((h_i)^{f\varphi'})_{1\le i\le m}, \\ \lambda &= \lambda_{h_i,\varphi'} = \gamma\gamma^{(1)} \dots \gamma^{(n)} \dots . \end{aligned}$$

Then

$$\varphi_{h_i x_i} = \varphi_{x_i}(\lambda_{h_i^{-1},\varphi_{x_i}}).$$

Proof. The representations $\varphi, \varphi'\colon G \to \mathcal{A}(m)$ are defined by

$$\begin{aligned} g^{\varphi} &= ((x_i g \cdot (x_{(i)g^{\pi}})^{-1})^{f\varphi})_{1\le i\le m} g^{\pi}, \\ g^{\varphi'} &= ((x'_i g \cdot (x'_{(i)g^{\pi}})^{-1})^{f\varphi'})_{1\le i\le m} g^{\pi}. \end{aligned}$$

The relationship between φ' and φ is established as follows,

$$\begin{aligned} g^{\varphi'} &= ((h_i x_i g \cdot (h_{(i)g^{\pi}} x_{(i)g^{\pi}})^{-1})^{f\varphi'})_{1\le i\le m} g^{\pi} \\ &= ((h_i (x_i g \cdot x^{-1}_{(i)g^{\pi}}) h^{-1}_{(i)g^{\pi}})^{f\varphi'})_{1\le i\le m} g^{\pi} \\ &= ((h_i)^{f\varphi'})_{1\le i\le m} \cdot ((x_i g . x^{-1}_{(i)g^{\pi}})^{f\varphi'})_{1\le i\le m} \cdot ((h^{-1}_{(i)g^{\pi}})^{f\varphi'})_{1\le i\le m} g^{\pi} \\ &= ((h_i)^{f\varphi'})_{1\le i\le m} \cdot ((x_i g \cdot x^{-1}_{(i)g^{\pi}})^{f\varphi'})_{1\le i\le m} g^{\pi} \cdot ((h_i)^{f\varphi'})^{-1}_{1\le i\le m}. \end{aligned}$$

Therefore

$$g^{\varphi'} = \gamma \cdot ((x_i g \cdot x^{-1}_{(i)g^\pi})^{f\varphi'})_{1\le i\le m} g^\pi \cdot \gamma^{-1},$$

where $\gamma = ((h_i)^{f\varphi'})_{1\le i\le m}$ is independent of g. Repeating this development for each $g_i = (x_i g \cdot x^{-1}_{(i)g^\pi})^f$, we find that

$$g^{\varphi'} = \gamma\gamma^{(1)} \cdot (((x_j g_i \cdot x^{-1}_{(j)g_i^\pi})^{f\varphi'})_{1\le j\le m} g_i^\pi)_{1\le i\le m} g^\pi \cdot \gamma^{-(1)}\gamma^{-1}.$$

Thus in the limit we obtain $\lambda = \gamma\gamma^{(1)}\dots\gamma^{(n)}\dots$ such that

$$g^{\varphi'} = \lambda g^{\varphi}\lambda^{-1} \quad \text{for all } g \in G,$$
$$\varphi = \varphi'\lambda.$$

Introducing the explicit dependence of φ, φ', λ on the transversals, the previous equation becomes

$$\varphi_{x_i} = (\varphi_{h_i x_i})(\lambda_{h_i,\varphi_{h_i x_i}}).$$

On replacing h_i by h_i^{-1} and on denoting $h_i^{-1}x_i$ by x_i', we obtain

$$\varphi_{h_i x_i'} = (\varphi_{x_i'})(\lambda_{h_i^{-1},\varphi_{x_i'}}).$$

□

Example 1. Let $G = C = \langle a\rangle$ be the infinite cyclic group, let $H = \langle a^2\rangle$ and let $f\colon H \to G$ be defined by $a^2 \to a$. Given $l, k \ge 0$, then on choosing the transversal $L_{k,l} = \{a^{2k}, a^{2l+1}\}$ for H in G, we obtain the representation $\varphi_{k,l}\colon G \to \mathcal{A}(m)$, where $\varphi_{k,l}\colon a \to \alpha = (\alpha^{k-l}, \alpha^{-k+l+1})\sigma$.

5.1.2. Subtriples, quotient triples. Given a triple (G, H, f) and given subgroups $V \le G, U \le H \cap V$ such that $(U)^f \le V$, we call $(V, U, f|_U)$ a *sub-triple* of G. If N is a normal semi-invariant subgroup of G, then $\bar f\colon \frac{HN}{N} \to \frac{G}{N}$ given by $\bar f\colon Nh \to Nh^f$ is well defined and $(\frac{G}{N}, \frac{HN}{N}, \bar f)$ is a *quotient triple*.

Let (G, H, f) be a simple triple where G is abelian and $[G : H] = m$. Then any sub-triple of G is simple. Let $T = \mathrm{tor}(G)$ denote the torsion subgroup of G and for $l \ge 1$ define $G(l) = \{g \in T \mid o(g) \,|\, l\}$, $H(l) = G(l) \cap H$. Then, clearly, $f\colon \mathrm{tor}(H) \to \mathrm{tor}(G)$ and $f\colon H(l) \to G(l)$. Therefore, $\mathrm{tor}(G)$ and $G(l)$ are semi-invariant and $(\mathrm{tor}(G), \mathrm{tor}(H), f|_{\mathrm{tor}(H)})$ and $(G(l), H(l), f|_{H(l)})$ are simple *sub-triples*.

Lemma 2. *Let (G, H, f) be a simple triple. The triple $\left(\frac{G}{G(l)}, \frac{HG(l)}{G(l)}, \bar f\right)$ is also simple.*

Proof. For suppose $K \le H$ is such that $G(l)K^f \le G(l)K$. Then

$$(G(l)K^f)^l = (K^f)^l = (K^l)^f \le (G(l)K)^l = (K)^l;$$

that is, K^l is f-invariant. Since f is simple, $K^l = \{e\}$, and so $K \le G(l)$. □

5.2. The torsion subgroup

Proposition 4. *Let A be transitive state-closed abelian group of degree m. Then* $\mathrm{tor}(A)$ *has finite exponent and is therefore a direct summand of A.*

Proof. Let $T = \mathrm{tor}(A)$, $A_1 = \mathrm{Stab}_A(1)$, $T_1 = T \cap A_1$and $[T : T_1] = m'$. Then the projection on the 1st coordinate of T_1 is a subgroup of T and the triple $(T, T_1, \pi_1|_{T_1})$ is simple of degree $m'|m$; let $m = m'm''$. Hence, in this representation T is a torsion transitive state-closed subgroup of $\mathcal{A}_{m'}$, the automorphism group of the tree $\mathcal{T}_{m'}$.

Fixing this last representation of T, let $Q = P(T)$ and let σ_i $(1 \leq i \leq k)$ be a minimal set of generators of Q and as before, let $\beta_i = \beta(\sigma_i) \in T$ be such that $\sigma(\beta_i) = \sigma_i$. Let r be the maximum order of the elements $\beta_1, \dots, \beta_k$. As any $\alpha \in T$ can be written in the form

$$\alpha = \prod_{1 \leq i \leq k} \beta_i^{r_{i1}} (\beta_i^{r_{i2}})^{(l_2)} \dots (\beta_i^{r_{ij}})^{(l_j)} \dots,$$

it follows that $\alpha^r = e$.

Since T has finite exponent, it is a pure bounded subgroup of A and therefore it is a direct summand of A ([6], Theorem 4.3.8). □

We recall a classic example of an abelian group G which does not split over its torsion subgroup (see [6], p. 108).

Example 2. Let G be the direct product of groups $\prod_{i \geq 1} C_i$, where $C_i = \langle c_i \rangle$ is cyclic of order p^i and let H be the direct sum $\sum_{i \geq 1} C_i$. Then $H \leq \mathrm{tor}(G) = \bigcup_{l \geq 1} G(p^l)$. Moreover, H is a basic subgroup of G and in particular, $\frac{G}{H}$ is p-divisible. This observation leads directly to a proof that G does not split over $\mathrm{tor}(G)$.

The proof of the previous proposition did not establish the exponent of $\mathrm{tor}(A)$. This we do in the next two lemmas.

Lemma 3. *Let m be a prime number and A an abelian transitive state-closed torsion group of degree m. Then A is conjugate by a tree automorphism to a subgroup of the diagonal-topological closure of $\langle \sigma \rangle$ and so has exponent m.*

Proof. We observe that $A(m)$ is not contained in $A_1 = \mathrm{Stab}_A(1)$. For otherwise, $A(m)$ would be invariant under the projection on the first coordinate. Choose $a \in A \backslash A_1$ of order m. Therefore, $A = A_1 \langle a \rangle$. On choosing $\{a^i \mid 0 \leq i \leq m-1\}$ as a transversal of A_1 in A, the image of a acquires the form $\sigma = (1, \dots, m)$ in this tree representation of A. Thus, we may suppose by Proposition 3 that $\sigma \in A$. Therefore, $\tilde{A}$ contains the subgroup $\widetilde{\langle \sigma \rangle} = \langle \sigma^{(i)} \mid i \geq 0 \rangle$. By Proposition 2, we have $C_{\mathcal{A}} \widetilde{\langle \sigma \rangle} = \langle \sigma \rangle^*$ and thus, $A \leq C_{\mathcal{A}}(A) \leq \langle \sigma \rangle^*$. □

Lemma 4. *Suppose that A is an abelian transitive state-closed torsion group of degree m. Then the exponent of A is equal to the exponent of $P(A)$.*

Proof. By induction on $|P(A)| = m$. The exponent of A is a multiple of the exponent of $P(A)$. By the previous lemma, we may assume m to be composite. Let p be a prime divisor of m and $A(p) = \{a \in A \mid a^p = e\}$. Then $A(p)$ is a nontrivial subgroup and $P(A(p)) \leq \{\sigma \in P \mid \sigma^p = e\}$. By Lemma 2, $\left(\frac{A}{A(p)}, \frac{A_1 A(p)}{A(p)}, \overline{\pi_0}\right)$ is simple; also, $P(\frac{A}{A(p)}) = \frac{P(A)}{P(A(p))}$. The proof follows by induction. □

Theorem 7. *Suppose that A is an abelian transitive state-closed torsion group of degree m. Then A is conjugate to a subgroup of the topological closure of*

$$\widetilde{P(A)} = \langle \sigma^{(i)} \mid \sigma \in P(A),\ i \geq 0 \rangle.$$

Proof. Let $P = P(A)$ have exponent r and let B be a maximal homogeneous subgroup of P of exponent r (that is, B is a direct sum of cyclic groups of order r), minimally generated by $\{\sigma_i \mid 1 \leq i \leq s\}$. Choose for each σ_i an element $\beta_i = \beta(\sigma_i) \in A$ and let $\dot{B} = \langle \beta_i \mid 1 \leq i \leq s \rangle$. Then, as the order of each β_i is a multiple of r, while the exponent of A is r, we conclude from the previous lemma that $o(\beta_i) = o(\sigma_i) = r$ for $1 \leq i \leq s$. Since $\beta_i \to \sigma_i$ defines a projection of $\dot{B}$ onto B we conclude that $\dot{B} \cong B$ and $\dot{B} \cap A_1 = \{e\}$, where $A_1 = \mathrm{Stab}_A(1)$.

Clearly $\dot{B}$ is a pure bounded subgroup and so it has a complement L in A, which may be chosen to contain A_1. Choose a right transversal W of A_1 in L. Then the set $W\dot{B}$ is a right transversal of A_1 in A. With respect to this transversal, the triple (A, A_1, π_1) produces a transitive state-closed representation φ where $\dot{B}^\varphi = B$. By Proposition 3, we may rewrite A^φ as A. Then the diagonal-topological closure A^* contains B^*. Let V be a complement of B in P. Each $\alpha \in A^*$ can be factored as $\alpha = \beta\gamma$, where $\beta \in B^*$ and γ is such that each of its states γ_u have activity $\sigma(\gamma_u) \in V$. Therefore, the set of these γ's is a group Γ such that $\Gamma = \Gamma^*$ and $A^* = \Gamma \oplus B^*$. Then $(\Gamma, \Gamma \cap A_1, \pi_1)$ is a simple triple with $P(\Gamma)$ having exponent smaller than r. The proof is finished by induction on the exponent. □

The example below illustrates some of the ideas developed so far.

Example 3. Let $m = 4$, $Y = \{1, 2, 3, 4\}$ and let σ be the cycle $(1, 2, 3, 4)$. Furthermore, let $\alpha = (e, e, e, \alpha^2)\sigma \in \mathcal{A}(4)$ and let $A = \langle \alpha \rangle$. Then

$$\alpha^2 = (\alpha^2, e, e, \alpha^2)(1, 3)(2, 4),$$
$$\alpha^4 = (\alpha^2)^{(1)} = \alpha^{2x}, \quad (\alpha^{2-x})^2 = e.$$

Thus A is cyclic, torsion-free, transitive and state-closed; it is, however, not diagonally closed because $\alpha^x \notin A$. Even though A is torsion-free, its diagonal closure $\tilde{A} = \langle \alpha^{x^i} \mid i \geq 0 \rangle$ is not; for $\kappa = \alpha^{2-x}$ has order 2. Let $K = \langle \kappa^{x^i} \mid i \geq 0 \rangle$.

Then $K \leq \mathrm{tor}(\tilde{A})$ and it is direct to check that $\tilde{A} = \langle \alpha, K \rangle$. Therefore, $K = \mathrm{tor}(\tilde{A})$ and

$$\tilde{A} = \mathrm{tor}(\tilde{A}) \oplus A.$$

Let $Y_1 = \{1, 3\}$, $Y_2 = \{2, 4\}$. Then $\{Y_1, Y_2\}$ is a complete block system for the action of α on Y. Also, α^2 induces the binary adding machine on both $\mathcal{T}(Y_1)$ and $\mathcal{T}(Y_2)$. The topological closure $\bar{A}$ of A is torsion-free and

$$\mathrm{tor}(A^*) = \mathrm{tor}(\tilde{A}), \quad A^* = \mathrm{tor}(A^*) \oplus \bar{A}.$$

Moreover, $\mathrm{tor}(A^*)$ induces a faithful state-closed, diagonally and topologically closed actions on the binary tree $\mathcal{T}(Y_1)$. Therefore, $\mathrm{tor}(A^*)$ is isomorphic to $\frac{\mathbb{Z}}{2\mathbb{Z}}[[x]]$. Furthermore, α is represented as the binary adding machine on $\mathcal{T}(\{Y_1, Y_2\})$ and $\bar{A}$ is represented on this tree as the topological closure of the image of A.

6. Cyclic $\mathbb{Z}_m[[x]]$-modules

Cyclic automorphism groups $\langle \alpha \rangle$ of the tree, for which their state-diagonal-topological closure is isomorphic to a cyclic $\mathbb{Z}_m$-module have the form

$$\alpha = (\alpha^{q_1}, \ldots, \alpha^{q_m})\sigma,$$

where $q_i \in \mathbb{Z}_m[[x]]$ for $1 \leq i \leq m$; here

$$q_i = \sum_{j \geq 0} q_{ij} x^j, \quad q_{ij} = \sum_{u \geq 0} q_{ij,u} m^u \in \mathbb{Z}_m.$$

We prove

Theorem 8. (i) *The expression*

$$\alpha = (\alpha^{\boldsymbol{q_1}}, \ldots, \alpha^{\boldsymbol{q_m}})\sigma$$

is a well-defined automorphism of the m-ary tree.

(ii) *Let A be the state closure of $\langle \alpha \rangle$. Then $A^*$is abelian, isomorphic to the quotient ring $\frac{\mathbb{Z}_m[[x]]}{(r)}$, where*

$$r = m - qx \quad \text{and} \quad q = q_{\mathbf{1}} + \cdots + q_{\boldsymbol{m}}.$$

Proof. (1) Let $\sigma(l)$ denote the permutation induced by α on the l-th level. Then the expression $\alpha = (\alpha^{\boldsymbol{q_1}}, \ldots, \alpha^{\boldsymbol{q_m}})\sigma$ represents

$$\sigma(1) = \sigma, \quad \sigma(l) = (\sigma(l-1)^{\overline{q_1}}, \ldots, \sigma(l-1)^{\overline{q_m}})\sigma,$$

where $\overline{q_i} = \overline{q_{i0}} + \overline{q_{i1}} x + \cdots + \overline{q_{i(l-1)}} x^{l-1}$ and $\overline{q_{ij}} = q_{ij,0} + q_{ij,1} m + \cdots + q_{ij,l-1} m^{l-1}$.

(2.1) The states of α are words in α^p for $p \in \mathbb{Z}_m[[x]]$. Let $v = \alpha^{l_1} \dots \alpha^{l_a}$, $w = \alpha^{n_1} \dots \alpha^{n_b} \in A^*$. Then clearly $[v, w] \in \mathrm{Stab}_A(1)$. We will prove that the entries of $[v, w]$ are products of conjugates of words in elements of the form $[\alpha^s, \alpha^t]$ where $s, t \in \mathbb{Z}_m[[x]]$.

Clearly $[v, w]$ can be developed into a word in conjugates of $[\alpha^{l_i}, \alpha^{n_j}]$.

Write $p = p_0 + p'x, n = n_0 + n'x$. We compute

$$\begin{aligned}[\alpha^p, \alpha^n] &= ([\alpha^{p_0}, \alpha^{n'x}][\alpha^{p_0}, \alpha^{n_0}]^{\alpha^{n'x}})^{\alpha^{p'x}}[\alpha^{p'}, \alpha^{n'}]^x[\alpha^{p'x}, \alpha^{n_0}]^{\alpha^{n'x}} \\ &= [\alpha^{p_0}, \alpha^{n'x}]^{\alpha^{p'x}}[\alpha^{p'}, \alpha^{n'}]^x[\alpha^{p'x}, \alpha^{n_0}]^{\alpha^{n'x}}.\end{aligned}$$

Therefore, we have to check $[\alpha^{\xi}, \alpha^{nx}]$ where $\xi \in \mathbb{Z}_m, n \in \mathbb{Z}_m[[x]]$. Write $\xi = \xi_0 + m\xi'$. Then

$$[\alpha^{\xi}, \alpha^{nx}] = [\alpha^{\xi_0 + m\xi'}, \alpha^{nx}] = [\alpha^{\xi_0}, \alpha^{nx}]^{\alpha^{m\xi'}}[\alpha^{m\xi'}, \alpha^{nx}].$$

Now

$$\alpha^{\xi_0} = (v_1, v_2, \dots, v_m)\sigma^{\xi_0},$$

where v_i are words in $\alpha^{q_1}, \dots, \alpha^{q_m}$ and

$$\alpha^m = (\alpha^{q_1} \dots \alpha^{q_m}, \alpha^{q_2} \dots \alpha^{q_m}\alpha^{q_1}, \dots, \alpha^{q_m}\alpha^{q_1} \dots \alpha^{q_{m-1}}).$$

Therefore,

$$[\alpha^{\xi_0}, \alpha^{nx}] = ([v_1, \alpha^n], \dots, [v_m, \alpha^n])$$

and similarly

$$[\alpha^{m\xi'}, \alpha^{nx}] = ([(\alpha^{q_1} \dots \alpha^{q_m})^{\xi'}, \alpha^n], \dots, [(\alpha^{q_m}\alpha^{q_1} \dots \alpha^{q_{m-1}})^{\xi'}, \alpha^n]).$$

Now we write $\beta = \alpha^{q_1} \dots \alpha^{q_m}$. Then $[\beta^{\xi'}, \alpha^n]$ can be developed further, as asserted. The same applies to the other entries.

(2.2) First, clearly $r\alpha = 0$. Now let $u = u(x)$ annul α; write $u = u_0 + u'x$ where $u_0 = u(0)$. Then $m | u_0$ and so

$$u = m\tfrac{u_0}{m} + u'x = (xq)\tfrac{u_0}{m} + u'x + vr = xw_1 + vr$$

for some $v = v(x)$ and $w_1 = q\frac{u_0}{m} + u'$. Then xw_1 annuls α and so does w_1. On repeating, we find w_i such that $u \equiv x^i w_i \bmod r$ and w_i annuls α for all $i \geq 1$.

In other words, $u \in \bigcap_{n \geq 1}(x\mathbb{Z})^n + (r) = (r)$. □

The group $D_m(j)$. Recall $\alpha = (e, \dots, e, \alpha^{x^{j-1}})\sigma \in \mathcal{A}_m$. Then $\alpha^m = \alpha^{x^j}$; that is, $\alpha^r = e$ where $r = m - x^j$. The states of α are $\alpha, \alpha^x, \dots, \alpha^{x^{j-1}}$ and

$$D_m(j) = \langle \alpha, \alpha^x, \dots, \alpha^{x^{j-1}} \rangle;$$

therefore $D_m(j)$ is diagonally closed. The topological closure $\overline{D_m(j)}$ is isomorphic to the quotient ring $S = \frac{\mathbb{Z}_m[[x]]}{(r)}$, which is clearly a free $\mathbb{Z}_m$-module of rank j.

6.1. The case $P(A)$ cyclic of prime order

Theorem 9. *Let m be a prime number. Let A be a torsion-free abelian transitive state-closed subgroup of $\mathcal{A}_m$. Let $\beta \in A \setminus \mathrm{Stab}_A(j)$. Then $\overline{A^*} = \overline{\langle\beta\rangle^*}$ and is topologically finitely generated. Furthermore, A^* is conjugate to $D_m(j)$ for some $j \geq 1$.*

The proof is developed in four steps.

Step 1. For $z \in A$, define $\zeta(z) = j$ such that $z^m \in \mathrm{Stab}(j) \setminus \mathrm{Stab}(j+1)$. As A is torsion-free, $\zeta(z)$ is finite for all nontrivial z and $z^m = (v)^{(j)}$, $v \in A \setminus \mathrm{Stab}_A(1)$.

Choose $\beta = (\beta_1, \beta_2, \ldots, \beta_m)\sigma \in A \setminus \mathrm{Stab}_A(1)$ having minimum $\zeta(\beta) = j$. If $z \in \mathrm{Stab}_A(1)$, $z \neq e$, then there exists $l > 0$ such that $z^m = (c)^{(l)}$ and $c \in A \setminus \mathrm{Stab}_A(1)$. Therefore, by minimality of β we have $\zeta(c) \geq \zeta(\beta)$ and $\zeta(z) > \zeta(\beta)$.

Lemma 5 (Uniform gap). *Let $z \in \mathrm{Stab}_A(1)$. Then $\zeta(z\beta) = \zeta(\beta)$.*

Proof. First note that

$$\beta^m = (\beta_1\beta_2 \ldots \beta_m)^{(1)},$$
$$\beta_1\beta_2 \ldots \beta_m = (\gamma)^{(j-1)}, \quad \gamma \in A \setminus \mathrm{Stab}_A(1).$$

We have $z = c^{(1)}$ and $z\beta = (c\beta_1, c\beta_2, \ldots, c\beta_m)\sigma$, $(z\beta)^m = (u)^{(1)}$, where $u = c^m\beta_1 \ldots \beta_m = c^m(\gamma)^{(j-1)}$. If $c \in A \setminus \mathrm{Stab}_A(1)$ then $\zeta(c) = n \geq j$, $c^m \in \mathrm{Stab}(n) \setminus \mathrm{Stab}(n+1)$, and so, $u \in \mathrm{Stab}_A(j-1) \setminus \mathrm{Stab}_A(j)$. If $c \in \mathrm{Stab}_A(1)$ then $\zeta(c) > j$ and so $c^m \in \mathrm{Stab}(k)$, where $k > j$ and again $u \in \mathrm{Stab}(j-1) \setminus \mathrm{Stab}(j)$. □

Step 2. Note that

$$\beta^m = (\gamma)^{(j)}, \quad \gamma^m = (\lambda)^{(j)},$$
$$\beta^{m^2} = (\lambda)^{(2j)},$$

where, by the uniform gap lemma above, $\gamma, \lambda \in A \setminus \mathrm{Stab}_A(1)$. Therefore, repeating this process, we find that β^{m^s} induces $\sigma^{(sj)}$ on the (sj)-th level of the tree for all $s \geq 0$. Now given a level $t \geq 0$, dividing t by j, we get $t = sj + i$ with $0 \leq i \leq j-1$, and then $(\beta^{(i)})^{m^s} = (\beta^{m^s})^{(i)}$ induces $(\sigma^{(sj)})^{(i)} = \sigma^{(sj+i)} = \sigma^{(t)}$ on the t-th level of the tree. It follows that the group A is a subgroup of the topological closure of $\langle\beta, \beta^{(1)}, \ldots, \beta^{(j-1)}\rangle$.

Step 3. We have for $\beta = (\beta_1, \beta_2, \ldots, \beta_m)\sigma$,

$$\beta_i = \beta^{p_i}, \;\; p_i = r_{i0} + r_{i1}x + \cdots + r_{i(j-1)}x^{j-1} \in \mathbb{Z}_m[x],$$

and

$$\beta^m = (\beta_1\beta_2 \ldots \beta_m)^{(1)},$$
$$\beta_1\beta_2 \ldots \beta_m = \beta^{p_1+\cdots+p_m},$$
$$p_1 + \cdots + p_m = q \cdot x^{j-1},$$

where q is an invertible element of $\mathbb{Z}_m[[x]]$.

Proposition 5. *The element* $\beta = (\beta_1, \beta_2, \ldots, \beta_m)\sigma$ *is conjugate in* $\mathcal{A}_m$ *to* $\alpha = (e, \ldots, e, \alpha^{x^{j-1}})\sigma$.

Proof. Let $h = (h_1, h_2, \ldots, h_m)$ be an automorphism of the tree. Then

$$\beta^h = (h_1^{-1}\beta_1 h_2, h_2^{-1}\beta_2 h_3, \ldots, h_m^{-1}\beta_m h_1)\sigma.$$

Therefore $\beta^h = \alpha$ holds if and only if

$$h_2 = \beta_1^{-1}h_1, \quad h_3 = \beta_2^{-1}h_2, \ \ldots, \ h_m = \beta_{m-1}^{-1}h_{m-1}, \quad h_1 = \beta_m^{-1}h_m\alpha^{x^{j-1}}.$$

These conditions can be rewritten as

$$h_2 = \beta_1^{-1}h_1, \quad h_3 = \beta_2^{-1}\beta_1^{-1}h_1, \ \ldots, \ h_m = \beta_{m-1}^{-1}\ldots\beta_1^{-1}h_1,$$
$$h_1 = \beta_m^{-1}\beta_{m-1}^{-1}\ldots\beta_1^{-1}h_1\alpha^{x^{j-1}},$$

or as

$$\begin{aligned} h &= (h_1, \beta_1^{-1}h_1, \beta_2^{-1}\beta_1^{-1}h_1, \ldots, \beta_{m-1}^{-1}\ldots\beta_1^{-1}h_1) \\ &= (e,\ \beta_1^{-1},\ \beta_2^{-1}\beta_1^{-1}, \ldots, \beta_{m-1}^{-1}\ldots\beta_1^{-1})(h_1)^{(1)}, \end{aligned}$$

and

$$(\beta_1\beta_2\ldots\beta_m)^{h_1} = \alpha^{x^{j-1}}.$$

Since

$$\beta_1\beta_2\ldots\beta_m = \beta^{q\cdot x^{j-1}},$$

we repeat the above procedure replacing β by β^q and replacing h_1 by $(h_1')^{x^{j-1}}$. This leads to the conjugation equation

$$(\beta^q)^{h_1'} = \alpha.$$

In this manner, we determine an automorphism h of the tree which effects the required conjugation

$$\beta^h = \alpha.$$

□

Example 4. Let $\beta = (e, \beta^q)\sigma$, where $q = 1 + x$. Then β is conjugate to the adding machine $\alpha = (e, \alpha)\sigma$. Note that from Example 1, β is not obtainable from α by simply choosing a different transversal. To exhibit the conjugator $h\colon \beta \to \alpha$ constructed in the proof, define the polynomial sequences

$$c_0 = 1, \quad c_1 = q, \quad c_n = 2c_{n-2} + c_{n-1};$$
$$c_{-1}' = 0, \quad c_0' = 0, \quad c_n' = c_{n-1} + c_{n-1}'.$$

Then

$$h = (e,e)^{(0)}(e, \beta^{-1})^{(1)}(e, \beta^{-(1+q)})^{(2)}\ldots(e, \beta^{-c_n'})^{(n)}\ldots.$$

Step 4. By Proposition 2, we have $A \leq \bar{A} = C_{\mathcal{A}}(\alpha)$ and

$$A^h \leq C_{\mathcal{A}}(\alpha^h) = C_{\mathcal{A}}(\beta) = \overline{D_m(j)}.$$

This finishes the proof of the theorem.

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