Codimension one subgroups and boundaries of hyperbolic groups

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Abstract. We construct hyperbolic groups with the following properties: The boundary of the group has big dimension, it is separated by a Cantor set, and the group does not split. This shows that Bowditch's theorem that characterizes splittings of hyperbolic groups over 2-ended groups in terms of the boundary cannot be extended to splittings over more complicated subgroups.

1. Introduction

Let G be a finitely generated group and let H be a subgroup of G. We say that H is a codimension 1 subgroup if for some finite index subgroup H' of H C_G/H' has more than 1 end, where C_G is the Cayley graph of G. If G splits over H then one easily sees that H is codimension 1. The converse is not true; for example, any closed geodesic on a surface group gives a cyclic codimension 1 subgroup of the fundamental group of the surface. On the other hand only simple closed geodesics correspond to splittings.

The surface example can be generalized to CAT(0) complexes to produce examples of codimension 1 subgroups: If X is a finite CAT(0) complex of (say) dimension 2 and if R is a locally geodesic track on X, then the subgroup of $G = \pi_1(X)$ corresponding to R is a codimension 1 free subgroup of G. Wise ([11]) exploited this idea producing codimension 1 subgroups for small cancellation groups. In the setting of small cancellation groups of course one needs some combinatorial analog for the convexity property of geodesics (or tracks), and Wise develops such a notion. Pride ([9]) has shown that there are small cancellation groups that have property (FA) (see [10], p. 58) , so such groups have codimension 1 subgroups but do not split.

Stallings showed that if a compact set separates the Cayley graph of a finitely generated group G into at least two unbounded components, then G splits over a

finite group. Bowditch ([1]) showed something similar for hyperbolic groups: If the boundary ∂G of a 1-ended hyperbolic group G has a local cut point, then the group splits over a 2-ended group, unless it is a triangle group. There have been other generalizations of Stallings theorem similar in spirit. The general idea is that if a 'small' set (coarsely) separates the Cayley graph of a group, then the group splits over a subgroup quasi-isometric to the 'small set'. For a precise conjecture see [7].

The main purpose of this paper is to show some limitations of this 'philosophy'. Given any $n > 0$, we produce an example of a hyperbolic group G such that $\dim(\partial G) > n$, ∂G is separated by a set of dimension 0 (a Cantor set) and G has property (FA) (so it does not split over any subgroup). Our example is based on Wise's construction which we generalize to the setting of small cancellation theory over free products.

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2. Preliminaries

Definition 1. A *diagram* is a finite connected planar graph. The *faces* of a diagram D are the closures of the bounded components of $\mathbb{R}^2 - D$.

In what follows we assume always that each interior vertex (i.e., not on ∂D) of a diagram has degree at least 3. We can always achieve this by ignoring all interior vertices of degree 2.

We will need some small cancellation results about diagrams shown by McCammond and Wise in [5]. For the reader's convenience and also because our setting is slightly different we include these results here. These results strengthen classical small cancellation results (see e.g. [4]).

We need some notation: If D is a diagram we denote by ∂D the boundary of the unbounded component of $\mathbb{R}^2 - D$ (so if U is the unbounded component of $\mathbb{R}^2 - D$, $\partial D = \bar{U} - U$). We say that the diagram is *non-singular* if ∂D is homeomorphic to S^1. We say that an edge of D is *interior* if it does not lie in ∂D.

If D is a diagram we denote by E, F, V respectively the total number of edges, faces and vertices of the diagram.

We denote by $E^\bullet$, E° respectively the number of edges of the diagram that lie (do not lie) in ∂D. We denote by V^+ the number of vertices in ∂D that lie in exactly one face and by V^- the number of vertices in ∂D that lie in more than one face. We denote by V° the number of vertices of D that do not lie on ∂D.

If f is a face of D then we can represent its boundary by a cycle $(e_1, \ldots, e_n)$, where the e_i's are oriented edges. We say that a diagram satisfies the $C(6)$ condition if the boundary cycle of each face of the diagram has at least 6 edges. We have the following version of Greedlinger's lemma (see [4]):

Lemma 2.1. *Let D be a non-singular diagram that satisfies the condition $C(6)$. Then $V^+ \geq V^- + 6$.*

Proof. We have the following inequalities. First,

$$6F \leq 2E^\circ + E^\bullet. \tag{1}$$

This is because each face has at least 6 edges and each interior edge appears twice as an edge of a face while each boundary edge appears once. Second,

$$2E \geq 3V^\circ + 3V^- + 2V^+. \tag{2}$$

This is because each edge yields 2 vertices and each interior vertex has degree at least 3.

Finally, using Euler's formula, inequality (1) and the equality $E = E^\circ + E^\bullet$ we obtain

$$V = E - F + 1 \geq E - \frac{E^\circ}{3} - \frac{E^\bullet}{6} + 1 = \frac{2E}{3} + \frac{E^\bullet}{6} + 1. \tag{3}$$

We note that

$$E^\bullet = V^- + V^+.$$

Substituting $E^\bullet$ in (3) and using inequality (2) for E we obtain

$$V \geq \frac{2}{3}\left(\frac{3}{2}V^\circ + \frac{3}{2}V^- + V^+\right) + \frac{V^- + V^+}{6} + 1.$$

Since

$$V = V^- + V^+ + V^\circ$$

we obtain

$$V^+ \geq V^- + 6. \qquad \square$$

We recall some definitions from [5]:

Definition 2. Let D be a non-singular diagram. A face F of the diagram is called an i-shell if the intersection of F with the boundary of D is connected and exactly i edges of F are interior edges of D.

Definition 3. We say that a diagram D is a *ladder* if there are at most two faces F_1, F_2 of D such that $D - F_1, D - F_2$ are connected while for every other face F of D, $D - F$ has exactly 2 components, and for every vertex v $D - v$ has at most 2 components.

We have the following corollary from Lemma 2.1.

Corollary 2.2. *Let D be a non-singular disc diagram which is $C(6)$ and which contains no 3-shells and at most two i-shells with $i \leq 2$. Then either D has a single face or it contains exactly two i-shells with $i \leq 2$ and it is a ladder.*
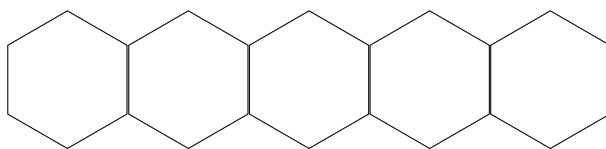
Figure 1. A ladder.

Proof. We modify D as follows: if a face F of D has more than 6 edges and it intersects the boundary we ignore successively vertices of F that do not lie on any other face till F has 6 edges (or there are no more vertices to erase). Let us call D_1 the new diagram. D_1 is still a $C(6)$ diagram. D_1 contains also no 3-shells and at most two i-shells for $i \leq 2$. We consider now a face F of D_1 that intersects the boundary and we see how it contributes to V^+, V^-. If F is not an i-shell then 2 vertices of F contribute to V^+, while at least 4 vertices of F contribute to V^-. So the total contribution of all such faces to the difference $V^+ - V^-$ is at most 0 (note that the contribution is not necessarily negative since we count twice the V^- vertices as they lie in at least 2 faces). The contribution of an i-shell to the difference $V^+ - V^-$ is $4 - i$.

Since D contains no 3-shells and at most 2 i-shells for $i \leq 2$, the inequality

$$V^+ - V^- \geq 6$$

implies that if D_1 has more than one face, then D_1 has exactly two 1-shells, say F_1, F_2. If we erase F_1 we obtain a diagram D_2 which is again $C(6)$. We note that F_1 intersects exactly one face of D_1, so after erasing it the diagram D_2 has still the other 1-shell of D_1 and at most one new i-shell for some $i \leq 3$. By the inequality $V^+ \geq V^- + 6$ again we conclude as before that either D_2 has only one face or it has exactly two 1-shells F_2, F_3. Inductively we see that D_1 is a ladder hence D is also a ladder. $\qquad \square$

We will need a more technical result. If v is a vertex in a diagram we denote by d_v the degree of v. The result below will be used to show that small cancellation products of word hyperbolic groups are word hyperbolic.

Lemma 2.3. *Let D be a non-singular diagram that satisfies the condition $C(7)$. Then*

$$\frac{1}{3} \sum_{v \in D^0} \frac{d_v}{2} - \frac{2E^\circ}{7} \leq V^\bullet + \frac{E^\bullet}{7}.$$

In particular

$$F \leq 3E^\bullet + 3V^\bullet,$$

i.e., D satisfies a linear isoperimetric inequality.

Proof. We denote by D^0 the set of vertices of D (the 0-skeleton). Clearly

$$\sum_{v \in D^0} \frac{d_v}{2} = E. \tag{4}$$

We also have the inequality

$$7F \leq 2E^\circ + E^\bullet.$$

This is because each face has at least 7 edges and each interior edge lies in at most 2 faces while boundary edges lie in one face.

Using Euler's formula and the inequality above we obtain

$$E + 1 = V + F \leq V^\bullet + \frac{E^\bullet}{7} + V^\circ + \frac{2E^\circ}{7}. \tag{5}$$

Since $d_v \geq 3$ for every v in the interior of D, it follows that

$$\sum_{v \in D^0} \frac{d_v}{2} - V^\circ \geq \frac{1}{3} \sum_{v \in D^0} \frac{d_v}{2}.$$

By (4) and (5) we have

$$\frac{1}{3} \sum_{v \in D^0} \frac{d_v}{2} - \frac{2E^\circ}{7} \leq V^\bullet + \frac{E^\bullet}{7}. \tag{6}$$

Since

$$\frac{1}{3} \sum_{v \in D^0} \frac{d_v}{2} \geq \frac{E^\circ}{3}$$

we have

$$\frac{1}{3} \sum_{v \in D^0} \frac{d_v}{2} - \frac{2E^\circ}{7} \geq \frac{E^\circ}{42} \geq \frac{3F}{7} - \frac{E^\bullet}{84}$$

and, using (6),

$$V^\bullet + \frac{2E^\bullet}{7} \geq \frac{3F}{7} - \frac{E^\bullet}{84},$$

which implies that $F \leq 3E^\bullet + 3V^\bullet$. $\qquad\qquad\square$

3. Small cancellation theory over free products

Small cancellation theory can be developed over free products (see [4]). We show in this section that small cancellation products have codimension 1 subgroups. This generalizes a result of Wise ([11]). We recall that the free product factors embed in small cancellation products ([4], Corollary 9.4, p. 278). Osin ([6], Lemma 4.4) showed that free product factors embed quasi-isometrically in small cancellation products (this also follows from [2]). For the reader's convenience we include a proof of this below.

Definition 4. Let $\langle S|R \rangle$ be a presentation of a group G. Here we consider R to be a sequence of elements rather than a set. We say that $\langle S|R \rangle$ is *symmetrized* if every $r \in R$ is cyclically reduced word and for any $r = y_1 \ldots y_n \in R$, R contains $2n$ distinct terms corresponding to the $2n$ cyclic permutations of r and r^{-1}. If $r_1 = cb$, $r_2 = ca$ and the words cb, ca are reduced we call c a *piece* of the presentation.

For example the symmetrized presentation that corresponds to the presentation $\langle a|a^2 \rangle$ is the presentation $\langle a|a^2, a^2, a^{-2}, a^{-2} \rangle$.

Let now $\langle S|R \rangle$ be a symmetrized presentation. We have then the following small cancellation conditions:

- *Condition $C'(\lambda)$*: If $r \in R$ and $r = cb$ with cb reduced word and c a piece then $|c| < \lambda |r|$.

- *Condition $C(p)$*: No element of R is a product of fewer than p pieces.

- *Condition $B(2p)$*: If $r = ab$ and a is a product of p pieces then $|a| \leq |r|/2$.

Wise showed in [11] that groups that admit a presentation in which all relators have even length and condition $B(6)$ is satisfied, have codimension 1 subgroups. Clearly condition $C'(1/6)$ is stronger than condition $B(6)$, so Wise's result holds for these groups too.

We now describe the small cancellation conditions on free products ([4], Chapter V, Section 9). Let F be the free product of the groups A_i.

A *word* in the free product $*A_i$ is a sequence of elements $a_1, \ldots, a_n$ where each a_i is an element of a free factor.

We say that a word $a_1 \ldots a_n$ is *reduced* if each a_j is an element of one of the A_i and a_j, a_{j+1} belong to different factors for any j. Any element $g \in F$ can be represented in a unique way as a reduced word (normal form of g). If $g = a_1 \ldots a_n$ is the normal form of g we define $\|g\| = n$. If $u = a_1 \ldots a_n, v = b_1 \ldots b_k$ are reduced words we say that the word $uv = a_1 \ldots a_n b_1 \ldots b_k$ is *semi-reduced* if $a_n b_1 \neq e$. Note however that a_n, b_1 might lie in the same factor. We say that a word $w = a_1 \ldots a_n$ is *weakly cyclically reduced* if it is reduced and $a_n a_1 \neq e$. We say that a sequence of words R is *symmetrized* if whenever $r \in R$ all weakly cyclically reduced conjugates of r and r^{-1} are in R. We remark here that if the A_i's are infinite groups a symmetrized set of relations is infinite if it has length greater than 1.

We say that c is a *piece* if there are distinct $r_1, r_2 \in R$ such that $r_1 = ca$, $r_2 = cb$ and the words ca, cb are semi-reduced. As before we have the condition $C'(\lambda)$:

Condition $C'(\lambda)$: If $r \in R$ and $r = cb$ with cb a semi-reduced word and c a piece then $\|c\| < \lambda \|r\|$.

Let now F be a free product $F = *A_i$ of non-trivial groups and let R be a symmetrized subset of F. The group G defined by the free product presentation $\langle F|R \rangle$ is the quotient

$$G = F/\langle\langle R \rangle\rangle$$

where $\langle\langle R \rangle\rangle$ is the normal closure of R in F.

We show that groups G with free product presentations $\langle F|R \rangle$ that satisfy the $C'(1/6)$ condition have codimension 1 subgroups. We start first by considering van Kampen diagrams over G. We consider a usual presentation of G with a set of generators S given by the generators of A_i's and a set of relators consisting of relators of the A_i's together with a set R' such that R is obtained by taking all weak cyclic conjugates of elements in R' and their inverses. If R' is finite we say that G has a *finite free product presentation.* Let now w be a word in S representing the identity in G. Let D be a reduced van Kampen diagram for w over the presentation given above. We remark that if p is a simple closed path in the 1-skeleton of D such that all edges of p lie in a single factor A_i then the word corresponding to p represents the identity in A_i (see [4], Corollary 9.4). Call such a simple closed path *maximal* if there is no other such simple closed path q in the interior of p. We modify now the diagram D as follows: For each maximal simple closed path p we erase all edges of p and all edges of D inside p and we introduce a new vertex v_p which we join with all vertices of p. Now each edge e of p has been replaced by two edges e_1, e_2. We label e_1, e_2 by elements of A_i so that the product of their labels is equal to the label of e. Note that we may label an edge by the identity. After this operation some of the edges of D are 'subdivided'. We subdivide the remaining edges of D so that the labels of the new edges lie in the same factor as the old ones and the product of their labels is equal to the label of the old edge. We call this a *van Kampen diagram over the free product.*

We remark now that the $C'(\lambda)$ condition holds for this new diagram, i.e., if p is a path in the diagram,which is contained in the intersection of two faces $R_1 \cap R_2$ and intersects no other face, except possibly at its endpoints, then

$$\text{length}(p) < \lambda \min(\text{length}(\partial R_1), \text{length}(\partial R_2)).$$

We will show that groups that satisfy the small cancellation condition $C'(1/6)$ over free products have codimension 1 free subgroups. The proof is similar to the proof in [11]. To simplify notation we show this below only in the case of free products with only two factors but the same proof applies in general.

Theorem 3.1. *Let G be a finitely generated group with a finite free product $C'(1/6)$ presentation $\langle F|R' \rangle$, where $F = A * B$. Assume further that each $r \in R'$ is a cyclically reduced word and $\|r\|$ is even. Then G has a codimension 1 subgroup.*

Proof. We construct a complex for G as usual. If K_1, K_2 are complexes with a single vertex respectively x_1, x_2 such that $\pi_1(K_1, x_1) = A, \pi_1(K_2, x_2) = B$ we take the wedge product $K_1 \vee K_2$ of identifying x_1, x_2 to a single vertex x. For each $r \in R'$ we glue a 2-cell, $c(r)$, to $K_1 \vee K_2$ in the obvious way to obtain a complex K such that $\pi_1(K, x) = G$. We argue now in a way similar to Wise ([11]). We slightly change the approach and we consider bouquets of circles that go through x rather than tracks. We explain now how to construct a bouquet of circles Γ which will generate a codimension 1 subgroup H. Algebraically H is the subgroup generated

by all 'halves' of relators in R'. So if $r \in R'$ and $r = c_1 c_2 \dots c_{2n}$ is the normal form of r in $A * B$ then $c_1 c_2 \dots c_n$ is a generator of H. We take also all cyclic permutations of the normal form of r: $c_2 \dots c_{2n} c_1, c_3 \dots c_{2n} c_1 c_2, \dots$, and we include the halves of these relations to the generating set of H.

Let $r \in R'$ and let $r = a_1 b_1 \dots a_n b_n$ be the normal form of r in F. We represent the 2-cell $c(r)$ corresponding to r as a polygon, where the a_i, b_i's are the labels of the sides of this polygon. We can represent the generators of H geometrically as 'diagonals' of the cells $c(r)$ ($r \in R'$). We remark now that since $c(r)$ has an even number of sides each vertex has a vertex opposite to it, so for each vertex there is a diagonal joining it with the opposite vertex. Note that this is slightly different from the way one usually represents r as a polygon, where the sides of the polygon correspond to the letters of r. In our context, however, we ignore most of these vertices and we keep only the ones that correspond to the normal form representation of r in the free product $A * B$. So if we see r in the usual way as a polygon, where the sides are letters, then, in our terminology, some vertices have *opposite* vertices, while others do not.
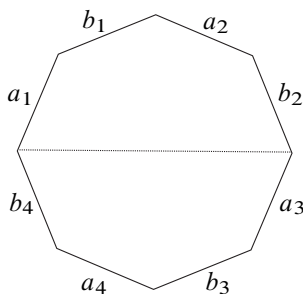

Figure 2. A generator (represented by a dotted line).

We will define now Γ as an abstract graph. Γ has a single vertex and the edges of Γ correspond to diagonals of the cells $c(r)$ ($r \in R'$).

We remark that Γ is a bouquet of circles if looked at as an abstract graph, but if we see it as immersed in K, some of its edges intersect each other in the middle points of the polygons.

If we see Γ as a subset of K, its edges (loops) generate H.

We will show that H is a codimension 1 subgroup of G.

Lemma 3.2. *There is a tree $\widetilde{\Gamma} \subset \widetilde{K}$ whose edges are diagonals of 2-cells such that $\widetilde{\Gamma}$ is invariant under H. $\widetilde{K} - \widetilde{\Gamma}$ has at least 2 components.*

Proof. We will show that the universal covering of Γ embeds in $\widetilde{K}$ as a graph $\widetilde{\Gamma}$. The graph $\widetilde{\Gamma}$ projects under the covering projection $\pi \colon \widetilde{K} \to K$ to the image of Γ in K.

Let $v \in \widetilde{K}$ be a vertex. We define a connected graph in $\widetilde{K}$ as follows: We say that two vertices are *related* if they are opposite. We take the equivalence relation

generated by this relation and we consider the equivalence class of v. Let $\widetilde{\Gamma}$ be the graph obtained by joining opposite vertices in this equivalence class by diagonals. We claim that $\widetilde{\Gamma}$ is a tree. If it is not a tree then there is a path p in $\widetilde{\Gamma}$ such that both endpoints of p lie on the same 2-cell of R' and p is not a single edge (a diagonal). Let us say that a, b are the endpoints of p and they lie on a 2-cell σ. Let q be a path on $\partial\sigma$ joining a, b. We may assume q to have minimal normal form length in the free product among the 2 possible paths. Now $p \cup q$ is a closed loop. We change now p by replacing each diagonal with the corresponding path on the boundary on which the diagonal lies. We note that we have two choices and we replace the diagonals so that the path we obtain by replacing all of them corresponds to a reduced word of F. Let p' be the path we obtain in this way. We may arrange also that $p' \cup q$ is reduced at the vertex a (unless $a = b$). We now consider the van Kampen diagram over the free product for $p' \cup q$ and observe that if it has an i-shell for $i \leq 2$, then the boundary of this i-shell contains a neighborhood of the vertex b. But this contradicts Corollary 2.2. It follows that $\widetilde{\Gamma}$ is a tree. By construction $\widetilde{\Gamma}$ is invariant under H and separates locally (hence also globally) $\widetilde{K}$. $\qquad\square$

The first part of the next lemma also follows from work of Osin ([6], see also [2]). We include a proof here for the sake of completeness.

Lemma 3.3. *The vertex groups A, B and H embed quasi-isometrically in G. Then H is a codimension 1 subgroup of G.*

Proof. Let a be a geodesic word in the Cayley graph of A (or B). We will show that a is a quasi-geodesic in $\widetilde{K}$. Let S be the generating set of G and let $|w|$ be the length of a word in S. Put

$$M = \max\{|r| : r \in R'\}.$$

If we think of w as an element of $F = A * B$ rather than as a word in S then we denote by $\|w\|$ its normal form length.

We define a new length function L for words in S:

$$L(w) = M\|w\| + |w|.$$

We refer to a path (word) of minimal L-length as an L-*geodesic*.

Obviously an L-geodesic is a quasi-geodesic.

We show now that a is a quasi-geodesic. Let p be an L-geodesic in the 1-skeleton of $\widetilde{K}$ with the same endpoints as a. We consider the van Kampen diagram over the free product for $a \cup p$. We may assume that $a \cap p$ is equal to the endpoints of a, p since along the intersection of a, p, the path a is quasi-geodesic (this is because p is a quasi-geodesic).

This diagram has at most 2 i-shells (with $i \leq 2$), because by the definition of L-geodesics it follows that an i-shell with $i = 1$ or 2 contains at least one endpoint of p. Thus by Corollary 2.2 this diagram is a ladder. By considering now the usual van Kampen diagram for $a \cup p$ we have that $|a| \leq M|p|$ so a is quasi-geodesic.

We prove now that H is quasi-isometrically embedded. Since H acts freely on $\widetilde{\Gamma}$ it is enough to show that $\widetilde{\Gamma}$ is quasi-isometrically embedded. Let p be a geodesic path in $\widetilde{\Gamma}$ joining two vertices v, u of $\widetilde{\Gamma}$.

We change p by replacing each diagonal by a corresponding path on the boundary of the cell on which lies the diagonal. We pick this path so that the word of F corresponding to the path is reduced (in the sense of normal forms). We note that this is possible since each diagonal can be replaced by two different paths, one starting with an a and the other with a b.

Let p' be the path we obtain in this way. Let q be an L-geodesic path joining v, u. As before we may assume that p', q intersect only at their endpoints. Again by the definition of p', q if we consider the van Kampen diagram over the free product for $p' \cup q$ we remark that it has at most 2 i-shells with $i \leq 2$.

Hence this diagram is a ladder. By considering the usual van Kampen diagram we have that $|p'| \leq M |q|$. Since q is a quasi-geodesic we have that p' is a quasi-geodesic, so H is quasi-isometrically embedded.

Finally we show that H is a codimension 1 subgroup. It suffices to show that $\widetilde{K} - \widetilde{\Gamma}$ has at least 2 components which are not contained in a finite neighborhood of $\widetilde{\Gamma}$. By its definition $\widetilde{\Gamma}$ separates locally $\widetilde{K}$. Since $\widetilde{K}$ is simply connected, $\widetilde{\Gamma}$ separates $\widetilde{K}$.

We introduce now some useful terminology. Let $r \in R'$ and let $c_1 c_2 \ldots c_n$ be the normal form of r in F. Recall that r is cyclically reduced. Let k be the smallest number such that

$$k \geq \frac{n}{6}.$$

We say then that $c_1 \ldots c_k$ is a *fragment* of r. Similarly we define fragments of all cyclic permutations of $c_1 c_2 \ldots c_n$.

Let R be a 2-cell in $\widetilde{K}$ which intersects $\widetilde{\Gamma}$ on an edge e. Let v, u be the vertices of e.

Let $c_1 c_2 \ldots c_n$ be the label of R starting from v and written in free product normal form. Let s be the vertex corresponding to the endpoint of a fragment $p_1 = c_1 \ldots c_k$ of R starting at v. We construct a path starting from v and lying in the same component of $\widetilde{X} - \widetilde{\Gamma}$ as s. The path starts at p_1. At s we continue by following p_1 with a fragment p_2 of another 2-cell R_2 corresponding to $r_2 \in R'$. We pick $R_2 \neq R$ and so that $p_1 p_2$ is reduced in F (in the sense of normal forms). We continue inductively in the same way picking each time a new 2-cell and a fragment so that the word we obtain is reduced in F. This is clearly possible, since at each step, we consider a new 2-cell and we have two possibilities for going around the boundary of the cell (corresponding to the two orientations of the boundary). At least one of the two possibilities produces a path reduced in the sense of normal forms.

Let $\beta = p_1 \ldots p_n$ be the path we obtain after n steps. If s_n is the endpoint of $p_1 \ldots p_n$ we claim that $d(s_n, \widetilde{\Gamma}) \to \infty$ as $n \to \infty$. Indeed, let q be a geodesic joining s_n to a closest vertex $t \in \widetilde{\Gamma}$. We consider a geodesic γ in $\widetilde{\Gamma}$ joining v to t.

We distinguish two cases. Assume first that u does not lie on γ. We change γ by replacing each diagonal by the corresponding path on the boundary of the 2-cell in
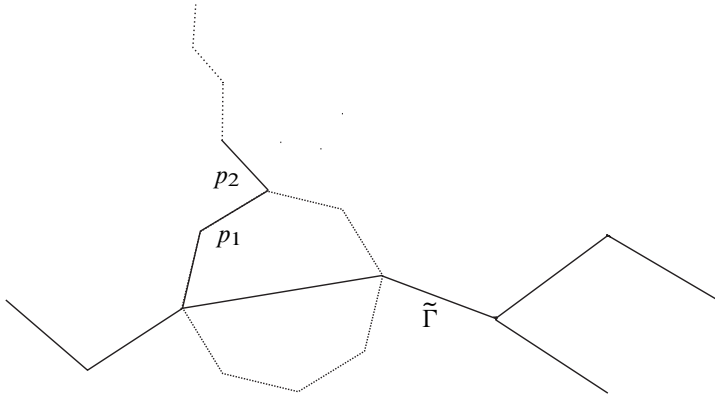
Figure 3. The tree $\widetilde{\Gamma}$ and the path $\beta = p_1 p_2 \ldots p_n$.

which the diagonal lies to obtain a path γ'. We make these replacements so that the word of F corresponding to the new path is reduced and $p_1^{-1}\gamma'$ corresponds also to a reduced word in F. Clearly this is possible since we have two choices for replacing each diagonal and the normal form of each starts from a different free factor. We now consider the loop

$$\beta \cup \gamma' \cup q.$$

Since q is geodesic, the van Kampen diagram for free products for this loop has at most 2 i-shells with $i \leq 2$ and these would have to appear around the endpoints of q. It follows that this diagram is a ladder (see Corollary 2.2); hence the lengths of q and $\beta \cup \gamma'$ are comparable, so the length of q goes to infinity as $n \to \infty$.

Now we deal with the second case, i.e., we assume that u lies on γ. We modify $p_1 \ldots p_n$ as follows. We replace p_1 by the path q_1 on the boundary of R joining s to u. We note that the new path $\beta' = q_1 p_2 \ldots p_n$ might not be reduced at the endpoint of q_1. We replace γ by a path γ' in the 1-skeleton of $\widetilde{K}$ as before so that $q_1^{-1}\gamma'$ is reduced in the free product F. We remark that the van Kampen diagram over free products for the loop

$$\beta' \cup \gamma' \cup q$$

is a ladder in this case too, hence the length of q goes to infinity as $n \to \infty$.

Similarly we see that the component of $R - \widetilde{\Gamma}$ that does not contain v is not contained in a finite neighborhood of $\widetilde{\Gamma}$. It follows that H is codimension 1. $\square$

This finishes the proof of Theorem 3.1.

4. The example

Theorem 4.1. *Given any $n > 0$, there is a one-ended hyperbolic group G such that*

- dim $\partial G \geq n$,
- ∂G *is separated by a Cantor set,*
- G *does not split.*

Proof. Let A be a torsion free 1-ended hyperbolic group with property T and such that $\dim(\partial A) \geq n$ (e.g., a lattice in $\mathrm{Sp}(n, 1)$). Let us denote $A = \langle a_1, \ldots, a_k \rangle$. We may assume that each a_i is non-trivial and that $a_i^m \neq a_j^r$ for any $i \neq j$ and $m, r > 0$. Let B denote a copy of A with corresponding generators $\langle b_1, \ldots, b_k \rangle$. We consider the free product $A * B$ and define G to be the small cancellation quotient of $A * B$ given by the relations

$$r_{i,j} = (a_i b_j)(a_i b_j^2) \ldots (a_i b_j^{19}), \quad 1 \leq i, j \leq k.$$

By Theorem 3.1, G has a free codimension 1 subgroup H. As shown in the proof of Theorem 3.1, H is quasi-isometrically embedded, so a Cantor set separates ∂G.

We show now that G has property (FA) (i.e., it does not split). Clearly G is not an HNN extension since the abelianization of A is finite, so the abelianization of G is finite. We show that G does not split as an amalgamated product. Suppose $G = X *_C Y$. Without loss of generality we may assume that $A \subset X$ and $B \subset gXg^{-1}$ or $B \subset gYg^{-1}$. Let $g = x_1 \ldots x_n$ be the normal form of g in the free product decomposition. By replacing A, B by conjugates we may assume that either $g = 1$ and $B \subset Y$, or $x_1 \notin X$. However, we see then that the word

$$r_{i,j} = (a_i b_j)(a_i b_j^2) \ldots (a_i b_j^{19})$$

is reduced in $X *_C Y$, unless a_i or b_j is in C. Since all $r_{i,j}$ are equal to the identity, this implies that $A = C$ and B is contained in Y, or $B = C$ and A is contained in X; but in both cases, the splitting would be trivial.

We claim finally that G is hyperbolic. Indeed this follows by Lemma 4.4 of [6], and [2]. For the reader's convenience we sketch a proof here using Lemma 2.3. It is enough to show that G satisfies a linear isoperimetric inequality. Let w be a word on the generators of G and let D be a reduced van Kampen diagram for G. As described in Section 3, one obtains from D a new diagram, say D_1, which is called the diagram for w over the free product. Since A, B are hyperbolic they satisfy some isoperimetric inequality of the form

$$A(p) \leq K l(p)$$

for any simple closed path p in the Cayley graph of A or of B.

It follows that if p is a simple closed path of D such that all edges of p lie in A (or in B) and if v is the vertex of D_1 that we obtain by collapsing p to a point, then

$$\mathrm{d}_v = l(p) \geq \frac{1}{K} A(p),$$

where d_v is the degree of v. It follows that

$$A(D) \leq A(D_1) + K \sum_{v \in D_1^0} d_v.$$

From Lemma 2.3 we have the following inequality for the diagram D_1:

$$\frac{1}{3} \sum_{v \in D_1^0} \frac{d_v}{2} - \frac{2E^\circ}{7} \leq V^\bullet + \frac{E^\bullet}{7}.$$

Moreover,

$$\sum_{v \in D_1^0} \frac{d_v}{2} \geq E^\circ \Rightarrow \frac{2E^\circ}{7} \leq \frac{2}{7} \sum_{v \in D_1^0} \frac{d_v}{2},$$

so

$$\frac{1}{3} \sum_{v \in D_1^0} \frac{d_v}{2} - \frac{2E^\circ}{7} \geq \frac{1}{42} \sum_{v \in D_1^0} \frac{d_v}{2}.$$

We also have $l(\partial D_1) \leq l(\partial D)$ and $V^\bullet, E^\bullet \leq l(\partial D)$. Thus, by Lemma 2.3, it follows that

$$A(D_1) \leq 6l(\partial D)$$

and

$$\sum_{v \in D_1^0} d_v \leq 42V^\bullet + 7E^\bullet,$$

hence

$$A(D) \leq (6 + 49K)l(\partial D).$$

In other words, G satisfies a linear isoperimetric inequality, so it is hyperbolic. $\square$

Remark 1. The above example also shows that for any n there is a finitely presented group G with asdim $G > n$, which is separated coarsely by a uniformly embedded set H of asdim $H = 1$ and which does not split. This answers a question in [7].

5. Boundaries and Cantor manifolds

We recall some notions from dimension theory.

Definition 5. We say that a closed subset F of a topological space X *separates* X if $X - F = U_1 \cup U_2$ with U_1, U_2 open disjoint sets.

Definition 6. A topological space X of dimension n is a *Cantor manifold* if it cannot be separated by a closed sets F such that dim $F \leq n - 2$.

Hurewicz and Tumarkin showed the following:

Theorem 5.1. *A compact metrizable topological space X of dimension n contains an n-dimensional Cantor manifold.*

Definition 7. A maximal Cantor manifold of dimension n of a compact metrizable topological space X of dimension n is called a *dimensional component* of X.

It is easy to see, using Zorn's lemma, that each Cantor manifold of X is contained in a dimensional component and that dimensional components are closed.

It was conjectured in [7] that if the boundary of a hyperbolic group G is not a Cantor manifold, then the group splits. This was motivated by a desire to generalize Stallings ends theorem to splittings over infinite groups. As we saw in the previous sections, this is false, but perhaps the group might split when the dimensional components themselves separate. So we have the following:

Question. Let G be a hyperbolic group such that ∂G is not a Cantor manifold and there is a dimensional component C of ∂G such that $\partial G - C$ is not connected. Is it true then that G splits?

We remark that in the example presented in the previous section the dimensional components correspond to the boundaries (of conjugates of the subgroups A, B). So we have infinitely (countably) many distinct dimensional components, but $\partial G - C$ is connected for any dimensional component in this example.

Proposition 5.2. *Let G be a hyperbolic group such that ∂G is not a Cantor manifold and there is a dimensional component C of ∂G such that $\partial G - C$ is not connected. Then G acts non-trivially, by isometries, on an $\mathbb{R}$-tree T.*

Proof. We give only a sketch since this type of argument is standard by now. Without loss of generality we assume that ∂G is connected. Let C be a dimensional component of ∂G for which $\partial G - C$ is not connected. Let R be a connected component of $\partial G - C$. Observe that
$$\partial R = \bar{R} - R$$
is not a single point since ∂G has no cut points.

Let
$$\mathcal{P} = \{gR : g \in G\}.$$
Note that if $A_1, A_2 \in \mathcal{P}$, then $A_1 \subset A_2$ or $A_2 \subset A_1$ or $A_1 \cap A_2 = \emptyset$.

We define now a betweeness relation in $\mathcal{P}$ that turns $\mathcal{P}$ into a pre-tree.

If x, y, z are three distinct elements of $\mathcal{P}$, we say that y is between x and z if one of the following four possibilities holds:

1. $z \subset y$ and $y \cap x = \emptyset$.

2. $z \subset y$ and $y \subset x$.

3. $x \subset y$ and $y \cap z = \emptyset$.

4. $x \subset y$ and $y \subset z$.

If y is between x, z we write xyz.

We show now the axioms of pre-trees hold (see [8]):

1. By definition xyx does not hold.

2. If xyz holds then by the symmetry between x, z in the definition above we see that zyx holds as well.

3. If xyz holds then by inspecting the four cases of the definition above we see that xzy does not hold.

4. If xyz holds and $w \neq y$ we will show that either xyw or zyw.

Since xyz holds, we may assume that $z \subset y$ (the case $x \subset y$ is treated similarly). We have now three cases: $y \subset w$, $w \subset y$ or $y \cap w = \emptyset$. If $y \subset w$ then zyw holds. If $w \subset y$ then xyw holds. Finally, if $y \cap w = \emptyset$ then zyw holds.

By the standard 'join the dots' construction (see eg [8]) one can promote $\mathcal{P}$ to an $\mathbb{R}$-tree T. The action of G on ∂G induces an action of G on T. It is easy to see that this action is non-nesting and that there is no fixed point. It follows [3] that G acts by isometries on an $\mathbb{R}$ tree T'. On the other hand it is not clear whether the action of G on T (hence also on T') is stable, so it is not clear whether G splits. □

One can easily construct continua with uncountably many dimensional components. However, when constructing hyperbolic groups using amalgamation or other standard constructions, the dimensional components correspond to quasi-isometrically embedded subgroups (so there are at most countably many such components). It is reasonable to ask:

Question. Let G be a hyperbolic group. Is it true that ∂G has at most countably many distinct dimensional components?

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