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Can Dehn surgery yield three connected summands?

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Abstract. A consequence of the Cabling Conjecture of Gonzalez-Acuña and Short is that Dehn surgery on a knot in S^3 cannot produce a manifold with more than two connected summands. In the event that some Dehn surgery produces a manifold with three or more connected summands, then the surgery parameter is bounded in terms of the bridge number by a result of Sayari. Here this bound is sharpened, providing further evidence in favour of the Cabling Conjecture.

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1. Introduction

The Cabling Conjecture of Gonzalez-Acuña and Short [4] asserts that Dehn surgery on a knot in S^3 can produce a reducible 3-manifold only if the knot is a cable knot and the surgery slope is that of the cabling annulus.

The Cabling Conjecture is known to hold in many special cases [2], [8], [9], [12], [13], [17], [19].

If k is the (p,q)-cable on a knot K, then the cabling annulus on k has slope pq, and the corresponding surgery manifold $M(k,pq)$ splits as a connected sum

$$M(K,p/q)\,\#\,L(p,q);$$

see [5]. (Here $L(p,q)$ is a lens space.) In particular both connected summands are prime [5]. Thus the Cabling Conjecture implies the weaker conjecture below:

Conjecture 1 (Two summands conjecture). *Let k be a knot in S^3 and $r\in\mathbb{Q}\cup\{\infty\}$ a slope. Then the Dehn surgery manifold $M(k,r)$ cannot be expressed as a connected sum of three non-trivial manifolds.*

Since any knot group has *weight* 1 (in other words, is the normal closure of a single element), the same is true for any homomorphic image of a knot group. Thus the two summands conjecture would follow from the group-theoretic conjecture below, which remains an open problem.

Conjecture 2. *A free product of three non-trivial groups has weight at least* 2*.*

The best known upper bound for the number of connected summands in $M(k,r)$ is 3, obtained by combining results of Sayari [14], Valdez Sánchez [18] and the author [11]. These results also show that, should some $M(k,r)$ have three connected summands, then two of these must be lens spaces (necessarily with fundamental groups of coprime orders) and the third must be a $\mathbb{Z}$-homology sphere. (See [11] for details.)

Suppose that k is a knot in S^3 with bridge number b, and that the 3-manifold M obtained by performing Dehn surgery on k with surgery parameter r has more than two connected summands. It is known from the work of Gordon and Luecke [5] that r must be an integer.

If ℓ_1, ℓ_2 are the orders of the fundamental groups of the lens spaces, then Sayari [15] has proved that $|r| = \ell_1\ell_2 \le (b-1)(b-2)$.

In this paper we shall prove the following inequality.

Theorem 1. *Let k be a knot in S^3 with bridge-number b. Suppose that r is a slope on k such that $M = M(k,r) = M_1 \# M_2 \# M_3$ where M_1, M_2 are lens spaces and M_3 is a homology sphere but not a homotopy sphere. Then*

$$|\pi_1(M_1)| + |\pi_1(M_2)| \le b + 1.$$

As an immediate consequence, we obtain a sharpening of Sayari's inequality.

Corollary 2. *Under the hypotheses of Theorem 1 we have*

$$|r| = |\pi_1(M_1)| \cdot |\pi_1(M_2)| \le b(b+2)/4.$$

We use the standard techniques of intersection graphs developed by Scharlemann [16] and by Gordon and Luecke [1], [6], [7]. In §2 below, we recall the construction of the intersection graphs in the particular context of this problem. A key feature of these is the existence of *Scharlemann cycles*, which correspond in a well-understood way to the lens space summands. In §3 we show that, should the inequality $\ell_1 + \ell_2 \le b + 1$ fail, then we can find, trapped between two Scharlemann cycles, a *sandwiched disk* (see Definition 3.3). We then show in §4 that sandwiched disks are impossible, which completes our proof.

2. The graphs

Throughout the remainder of the paper, we assume that the manifold $M = M(k,r)$ obtained by r-Dehn surgery on $k \subset S^3$ is a connected sum of three factors M_1, M_2, M_3, where M_1 and M_2 are lens spaces while M_3 is a (prime) integer homology sphere. Note that, since $\pi_1(M)$ has weight 1, the orders ℓ_1, ℓ_2 of $\pi_1(M_1)$ and

$\pi_1(M_2)$ are necessarily coprime. It follows that the factors M_1, M_2, M_3 are pairwise non-homeomorphic.

An essential embedded sphere $\Sigma \subset M$ necessarily separates, with one component of $M \smallsetminus \Sigma$ homeomorphic to a punctured M_s and the other to a punctured $M_t \# M_u$, where $\{s, t, u\} = \{1, 2, 3\}$. We will say that such a Σ *separates* M_s and M_t (and also separates M_s and M_u).

For $i = 1, 2$ let P_i be a planar surface in the exterior $X(k)$ of k (the complement of an open regular neighbourhood of k in S^3) that extends to an essential sphere $\widehat{P_i} \subset M$ such that $\widehat{P_i}$ separates M_i and M_3. Assume also that P_i has the smallest possible number of boundary components amongst all such planar surfaces.

A standard argument ensures that we may also choose P_1, P_2 to be disjoint (without increasing the number of boundary components of either).

Following Gabai [3], Section 4 (A), we put k in thin position, find a level surface Q for k and isotope $P := P_1 \cup P_2$ such that P meets Q transversely, and such that no component of $Q \cap P$ is an arc that is boundary-parallel in P. (The minimality condition in the definition of P_1 and P_2 ensures also that no component of $Q \cap P$ is a boundary-parallel arc in Q.)

The number q of boundary components of Q is necessarily even, and is bounded above by twice the bridge number, $q \leq 2b$. We can complete Q to a sphere $\widehat{Q} \subset S^3$ by attaching q meridional disks.

We denote the intersection graph of P_i and Q in $\widehat{Q}$ by G_i for $i = 1, 2$. The (*fat*) vertices of G_i are the meridional disks $\widehat{Q} \smallsetminus Q$, and the edges are the components of $P_i \cap Q$ (some of which may be closed curves rather than arcs). Each fat vertex contains precisely one point of intersection of k with $\widehat{Q}$, so a choice of orientation for k and for Q induces an orientation on the collection of fat vertices – that is, a partition of fat vertices into two types, which we call *positive* and *negative*. There are precisely $q/2$ vertices of each type.

Note that the graphs G_1 and G_2 have the same vertex set but disjoint edges sets. Let G_Q denote their union: $G_Q := G_1 \cup G_2$.

Similarly, we denote the intersection graph of P and Q in $\widehat{P} = \widehat{P_1} \cup \widehat{P_2}$ by G_P (noting that this graph is the union of two disjoint non-empty subgraphs $G_{P_i} := G_P \cap \widehat{P_i}$, $i = 1, 2$, and hence is not connected).

The edges incident at a vertex v of G_Q are labelled by the boundary components of P. These labels always occur in the same cyclic order around v (subject to change of orientation). We choose a numbering $1, \dots, p$ of $\pi_0(\partial P)$ in such a way that the labels $1, \dots, p$ always occur in that cyclic order around each vertex of G_Q (without loss of generality, clockwise for positive vertices and anti-clockwise for negative vertices).

The corner at a vertex v of G_Q between the edges labelled x and $x + 1$ (modulo p) is also given a label: g_x if v is positively oriented, and g_x^{-1} if v is negatively oriented. Note that corners are arcs in $\partial X(k)$ with endpoints in P. In the usual set-up for intersection disks, P is connected, and one can interpret the labels $g_x^{\pm 1}$ as elements of $\pi_1(M)$ (relative to a base-point on P). In our context it is more natural to interpret $g_x^{\pm 1}$ as an element of the path-groupoid $\Pi = \pi(M, P)$, whose elements are (free) homotopy classes of maps of pairs from $([0,1], \{0,1\})$ to (M, P). Thus Π is a connected 2-vertex groupoid whose vertex groups are isomorphic to $\pi_1(M)$.

Let $T \subset M$ denote the Dehn-filling solid torus, and $k' \subset T$ its core (a knot in M).

A *Scharlemann cycle* in G_i is a cycle C bounding a disk-component Δ of $\widehat{Q} \smallsetminus G_i$ (which we call a *Scharlemann disk*), such that each edge of C, regarded as an arc in P_i, joins two fixed components of ∂P_i (x and y, say). Thus each edge of C has label x at one end, and y at the other. Since x, y are consecutive edges of G_i at each vertex of C, the edges of $G_Q \cap \Delta$ between x and y at v belong to G_{3-i} and correspond to intersection points of k' with P_{3-i}. Since P_{3-i} is separating, it follows that $x - y$ is odd, and hence from the *parity rule* (see for example [6], p. 386) that all vertices of C have the same orientation.

It is well known (see for example [1], [6]) that any Scharlemann cycle in G_i corresponds to a lens-space summand of M. We have set things up in such a way that this summand is necessarily isotopic to M_i, which leads to the following observation. (Compare also [10], Lemma 2.1, which states a similar conclusion under slightly different hypotheses.)

Lemma 2.1. *Any Scharlemann cycle in G_i has length $\ell_i := |\pi_1(M_i)|$.*

Proof. Without loss of generality, we may assume that $i = 1$. Let C be a Scharlemann cycle in G_1, and Δ the corresponding Scharlemann disk. Assume that x, y are the labels on the edges of C.

Following [1], [6], we construct a twice punctured lens space in M as follows. The fat vertices of G_{P_1} can be regarded as meridional slices of the filling solid torus T. The fat vertices x and y divide T into two 1-handles, one of which – H, say – satisfies $\partial\Delta \subset P_1 \cup \partial H$.

Then a regular neighbourhood L of $\widehat{P_1} \cup H \cup \Delta$ is a twice-punctured lens space, with $\pi_1(L) \cong \mathbb{Z}_\ell$, where ℓ is the length of C.

One component of ∂L is $\widehat{P_1}$. The second component Σ has precisely two fewer points of intersection with k' than $\widehat{P_1}$.

By the uniqueness of the prime decomposition $M = M_1 \# M_2 \# M_3$, L is homeomorphic to a twice-punctured copy of M_1 or of M_2. In the latter case, Σ also separates M_1 from M_3, which contradicts the minimality hypothesis on P_1. Hence L is homeomorphic to a twice-punctured copy of M_1, whence $\ell = \ell_1$ as claimed. □

More generally, we have the following essentially well-known result, which is an important tool in our proof.

Define the 2-complex K as follows. K has two vertices, labelled 1 and 2, and p edges, labelled $g_1, \dots, g_p$. The initial (resp. terminal) vertex of g_i is 1 or 2 depending on whether the vertex i (resp. $i+1$) of G_P is contained in P_1 or in P_2. The 2-cells of K are in one-to-one correspondence with the disk-regions of G_Q; the attaching map for a 2-cell being read off from the corner-labels of the corresponding region of G_Q.

Lemma 2.2. *Let K_0 be a subcomplex of K with $H^1(K_0, \mathbb{Z}) = \{0\}$. If K_0 is connected then M has a connected summand with fundamental group isomorphic to $\pi_1(K_0)$. If K_0 is disconnected, then M has a connected summand with fundamental group isomorphic to $\pi_1(K_0, 1) * \pi_1(K_0, 2)$.*

Proof. The intersection of $\widehat{P}$ with the filling solid torus T is precisely the set of fat vertices of G_P, each of which is a meridional disk in T. These disks divide T into 1-handles $H_1, \dots, H_p$, where H_i is the section of T between the fat vertices i and $i + 1$ (modulo p).

Suppose first that K_0 is connected. Define K' to be the union of the following subsets of M:

(1) P_1 if K_0 contains the vertex 1 of K;
(2) P_2 if K_0 contains the vertex 2 of K;
(3) the one-handle H_i for each edge $g_i \in K_0$;
(4) the disk-region of G_Q corresponding to each 2-cell of K_0.

It is easy to check that K' is connected, and that $\pi_1(K') \cong \pi_1(K_0)$. Let N be a regular neighbourhood of K' in M

Then N is a compact, connected, orientable 3-manifold with $\pi_1(N) \cong \pi_1(K_0)$ and hence $H^1(N, \mathbb{Z}) = \{0\}$. It follows that ∂N consists entirely of spheres, by Poincaré duality.

Capping off each boundary component of N by a ball yields a closed manifold $\widehat{N}$ with $\pi_1(\widehat{N}) \cong \pi_1(N) \cong \pi_1(K_0)$, and $\widehat{N}$ is a connected summand of M since $N \subset M$.

Next suppose that K_0 is disconnected. Then K_0 contains both vertices 1, 2 of K, but no edge from 1 to 2. Choose an edge g_z of K joining 1 to 2, and define $K_1 = K_0 \cup \{g_z\}$. Then K_1 is connected and $\pi_1(K_1) \cong \pi_1(K_0, 1) * \pi_1(K_0, 2)$. Replacing K_0 by K_1 in the above gives the result. □

Corollary 2.3. *No subcomplex of K has fundamental group which is a free product of three or more finite cyclic groups.*

Proof. Suppose that K has such a subcomplex. Then by Lemma 2.2 M has a connected summand which is the connected sum of three lens spaces. This contradicts [11], Corollary 5.3. □

Finally, the element $R = g_1 g_2 \dots g_p \in \pi_1(M)$ is a *weight element* – that is, its normal closure is the whole of $\pi_1(M)$ – since it is represented by a meridian in $S^3 \setminus k$. This leads to the following observation, which will be useful later.

Lemma 2.4. *Let $x \in \{1, \dots, p\}$. There is at least one integer $i \in \{1, \dots, (p-2)/2\}$ such that no 2-gonal region of G_Q has corners g_{x+i} and g_{x-i} (or g_{x+i}^{-1} and g_{x-i}^{-1}).*

Proof. Otherwise we have $g_{x+i} = g_{x-i}^{-1}$ in $\pi_1(M)$ for each $i = 1, \dots, (p-2)/2$, and hence the weight element $W = g_1 \dots g_p$ is conjugate to a word of the form $g_x U g_y U^{-1}$ (where $U = g_{x+1} \dots g_{x+(p-2)/2}$ and $y = x + \frac{p}{2}$ modulo p). Moreover, g_x is conjugate in $\pi_1(M)$ to an element of $\pi_1(M_i)$ for some $i \in \{1, 2, 3\}$, and a similar statement holds for g_y. Hence W belongs to the normal closure in $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2) * \pi_1(M_3)$ of the free factors containing conjugates of g_x and g_y. Since all three free factors are non-trivial, this normal subgroup is proper, which contradicts the fact that W is a weight element. □

3. Analysis of Scharlemann cycles

By [6], Proposition 2.8.1, there are Scharlemann cycles in G_1 and in G_2. In this section we show that, if $\ell_1 + \ell_2$ is big enough, then these form a configuration we call a *sandwiched disk* (which we will show in the next section to be impossible). Our next two results should be compared to [15], Lemmas 3.2 and 5.3, and [7], Theorem 2.4, respectively, where the conclusions are similar but the hypotheses slightly different.

Lemma 3.1. *If Δ is a Scharlemann disk in bounded by a Scharlemann cycle in G_1 (resp. G_2) then Δ contains no edges of G_2 (resp. G_1).*

Proof. Suppose that Δ is bounded by a Scharlemann cycle C in G_1, and that it contains edges of G_2. By [6], Proposition 2.8.1, we know that there exists a Scharlemann cycle in $G_2 \cap \Delta$. We will find such a Scharlemann cycle explicitly, and use it to obtain a contradiction.

Recall that C has length ℓ_1, by Lemma 2.1. Let $v_1, \dots, v_{\ell_1}$ denote the vertices of C in cyclic order. Each edge of C has labels x and $x+2t+1$, say, which correspond to vertices in G_{P_1}, and the intermediate labels $x+1, \dots, x+2t$ correspond to vertices of G_{P_2}. (Necessarily, these are even in number and alternating in orientation, since they correspond to consecutive intersection points of k' with $\widehat{P_2}$ between two consecutive intersection points of k' with $\widehat{P_1}$.)

The graph $Y := G_2 \cap \Delta$ has ℓ_1 vertices, each of valence $2t$ and each of the same orientation (which we assume to be positive).

If $\ell_1 = 2$, then every edge of Y joins v_1 to v_2. Such an edge has labels $x + j$ at one end and $x + 2t + 1 - j$ at the other, for some j. The two edges whose labels are $x + t$ and $x + t + 1$ bound a 2-gonal region, and hence form a Scharlemann cycle of length 2. But then $\ell_1 = \ell_2 = 2$, contradicting the fact that ℓ_1, ℓ_2 are coprime.

Suppose then that $\ell_1 > 2$. There must be a vertex v_j in C that is joined only to v_{j-1} and v_{j+1} (subscripts modulo ℓ_1) by edges of Y. In particular there are two consecutive vertices of C that are joined by $s \geq t$ edges of Y. The resulting s 2-gonal regions of $G_Q \cap \Delta$ give rise to relations $g_{x+j} g_{x+2t-j} = 1$ for $0 \leq j \leq s-1$ in the path-groupoid $\Pi = \pi(M, P)$. But all the corners of the Scharlemann disk Δ have label $h := g_x g_{x+1} \dots g_{x+2t}$, so h has order ℓ_1 in Π. Hence g_{x+t} also has order $\ell_1 > 2$. Hence also $s = t$ in the above, for otherwise $g_{x+t}^2 = 1$ in Π.

Choose a pair v_i, v_j of vertices of C with $i < j - 1$ with $j - i$ minimal subject to the condition that v_i, v_j are joined by an edge of Y. Then each pair $(v_i, v_{i+1}), \dots, (v_{j-1}, v_j)$ is joined by *precisely* t edges of Y, so there is an edge joining v_i and v_j that has labels $x + t$ and $x + t + 1$, and this forms part of a Scharlemann cycle of length $j + 1 - i$ in G_2. (See Figure 1.)

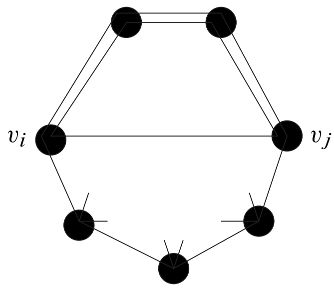

Figure 1

Since g_{x+t} has order ℓ_1 in Π, we deduce that $\ell_1 = \ell_2$, which again contradicts the fact that ℓ_1, ℓ_2 are coprime. □

In particular, if C is a Scharlemann cycle in G_1 or G_2, then the two labels appearing on the edges of C are consecutive (modulo p): say $x, x + 1$. We call x the *label* of C. Note that all the corners of the corresponding Scharlemann disk have the same label g_x or g_x^{-1}.

Corollary 3.2. *Any two Scharlemann cycles in G_1 (respectively in G_2) have the same label.*

Proof. Let C, C' be Scharlemann cycles in G_1, bounding Scharlemann disks Δ, Δ' respectively. By Lemma 3.1, Δ and Δ' contain no edges of G_2, so are Scharlemann disks of G_Q. By Lemma 2.1 each of C, C' has length ℓ_1. Suppose that C has label x and C' has label $y \neq x$. Then K has a subcomplex K_0 with one vertex 1, two edges g_x, g_y and two 2-cells Δ, Δ', so that

$$\pi_1(K_0) = \langle g_x, g_y \mid g_x^{\ell_1} = g_y^{\ell_1} = 1 \rangle \cong \mathbb{Z}_{\ell_1} * \mathbb{Z}_{\ell_1}.$$

In particular, $\pi_1(K_0)$ has weight 2, so cannot be isomorphic to a free factor of $\pi_1(M)$, which contradicts Lemma 2.2. □

Definition 3.3. A *sandwiched disk* in $\widehat{Q}$ is a disk $D \subset \widehat{Q}$ such that

(a) ∂D is the union of a subpath a_1 of a Scharlemann cycle $C_1 \subset G_1$ and a subpath a_2 of a Scharlemann cycle $C_2 \subset G_2$, with $a_1 \cap a_2 = \partial a_1 = \partial a_2$;

(b) there are no vertices of G_Q in the interior of D.

Lemma 3.4. *If* $|\pi_1(M_1)| + |\pi_1(M_2)| > (q+2)/2$*, then there exists a sandwiched disk* $D \subset \widehat{Q}$.

Proof. As observed in [10], p. 551, and [15], Lemma 6.1, we know that there are at least two Scharlemann cycles in G_1 – necessarily with disjoint sets of vertices, since they have the same label (Corollary 3.2). Similarly there are at least two Scharlemann cycles in G_2 – again with the same label and hence with disjoint sets of vertices.

By hypothesis, at least one of $\ell_1 = |\pi_1(M_1)|, \ell_2 := |\pi_1(M_2)|$ is greater than $q/4$. Without loss of generality, assume that $\ell_1 > q/4$. If G_1 contained two Scharlemann cycles with the same (say, positive) orientation, then G_1 would have at least $2\ell_1 > q/2$ positive vertices, contradicting the fact that G_1 has precisely $q/2$ vertices of each orientation.

Hence G_1 must contain precisely two Scharlemann cycles, one of each possible orientation. Let us call them C_1^+ and C_1^-, and let $\Delta_1^{\pm}$ denote the Scharlemann disks bounded by $C_1^{\pm}$.

Now let C_2, C_2' denote two disjoint Scharlemann cycles in G_2, and Δ_2, Δ_2' the corresponding Scharlemann disks. Since $\ell_1 + \ell_2 > q/2$, C_2 must intersect C_1^+ (if the vertices of C_2 are positive) or C_1^- (if the vertices of C_2 are negative). On the other hand, consideration of vertex orientations shows that C_2 cannot intersect both C_1^+ and C_1^-. Similar remarks apply to C_2'.

Now $(C_1^+ \cup C_1^-) \cap (C_2 \cup C_2')$ consists only of some set V (of cardinality t, say) of vertices.

Then $\Delta := \Delta_1^+ \cup \Delta_1^- \cup \Delta_2 \cup \Delta_2'$ has precisely two components, $2\ell_1 + 2\ell_2 - t$ vertices, $2\ell_1 + 2\ell_2$ edges, and four 2-cells. The complement of Δ in $\widehat{Q}$ thus contains $t-1$ components, one of which is an annulus and $t-2$ are disks. But $\widehat{Q} \smallsetminus \Delta$ also contains precisely $q - 2\ell_1 - 2\ell_2 + t$ vertices. Since $2\ell_1 + 2\ell_2 \geq q+4$, this number is at most $t-4$. Hence there are at least two disk-components of $\widehat{Q} \smallsetminus \Delta$ that contain no vertices of G_Q.

Moreover, each vertex of V appears twice in $\partial(\widehat{Q} \smallsetminus \Delta)$, so there are $2t$ such occurrences in total. Each occurrence separates an arc of $C_1^+ \cup C_1^-$ from an arc of $C_2 \cup C_2'$ in $\partial(\widehat{Q} \smallsetminus \Delta)$, so each component of $\partial(\widehat{Q} \smallsetminus \Delta)$ contains an even number of occurrences of vertices from V.

The number of boundary components of $\widehat{Q} \smallsetminus \Delta$ is precisely t. If the vertices in C_2 and those in C_2' have the same orientation, then one of C_1^+, C_1^- is a boundary component of (the annulus component of) $\widehat{Q} \smallsetminus \Delta$ and contains no vertices from V. With that exception, each boundary component of $\widehat{Q} \smallsetminus \Delta$ contains at least two occurrences of vertices from V. Hence at most one boundary component of $\widehat{Q} \smallsetminus \Delta$ can contain more than two occurrences of vertices from V.

Hence, of the two (or more) disk-components of $\widehat{Q} \smallsetminus \Delta$ that contain no vertices of G_Q, each contains at least two occurrences of vertices from V, while at most one of these disk-components contains more than two occurrences of vertices from V. It follows that there is at least one disk component D of $\widehat{Q} \setminus \Delta$ whose boundary contains precisely two occurrences of vertices from V and whose interior contains no vertices of G_Q.

Any such D is, by definition, a sandwiched disk. □

4. Analysis of sandwiched disks

In this section we complete the proof of our upper bound on $|r|$ by showing that sandwiched disks do not exist. This result holds with no assumptions on ℓ_1 or ℓ_2, so may have wider applications.

We assume throughout that G_1, G_2 contain Scharlemann cycles of length ℓ_1, ℓ_2, respectively, with labels x_1, x_2 respectively.

Lemma 4.1. *Let D be a sandwiched disk with $\partial D = a_1 \cup a_2$, where a_1, a_2 are sub-paths of Scharlemann cycles in G_1, G_2, respectively. Then no two consecutive vertices of a_1 (or of a_2) are joined by $p/2$ edges in G_Q.*

Proof. Suppose that two vertices of (say) a_1 are joined by $p/2$ edges. Then there are 2-gonal regions D_i in $G_Q \cap D$ such that the corner labels of D_i are g_{x_1+i} and g_{x_1-i}. This contradicts Lemma 2.4. □

Corollary 4.2. *Let D, a_1, a_2 be as in Lemma 4.1. If two vertices of a_1 (or of a_2) are connected by an edge in G_Q, then they are consecutive vertices of a_1 (respectively of a_2).*

Proof. Let $w_0, \ldots, w_t$ be the vertices of a_1, in order. Suppose that w_i, w_j are joined by an edge in G_Q, where $j > i+1$, and that $j-i$ is minimal for such pairs of vertices. Then w_{i+1} has precisely two neighbours in G_Q: w_i and w_{i+2}. By Lemma 4.1 it is connected to each by fewer than $p/2$ edges, contradicting the fact that it has valence p. □

Corollary 4.3. *Let D, a_1, a_2 be as in Lemma 4.1. Each of a_1, a_2 has length greater than 1, and each interior vertex of a_1 (respectively a_2) is joined to an interior vertex of a_2 (respectively a_1) by an edge of $G_Q \cap D$.*

Proof. If a_1, a_2 both have length 1, then every edge of $G_Q \cap D$ joins the two common endpoints u, v of a_1 and a_2. Without loss of generality, the edges of $G_Q \cap D$ incident at u have labels $x_1+1, x_1+2, \ldots, x_2$, while those incident at v have labels $x_2+1, x_2+2, \ldots, x_1$. Hence $|x_1 - x_2| = p/2$, and D contains precisely $p/2$ arcs joining u to v. But this contradicts Lemma 4.1.

If w is an interior vertex of (say) a_1, then w has two neighbours in ∂D. It is joined to each of these by strictly fewer than $p/2$ arcs, by Lemma 4.1, and hence is also joined to a third vertex in G_Q. Since all the edges of G_Q incident at w are contained in D, this third vertex is also in ∂D. By Corollary 4.2 it cannot be a vertex of a_1, so it must be an interior vertex of a_2. □

Lemma 4.4. *Let D be a sandwiched disk in G_Q. Then there are no Scharlemann cycles in $G_Q \cap D$.*

Proof. Any Scharlemann cycle C in $G_Q \cap D$ is a Scharlemann cycle in G_1 or in G_2, so has label x_1 or x_2 by Corollary 3.2. Assume without loss of generality that C has label x_2. For any vertex v of a_2, the corner labelled g_{x_2} does not lie in D, so the vertices of C are interior vertices of a_1.

By Corollary 4.2, the vertices of C must be pairwise consecutive vertices of a_1, and hence C has length 2. Moreover, if v_1, v_2 are the vertices of C, then v_1, v_2 are connected by edges labelled $x_1+1,\ldots,x_2$ at one end (say the v_1 end), and by edges labelled $x_2+1,\ldots,x_1$ at the other (v_2) end. In particular, they are joined by at least $p/2$ edges, contradicting Lemma 4.1. □

Corollary 4.5. *If there is a sandwiched disk D in G_Q such that $\partial D = a_1 \cup a_2$ where a_i is a subpath of a Scharlemann cycle with label x_i, then $|x_1 - x_2| = p/2$.*

Proof. Let $a_1 \cap a_2 = \{u, v\}$. Without loss of generality, $x_1 = p$ and the edges of $G_Q \cap D$ meeting u are labelled $1,\ldots,x_2$ at u, while those meeting v are labelled $x_2+1,\ldots,p$ at v. If (say) $x_2 < p/2$, then there is a label y with $x_2 < y \leq p$ such that y does not appear as either label of any edge meeting u that is contained in D. Consider the subgraph Γ of $G_Q \cap D$ that is obtained by removing u and its incident edges. At each vertex of Γ, the edge labelled y leads to another vertex of Γ. Since all vertices of Γ are positive, it follows that Γ contains a great y-cycle, and hence a Scharlemann cycle by [1], Lemma 2.6.2. This contradicts Lemma 4.4. □

Theorem 4.6. *There are no sandwiched disks in G_Q.*

Proof. We assume that there is a sandwiched disk D in G_Q, and derive a contradiction. Suppose that $\partial D = a_1 \cup a_2$, where a_i is a subpath of a Scharlemann cycle C_i. Let x_i be the label of C_i. By Corollary 4.5, it follows that $|x_1 - x_2| = p/2$.

Let u, v denote the common vertices of a_1, a_2. By Corollary 4.2 each of a_1, a_2 has length greater than 1. Let s_1, s_2 be the vertices of a_1, a_2, respectively, which are adjacent to u, and let t_1, t_2 be the vertices of a_1, a_2, respectively, which are adjacent to v. (Note that neither of the possibilities $s_1 = t_1$, $s_2 = t_2$ is excluded at this stage.)

By Corollary 4.2 again, s_1 is connected to a vertex of a_2 other than u, v by an edge contained in D. Similarly, s_2 is connected to a vertex of a_1 other than u, v by an edge contained in D. These edges cannot cross; hence s_1 and s_2 are joined by an edge. Similarly t_1 and t_2 are joined by an edge. Hence each of u, v is incident at a triangular region of $G_Q \cap D$: call them Δ_u and Δ_v.

Suppose that the edges of $G_Q \cap D$ that are incident at u have labels $x_1+1,\dots,x_2$ at u, and suppose that i of these edges (namely those with labels $x_1+1,\dots,x_1+i$) are connected to s_1. Then these edges have labels $x_1, x_1-1,\dots,x_1-i+1$ at s_1, and together they bound $i-1$ 2-gonal faces of G_Q, of which the j'th has corner labels g_{x_1+j} and g_{x_1-j}.

The remaining $(p-2i)/2$ edges of $G_Q \cap D$ incident at u join u to s_2. They have labels $x_1+i+1,\dots,x_2$ at u, and $x_1-i,\dots,x_2+1$ at s_2. Together they bound $(p-2i-2)/2$ 2-gonal regions of G_Q, the j'th of which has corner labels g_{x_2-j} and g_{x_2+j}. Thus the triangular region Δ_u of $D\cap G_Q$ that is incident at u has corner labels g_y at u and g_z at each of s_1 and s_2, where $y=x_1+i$ and $z=x_1-i$ (modulo p).

We can now perform a similar analysis on the edges of $G_Q\cap D$ that are incident at v. Note, however, that for all $j\in\{1,\dots,(p-2)/2\}\smallsetminus\{i\}$ there is a 2-gonal region of $G_Q\cap D$ with corner labels g_{x_1-j} and g_{x_1+j}. By Lemma 2.4 there cannot be a 2-gonal region of $G_Q\cap D$ with corner labels g_{x_1-i} and g_{x_1+i}. It follows that there are also precisely i edges joining v to t_1, and $(p-2i)/2$ joining v to t_2. The triangular region Δ_v of $D\cap G_Q$ that is incident at v then has corner labels g_z at v and g_y at each of t_1, t_2, where $y=x_1+i$ and $z=x_1-i$ as above (see Figure 2).

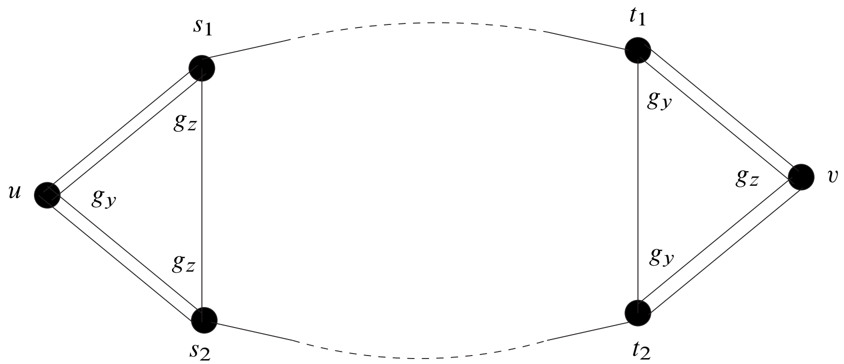

Figure 2

Finally, let K_0 denote the (disconnected) subcomplex of K with vertices $\{0,1\}$, edges $\{g_{x_1}, g_{x_2}, g_y, g_z\}$ and 2-cells $\{\Delta_1,\Delta_2,\Delta_u,\Delta_v\}$.

Then by Lemma 2.2, M has a connected summand with fundamental group

$$\begin{aligned}\pi_1(K_0,1)*\pi_1(K_0,2) &\cong \langle g_{x_1}, g_{x_2}, g_y, g_z \mid g_{x_1}^{\ell_1} = g_{x_2}^{\ell_2} = g_y g_z^2 = g_z g_y^2 = 1\rangle\\ &\cong \mathbb{Z}_{\ell_1} * \mathbb{Z}_{\ell_2} * \mathbb{Z}_3.\end{aligned}$$

But this contradicts Corollary 2.3, which completes the proof. □

Theorem 4.7 (= Theorem 1). *Let k be a knot in S^3 with bridge-number b. Suppose that r is a slope on k such that $M=M(k,r)=M_1\,\#\,M_2\,\#\,M_3$ where M_1, M_2 are* *lens spaces and M_3 is a homology sphere but not a homotopy sphere. Then*

$$|\pi_1(M_1)| + |\pi_1(M_2)| \leq b + 1.$$

Proof. As discussed in §2, we put k in thin position, and choose a level surface Q and disjoint planar surfaces P_1, P_2 such that

- P_i extends to a sphere in M separating M_i from M_3, and has fewest boundary components among all such;
- no component of $Q \cap P_i$ is a boundary-parallel arc in Q or P_i.

By Gordon and Luecke [5], there are Scharlemann cycles C_i in G_i for $i = 1, 2$. Moreover, the Scharlemann cycle C_i has length $\ell_i := |\pi_1(M_i)|$ and bounds a disk-region Δ_i of G_Q. If $\ell_1 + \ell_2 > b + 1 \geq (q+2)/2$, then by Lemma 3.4 there is at least one sandwiched disk D in G_Q. But this contradicts Theorem 4.6.

Hence $\ell_1 + \ell_2 \leq b + 1$ as claimed. □

Corollary 4.8 (= Corollary 2). *With the hypotheses and notation of Theorem 4.7, we have*

$$|r| = |\pi_1(M_1)| \cdot |\pi_1(M_2)| \leq \frac{b(b+2)}{4}.$$

Proof. Let $\ell_1 = |\pi_1(M_1)|$ and $\ell_2 = |\pi_1(M_2)|$. The equation $|r| = \ell_1 \cdot \ell_2$ comes from computing $|H_1(M, \mathbb{Z})|$ in two different ways.

Given that ℓ_1, ℓ_2 are distinct positive integers, the inequality $\ell_1 \cdot \ell_2 \leq b(b+2)/4$ follows easily from Theorem 4.7. □

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