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Free subgroups in groups acting on rooted trees

Volodymyr Nekrashevych*

Abstract. We show that if a group G acting faithfully on a rooted tree T has a free subgroup, then either there exists a point w of the boundary ∂T and a free subgroup of G with trivial stabilizer of w, or there exists $w \in \partial T$ and a free subgroup of G fixing w and acting faithfully on arbitrarily small neighborhoods of w. This can be used to prove the absence of free subgroups for different known classes of groups. For instance, we prove that iterated monodromy groups of expanding coverings have no free subgroups and give another proof of a theorem by S. Sidki.

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1. Introduction

It is well known that free groups are ubiquitous in the automorphism group of an infinite rooted spherically homogeneous tree, see for instance [Bha95], [AV05], though explicit examples (especially ones generated by finite automata) were not so easy to construct; see [Ale83], [BS98], [Oli99], [GM05], [VV07].

On the other hand, many famous groups which are defined by their action on rooted trees do not have free subgroups. Absence of free subgroups is proved in different ways. In some cases it follows from torsion or sub-exponential growth (as in the Grigorchuk groups [Gri80], [Gri85] and Gupta–Sidki groups [GS83]). In other cases it is proved using some contraction arguments (see, for instance, [GŻ02]).

S. Sidki has proved in [Sid04] absence of free groups generated by "automata of polynomial growth", which covers many examples.

An important class of groups acting on rooted trees consists of *contracting self-similar groups*. They appear naturally as iterated monodromy groups of expanding dynamical systems (see [Nek05]). There are no known examples of contracting self-similar groups with free subgroups and it was a folklore conjecture that they do not exist.

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The intuition behind this conjecture and the theorems mentioned above is that there is no sufficient "room" for free subgroups. The action of the elements of the groups in all examples are concentrated on small portions of the tree and the graphs of the action of the groups on the boundary of the rooted tree are also small in some sense. For instance, the graphs of the action of contracting groups have polynomial growth.

We prove in our paper the following theorem formalizing this intuition.

Theorem 3.3. *Let G be a group acting faithfully on a locally finite rooted tree T. Then one of the following holds.*

(1) *G has no free non-abelian subgroups.*
(2) *There is a free non-abelian subgroup $F < G$ and a point $w \in \partial T$ such that the stabilizer F_w is trivial.*
(3) *There is a point $w \in \partial T$ such that the group of G-germs $G_{(w)}$ has a free non-abelian subgroup.*

Here the group of G-germs $G_{(w)}$ is the quotient of the stabilizer G_w by the subgroup of automorphisms g of the tree T acting trivially on a neighborhood $U_g \subset \partial T$ of w.

Theorem 3.3 was inspired by a result of M. Abért implying that if F is a free group acting faithfully and level transitively on a rooted tree, then there exists a point of the boundary of the tree having trivial stabilizer in F; see [Abé07].

Even though the theorem itself is not very complicated, it gives a way to find simple proofs of absence of free subgroups in many groups acting on rooted trees. One has to show that the graphs of the action of the group on the boundary are so small that a free action of a free subgroup is not possible, and then to analyze the action of the elements of the group on neighborhoods of fixed points in order to show that the groups of germs of the action are also small and have no free subgroups.

In particular, we confirm the conjecture on contracting groups.

Theorem 4.2. *Contracting groups have no free subgroups.*

This theorem implies, for instance, that the iterated monodromy groups of post-critically finite rational functions and other expanding dynamical systems (see [Nek05], [BGN03]) have no free subgroups. It is an interesting open question if all contracting groups are amenable.

We also generalize (in Theorem 4.4) the fact that there are no free groups generated by bounded automorphisms of a rooted tree. For the notion of bounded automorphisms see Definition 4.3 of our paper and the articles [Sid00], [BN03], [BKN10]. It is known that groups generated by bounded automorphisms *defined by finite automata* have no free subgroups [Sid04] and that they are even amenable [BKN10]. Absence of free subgroups for the general case of bounded automorphisms is proved here for the first time. It is not known if they are all amenable.

We also give a shorter proof of the theorem of S. Sidki [Sid04] about automata of polynomial growth (only for the case of finite alphabets).

A very intriguing open question now is to see if all the groups covered by these theorems are amenable and to prove a theorem on amenability of groups acting on rooted trees similar to Theorem 3.3. The first more or less general result in this direction is the proof of amenability of groups generated by bounded automata in [BKN10].

2. Preliminaries on rooted trees

Here we recall the basic notions related to rooted trees and fix notation. The reader can find more on this in [BORT96], [Sid98], [BGŠ03], [GNS00].

A *rooted tree* is a tree with a fixed vertex called the *root* of the tree. We consider only locally finite trees in our paper, i.e., trees in which every vertex belongs to a finite number of edges.

We say that a vertex v of a tree T is *below* a vertex u if the path from the root of T to v goes through u. We denote by T_v the subtree of all vertices which are below v together with v serving as a root of T_v. See Figure 1.

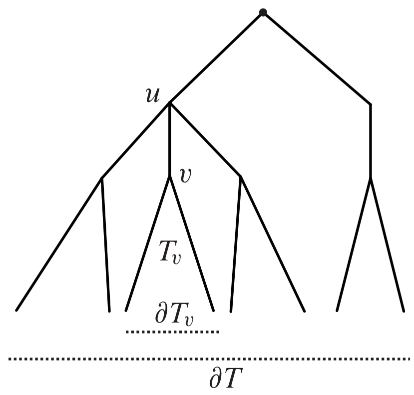

Figure 1. Rooted tree.

The *boundary* ∂T of the tree T is the set of simple infinite paths starting at the root of T. The boundary ∂T_v of a subtree T_v consists then of the paths going through the vertex v. The collection of subsets ∂T_v of ∂T is a basis of topology on ∂T. The topological space ∂T is compact and totally disconnected. If $\widetilde{T}$ is a subtree of T, then $\partial\widetilde{T}$ is a closed subset of ∂T in the natural way.

An automorphism of the rooted tree T is an automorphism of the tree T fixing the root vertex. Every automorphism of T fixes also the *levels* of the tree as sets. Here a *level number* n of the rooted tree T is the set L_n of the vertices on distance n from the root.

A rooted tree T is said to be *spherically homogeneous* if the automorphism group of T is transitive on the levels.

If T is spherically homogeneous, then ∂T is equipped with a natural probability measure m_T defined by the condition that measure of ∂T_v is equal to $1/|L_n|$, where L_n is the level of the vertex v. This is the unique probability measure invariant under the action of the automorphism group of T.

2.1. Trees of words and almost finitary automorphisms. Let X be a finite set, called *alphabet* and let X^* be the free monoid generated by X, i.e., the set of finite words $x_1x_2\dots x_n$ over the alphabet X, including the empty word $\varnothing$. The set X^* has a natural structure of a rooted tree, where a vertex $v \in X^*$ is connected to the vertices of the form vx and the empty word is the root.

We denote by vX^* the sub-tree of words starting by v, i.e., the sub-tree X_v^* of vertices below the vertex v.

The boundary ∂X^* is naturally homeomorphic to the set of infinite sequences $X^\omega = \{x_1x_2\dots : x_i \in X\}$ with the product topology (where X is discrete).

More generally, if

$$\mathsf{X} = (X_1, X_2, \dots)$$

is a sequence of finite sets, then we denote

$$\mathsf{X}^* = \bigcup_{n\geq 0} \mathsf{X}^n,$$

where $\mathsf{X}^n = X_1 \times X_2 \times \cdots \times X_n$ for $n \geq 1$ and $\mathsf{X}^0 = \{\varnothing\}$. We denote by $|v|$ the length of a word $v \in \mathsf{X}^*$, i.e., number such that $v \in \mathsf{X}^{|v|}$.

The set X^* is a rooted tree with the root $\varnothing$ in which an element $v \in \mathsf{X}^n$ is connected to all elements of the form vx for $x \in X_{n+1}$.

The boundary of the tree X^* is homeomorphic to the direct product

$$\mathsf{X}^\omega = X_1 \times X_2 \times \cdots$$

of discrete sets.

The invariant measure on X^ω coincides in this case with the *uniform Bernoulli* measure defined as the direct product of the uniform distributions on X_n.

We denote by X_n the sequence

$$\mathsf{X}_n = (X_{n+1}, X_{n+2}, \dots).$$

If g is an automorphism of the tree X^*, then for every $v \in \mathsf{X}^*$ there exists a unique automorphism $g|_v$ of the tree $\mathsf{X}_{|v|}^*$ such that

$$g(vu) = g(v)g|_v(u)$$

for all $u \in \mathsf{X}_{|v|}^*$.

It is easy to see that the following properties of this operation hold

$$(g_1g_2)|_v = g_1|_{g_2(v)}g_2|_v, \quad (g|_v)^{-1} = g^{-1}|_{g(v)}, \quad g|_{v_1v_2} = g|_{v_1}|_{v_2}. \tag{1}$$

2.2. Almost finitary automorphisms of X^*

Definition 2.1. Let g be an automorphism of the tree X^*. A sequence $w \in \mathsf{X}^\omega$ is *g-regular* if there exists a beginning $v \in \mathsf{X}^*$ of w such that $g|_v$ is trivial. We say that $w \in \mathsf{X}^\omega$ is *g-singular* if it is not g-regular.

The set of g-singular points of X^ω can be measured using the *growth* of the automorphism g.

Definition 2.2. Let g be an automorphism of the spherically homogeneous rooted tree X^*. Then its *growth function* is

$$\theta_g(n) = |\{v \in \mathsf{X}^n : g|_v \neq \mathrm{id}\}|.$$

If $T \subset \mathsf{X}^*$ is a *rooted* subtree of X^*, i.e., a sub-tree containing the root of X^*, then the *relative growth function* is

$$\theta_{g,T}(n) = |\{v \in L_n : g|_v \neq \mathrm{id}\}|;$$

and g is *almost finitary on T* if

$$\lim_{n\to\infty} \frac{\theta_{g,T}(n)}{|L_n|} = 0,$$

where $L_n = T \cap \mathsf{X}^n$ is the nth level of the tree T.

Proposition 2.1. *Let T be a spherically homogeneous rooted subtree of X^* and let g be an automorphism of X^*. Then*

$$\lim_{n\to\infty} \frac{\theta_{g,T}(n)}{|L_n|} = m_T(\Sigma_g),$$

where L_n is the nth level of the tree T and Σ_g is the set of g-singular points $w \in \partial T \subset \mathsf{X}^\omega$.

Proof. It is easy to see that the set of g-regular points $w \in \mathsf{X}^\omega$ is open, hence the set Σ_g is closed in ∂T. It follows from by definitions of $\theta_{g,T}$ and the measure m_T that the number $\frac{\theta_{g,T}(n)}{|L_n|}$ is equal to $m_T(\Sigma_{g,n})$, where

$$\Sigma_{g,n} = \bigcup_{v\in L_n, g|_v\neq \mathrm{id}} v\mathsf{X}^\omega \cap \partial T.$$

The sequence of the sets $\Sigma_{g,n}$ for $n = 1, 2, \ldots$ is decreasing and their intersection is Σ_g. This finishes the proof, by continuity of the measure. □

Recall that if we have a group G generated by a finite symmetric set S and acting on a set M, then the corresponding Schreier graph is the graph with the set of vertices M and the set of edges $S \times M$, where an edge $(s, x) \in S \times M$ starts in x and ends in $s(x)$.

A locally finite graph is said to be *amenable* if for every ϵ there exists a finite set F_ϵ (called *Følner set*) such that

$$\frac{|\partial F_\epsilon|}{|F_\epsilon|} < \epsilon,$$

where ∂F_ϵ is the set of edges beginning in F_ϵ and ending outside of F_ϵ. A regular tree, in particular the Cayley graph of a free non-abelian group, is an example of a *non*-amenable graph.

Proposition 2.2. *Let G be a finitely generated automorphism group of X^* and let $T \subset \mathsf{X}^*$ be a G-invariant rooted subtree on which G acts level transitively. If all elements of G are almost finitary on T, then all components of the Schreier graph of the action of G on ∂T are amenable.*

Proof. The proposition follows directly from a theorem of [GN05] (which in turn is a corollary of a result of V. Kaimanovich [Kai01]). But we prefer to give here a direct proof constructing the sets F_ϵ.

Fix a finite symmetric generating set S of G and consider the Schreier graph Γ_n of the action of G on the nth level L_n of the tree T.

Let Γ'_n be the subgraph of Γ_n which consists only of the edges (s, v) such that $s|_v$ is trivial. Note that if (s, v) belongs to Γ'_n, then the inverse edge $(s^{-1}, s(v))$ also belongs to Γ'_n. The number of edges in the difference $\Gamma_n \setminus \Gamma'_n$ is equal to $\sum_{s\in S} \theta_{s,T}(n)$.

Let $\Phi_1, \Phi_2, \ldots, \Phi_k$ be the connected components of Γ'_n (as sets of vertices). Then in Γ_n we have

$$|\partial\Phi_1| + |\partial\Phi_2| + \cdots + |\partial\Phi_k| \le \textstyle\sum_{s\in s} \theta_{s,T}(n),$$

$$|\Phi_1| + |\Phi_2| + \cdots + |\Phi_k| = |L_n|$$

hence there exists a component Φ_i such that

$$\frac{|\partial\Phi_i|}{|\Phi_i|} \le \frac{\sum_{s\in S} \theta_{s,T}(n)}{|L_n|}.$$

Consider an orbit O of the action of G on ∂T. Since the action of G on T is level-transitive, there exists $w \in O$ such that the beginning v_1 of length n of w belongs to Φ_i. Let $u \in \mathsf{X}_n^\omega$ be such that $w = v_1 u$. By the definition of Φ_i, for every $v \in \Phi_i$ there exists an element $g \in G$ such that $g(v_1) = v$ and $g|_{v_1} = \mathrm{id}$ (see the first two equalities of (1) in Section 2.1). Then $g(v_1 u) = vu$, i.e., the point vu belongs to the orbit O. Let

$$F_n = \{vu : v \in \Phi_i\} \subset O.$$

If the edge (s, v) from $v \in \Phi_i$ to $s(v)$ belongs to Γ'_n, then $s|_v = \mathrm{id}$, hence $s(vu)$ belongs to Φ_i. Consequently, in the Schreier graph of the action of G on O we have

$$\frac{|\partial F_n|}{|F_n|} \leq \frac{|\partial \Phi_i|}{|\Phi_i|} \leq \frac{\sum_{s \in S} \theta_{s,T}(n)}{|L_n|} \to 0,$$

as $n \to \infty$. This means that F_n are Følner sets, i.e., that the Schreier graph of O is amenable. □

3. Main theorem

We will use the following simple lemmata. Their proofs are known (see, for instance Proposition 1 of [Sid04] for Lemma 3.2), but we provide them for completeness.

Lemma 3.1. *Let F be a free non-abelian group and let $H < F$ be a cyclic subgroup. Then there exists a free non-abelian subgroup $\widetilde{F} < F$ such that $\widetilde{F} \cap H$ is trivial.*

Proof. We can find a subgroup of F freely generated by the generator h of H and two other elements g_1, g_2 of F (take, for instance, any element g which does not commute with h and then take the index 2 subgroup of the free group $\langle g, h\rangle$ generated by h, g^2 and $g^{-1}hg$). Then we can take $\widetilde{F} = \langle g_1, g_2\rangle$. □

Lemma 3.2. *Let F be a free non-abelian group and let*

$$\phi\colon F \to G = G_1 \times G_2 \times \cdots \times G_n$$

be a homomorphism into a finite direct product of groups. If every composition ϕ_i of ϕ with the projection $G \to G_i$ has a non-trivial kernel, then ϕ has a non-trivial kernel.

Proof. It is sufficient to prove the lemma for $n = 2$. The general statement will follow then by induction. Let r_1 and r_2 be non-trivial elements of the kernels of ϕ_1 and ϕ_2, respectively. If r_1 and r_2 belong to one cyclic subgroup of F, then there exist $m_1, m_2 \in \mathbb{Z}$ such that $r_1^{m_1} = r_2^{m_2}$ is a non-trivial element of the kernel of ϕ. If they do not belong to a common cyclic subgroup, then they do not commute and $[r_1, r_2]$ is a non-trivial element of the kernel of ϕ. □

Definition 3.1. Let G be a group acting on a topological space $\mathcal{X}$. The *group of G-germs* of a point $x \in \mathcal{X}$ is the quotient of the stabilizer of x in G by the subgroup of elements acting trivially on a neighborhood of x. We denote it by $G_{(x)}$.

Informally speaking, $G_{(x)}$ describes the action of the stabilizer G_x locally on neighborhoods of x.

See an application of groups of germs to growth of groups in [Ers04].

Theorem 3.3. *Let G be a group acting faithfully on a locally finite rooted tree T. Then one of the following holds.*

(1) *G has no free non-abelian subgroups.*
(2) *There is a free non-abelian subgroup $F < G$ and a point $w \in \partial T$ such that the stabilizer F_w is trivial.*
(3) *There is a point $w \in \partial T$ such that the group of G-germs $G_{(w)}$ has a free non-abelian subgroup.*

Proof. Suppose that on the contrary, G has a free subgroup F, the groups of G-germs $G_{(w)}$ for all $w \in \partial T$ have no free subgroups, and there is no free subgroup $\widetilde{F}$ and a point $w \in \partial T$ such that the stabilizer $\widetilde{F}_w$ is trivial.

For every point $w \in \partial T$ the stabilizer F_w is a free non-abelian subgroup since otherwise F_w is cyclic or trivial, and then we can find by Lemma 3.1 a free non-abelian subgroup $\widetilde{F} < F$ such that $\widetilde{F}_w = \widetilde{F} \cap F_w$ is trivial. But the group $G_{(w)}$ has no free subgroups, consequently the natural homomorphism $F_w \to G_{(w)}$ has a non-trivial kernel. This means that there exists a vertex v_w on the path w and a non-trivial element of F acting trivially on the sub-tree T_{v_w}.

We get a covering of ∂T by open subsets ∂T_{v_w}. The boundary of the tree is compact, hence there exists a finite sub-covering. This means that there exists a finite set of vertices V such that $\partial T = \bigcup_{v \in V} \partial T_v$ and for every $v \in V$ the group of the elements of F acting trivially on T_v is non-trivial, hence infinite.

Denote by F_V the intersection of the stabilizers of the vertices of V in F. It is a subgroup of finite index in F since the levels are finite sets. Consider the homomorphism

$$\phi \colon F_V \to \prod_{v \in V} \mathrm{Aut}(T_v)$$

mapping an automorphism $g \in F_V$ to the automorphisms of the trees T_v that it induces. Composition of ϕ with the projection onto $\mathrm{Aut}(T_v)$ has non-trivial kernel for every $v \in V$ since F_V has finite index in F and the group of elements of F acting trivially on T_v is infinite for each $v \in V$. Consequently, by Lemma 3.2, the kernel of the homomorphism ϕ is non-trivial. But this is not possible since we assume that F acts faithfully on ∂T and $\partial T = \bigcup_{v \in V} \partial T_v$. □

Example 1. Let F be a free group. Then there exists a descending series of finite index subgroups $F = G_0 > G_1 > G_2 > \cdots$ with trivial intersection $\bigcap_{n \geq 0} G_n$. Every such series defines an action of F on the *coset tree*. It is the rooted tree with the set of vertices equal to the union $\bigcup_{n \geq 0} F/G_n$ of the sets of cosets. The set F/G_n is the nth level of the tree and a vertex $gG_n \in F/G_n$ is connected by an edge with a vertex $hG_{n+1} \in F/G_{n+1}$ if and only if $hG_{n+1} \subset gG_n$. The group F acts then on the coset tree by the natural action on the cosets:

$$g \cdot (hG_n) = (gh)G_n.$$

The action is level transitive and the stabilizer of the path $G_0, G_1, G_2, \ldots$ is equal to the intersection of the subgroups G_n, i.e., is trivial.

It is not hard to prove (see, for instance, Proposition 4.4 of [LN02]) that every level transitive action of F on a rooted tree T for which there exists a path $w \in \partial T$ with trivial stabilizer is conjugate to the action on a coset tree.

Example 2. Let T be a rooted tree and $w = (v_0, v_1, \ldots) \in \partial T$ be an infinite simple path starting at the root, where v_i are the vertices that it goes through. Then the stabilizer of w in the $\mathrm{Aut}(T)$ is isomorphic to the Cartesian product of the automorphism groups of the sub-trees $\widetilde{T}_{v_i} = T_{v_i} \setminus T_{v_{i+1}}$ "hanging" from the path w. For any free group F consider a sequence of homomorphisms $\phi_i : F \to \mathrm{Aut}(\widetilde{T}_{v_i})$ such that the intersection of kernels $\bigcap_{i \geq 0} \ker \phi_i$ is trivial. Then we get a faithful action of F on the tree T for which w is a fixed point. If the intersection $\bigcap_{i \geq n} \ker \phi_i$ is trivial for every n, then the image of F in the group of germs $\mathrm{Aut}(T)_{(w)}$ is faithful.

4. Applications

4.1. Contracting groups.

Let $\mathsf{X} = (X, X, \ldots)$ be a constant sequence.

Definition 4.1. A *self-similar group* is a group $G < \mathrm{Aut}(\mathsf{X}^*)$ such that for every $g \in G$ and $v \in \mathsf{X}^*$ we have $g|_v \in G$.

It is sufficient to check that $g|_x \in G$ for every $x \in X$ and every generator g of G, due to the properties (1) in Section 2.1.

Definition 4.2. A self-similar group G of automorphisms of X^* is called *contracting* if there exists a finite set $\mathcal{N} \subset G$ such that for every $g \in G$ there exists n such that $g|_v \in \mathcal{N}$ for all words v of length at least n. The smallest set $\mathcal{N}$ satisfying this condition is called the *nucleus* of the contracting group.

It is proved in [Nek05], Proposition 2.13.8, that if a finitely generated group G is contracting then the growth of the components of the Schreier graph of the action of G on X^ω is polynomial.

Proposition 4.1. *If G is a contracting group then for every $w \in \mathsf{X}^\omega$ the group of germs $G_{(w)}$ is finite of cardinality not greater than the size of the nucleus.*

Proof. Let $A \subset G_w$ be a subset such that such that $|A| > |\mathcal{N}|$, where $\mathcal{N}$ is the nucleus of the group. There is n such that $g|_v \in \mathcal{N}$ for all $g \in A$ and all $v \in \mathsf{X}^n$. In particular, there exist $g, h \in A$ such that $g|_u = h|_u$, where u is the beginning of the length n of the word w. This means that $g^{-1}h$ acts trivially on the subtree $u\mathsf{X}^*$, i.e., that the images of g and h in $G_{(w)}$ are equal. Consequently, $G_{(w)}$ has no finite subsets of size more than $|\mathcal{N}|$, i.e., $|G_{(w)}| \leq |\mathcal{N}|$. □

Theorem 4.2. *Contracting groups have no free subgroups.*

Proof. We have to eliminate the possibilities (2) and (3) of Theorem 3.3. Note that it is sufficient to prove this theorem for finitely generated contracting groups since every finitely generated subgroup $H = \langle S \rangle$ of a contracting group is a subgroup of a finitely generated contracting group. One has to take the group generated by all elements of the form $s|_v$ for $s \in S$ and $v \in \mathsf{X}^*$.

Condition (2) cannot be true since the orbits of the action of a contracting group have polynomial growth.

Condition (3) is not possible by Proposition 4.1. □

Examples 3. Iterated monodromy groups (see definitions in [BGN03], [Nek05]) of expanding self-coverings of orbispaces, in particular, iterated monodromy groups of post-critically finite rational functions, are contracting hence have no free subgroups. In some cases (such as for the polynomial z^2+i, see [BP06]) the iterated monodromy groups have sub-exponential growth, which obviously implies that they have no free subgroups. In some other cases absence of free groups was proved separately using some contraction arguments. See for instance a proof of absence of free subgroups in the iterated monodromy group of $z^2 - 1$ in [GŻ02].

For more on contracting groups and their properties see the monograph [Nek05], especially Sections 2.11 and 2.13.

4.2. Groups generated by bounded automorphisms. Let X^* be a level-transitive rooted tree defined by a sequence

$$\mathsf{X} = (X_1, X_2, \dots)$$

of finite sets.

We say that an automorphisms g of X^* is *finitary* if there exists n such that $g|_v$ is trivial for all $v \in \mathsf{X}^n$. (Recall that $\mathsf{X}^n = X_1 \times \cdots \times X_n$.) The smallest n with this property is called the *depth* of g.

Note that the set of all automorphisms of depth at most n is a finite subgroup of $\mathrm{Aut}(\mathsf{X}^*)$. In particular, the set of all finitary automorphisms of X^* is a locally finite group.

Definition 4.3. An automorphism g of X^* is *bounded* if there exists a finite set of sequences $W = \{w_1, w_2, \dots, w_m\} \subset \mathsf{X}^\omega$ and a number n such that if $v \in \mathsf{X}^*$ is not a beginning of any sequence w_i then $g|_v$ is finitary of depth at most n. The number n is called the *finitary depth* of g and the set W is called the set of *directions* of g.

Informally, an automorphism $g \in \mathrm{Aut}(\mathsf{X}^*)$ is bounded if its activity is concentrated in strips of bounded width around a finite number of paths in X^*.

Note that if g is bounded, then the set Σ_g of g-singular points is a subset of the set of directions of g and the sequence $\theta_g(n)$ is bounded.

Proposition 4.3. *If g_1, g_2 are bounded automorphisms of X^* of finitary depth $\le n$, then g_1^{-1} and $g_1 g_2$ are bounded of finitary depth $\le n$.*

In particular, the set of bounded automorphisms of X^* is a group. Note that this group is uncountable.

Proof. It is a direct corollary of the properties (1) from Section 2.1 and the fact that the set of finitary automorphisms of depth $\le n$ is a group. □

Theorem 4.4. *The group of bounded automorphisms of a spherically homogeneous tree of bounded degree of vertices does not contain free non-abelian subgroups.*

Proof. Since the degree of vertices of the tree X^* is bounded, the sequence $(|X_i|)_{i\ge 1}$ is bounded.

Suppose that the group generated by bounded automorphisms contains a finitely generated free non-abelian group. Then by Theorem 3.3 either there exists a free group F generated by bounded automorphisms and having trivial stabilizer of a sequence $w \in \mathsf{X}^\omega$, or there exists a free group F fixing a point $w \in \mathsf{X}^\omega$ such that F acts faithfully on every neighborhood $v\mathsf{X}^\omega_{|v|}$ of w.

Suppose that the first case holds. Let T be the sub-tree of X^* such that $w \in \partial T$ and the free group F acts level transitively on T. (The tree T is the union of the F-orbits of beginnings of w.) The boundary ∂T has to be infinite, otherwise the stabilizer of $w \in \partial T$ is of finite index. But for every $g \in F$ the set of g-singular points of ∂T is finite, hence of zero measure. Consequently, by Proposition 2.1 and 2.2, the components of the Schreier graph of the action of F on ∂T are amenable, which contradicts with triviality of the stabilizer of w.

Suppose now that there exists w such that a free 2-generated group $F = \langle g, h\rangle$ fixes w and acts faithfully on neighborhoods of w. Let W_g and W_h be the sets of directions of g and h, respectively. Then $w \in W_g \cap W_h$ since otherwise there would exist a neighborhood $v\mathsf{X}^\omega_{|v|}$ of w such that $g|_v$ or $h|_v$ is trivial.

There exists a beginning u of w such that all elements of W_g and W_h except for w have beginning of length $|u|$ different from u. Then $g|_u$ and $h|_u$ are bounded automorphisms of $\mathsf{X}^*_{|u|}$ with one direction w', where $w = uw'$. They generate a free group, since F acts faithfully on $u\mathsf{X}^\omega_{|u|}$.

Let $w' = x_1x_2\dots$. Let m be a number greater than the finitary depths of $g' = g|_u$ and $h' = h|_u$. Then g' and h' may, for $y_{n+1} \ne x_{n+1}$, change in a word $x_1x_2\dots x_n y_{n+1} y_{n+2}\dots$ only the letters $y_{n+1}, y_{n+2}, \dots, y_{n+m}$ and cannot change y_{n+1} to x_{n+1}. Since the size of the alphabets X_i is uniformly bounded, this implies that the automorphisms g' and h' have finite order, which is a contradiction. □

Examples 4. It is proved in [BKN10] that groups generated by *finite-state* (see Definition 4.4) bounded automorphisms of a rooted tree X^* are amenable, which implies that they have no free subgroups. Some examples of non-finite-state groups of bounded automorphisms of X^* are studied in [Gri85] and [Nek07].

4.3. Theorem of S. Sidki. Let again $\mathsf{X} = (X, X, \dots)$ be a constant sequence of finite alphabets.

Definition 4.4. An automorphism g of the tree X^* is said to be *finite-state* if the set

$$\{g|_v : v \in \mathsf{X}^*\} \subset \mathrm{Aut}(\mathsf{X}^*)$$

is finite.

The set of all finite state automorphisms of the tree X^* is a countable group. A version of the growth function $\theta_g(n)$ (given in Definition 2.2) was defined and studied by S. Sidki in [Sid00]. It follows from the results of [Sid00] that if g is finite state then the function $\theta_g(n)$ has either exponential or polynomial growth. The set $P_d(\mathsf{X})$ of all finite state automorphisms of X^* for which $\theta_g(n)$ is bounded by a polynomial of degree d is a group. Later in [Sid04] S. Sidki proved that the groups $P_d(\mathsf{X})$ have no free subgroups (actually, his theorem is more general since it also covers some cases of infinite alphabet X). Let us show how his theorem (for the case of finite alphabet) follows from Theorem 3.3.

The following inductive description of elements of $P_d(\mathsf{X})$ is given in [Sid00]. We denote by $P_{-1}(\mathsf{X})$ the group of finitary automorphisms of X^*, i.e., the automorphisms g for which the sequence $\theta_g(n)$ is eventually zero.

Proposition 4.5. *Let d be a non-negative integer. A finite-state automorphism g of X^* belongs to $P_d(\mathsf{X})$ if and only if there exists a finite number of almost periodic words $W = \{v_i u_i^{\omega}\} \subset \mathsf{X}^{\omega}$ such that the restrictions $g_{i,n} = g|_{v_i u_i^n}$ do not depend on n, and $g|_v \in P_{d-1}(\mathsf{X})$ if v is not a beginning of any $v_i u_i^{\omega}$.*

Corollary 4.6. *Let $G < P_d(\mathsf{X})$ be finitely generated. Then every component of the Schreier graph of the action of G on X^{ω} is amenable.*

It seems that the Schreier graphs of subgroups of $P_d(\mathsf{X})$ acting on X^{ω} have sub-exponential growth. That would of course imply their amenability. See, for instance the graphs considered in [CSFS04], [BH05], which are Schreier graphs of a sub-group of $P_1(\{0, 1\})$ (see Example 5).

Proof. Let us show that for every $g \in P_d(\mathsf{X})$ the set of g-singular points of X^{ω} is at most countable. We argue by induction on d. If $g \in P_{-1}(\mathsf{X})$, i.e., if g is finitary, then the set of g-singular points is empty. Suppose that for every $g \in P_{d-1}(\mathsf{X})$ the set of g-singular points is at most countable. Take arbitrary $g \in P_d(\mathsf{X})$. Then by Proposition 4.5, there is a finite set $W \subset \mathsf{X}^{\omega}$ such that $g|_v \in P_{d-1}(\mathsf{X})$ for every $v \in \mathsf{X}^*$ which is not a beginning of an element of W. Then the set Σ_g of g-singular points of X^{ω} is a subset of

$$W \cup \bigcup_{\substack{v\in\mathsf{X}^* \text{ is not a} \\ \text{beginning of any} \\ w\in W}} v\Sigma_{g|_v}.$$

The sets $\Sigma_{g|_v}$ are at most countable by the inductive hypothesis. Hence Σ_g is at most countable.

Let G be a group generated by a finite set $S \subset P_d(\mathsf{X})$. Let T be a G-invariant rooted subtree of X^* on which G acts level transitively. Then the measure space $(\partial T, m_T)$ is either a finite set, or it is isomorphic to the standard Lebesgue space. In the first case the Schreier graph of the action of G on ∂T is finite. In the second case the components of the Schreier graph of the action of G on ∂T are amenable by Propositions 2.1 and 2.2 since the sets of g-singular points of ∂T for $g \in S$ are at most countable, hence have zero measure. □

Theorem 4.7 (S. Sidki). *The group $P_d(\mathsf{X})$ has no free subgroups.*

Proof. Note that $P_{-1}(\mathsf{X})$ is locally finite, hence has no free subgroups. The case of $P_0(\mathsf{X})$ is covered by Theorem 4.4.

Condition (2) of Theorem 3.3 cannot hold for $P_d(\mathsf{X})$ due to Corollary 4.6.

Let us investigate now the groups of germs in $P_d(\mathsf{X})$. Assume that we have proved that $P_{d-1}(\mathsf{X})$ has no free subgroups. Suppose that elements a and b of $P_d(\mathsf{X})$ generate a free subgroup of the group of $P_d(\mathsf{X})$-germs of $w \in \mathsf{X}^\omega$. It follows from Proposition 4.5 that w can be represented in the form $w = v(u^\omega)$ for some finite words v and u so that $a|_{vu} = a|_v$, $b|_{vu} = b|_v$ and $a|_v$ and $b|_v$ do not move u. Moreover, $a|_{vu^n u'}, b|_{vu^n u'} \in P_{d-1}(\mathsf{X})$ for $u' \in \mathsf{X}^*$ which are not beginnings of the word u^ω. Since a and b generate a free group of germs, their restrictions $a_1 = a|_v$ and $b_1 = b|_v$ onto $v\mathsf{X}^*$ generate a free group.

Since $a_1|_u = a_1$ and $b_1|_u = b_1$, restrictions of the action of a_1 and b_1 onto $\mathsf{X}^* \setminus u\mathsf{X}^*$ generate a free group F (everything is periodic along the path $uuu\ldots$). Let $u = x_1x_2\ldots x_m$ and let $U \subset \mathsf{X}^*$ be the set of words of the form $x_1x_2\ldots x_k y_{k+1}$, where $y_{k+1} \neq x_{k+1}$ and $0 \le k \le m-1$, i.e., the set of vertices adjacent to the path from the root to u. Denote by $\widetilde{F}$ the intersection of the stabilizers in F of the elements of U. See Figure 2, where the elements of U are circled.

Then $\widetilde{F}$ is a subgroup of finite index in F and we get a monomorphism

$$\phi\colon \widetilde{F} \to (\mathrm{Aut}(\mathsf{X}^*))^U$$

mapping g to $(g|_r)_{r \in U}$. The homomorphism ϕ takes values in $(P_{d-1}(\mathsf{X}))^U$, hence it has a non-trivial kernel on each coordinate. Then, by Lemma 3.2, the homomorphism ϕ has a non-trivial kernel, which is a contradiction. □

Example 5. Consider the permutations a and b of the set of integers given by

$$a(n) = n + 1$$

and

$$b(0) = 0, \quad b(2^k(2n+1)) = 2^k(2n+3)$$

for $k \ge 0$ and $n \in \mathbb{Z}$. Let G be the group generated by a and b.

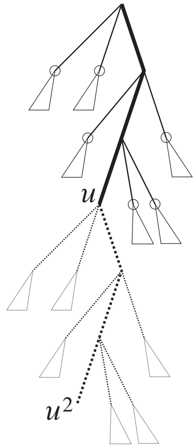

Figure 2. The set U and the path u^ω.

It is easy to see (using the binary numeration system on $\mathbb{Z}$) that G is isomorphic to the group of automorphisms of the binary rooted tree $\{0, 1\}^*$ given by the recurrent rules

$$a(0v) = 1v, \qquad a(1v) = 0a(v)$$

and

$$b(0v) = 0b(v), \quad b(1v) = 1a(v).$$

A direct check (for instance using Proposition 4.5) shows that $a \in P_0(\{0, 1\})$ and $b \in P_1(\{0, 1\})$, hence the group G has no free subgroups by theorem of S. Sidki.

The Schreier graphs of the action of G on X^ω coincide with the graphs considered by T. Ceccherini-Silberstein, F. Fiorenzi, F. Scarabotti [CSFS04] and I. Benjamini, C. Hoffman [BH05] and have intermediate growth.

It was proved recently by G. Amir, O. Angel, and B. Virag [AAV09] that the group $P_1(\mathsf{X})$ is amenable for every finite alphabet X. This implies, in particular, that the group from the last example is amenable. Amenability of the groups $P_d(\mathsf{X})$ for d greater than 1 remains to be open.

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V. Nekrashevych, Department of Mathematics, Texas A&M University, MS-3368, College Station, TX 77843-3368, U.S.A.

E-mail: nekrash@math.tamu.edu