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Groups, Geometry, and Dynamics

Universal diagram groups with identical Poincaré series

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Abstract. For a diagram group G, the first derived quotient G_1/G_2 is always free abelian (as proved by M. Sapir and V. Guba). However the second derived quotient G_2/G_3 may contain torsion. In fact, we show that for any finite or countably infinite direct product of cyclic groups A, there is a diagram group with second derived quotient A. We use that to construct families with the properties of the title.

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1. Introduction

A *rewriting system* $\mathcal{K}$ is a pair $\langle X\,;\Phi\rangle$, where X is a set (alphabet), and each $f \in \Phi$ is a pair of words on X, written as $f_{+1} = f_{-1}$. The elements of Φ are called *rewriting rules*. It is allowed that f_{+1} and f_{-1} are the same word. It is also allowed that for two rewriting rules f, g the words f_{+1}, g_{+1} are the same, and the words f_{-1}, g_{-1} are the same. For example, the rewriting system $\langle x; x^2 = x, x = x, x = x\rangle$ has three rules.

Associated with $\mathcal{K}$ is a (combinational) 2-complex $\mathcal{C}(\mathcal{K})$. The underlying 1-skeleton is a graph in the sense of Serre [8]: the vertices are all the words on X (i.e., X^*); the edges are triples of the form $(u, f_{+\varepsilon} \to f_{-\varepsilon}, v)$ $(u, v \in X^*, f \in \Phi, \varepsilon \pm 1)$; the inverse e^{-1} of the edge $e = (u, f_{+\varepsilon} \to f_{-\varepsilon}, v)$ is $(u, f_{-\varepsilon} \to f_{+\varepsilon}, v)$. Note that if f_{+1} and f_{-1} are the same word, we will still regard e^{+1} and e^{-1} as different edges. The 2-cells are the quintuples (u, f, v, g, w) $(u, v, w \in X^*, f, g \in \Phi)$. The boundary of the 2-cell is the closed path $(uf_{+1}v, g, w)(u, f, vg_{-1}w)(uf_{-1}v, g, w)^{-1}$ $(u, f, vg_{+1}w)^{-1}$. The connected components of $\mathcal{C}(\mathcal{K})$ are in one-to-one correspondence with the elements of the semigroup defined by the rewriting system $\mathcal{K}$. The *diagram group* $\mathcal{D}(\mathcal{K}, v)$ of $\mathcal{K}$ is the fundamental group of the connected component of $\mathcal{C}(\mathcal{K})$ with v as the basepoint.

These groups have been studied comprehensively by V. Guba and M. Sapir in the monograph [4], and the papers [6], [7]. Other interesting work about these groups can be found in [1], [2], [3], [5], [9]. There is a subclass of the class of diagram groups which is of interest, namely those which are *universal*. These groups have the property that they contain a copy of every countable diagram group (see [6], §5).

For any group G we have the lower central series: $G_1 = G$, and inductively, $G_{n+1} = [G_n, G]$. It is shown in [4], §11, that for a diagram group D, D_1/D_2 is always free abelian. Since "diagram groups can be considered as 2-dimensional versions of free groups" [6], p. 2, line 11, it is reasonable to think that D_2/D_3, would also be free abelian. However, we will show that for any abelian group A which is a finite or countably infinite direct product of cyclic groups, there is a diagram group D such that D_2/D_3 is isomorphic to A (Theorem 1).

For diagram groups arising from *finite complete* rewriting systems, the form of the canonical (minimal) presentation makes it easy to compute D_2/D_3. We will use this to exhibit, for any k, a family of universal diagram groups $\{D(i) : i = 1, 2, \ldots, k\}$ which have very similar canonical presentations and the same integral homology, but which can be separated via their second derived quotients (Theorems 2, 3).

Throughout, a^b will mean $b^{-1}ab$, and $[a,b]$ will mean $a^{-1}b^{-1}ab$.

For any elements $a, b \in G$ we write $\langle a, b\rangle$ for the element $[a,b]G_3 \in G_2/G_3$. From the commutator calculus we have for any $a, b, c \in G$:

(i) $\langle a, bc\rangle = \langle a, b\rangle\langle a, c\rangle$;
(ii) $\langle a, b\rangle = 1$ if a or b is in G_2;
(iii) $\langle a, b\rangle = \langle b, a\rangle^{-1}$;
(iv) $\langle a^{-1}ba, c\rangle = \langle b, c\rangle = \langle b, aca^{-1}\rangle$;
(v) $\langle a^{-1}, b\rangle = \langle a, b\rangle^{-1} = \langle a, b^{-1}\rangle$.

These formulae will be used without further mention.

We will make use of the method (and notation) found in [6], pp. 25–26, (see also [4], p. 53) for computing minimal presentations of diagram groups arising from complete rewriting systems. This requires the use of left and right forests of $\mathcal{C}(\mathcal{K})$. For our situation there will be a unique left (respectively right) forest consisting of the edges $(u, f_{+1} \to f_{-1}, v)^{\pm 1}$, where $f \in F$, f_{+1} and f_{-1} are distinct words, and every proper prefix of uf_{+1} (respectively every proper suffix of $f_{+1}v$)) is irreducible (see [6], Lemma 6.3, Remark 6.4).

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2. The second derived quotients of some diagram groups

Let F be the free group with basis X, linearly ordered by $>$. The group F_2/F_3 is a free abelian group with free basis the elements $\langle x, x'\rangle$ $(x, x' \in X, x > x')$.

Let $G = F/N$, where $N \lhd F_2$. Then $G_2 = F_2/N$, and $G_3 = F_3N/N$. Then

$$G_2/G_3 = (F_2/N)/(F_3N/N) \cong F_2/(F_3N) \cong (F_2/F_3)/(F_3N/F_3).$$

Suppose that N is the normal closure of a set $R \subset F_2$. Then F_3N/F_3 is generated as a *group* by the elements rF_3 ($r \in R$). For N is generated as a group by elements of the form $u^{-1}ru$ ($u \in F, r \in R$), so F_3N/N is generated as a group by the elements $u^{-1}ruF_3 = (rF_3)([r,u]F_3) = (rF_3)$. Thus if we can express each element rF_3 as an expression ξ_r in terms of the basis elements $\langle x, x'\rangle$, then G_2/G_3 is isomorphic to the free abelian group on the elements $\langle x, x'\rangle$ ($x, x' \in X$, $x > x'$) factored by the subgroup generated by the elements ξ_r ($r \in R$). Here is a couple of examples of how this works.

Example 1 (Generalised Thompson groups). Denote $\mathcal{P} = \mathcal{P}_n = \langle y; y^n = y\rangle$ ($n > 1$). As shown in [4], pp. 56–57, the diagram group $G = \mathcal{D}(\mathcal{P}, y)$ has minimal presentation with generators $z_0, z_1, \ldots, z_{n-1}$, and defining relators as follows:

(i) $r_{i,k} = [z_k, z_0]^{-1}[z_k, z_i]$, $1 \le i < k \le n-1$;

(ii) $s_{i,k} = [z_k, z_0^2]^{-1}[z_k, z_0 z_i]$, $1 \le i, k \le n-1$, $k-1 \le i$;

(iii) $p = [z_1, z_0^3]^{-1}[z_1, z_0^2 z_{n-1}]$.

Then in F_2/F_3 we have

(i′) $\xi_{r_{i,k}} = \langle z_k, z_0\rangle^{-1}\langle z_k, z_i\rangle$, $1 \le i < k \le n-1$;

(ii′) $\xi_{s_{i,k}} = \langle z_k, z_0\rangle^{-2}\langle z_k, z_0\rangle\langle z_k, z_i\rangle = \langle z_k, z_0\rangle^{-1}\langle z_k, z_i\rangle$, $1 \le i,k \le n-1$, $k-1 \le i$;

(iii′) $\xi_p = \langle z_1, z_0\rangle^{-3}\langle z_1, z_0\rangle^2\langle z_1, z_{n-1}\rangle = \langle z_1, z_0\rangle^{-1}\langle z_1, z_{n-1}\rangle$.

Let K be the subgroup generated by (i′), (ii′), (iii′). For $2 \le k \le n-1$ (and writing $=_K$ for equal modulo K) we have from (i′):

$$\langle z_k, z_0\rangle =_K \langle z_k, z_i\rangle \quad (1 \le i < k).$$

Taking $i = k$ in (ii′) we have

$$\langle z_k, z_0\rangle =_K \langle z_k, z_k\rangle =_K 1 \quad (1 \le i < k).$$

Thus, for $2 \le k \le n-1$, we have $\langle z_k, z_i\rangle =_K 1$ ($0 \le i < k \le n-1$). Then from (iii′) we have $\langle z_1, z_0\rangle =_K \langle z_{n-1}, z_1\rangle^{-1}$, and since from above $\langle z_{n-1}, z_1\rangle =_K 1$, we obtain $\langle z_1, z_0\rangle =_K 1$. Thus G_2/G_3 is trivial.

Example 2 (The •-product of Guba/Sapir). Let G, H be groups and let Z be an infinite cyclic group generated by z. Then $G \bullet H$ is the free product $G * H * Z$ factored out by the normal closure of $\{[g^{z^n}, h] : g \in G,\ h \in H,\ n = 0, 1, 2, \ldots\}$. This product is closed on the class of diagram groups [4], Theorem 8.6. The diagram group $\mathbb{Z} \bullet \mathbb{Z}$ is finitely generated but not finitely presented [4], Theorem 10.5. Let $G \cong F/N$ and $H \cong \hat{F}/M$, where F is free on X, $\hat{F}$ is free on Y, and M, N are the normal closures of sets $R \subseteq F_2$, $S \subseteq \hat{F}_2$, respectively. Then $P = G \bullet H$ has group presentation

$$\langle X, Y, z; R, S, [w^{z^n}, v]\ (w \in F,\ v \in \hat{F},\ n = 0, 1, 2, \ldots)\rangle.$$

Thus P_2/P_3 is the free abelian group on the generators $\langle x, x'\rangle$ $(x, x' \in X, x > x')$, $\langle y, y'\rangle$ $(y, y' \in Y, y > y')$, $\langle x, y\rangle$ $(x \in X, y \in Y)$, $\langle x, z\rangle$ $(x \in X)$, $\langle y, z\rangle$ $(y \in Y)$, with relators ξ_r $(r \in R)$, ξ_s $(s \in S)$, $\xi_{[w^{z^n}, v]}$ $(= \xi_{[w,v]})$ $(w \in F, v \in \widehat{F}, n = 0, 1, 2, \dots)$. Thus

$$P_2/P_3 \cong G_2/G_3 \times H_2/H_3 \times A(X) \times A(Y),$$

where $A(X)$, $A(Y)$ are the free abelian groups on the generating sets $\{\langle x, z\rangle : x \in X\}$, $\{\langle y, z\rangle : y \in Y\}$, respectively.

Theorem 1. *For any abelian group A which is a finite or countably infinite direct product of cyclic groups, there is a diagram group G such that $G_2/G_3 \cong A$.*

Proof. Consider the complete rewriting system

$$\mathcal{K} = \mathcal{K}_{n,m} = \langle a, y, c; h, f, g\rangle,$$
$$h\colon ay = a, \quad f\colon y^n = y^m, \quad g\colon yc = c,$$

$0 < m \le n$. The generators of the groups $G(n, m) = \mathcal{D}(\mathcal{K}, ac)$ are $z = (a, f, c)$ and $t = (a, g, 1)$, and the defining relators r_i are coded by the quintuples (a, f, y^i, f, c), $0 \le i < n$. Then according to [6], p. 25, r_i is the word

$$[a, f, y^{i+n}c]^{-1}[a, f, y^{i+m}c]^{[a,f,c]}.$$

Now the negative edge from the right tree that is assigned to y^pc is $(y^{p-1}, g, 1)$. Thus

$$[a, f, y^pc] = [a, f, y^{p-1}c]^{[a,g,1]},$$

and so by induction $[a, f, y^pc]$ is the word z^{t^p}. Hence r_i is the word

$$t^{-(i+n)}z^{-1}t^{i+n}z^{-1}t^{-(i+m)}zt^{(i+m)}z.$$

Now r_i is freely equivalent to $[t^{i+n}, z]z^{-1}[z, t^{i+m}]z$ and thus freely equivalent to $[z, t^{i+n}]^{-1}[z, t^{i+m}][[z, t^{i+m}], z]$. It thus follows that $r_iF_3 = \langle z, t\rangle^{m-n}$, so $G(n, m)_2/G(n, m)_3$ is cyclic of order $n - m$ (infinite cyclic if $n - m = 0$).

Now the class of diagram groups is closed under countable direct products [7], Theorem 2.5. Also, if $G = \prod_{j \in J} G(j)$, then $G_2/G_3 \cong \prod_{j \in J} G(j)_2/G(j)_3$. So using the groups above we obtain our result. □

3. Some universal diagram groups

In this section we will need to regard rewriting systems as *directed* 2-*complexes* with one vertex (see [6], §1 and Theorem 4.3). Also, the material on *morphisms* of directed 2-complexes, [6], p. 17, and the concept of a *universal* directed 2-complex, [6], p. 18, will be needed.

In [6], Theorem 5.5, it was shown that $\mathcal{H} = \langle x; x = x,\ x^2 = x\rangle$ is a universal directed 2-complex. The authors also showed that there is a non-singular map from $\mathcal{H}$ into $\mathcal{V}_{3,2} = \langle y; y^3 = y^2\rangle$, and thus $\mathcal{V}_{3,2}$ is universal [6], Theorem 5.6. The proof can be generalised:

Theorem 2. *$\mathcal{V}_{n,m} = \langle y; y^n = y^m\rangle$ $(n > m > 1)$ is universal for $\frac{3n}{5} < m \le \frac{5n}{7}$.*

Proof. Consider the map

$$\phi\colon \mathcal{H} \to \mathcal{V} = \mathcal{V}_{n,m},$$

where the edge x is sent to the path $y^{4(n-m)}$, and the two 2-cells $f\colon x = x$ and $g\colon x^2 = x$ are sent respectively to the 2-paths A, B in Diagram 1 in Figure 1 (where a directed line labelled by a positive integer k is shorthand for the directed path y^k, and where $\varepsilon = 0$ if m is even, and $\varepsilon = 1$ if m is odd). The proof is to check that for any reduced diagram Δ over $\mathcal{H}$, when we fill in each f-cell by a copy of A, and fill in each g-cell with a copy of B, no dipole can arise.

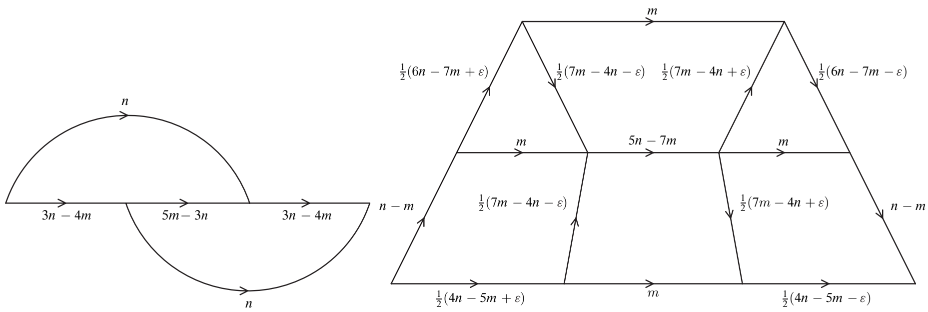

Figure 1. Diagram 1

(1) The only interaction of two f 2-cells in a reduced $\mathcal{H}$ diagram is if the two copies of f are both positive, or both are negative. Assuming they are both positive (the negative case is similar), then when these cells are filled in by copies of A, the potential pair (shaded) is not a dipole (see the diagram in Figure 2).

(2) The possible interactions of two g 2-cells in a reduced diagram over $\mathcal{H}$ are as in Diagram 3 in Figure 3 (up to rotation by π around the central line; or, by reflection in a mirror perpendicular to the central line and the arrows reversed). When these cells are filled in by the appropriate copies of B, the only potential pairs for dipoles (shaded in the diagram) cannot be such.

(3) Clearly, no dipole can arise from an interaction between an A diagram and a B diagram. □

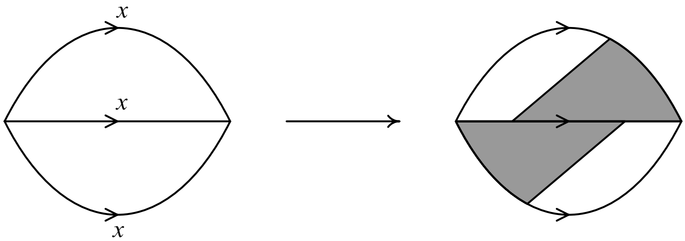

Figure 2. Diagram 2

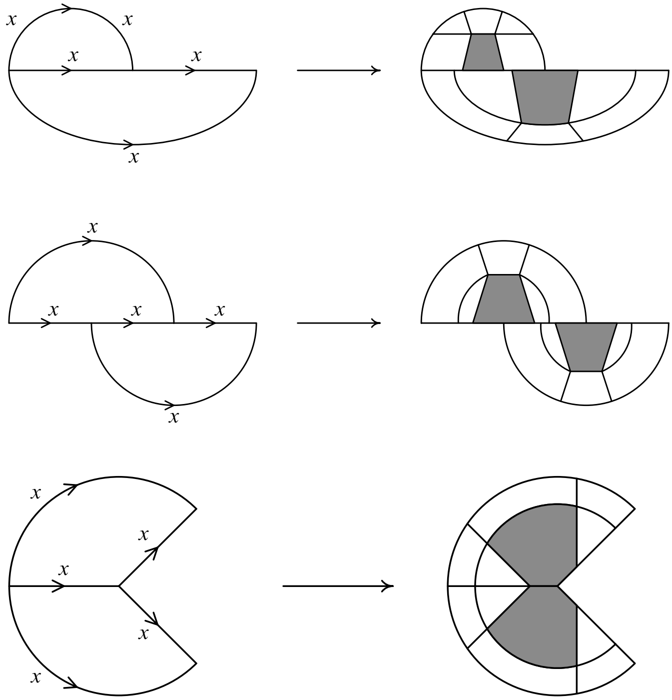

Figure 3. Diagram 3

Theorem 3. *For a fixed n there is a family of diagram groups* $G(n, m)$ $(\frac{3n}{5} < m \leq \frac{5n}{7})$ *with the following properties:*

(i) *The groups are universal.*

(ii) *Their minimal presentations each have the same number of generators, and each have the same number of defining relators.*

(iii) *The groups are all of type* FP_∞*, and all have the same Poincaré series.*

However, no two groups in the family are isomorphic.

Proof. By [6], Lemma 5.1 (2), the inclusion

$$\varphi\colon \mathcal{V}_{n,m} \to \mathcal{K}_{n,m}$$

is non-singular. Thus we have the induced group homomorphism

$$\varphi_*\colon \mathcal{D}(\mathcal{V}_{n,m}, y^{4(n-m)}) \to \mathcal{D}(\mathcal{K}_{n,m}, y^{4(n-m)}),$$

which is injective. Then, by [6], Corollary 3.6,

$$\mathcal{D}(\mathcal{K}_{n,m}, a) \times \mathcal{D}(\mathcal{K}_{n,m}, y^{4(n-m)}) \times \mathcal{D}(\mathcal{K}_{n,m}, c)$$

is imbedded into $\mathcal{D}(\mathcal{K}_{n,m}, ay^{4(n-m)}c)$. By [6], Corollary 3.5, $\mathcal{D}(\mathcal{K}_{n,m}, ay^{4(n-m)}c)$ is isomorphic to $\mathcal{D}(\mathcal{K}_{n,m}, ac)$. Thus $G(n,m) = \mathcal{D}(\mathcal{K}_{n,m}, ac)$ is universal.

Note that for a fixed n, the minimal presentations of the groups $\mathcal{D}(\mathcal{K}_{n,m}, ac)$ $(1 \leq m < n)$ all have the same number of generators (namely 2) and the same number of relators (namely n). See Theorem 1.

Also, for the minimal free resolution (see [6], Theorem 9.2)

$$0 \leftarrow \mathbb{Z} \leftarrow F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \cdots \leftarrow F_p \leftarrow \cdots$$

the basis of F_p is:

(i) (ac) if $p = 0$;
(ii) $(a, f, c), (a, g, 1)$ if $p = 1$;
(iii) (a, f, y^i, f, c) $(1 \leq i < n)$ and $(a, f, y^{n-1}, g, 1)$ if $p = 2$;

and

(iv) for $p \geq 3$ the basis is:

$$(a, f, y^{i_1}, f, y^{i_2}, \ldots, y^{i_{p-1}}, f, c), \quad 1 \leq i_k \leq n-1,\ 1 \leq k \leq p-1,$$

and

$$(a, f, y^{i_1}, f, y^{i_2}, \ldots, y^{i_{p-2}}, f, y^{n-1}, g, 1), \quad 1 \leq i_k \leq n-1,\ 1 \leq k \leq p-2.$$

Thus the groups have the same Poincaré series, namely

$$\begin{aligned} P(t) &= 1 + 2t + nt^2(1 + (n-1)t + ((n-1)t)^2 + ((n-1)t)^3 + \cdots) \\ &= 1 + 2t + \frac{nt^2}{1-(n-1)t}. \end{aligned}$$

However, as shown in §2, $G(n,m)_2/G(n,m)_3 \cong \mathbb{Z}_{n-m}$ $(1 \leq m \leq n)$, so for a fixed n, the groups $G(n,m)$ $(1 \leq m \leq n)$ are distinct. □

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