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**Groups, Geometry, and Dynamics**


# The Recognition Theorem for Out($F_n$)

Mark Feighn[1] and Michael Handel[2]

**Abstract.** Our goal is to find dynamic invariants that completely determine elements of the outer automorphism group Out($F_n$) of the free group $F_n$ of rank $n$. To avoid finite order phenomena, we do this for *forward rotationless* elements. This is not a serious restriction. For example, there is $K_n > 0$ depending only on $n$ such that, for all $\phi \in \mathrm{Out}(F_n)$, $\phi^{K_n}$ is forward rotationless. An important part of our analysis is to show that rotationless elements are represented by particularly nice relative train track maps.

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## Contents



## 1. Introduction

The Thurston classification theorem for mapping class groups of surfaces inspired a surge in research on the outer automorphism group Out($F_n$) of the free group of rank $n$. One direction of this research is the development and use of relative train track maps which are the analog of the normal forms for mapping classes. Thurston's normal forms give rise to invariants that completely determine a mapping class, perhaps after

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passing to a finite power. In this paper we provide similar invariants for elements of $\mathrm{Out}(F_n)$ and we add an important feature to relative train track maps.

We begin by recalling the invariants associated to a mapping class. After passing to a finite power, there is a decomposition of the surface into invariant subsurfaces on which the action of the mapping class is either a Dehn twist in an annulus, trivial or pseudo-Anosov; in the pseudo-Anosov case, each singular ray of the associated measured foliations is invariant. The mapping class is completely determined by the (isotopy classes of the) core curves of the annuli, the Dehn twist degrees, the pseudo-Anosov measured foliations and the expansion factors on the pseudo-Anosov measured foliations.

To see how this might generalize to $\mathrm{Out}(F_n)$, consider the special case in which the mapping class $\psi$ is a composition of disjoint Dehn twists and so is determined by the twisting circles and the degrees of twist. The dual point of view is useful here. The complementary components of the twisting curves are maximal subsurfaces on which $\psi$ acts trivially. Viewing the mapping class group as the group of outer automorphism of the fundamental group $\pi_1(S)$ of the surface $S$, each complementary component determines, up to conjugacy, a subgroup of $\pi_1(S)$ of rank at least two that is the fixed subgroup $\mathrm{Fix}(\Psi)$ of some automorphism $\Psi$ of $\pi_1(S)$ that represents the outer automorphism $\psi$. If $\Psi_1$ and $\Psi_2$ are two distinct such automorphisms, corresponding to either the same or distinct subsurfaces, then $\mathrm{Fix}(\Psi_1) \cap \mathrm{Fix}(\Psi_2)$ is either trivial or a maximal cyclic group $\langle a \rangle$. In the latter case $\Psi_2 = i_a^d \Psi_1$ for some $d \neq 0$ where $i_a$ is the inner automorphism determined by $a$; moreover, the conjugacy class determined by $a$ represents a twisting curve for $\mu$ with twisting degree $\pm d$.

This point of view, focusing on fixed subgroups of rank at least two and their intersections, is sufficient [6] if one restricts to elements of $\mathrm{Out}(F_n)$ that have linear growth.[1] For general $\phi \in \mathrm{Out}(F_n)$, we must also account for exponential and non-linear polynomial growth. To do this, we consider the boundary of the free group. An automorphism $\Phi$ induces a homeomorphism $\hat{\Phi}\colon \partial F_n \to \partial F_n$. In the general case there are isolated points, and these contain essential information about the automorphism. For example, the set of attracting laminations associated to $\phi$ can be recovered from the isolated attracting points.

The idea then is to replace $\mathrm{Fix}(\Phi)$ with the set $\mathrm{Fix}_N(\hat{\Phi})$ of non-isolated points and attractors in $\mathrm{Fix}(\hat{\Phi})$; see [12], [14], [3] and [15] where this same idea has been used effectively. In Section 3.2, we define the set $\mathrm{P}(\phi) \subset \mathrm{Aut}(F_n)$ of *principal automorphisms* representing $\phi$. In the case of linear growth, $\Phi$ is principal if and only if $\mathrm{Fix}(\Phi)$ has rank at least two. The invariants that determine $\phi$, after possibly passing to a finite power, are the sets $\mathrm{Fix}_N(\hat{\Phi})$ as $\Phi$ varies over $\mathrm{P}(\phi)$, the expansion factors for the attracting laminations of $\phi$ and twisting coordinates associated to pairs of principal automorphisms whose fixed points sets intersect non-trivially.

[1] $\phi \in \mathrm{Out}(F_n)$ has *linear growth* if, for all conjugacy classes of elements $a \in F_n$, the cyclically reduced word length of the conjugacy class of $\phi^m(a)$ is bounded by a linear function (depending on $a$) of $m$.

It is common when studying elements $\phi \in \mathrm{Out}(F_n)$ to 'stabilize' $\phi$ by replacing it with a power $\phi^k$. In Section 3, we specify a subset of $\mathrm{Out}(F_n)$ whose elements require no stabilization. These outer automorphisms are said to be *forward rotationless*. In Lemma 4.42 we prove that there is $K_n > 0$, depending only on $n$, so that $\phi^{K_n}$ is forward rotationless for all $\phi \in \mathrm{Out}(F_n)$. We also define what it means for a relative train track map $f : G \to G$ to be *rotationless* and prove (Proposition 3.29) that $\phi$ is forward rotationless if and only if some (every) relative train track map representing it is rotationless. (There is no need to add 'forward' to this terminology because $f^k$ is only defined for $k \geq 1$.) It is easy to check if $f : G \to G$ is rotationless and if not to find the minimal $k$ such that $f^k$ is rotationless.

We can now state our main result. Complete details and further motivation are supplied in Section 5. In addition to being of intrinsic interest this theorem is needed in [10]. The set of attracting laminations for $\phi$ is denoted $\mathcal{L}(\phi)$ and the expansion factor for $\phi$ on $\Lambda \in \mathcal{L}(\phi)$ is denoted $\mathrm{PF}_\Lambda(\phi)$.

**Theorem 5.3** (Recognition Theorem). *Suppose that* $\phi, \psi \in \mathrm{Out}(F_n)$ *are forward rotationless and that*

(1) $\mathrm{PF}_\Lambda(\phi) = \mathrm{PF}_\Lambda(\psi)$, *for all* $\Lambda \in \mathcal{L}(\phi) = \mathcal{L}(\psi)$*; and*

(2) *there is bijection* $B\colon\ \mathrm{P}(\phi) \to \mathrm{P}(\psi)$ *such that:*

  (i) **(fixed sets preserved)** $\mathrm{Fix}_N(\hat{\Phi}) = \mathrm{Fix}_N(\widehat{B(\Phi)})$*; and*

  (ii) **(twist coordinates preserved)** *if* $w \in \mathrm{Fix}(\Phi)$ *and* $\Phi, i_w\Phi \in \mathrm{P}(\phi)$*, then* $B(i_w\Phi) = i_w B(\Phi)$.

*Then* $\phi = \psi$.

In the course of proving Theorem 5.3, we construct relative train track maps that are better than the those constructed in [2]. We also reorganize elements of the theory to make future modification and referencing of results easier.

The idea behind relative train track maps is that one can study the action of an outer automorphism $\phi$ on conjugacy classes in $F_n$ or on $\partial F_n$ by analyzing the action of a homotopy equivalence $f : G \to G$ of a marked graph $G$ representing $\phi$ on paths, circuits and lines in $G$. For simplicity, suppose that $\sigma$ is a finite path in $G$, which by convention is always assumed to be the immersed image of a compact interval. The image $f(\sigma)$ of $\sigma$ is homotopic rel endpoints to a path denoted $f_\#(\sigma)$. Replacing $f(\sigma)$ with $f_\#(\sigma)$ is called *tightening* and is analogous to replacing a word in a set of generators for $F_n$ with a reduced word in those generators. A decomposition of $\sigma$ into subpaths $\sigma = \sigma_1 \dots \sigma_m$ is a *splitting* if $f_\#^k(\sigma) = f_\#^k(\sigma_1) \dots f_\#^k(\sigma_m)$ for all $k \geq 0$; i.e. if one can tighten the image of $\sigma$ under any iterate of $f$ by tightening the images of the $\sigma_i$'s. The more one can split $\sigma$ and the better one can understand the subpaths $\sigma_i$, the more effectively one can analyze the iterates $f_\#^k(\sigma)$.

Relative train track maps were defined and constructed in [4] with exponentially growing strata in mind. Few restrictions were placed on the non-exponentially growing strata. This was rectified in [2] where *improved relative train tracks* (*IRTs*) are defined and shown to exist for a sufficiently high, but unspecified, iterate of $\phi$. For our current application, IRTs are inadequate. In Theorem 4.28, we prove that every forward rotationless $\phi$ is represented by a relative train track map $f: G \to G$ that has all the essential properties of an IRT (see Section 4.3) and has the additional feature that, for all $\sigma$ and all sufficiently large $k$, there is a canonical splitting (called the *complete splitting*) of $f_\#^k(\sigma)$ into simple, explicitly described subpaths. (Splittings in an IRT are not canonical and the subpaths $\sigma_i$ are understood more inductively than explicitly.) Such $f: G \to G$, called *CTs*, are used in the proof of the Recognition Theorem and in the classification of abelian subgroups given in [10]. It is likely that the existence of complete splittings will be useful in other contexts as well. For example, complete splittings are *hard splittings* as defined in [5].

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## 2. Preliminaries

Fix $n \geq 2$ and let $F_n$ be the free group of rank $n$. Denote the automorphism group of $F_n$ by $\mathrm{Aut}(F_n)$, the group of inner automorphisms of $F_n$ by $\mathrm{Inn}(F_n)$ and the group of outer automorphisms of $F_n$ by $\mathrm{Out}(F_n) = \mathrm{Aut}(F_n)/\mathrm{Inn}(F_n)$. We follow the convention that elements of $\mathrm{Aut}(F_n)$ are denoted by upper case Greek letters and that the same Greek letter in lower case denotes the corresponding element of $\mathrm{Out}(F_n)$. Thus $\Phi \in \mathrm{Aut}(F_n)$ represents $\phi \in \mathrm{Out}(F_n)$.

**2.1. Marked graphs and outer automorphisms.** Identify $F_n$ with $\pi_1(R_n, *)$ where $R_n$ is the rose with one vertex $*$ and $n$ edges. A *marked graph* $G$ is a graph of rank $n$, all of whose vertices have valence at least two, equipped with a homotopy equivalence $m: R_n \to G$ called a *marking*. Letting $b = m(*) \in G$, the marking determines an identification of $F_n$ with $\pi_1(G, b)$. It is often assumed that $G$ does not have valence two vertices, but such vertices occur naturally in relative train track theory so we allow them.

A homotopy equivalence $f: G \to G$ and a path $\sigma$ from $b$ to $f(b)$ determines an automorphism of $\pi_1(G, b)$ and hence an element of $\mathrm{Aut}(F_n)$. If $f$ fixes $b$ and no path is specified, then we use the trivial path. This construction depends only on the homotopy class of $\sigma$ and, as the homotopy class varies, the automorphism ranges over all representatives of the associated outer automorphism $\phi$. We say that $f: G \to G$ *represents* $\phi$. We always assume that $f$ maps vertices to vertices and that the restriction of $f$ to any edge is an immersion.

**2.2. Paths, circuits and edge paths.** Let $\Gamma$ be the universal cover of a marked graph $G$ and let $\mathrm{pr}: \Gamma \to G$ be the covering projection. A proper map $\tilde{\sigma}: J \to \Gamma$ with domain a (possibly infinite) closed interval $J$ will be called a *path in* $\Gamma$ if it is an embedding or if $J$ is finite and the image is a single point; in the latter case we say that $\tilde{\sigma}$ is *a trivial path*. If $J$ is finite, then any map $\tilde{\sigma}\colon J \to \Gamma$ is homotopic rel endpoints to a unique (possibly trivial) path $[\tilde{\sigma}]$; we say that $[\tilde{\sigma}]$ *is obtained from $\tilde{\sigma}$ by tightening*. If $\tilde{f} : \Gamma \to \Gamma$ is a lift of a homotopy equivalence $f : G \to G$, we denote $[\tilde{f}(\tilde{\sigma})]$ by $\tilde{f}_{\#}(\tilde{\sigma})$.

We will not distinguish between paths in $\Gamma$ that differ only by an orientation preserving change of parametrization. Thus we are interested in the oriented image of $\tilde{\sigma}$ and not $\tilde{\sigma}$ itself. If the domain of $\tilde{\sigma}$ is finite, then the image of $\tilde{\sigma}$ has a natural decomposition as a concatenation $\widetilde{E}_1\widetilde{E}_2\dots\widetilde{E}_{k-1}\widetilde{E}_k$ where $\widetilde{E}_i$, $1 < i < k$, is an edge of $\Gamma$, $\widetilde{E}_1$ is the terminal segment of an edge and $\widetilde{E}_k$ is the initial segment of an edge. If the endpoints of the image of $\tilde{\sigma}$ are vertices, then $\widetilde{E}_1$ and $\widetilde{E}_k$ are full edges. The sequence $\widetilde{E}_1\widetilde{E}_2\dots\widetilde{E}_k$ is called *the edge path associated to $\tilde{\sigma}$*. This notation extends naturally to the case that the interval of domain is half-infinite or bi-infinite. In the former case, an edge path has the form $\widetilde{E}_1\widetilde{E}_2\dots$ or $\dots\widetilde{E}_{-2}\widetilde{E}_{-1}$ and in the latter case has the form $\dots\widetilde{E}_{-1}\widetilde{E}_0\widetilde{E}_1\widetilde{E}_2\dots$.

A *path in $G$* is the composition of the projection map $pr$ with a path in $\Gamma$. Thus a map $\sigma\colon J \to G$ will be called a *non-trivial* path if $J$ is a (possibly infinite) closed non-trivial interval and $\sigma$ is an immersion which lifts to a proper map $\tilde{\sigma}\colon J \to \Gamma$. A map $\sigma\colon J \to G$ will be called a *trivial* path if $J$ is finite and $\sigma(J)$ is a single point. If $J$ is finite, then every map $\sigma\colon J \to G$ is homotopic rel endpoints to a unique (possibly trivial) path $[\sigma]$; we say that $[\sigma]$ *is obtained from $\sigma$ by tightening*. For any lift $\tilde{\sigma}\colon J \to \Gamma$ of a path $\sigma$, $[\sigma] = pr[\tilde{\sigma}]$. We denote $[f(\sigma)]$ by $f_{\#}(\sigma)$. We do not distinguish between paths in $G$ that differ by an orientation preserving change of parametrization. The *edge path associated to a path $\sigma$* is the projected image of the edge path associated to a lift $\tilde{\sigma}$. Thus the edge path associated to a path with finite domain has the form $E_1E_2\dots E_{k-1}E_k$ where $E_i$, $1 < i < k$, is an edge of $G$, $E_1$ is the terminal segment of an edge and $E_k$ is the initial segment of an edge. We will identify paths with their associated edge paths whenever it is convenient.

We reserve the word *circuit* for an immersion $\sigma\colon S^1 \to G$. Any homotopically non-trivial map $\sigma\colon S^1 \to G$ is homotopic to a unique circuit $[\sigma]$. As was the case with paths, we do not distinguish between circuits that differ only by an orientation preserving change in parametrization and we identify a circuit $\sigma$ with a *cyclically ordered edge path $E_1E_2\dots E_k$*.

A path or circuit *crosses* or *contains* an edge if that edge occurs in the associated edge path. For any path $\sigma$ in $G$ define $\bar{\sigma}$ to be '$\sigma$ with its orientation reversed'. For notational simplicity, we sometimes refer to the inverse of $\tilde{\sigma}$ by $\tilde{\sigma}^{-1}$.

A decomposition of a path or circuit into subpaths is a *splitting* for $f : G \to G$ and is denoted $\sigma = \dots\cdot\sigma_1\cdot\sigma_2\cdot\dots$ if $f_{\#}^k(\sigma) = \dots f_{\#}^k(\sigma_1)f_{\#}^k(\sigma_2)\dots$ for all $k \geq 0$. In other words, a decomposition of $\sigma$ into subpaths $\sigma_i$ is a splitting if one can tighten the image of $\sigma$ under any iterate of $f_{\#}$ by tightening the images of the $\sigma_i$'s.

A path $\sigma$ is a *periodic Nielsen path* if $\sigma$ is non-trivial and $f_{\#}^k(\sigma) = \sigma$ for some $k \geq 1$. The minimal such $k$ is the *period* of $\sigma$ and if the period is one then $\sigma$ is a *Nielsen path*. A (periodic) Nielsen path is *indivisible* if it does not decompose as a concatenation of non-trivial (periodic) Nielsen subpaths. A path or circuit is *root-free* if it is not equal to $\mu^k$ for some path $\mu$ and some $k > 1$.

**2.3. Automorphisms and lifts.** Section 1 of [12] and Section 2.1 of [3] are good sources for facts that we record below without specific references. The universal cover $\Gamma$ of a marked graph $G$ with marking $m\colon R_n \to G$ is a simplicial tree. We always assume that a base point $\tilde{b} \in \Gamma$ projecting to $b = m(*) \in G$ has been chosen, thereby defining an action of $F_n$ on $\Gamma$. The set of ends $\mathcal{E}(\Gamma)$ of $\Gamma$ is naturally identified with the boundary $\partial F_n$ of $F_n$ and we make implicit use of this identification throughout the paper.

Each non-trivial $c \in F_n$ acts by a *covering translation* $T_c\colon \Gamma \to \Gamma$ and each $T_c$ induces a homeomorphism $\hat{T}_c\colon \partial F_n \to \partial F_n$ that fixes two points, a sink $T_c^+$ and a source $T_c^-$. The line in $\Gamma$ whose ends converge to $T_c^-$ and $T_c^+$ is called the *axis of* $T_c$ and is denoted $A_c$. The image of $A_c$ in $G$ is the circuit corresponding to the conjugacy class of $c$.

If $f : G \to G$ represents $\phi \in \mathrm{Out}(F_n)$ then a path $\sigma$ from $b$ to $f(b)$ determines both an automorphism representing $\phi$ and a lift of $f$ to $\Gamma$. This defines a bijection between the set of lifts $\tilde{f}: \Gamma \to \Gamma$ of $f : G \to G$ and the set of automorphisms $\Phi\colon F_n \to F_n$ representing $\phi$. Equivalently, this bijection is defined by $\tilde{f} T_c = T_{\Phi(c)} \tilde{f}$ for all $c \in F_n$. We say that $\tilde{f}$ *corresponds to* $\Phi$ or *is determined by* $\Phi$ and vice versa. Under the identification of $\mathcal{E}(\Gamma)$ with $\partial F_n$, a lift $\tilde{f}$ determines a homeomorphism $\hat{f}$ of $\partial F_n$. An automorphism $\Phi$ also determines a homeomorphism $\hat{\Phi}$ of $\partial F_n$ and $\hat{f} = \hat{\Phi}$ if and only if $\tilde{f}$ corresponds to $\Phi$. In particular, $\hat{i}_c = \hat{T}_c$ for all $c \in F_n$ where $i_c(w) = cwc^{-1}$ is the inner automorphism of $F_n$ determined by $c$. We use the notation $\hat{f}$ and $\hat{\Phi}$ interchangeably depending on the context.

We are particularly interested in the dynamics of $\hat{f} = \hat{\Phi}$. The following two lemmas are contained in Lemmas 2.3 and 2.4 of [3] and in Proposition 1.1 of [12].

**Lemma 2.1.** *Assume that* $\tilde{f} : \Gamma \to \Gamma$ *corresponds to* $\Phi \in \mathrm{Aut}(F_n)$. *Then the following are equivalent:*

(i) $c \in \mathrm{Fix}(\Phi)$.

(ii) $T_c$ *commutes with* $\tilde{f}$.

(iii) $\hat{T}_c$ *commutes with* $\hat{f}$.

(iv) $\mathrm{Fix}(\hat{T}_c) \subset \mathrm{Fix}(\hat{f}) = \mathrm{Fix}(\hat{\Phi})$.

(v) $\mathrm{Fix}(\hat{f}) = \mathrm{Fix}(\hat{\Phi})$ *is* $\hat{T}_c$*-invariant.*

**Remark 2.2.** It is not hard to see that $T_c^+ \in \mathrm{Fix}(\Phi)$ if and only if $T_c^- \in \mathrm{Fix}(\Phi)$.

A point $P \in \partial F_n$ is an *attractor* for $\hat{\Phi}$ if it has a neighborhood $U$ such that $\hat{\Phi}(U) \subset U$ and such that $\bigcap_{n=1}^{\infty} \hat{\Phi}^n(U) = P$. If $Q$ is an attractor for $\hat{\Phi}^{-1}$ then we say that it is a *repeller* for $\hat{\Phi}$.

**Lemma 2.3.** *Assume that* $\tilde{f}: \Gamma \to \Gamma$ *corresponds to* $\Phi \in \mathrm{Aut}(F_n)$ *and that* $\mathrm{Fix}(\hat{\Phi}) \subset \partial F_n$ *contains at least three points. Denote* $\mathrm{Fix}(\Phi)$ *by* $\mathbb{F}$ *and the corresponding subgroup of covering translations of* $\Gamma$ *by* $\mathbb{T}(\Phi)$. *Then*

(i) $\partial\mathbb{F}$ *is naturally identified with the closure of* $\{T_c^{\pm} : T_c \in \mathbb{T}(\Phi)\}$ *in* $\partial F_n$. *None of these points is isolated in* $\mathrm{Fix}(\hat{\Phi})$.
(ii) *Each point in* $\mathrm{Fix}(\hat{\Phi}) \setminus \partial\mathbb{F}$ *is isolated and is either an attractor or a repeller for the action of* $\hat{\Phi}$.
(iii) *There are only finitely many* $\mathbb{T}(\Phi)$*-orbits in* $\mathrm{Fix}(\hat{\Phi}) \setminus \partial\mathbb{F}$.

**2.4. Lines and laminations.** Suppose that $\Gamma$ is the universal cover of a marked graph $G$. An unoriented bi-infinite path in $\Gamma$ is called a *line in* $\Gamma$. The *space of lines in* $\Gamma$ is denoted $\widetilde{\mathcal{B}}(\Gamma)$ and is equipped with what amounts to the compact-open topology. Namely, for any finite path $\tilde{\alpha}_0 \subset \Gamma$ (with endpoints at vertices if desired), define $N(\tilde{\alpha}_0) \subset \widetilde{\mathcal{B}}(\Gamma)$ to be the set of lines in $\Gamma$ that contain $\tilde{\alpha}_0$ as a subpath. The sets $N(\tilde{\alpha}_0)$ define a basis for the *weak topology on* $\widetilde{\mathcal{B}}(\Gamma)$.

An unoriented bi-infinite path in $G$ is called a *line in* $G$. *The space of lines in* $G$ is denoted $\mathcal{B}(G)$. There is a natural projection map from $\widetilde{\mathcal{B}}(\Gamma)$ to $\mathcal{B}(G)$ and we equip $\mathcal{B}(G)$ with the quotient topology.

A line in $\Gamma$ is determined by the unordered pair of its endpoints $(P, Q)$ and so corresponds to a point in the *space of abstract lines* defined to be $\widetilde{\mathcal{B}} := ((\partial F_n \times \partial F_n) \setminus \Delta)/\mathbb{Z}_2$, where $\Delta$ is the diagonal and where $\mathbb{Z}_2$ acts on $\partial F_n \times \partial F_n$ by interchanging the factors. The action of $F_n$ on $\partial F_n$ induces an action of $F_n$ on $\widetilde{\mathcal{B}}$ whose quotient space is denoted $\mathcal{B}$. The 'endpoint map' defines a homeomorphism between $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{B}}(\Gamma)$ and we use this implicitly to identify $\widetilde{\mathcal{B}}$ with $\widetilde{\mathcal{B}}(\Gamma)$ and hence $\widetilde{\mathcal{B}}(\Gamma)$ with $\widetilde{\mathcal{B}}(\Gamma')$ where $\Gamma'$ is the universal cover of any other marked graph $G'$. There is a similar identification of $\mathcal{B}(G)$ with $\mathcal{B}$ and with $\mathcal{B}(G')$. We sometimes say that the line in $G$ or $\Gamma$ corresponding to an abstract line is *the realization* of that abstract line in $G$ or $\Gamma$.

A closed set of lines in $G$ or a closed $F_n$-invariant set of lines in $\Gamma$ is called a *lamination* and the lines that compose it are called *leaves*. If $\Lambda$ is a lamination in $G$ then we denote its pre-image in $\Gamma$ by $\tilde{\Lambda}$. Conversely, if $\tilde{\Lambda}$ is an $F_n$-invariant lamination in $\Gamma$ then its image in $G$ is denoted $\Lambda$.

Suppose that $f: G \to G$ represents $\phi$ and that $\tilde{f}$ is a lift of $f$. If $\tilde{\gamma}$ is a line in $\Gamma$ with endpoints $P$ and $Q$, then there is a bounded homotopy from $\tilde{f}(\tilde{\gamma})$ to the line $\tilde{f}_\#(\gamma)$ with endpoints $\hat{f}(P)$ and $\hat{f}(Q)$. This defines an action $\tilde{f}_\#$ of $\tilde{f}$ on lines in $\Gamma$. If $\Phi \in \mathrm{Aut}(F_n)$ corresponds to $\tilde{f}$ then $\Phi_\# = \tilde{f}_\#$ is described on abstract lines by $(P, Q) \mapsto (\hat{\Phi}(P), \hat{\Phi}(Q))$. There is an induced action $\phi_\#$ of $\phi$ on lines in $G$ and in particular on laminations in $G$.

To each $\phi \in \mathrm{Out}(F_n)$ is associated a finite $\phi$-invariant set of laminations $\mathcal{L}(\phi)$ called the set of *attracting laminations* for $\phi$. The individual laminations need not be $\phi$-invariant. By definition (see Definition 3.1.5 of [2]) $\mathcal{L}(\phi) = \mathcal{L}(\phi^k)$ for all $k \geq 1$ and each $\Lambda \in \mathcal{L}(\phi)$ contains birecurrent leaves $\ell$ whose weak closure (that is, its closure in the weak topology) is all of $\Lambda$; any such $\ell$ is called a *generic leaf* of $\Lambda$. A birecurrent leaf $\ell$ is a generic leaf of some $\Lambda \in \mathcal{L}(\phi)$ if and only if it has a weak neighborhood $U$, called an attracting neighborhood, such that $\{\phi^{pk}(U) : k \geq 0\}$ is a weak neighborhood basis for $\ell$ for some $p \geq 1$. Complete details on $\mathcal{L}(\phi)$ can be found in Section 3 of [2].

A point $P \in \partial F_n$ determines a lamination $\Lambda(P)$, called *the accumulation set of* $P$, as follows. Let $\Gamma$ be the universal cover of a marked graph $G$ and let $\widetilde{R}$ be any ray in $\Gamma$ converging to $P$. A line $\tilde{\sigma} \subset \Gamma$ belongs to $\widetilde{\Lambda(P)}$ if every finite subpath of $\tilde{\sigma}$ is contained in some translate of $\widetilde{R}$. Suppose that $\widetilde{R} = \tilde{\rho}\widetilde{R}'$ where $\tilde{\rho}$ is an initial subpath of $\widetilde{R}$ with finite length, say $N$. Given a finite subpath $\tilde{\alpha}_2$ of $\tilde{\sigma}$, there is a finite subpath $\tilde{\alpha}$ of $\tilde{\sigma}$ where $\tilde{\alpha} = \tilde{\alpha}_1\tilde{\alpha}_2$ and where $\tilde{\alpha}_1$ has length greater than $N$. Since $\tilde{\alpha}$ is contained in some translate of $\widetilde{R}$, $\tilde{\alpha}_2$ is contained in some translate of $\widetilde{R}'$. This proves that $\widetilde{\Lambda(P)}$ is unchanged if $\widetilde{R}$ is replaced by any subray. Since any two rays converging to $P$ have a common infinite end, $\widetilde{\Lambda(P)}$ is independent of the choice of $\widetilde{R}$. The bounded cancellation lemma implies (cf. Lemma 3.1.4 of [2]) that this definition is independent of the choice of $G$ and $\Gamma$ and that $\hat{\Phi}_\#(\widetilde{\Lambda(P)}) = \widetilde{\Lambda(\hat{\Phi}(P))}$. In particular, if $P \in \mathrm{Fix}(\hat{\Phi})$ then $\Lambda(P)$ is $\phi_\#$-invariant.

**2.5. Free factor systems.** The conjugacy class of a free factor $F^i$ of $F_n$ is denoted $[[F^i]]$. If $F^1, \dots, F^k$ are non-trivial free factors and if $F^1 * \dots * F^k$ is a free factor then we say that the collection $\{[[F^1]], \dots, [[F^k]]\}$ is a *free factor system*. For example, if $G$ is a marked graph and $G_r \subset G$ is a subgraph with non-contractible components $C_1, \dots, C_k$ then the conjugacy class $[[\pi_1(C_i)]]$ of the fundamental group of $C_i$ is well defined and the collection of these conjugacy classes is a free factor system denoted $\mathcal{F}(G_r)$; we say that $G_r$ *realizes* $\mathcal{F}(G_r)$.

The image of a free factor $F$ under an element of $\mathrm{Aut}(F_n)$ is a free factor. This induces an action of $\mathrm{Out}(F_n)$ on the set of free factor systems. We sometimes say that a free factor is $\phi$-invariant when we really mean that its conjugacy class is $\phi$-invariant. If $[[F]]$ is $\phi$-invariant then $F$ is $\Phi$-invariant for some automorphism $\Phi$ representing $\phi$ and $\Phi|F$ determines a well defined element $\phi|F$ of $\mathrm{Out}(F)$.

The conjugacy class $[a]$ of $a \in F_n$ is *carried by* $[[F^i]]$ if $F^i$ contains a representative of $[a]$. Sometimes we say that $a$ is carried by $F^i$ when we really mean that $[a]$ is carried by $[[F^i]]$. If $G$ is a marked graph and $G_r$ is a subgraph of $G$ such that $[[F^i]] = [[\pi_1(G_r)]]$, then $[a]$ is carried by $[[F^i]]$ if and only if the circuit in $G$ that represents $[a]$ is contained in $G_r$. We say that *an abstract line $\ell$ is carried by* $[[F^i]]$ if its realization in $G$ is contained in $G_r$ for some, and hence any, $G$ and $G_r$ as above. Equivalently, $\ell$ is the limit of periodic lines corresponding to $[c_i]$ where each $[c_i]$ is carried by $[[F^i]]$. A collection $W$ of abstract lines and conjugacy classes in $F_n$ is carried by a free factor system $\mathcal{F} = \{[[F^1]], \dots, [[F^k]]\}$ if each element of $W$ is carried by some $F^i$.

There is a partial order $\sqsubset$ on free factor systems generated by inclusion. More precisely, $[[F^1]] \sqsubset [[F^2]]$ if $F^1$ is conjugate to a free factor of $F^2$ and $\mathcal{F}_1 \sqsubset \mathcal{F}_2$ if for each $[[F^i]] \in \mathcal{F}_1$ there exists $[[F^j]] \in \mathcal{F}_2$ such that $[[F^i]] \sqsubset [[F^j]]$.

The *complexity* of a free factor system is defined on page 531 of [2]. Rather than repeat the definition, we recall the three properties of complexity that we use. The first is that if $\mathcal{F}_1 \sqsubset \mathcal{F}_2$ for distinct free factor systems $\mathcal{F}_1$ and $\mathcal{F}_2$ then the complexity of $\mathcal{F}_1$ is less than the complexity of $\mathcal{F}_2$. This is immediate from the definition. The second is Corollary 2.6.5 of [2].

**Lemma 2.4.** *For any collection $W$ of abstract lines there is a unique free factor system $\mathcal{F}(W)$ of minimal complexity that carries every element of $W$. If $W$ is a single element then $\mathcal{F}(W)$ has a single element.*

The third is an immediate consequence of the uniqueness of $\mathcal{F}(W)$.

**Corollary 2.5.** *If a collection $W$ of abstract lines and conjugacy classes in $F_n$ is $\phi$-invariant then $\mathcal{F}(W)$ is $\phi$-invariant.*

Further details on free factor systems can be found in Section 2.6 of [2].

**2.6. Relative train track maps.** In this section we review and set notation for relative train track maps as defined in [4].

Suppose that $G$ is a marked graph and that $f: G \to G$ is a homotopy equivalence representing $\phi \in \mathrm{Out}(F_n)$. A *filtration* of $G$ is an increasing sequence $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_N = G$ of subgraphs, each of whose components contains at least one edge. If $f(G_i) \subset G_i$ for all $i$ then we say that $f: G \to G$ *respects the filtration* or that the *filtration is $f$-invariant*. A path or circuit has *height* $r$ if it is contained in $G_r$ but not $G_{r-1}$. A lamination has height $r$ if each leaf in its realization in $G$ has height at most $r$ and some leaf has height $r$.

The $r^{\text{th}}$ *stratum* $H_r$ is defined to be the closure of $G_r \setminus G_{r-1}$. To each stratum $H_r$ there is an associated square matrix $M_r$, called the *transition matrix for* $H_r$, whose $ij^{\text{th}}$ entry is the number of times that the $f$-image of the $i^{\text{th}}$ edge (in some ordering of the edges of $H_r$) crosses the $j^{\text{th}}$ edge in either direction. By enlarging the filtration, we may assume that each $M_r$ is either irreducible or the zero matrix. We say that $H_r$ *is an irreducible stratum* if $M_r$ is irreducible and is *a zero stratum* if $M_r$ is the zero matrix.

If $M_r$ is irreducible and the Perron–Frobenius eigenvalue of $M_r$ is 1, then $M_r$ is a permutation matrix and $H_r$ is *non-exponentially growing* or simply *NEG*. After subdividing and replacing the given NEG stratum with a pair of NEG strata if necessary, the edges $\{E_1, \dots, E_l\}$ of $H_r$ can be oriented and ordered so that $f(E_i) = E_{i+1}u_i$ where $u_i \subset G_{r-1}$ and where indices are taken mod $l$. We always assume that edges in an NEG stratum have been so oriented and ordered. If each $u_i$ is trivial then $f^l(E_i) = E_i$ for all $i$ and we say that $E_i$ $[H_r]$ is a *periodic edge* [*stratum*] with period $l$ or a *fixed edge* [*stratum*] if $l = 1$. If each $u_i$ is a Nielsen path then the combinatorial length of $f_\#^k(E_i)$ is bounded by a linear function of $k$ for all $i$ and we say that $E_i$ $[H_r]$ is a *linear edge* [*stratum*].

If $M_r$ is irreducible and if the Perron–Frobenius eigenvalue of $M_r$ is greater than 1 then $H_r$ is an *exponentially growing stratum* or simply an *EG stratum*. If $H_r$ is EG and $\alpha \subset G_{r-1}$ is a non-trivial path with endpoints in $H_r \cap G_{r-1}$ then we say that $\alpha$ is a *connecting path for $H_r$*. If $H_r$ is EG and $\sigma$ is a path with height $r$ then we sometimes say that $\sigma$ *has EG height*.

A *direction $d$* at $x \in G$ is the germ of an initial segment of an oriented edge (or partial edge if $x$ is not a vertex) based at $x$. There is an $f$-induced map $Df$ on directions and we say that $d$ is *a periodic direction* if it is periodic under the action of $Df$; if the period is one then $d$ is a *fixed direction*. Thus, the direction determined by an oriented edge $E$ is fixed if and only if $E$ is the initial edge of $f(E)$. Two directions with the same basepoint belong to the same *gate* if they are identified by some iterate of $Df$. If $x$ is a periodic point then the number of gates based at $x$ is equal to the number of periodic directions based at $x$.

A *turn* is an unordered pair of directions with a common base point. The turn is *nondegenerate* if is defined by distinct directions and is *degenerate* otherwise. If $E_1 E_2 \dots E_{k-1} E_k$ is the edge path associated to a path $\sigma$, then we say that $\sigma$ *contains the turns* $(\overline{E}_i, E_{i+1})$ *for* $1 \le i \le k-1$. A turn is *illegal* with respect to $f : G \to G$ if its image under some iterate of $Df$ is degenerate; a turn is *legal* if it is not illegal. Equivalently, a turn is legal if and only if it is defined by directions that belong to distinct gates. A *path or circuit $\sigma \subset G$ is legal* if it contains only legal turns. A turn whose two defining directions belong to the same stratum $H_r$ is said to be a *turn in $H_r$*. If $\sigma \subset G_r$ does not contain any illegal turns in $H_r$, then $\sigma$ is *$r$-legal*. It is immediate from the definitions that $Df$ maps legal turns to legal turns and that the restriction of $f$ to a legal path is an immersion.

We recall the definition of relative train track map from page 38 of [4].

**Definition 2.6.** A homotopy equivalence $f : G \to G$ representing $\phi$ is a *relative train track map* if it satisfies the following conditions for every EG stratum $H_r$ of an $f$-invariant filtration $\mathcal{F}$.

(RTT-i) $Df$ maps the set of directions in $H_r$ with basepoints at vertices to itself; in particular every turn with one direction in $H_r$ and the other in $G_{r-1}$ is legal.

(RTT-ii) If $\alpha$ is a connecting path for $H_r$ then $f_\#(\alpha)$ is a connecting path for $H_r$; in particular, $f_\#(\alpha)$ is nontrivial.

(RTT-iii) If $\alpha \subset G_r$ is $r$-legal then $f_\#(\alpha)$ is $r$-legal.

**Remark 2.7.** If $f : G \to G$ is a relative train track map, then so is $f^k$ for $k > 0$.

A subgraph $C$ of $G$ is *wandering* if $f^k(C)$ is contained in the closure of $G \setminus C$ for all $k > 0$; otherwise $C$ is *non-wandering*. Each edge in a wandering subgraph is contained in a zero stratum. If $C$ is a component of a filtration element then $C$ is non-wandering if and only if $f^i(C) \subset C$ for some $i > 0$.

**Remark 2.8.** Suppose that $H_r$ is an EG stratum and that $f : G \to G$ satisfies (RTT-i) for $H_r$. Then $f(H_r \cap G_{r-1}) \subset H_r \cap G_{r-1}$ and verifying (RTT-ii) for $H_r$ reduces to showing that $f_\#(\alpha)$ is nontrivial for each connecting path $\alpha$ for $H_r$. Suppose that the component $C$ of $G_{r-1}$ that contains $\alpha$ is non-wandering. In checking (RTT-ii) there is no loss in replacing $f$ by $f^i$ so we may assume that $f(C) \subset C$. If $f$ permutes the elements of the finite set $H_r \cap C$, then $f_\#(\alpha)$ is nontrivial for each $\alpha \subset C$. If $f$ identifies two of these points then they can be connected by an arc $\alpha \subset C$ such that $f_\#(\alpha)$ is trivial (because $f|C : C \to C$ is a homotopy equivalence). This proves that if $C$ is a non-wandering component of $G_{r-1}$ then (RTT-ii) holds for all $\alpha \subset C$ if and only if $H_r \cap C \subset \mathrm{Per}(f)$.

The most common applications of the relative train track properties are contained in following lemma.

**Lemma 2.9.** *Suppose that $f : G \to G$ is a relative train track map and that $H_r$ is an EG stratum.*

(1) *Suppose that a vertex $v$ of $H_r$ is contained in a component $C$ of $G_{r-1}$ that is non-contractible or more generally satisfies $f^i(C) \subset C$ for some $i > 0$. Then $v$ is periodic and has at least one periodic direction in $H_r$.*

(2) *If $\sigma$ is an $r$-legal circuit or path of height $r$ with endpoints, if any, at vertices of $H_r$ then the decomposition of $\sigma$ into single edges in $H_r$ and maximal subpaths in $G_{r-1}$ is a splitting.*

*Proof.* The first item follows from Remark 2.8. The second item is contained in Lemma 5.8 of [4]. $\square$

**Lemma 2.10.** *If $H_r$ is an EG stratum of a relative train track map $f : G \to G$ and $x \in H_r$ is either a vertex or a periodic point then there is a legal turn in $G_r$ that is based at $x$. In particular, there are at least two gates in $G_r$ that are based at $x$.*

*Proof.* There exists $j > 0$ and a point $y$ in the interior of an edge $E$ of $H_r$ so that $x = f^j(y)$. By (RTT-iii), $x$ is in the interior of an $r$-legal path. Moreover the turn at $x$ determined by this path is legal by Properties (RTT-iii) and (RTT-i). $\square$

The following lemma describes indivisible periodic Nielsen paths with EG height.

**Lemma 2.11.** *Suppose that $f : G \to G$ is a relative train track map and that $H_r$ is an EG stratum.*

(1) *There are only finitely many indivisible periodic Nielsen paths of height $r$.*

(2) *If $\sigma$ is an indivisible periodic Nielsen path of height $r$ then $\sigma = \alpha\beta$ where $\alpha$ and $\beta$ are $r$-legal paths that begin and end with directions in $H_r$ and the turn $(\bar\alpha, \beta)$ is illegal. Moreover, if $\bar\alpha(k)$ and $\beta(k)$ are the initial segments of $\bar\alpha$ and $\beta$ that are identified by $f_\#^k$ then the $\alpha(k)\beta(k)$'s form an increasing sequence of subpaths whose union is the interior of $\sigma$.*

(3) *An indivisible periodic Nielsen path $\sigma$ of height $r$ has period* 1 *if and only if the initial and terminal directions of $\sigma$ are fixed.*

*Proof.* In proving (2) there is no loss in replacing $f$ by an iterate so we may assume that $\sigma$ is an indivisible Nielsen path. The first part of (2) is therefore contained in the statement of Lemma 5.11 of [4] and the moreover part of (2) is contained in the proof of that lemma. Item (1) follows from Lemma 4.2.5 of [2]. We now turn our attention to (3).

Suppose that $\sigma$ is an indivisible periodic Nielsen path of height $r$ and period $p$ and that $\sigma = \alpha\beta$ as in (2). After subdividing at the endpoints of $f_\#^j(\sigma)$ for $0 \leq j \leq p-1$ (which clearly preserves the property of being a relative train track map) we may assume that the endpoints of $\sigma$ are vertices. Since $\alpha$ and $\bar\beta$ are $r$-legal and begin with edges in $H_r$, Lemma 2.9 (2) implies that $Df$ maps the initial directions of $\alpha$ and $\bar\beta$ to the initial direction of $f_\#(\alpha)$ and $f_\#(\bar\beta)$ respectively. The only if part of (3) therefore follows from the fact that $f_\#(\sigma)$ is obtained from $f_\#(\alpha)$ and $f_\#(\bar\beta)$ by cancelling their maximal common terminal segment.

Assume now that the initial edges of $\alpha$ and $\bar\beta$ determine fixed directions and write $f_\#(\sigma) = \alpha_1\beta_1$ as in (2). In particular, the first edge $E$ of $\alpha$ is also the first edge in $\alpha_1$. The moreover part of (2) implies that both $\alpha$ and $\alpha_1$ are initial segments of $f_\#^{Np}(E)$ for all sufficiently large $N$. Thus either $\alpha$ is an initial segment of $\alpha_1$ or $\alpha_1$ is an initial segment of $\alpha$. For concreteness assume the former.

We claim that $\alpha_1 = \alpha$. If not then $\alpha_1 = \alpha\gamma$ for some non-trivial $\gamma$. The path $\alpha_2 := \bar\beta\gamma$ is a subpath of $f_\#^{Np}(\bar\beta)$ for large $N$ and so is $r$-legal. The path $\alpha_2\beta_1 = [(\bar\beta\bar\alpha)(\alpha_1\beta_1)]$ is a non-trivial periodic Nielsen path with exactly one illegal turn in $H_r$ and is therefore indivisible. By construction, $\alpha_2$ and $\bar\beta$, and hence $\alpha_2$ and $\bar\beta_1$, have a non-trivial common initial subpath in $H_r$. This implies as above that $\alpha_2$ and $\bar\beta_1$ are initial subpaths of a common path $\delta$. They cannot be equal so one is a proper initial subpath of the other. But then the difference between the number of $H_r$ edges in $f_\#^{Np}(\alpha_2)$ and the number of $H_r$ edges in $f_\#^{Np}(\bar\beta_1)$ grows exponentially in $N$ in contradiction to the fact that $\alpha_2\beta_1$ is a periodic Nielsen path. This contradiction verifies the claim that $\alpha = \alpha_1$. The symmetric argument implies that $\beta = \beta_1$ so $p = 1$. □

We extend Lemma 2.11 (1) as follows.

**Lemma 2.12.** *For any relative train track map $f : G \to G$ there are only finitely many points that are endpoints of an indivisible periodic Nielsen path $\sigma$.*

*Proof.* After replacing $f$ with an iterate and perhaps subdividing some NEG edges, we may assume that each NEG stratum is a single edge whose initial direction is fixed.

Let $S_r$ be the set of endpoints of indivisible periodic Nielsen paths $\sigma$ with height at most $r$. We prove that $S_r$ is finite for all $r$ by induction on $r$. Since $S_0$ is empty we assume that $S_{r-1}$ is finite and prove that $S_r$ is finite. This is obvious if $H_r$ is a zero stratum (because $S_r = S_{r-1}$) and follows from Lemma 2.11 (1) if $H_r$ is EG. We may therefore assume that $H_r$ is single NEG edge $E$.

Suppose that $\sigma$ is an indivisible Nielsen path of height $r$. If $E$ is not periodic then no point in its interior is periodic. If $E$ is periodic, then no point in its interior can be an endpoint of $\sigma$ because $\sigma$ is indivisible. We conclude that the endpoints of $\sigma$ are not in the interior of $E$. Lemma 4.1.4 of [2] implies that $\sigma$ may begin with $E$ and/or end with $\bar{E}$ but all other edges must be contained in $G_{r-1}$. If $\sigma$ begins with $E$ and ends with $\bar{E}$ then it is a closed path with endpoint equal to the initial endpoint of $E$. It therefore suffices to consider only those $\sigma$ that begin with $E$ or end with $\bar{E}$ but not both. By symmetry we may assume that $\sigma = Eu$ for some path $u \subset G_{r-1}$. If $\sigma' = Eu'$ is another such indivisible periodic Nielsen path then $[\bar{u}u']$ is an indivisible periodic Nielsen path with height less than $r$ and so its endpoints are contained in $S_{r-1}$. Thus $S_r$ is obtained from $S_{r-1}$ by adding at most two points for each edge in $H_r$. This completes the inductive step. $\square$

An irreducible matrix $M$ is *aperiodic* if $M^k$ has all positive entries for some $k \geq 1$. For example, if some diagonal element of $M$ is non-zero then $M$ is aperiodic. If $H_r$ is an EG stratum of a relative train track map $f : G \to G$ and if the transition matrix $M_r$ is aperiodic, then $H_r$ is said to be an *aperiodic EG stratum*. For each aperiodic EG stratum there is a unique $\phi$-invariant attracting lamination $\Lambda_r \in \mathcal{L}(\phi)$ of height $r$. If every EG stratum is aperiodic then every element of $\mathcal{L}(\phi)$ is related to an EG stratum in this way. See Definition 3.1.12 of [2] and the surrounding material for details.

The following lemma produces rays and lines associated to aperiodic EG strata of a relative train track map.

**Lemma 2.13.** *Suppose that $H_r$ is an EG stratum of a relative train track map $f : G \to G$, that $\tilde{f} : \Gamma \to \Gamma$ is a lift and that $\tilde{v} \in \mathrm{Fix}(\tilde{f})$.*

(1) *If $E$ is an oriented edge in $H_r$ and $\tilde{E}$ is a lift that determines a fixed direction at $\tilde{v}$, then there is a unique $r$-legal ray $\tilde{R} \subset \Gamma$ of height $r$ that begins with $\tilde{E}$, intersects $\mathrm{Fix}(\tilde{f})$ only in $\tilde{v}$ and that converges to an attractor $P \in \mathrm{Fix}(\hat{f})$. The accumulation set of $P$ is the (necessarily $\phi$-invariant) element $\Lambda_r$ of $\mathcal{L}(\phi)$ whose realization in $G$ has height $r$.*

(2) *Suppose that $E' \neq E$ is another oriented edge in $H_r$, that $\tilde{E}'$ determines a fixed direction at $\tilde{v}$ and that $\tilde{R}'$ is the ray associated to $\tilde{E}'$ as in (1). Suppose further that the turn $(\bar{E}, E')$ is contained in the path $f^k_\#(E'')$ for some $k \geq 1$ and some edge $E''$ of $H_r$. Then the line $\tilde{R}^{-1}\tilde{R}'$ is a generic leaf of $\tilde{\Lambda}_r$.*

*Proof.* Lemma 2.9 (2) and (RTT-i) imply that $\tilde{f}(\tilde{E}) = \tilde{E} \cdot \tilde{\mu}_1$ for some non-trivial $r$-legal subpath $\tilde{\mu}_1$ of height $r$ that ends with an edge of $H_r$. Applying Lemma 2.9 (2) again, we have $\tilde{f}^2_\#(\tilde{E}) = \tilde{E} \cdot \tilde{\mu}_1 \cdot \tilde{\mu}_2$ for some $r$-legal subpath $\tilde{\mu}_2$ of height $r$ that ends with an edge of $H_r$. Iterating this produces a nested increasing sequence of paths $\widetilde{E} \subset \tilde{f}(\widetilde{E}) \subset \tilde{f}^2_\#(\widetilde{E}) \subset \cdots$ whose union is a ray $\widetilde{R}$ that converges to some attractor $P \in \mathrm{Fix}_N(\hat{f})$.

If $\widetilde{R}'$ is another $r$-legal height $r$ ray that begins with $\widetilde{E}$ and converges to some $P' \in \mathrm{Fix}(\hat{f})$ then $\widetilde{R}'$ has a splitting into terms that project to either edges in $H_r$ or maximal subpaths in $G_{r-1}$. In particular $\widetilde{E}$ is a term in this splitting. Since $f_\#(\widetilde{R}') = \widetilde{R}'$, the obvious induction argument implies that $\tilde{f}^k_\#(\widetilde{E})$ is an initial segment of $\widetilde{R}'$ for all $k$ and hence that $\widetilde{R}' = \widetilde{R}$. This proves the uniqueness part of (1).

The transition matrix for $H_r$ is aperiodic since $f_\#(E)$ contains $E$. Let $\beta$ be a generic leaf of $\tilde{\Lambda}_r$. Following Definition 3.1.7 of [2], a path of the form $f^k_\#(E_i)$ where $k \geq 0$ and $E_i$ is an edge of $H_r$ is called a *tile*. By Lemma 2.9 and by construction, every tile occurs infinitely often in $\widetilde{R}$. By Corollary 3.1.11 of [2], each subpath of $\beta$ is contained in some tile and each tile occurs as a subpath of $\beta$. It follows that the accumulation set of $P$ is equal to the weak limit of $\beta$ which is $\Lambda^+_r$ because $\beta$ is generic. This proves (1).

Assuming now the notation of (2), each finite subpath of $\widetilde{R}^{-1}\widetilde{R}'$ is contained in a tile. This implies, as in the previous case, that $\widetilde{R}^{-1}\widetilde{R}'$ is contained in $\Lambda_r$ and that $\widetilde{R}^{-1}\widetilde{R}'$ is birecurrent. Lemma 3.1.15 of [2] implies that $\widetilde{R}^{-1}\widetilde{R}'$ is generic. □

As noted in Section 2.4, $\mathrm{Out}(F_n)$ acts on the set of laminations in $G$. The stabilizer $\mathrm{Stab}(\Lambda)$ of a lamination $\Lambda$ is the subgroup of $\mathrm{Out}(F_n)$ whose elements leave $\Lambda$ invariant. We recall Corollary 3.3.1 of [2].

**Lemma 2.14.** *For each $\Lambda \in \mathcal{L}(\phi)$, there is a homomorphism* $\mathrm{PF}_\Lambda : \mathrm{Stab}(\Lambda) \to \mathbb{Z}$ *such that $\psi \in \mathrm{Ker}(\mathrm{PF}_\Lambda)$ if and only if $\Lambda \notin \mathcal{L}(\psi)$ and $\Lambda \notin \mathcal{L}(\psi^{-1})$.*

We refer to $\mathrm{PF}_\Lambda$ as the *expansion factor homomorphism* associated to $\Lambda$. The notation is chosen to remind readers that the expansion factor is realized as the logarithm of a Perron–Frobenius eigenvalue in a natural way.

**2.7. Modifying relative train track maps.** To simplify certain arguments in Section 3 and as a step toward our ultimate existence theorem (Theorem 4.28), we add properties to the relative train track maps produced in [4]. We can not simply quote results from [2] because, unlike in [2], here we do not allow iteration and we make no assumptions on $\phi$.

We need some further notation. The *set of periodic points of $f$* is denoted $\mathrm{Per}(f)$. A path $\alpha$ is *pre-trivial* if $f^k_\#(\alpha)$ is trivial for some $k > 0$. We say that a *nested sequence $\mathcal{C}$ of free factor systems $\mathcal{F}^1 \sqsubset \mathcal{F}^2 \sqsubset \cdots \sqsubset \mathcal{F}^m$ is realized* by a relative train track map $f : G \to G$ and filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_N = G$ if each $\mathcal{F}^j$ is realized by some $G_{l(j)}$. For any finite graph $K$, the *core of $K$* is the subgraph of $K$ consisting of edges that are crossed by some circuit in $K$.

The transition matrix $M_r$ for an EG stratum $H_r$ of a topological representative $f : G \to G$ of $\phi$ has a Perron–Frobenius eigenvalue $\lambda_r > 1$. The set $\{\log(\lambda_r) : H_r \text{ is EG}\}$, listed in non-increasing order, is denoted $\mathrm{PF}(f)$. (In [4] this set is denoted

$\Lambda(f)$ but $\Lambda$ is now usually reserved for laminations.) The set $\{\mathrm{PF}_\Lambda(\phi) : \Lambda \in \mathcal{L}(\phi)\}$ of expansion factors for $\phi$, listed in non-increasing order, is denoted $\mathrm{EF}(\phi)$. Then $\mathrm{PF}(f) \geq \mathrm{EF}(\phi)$ in the lexicographical order for all $f$ representing $\phi$ and $\mathrm{PF}(f) = \mathrm{EF}(\phi)$ if $f : G \to G$ is a relative train track map representing $\phi$ by Proposition 3.3.3 (4) of [2].

The number of indivisible Nielsen paths for $f : G \to G$ with height $r$ is denoted $N_r(f)$.

**Remark 2.15.** If $f : G \to G$ satisfies (RTT-i) and if $\mathrm{PF}(f) = \mathrm{EF}(\phi)$ then $f : G \to G$ satisfies (RTT-iii) by Lemma 5.9 of [4]. Thus any topological representative that satisfies (RTT-i), (RTT-ii) and $\mathrm{PF}(f) = \mathrm{EF}(\phi)$ is a relative train track map.

**Lemma 2.16.** *Suppose that $f : G \to G$ and $f' : G' \to G'$ are relative train track maps with EG strata $H_r$ and $H'_s$ respectively and that $p : G \to G'$ is a homotopy equivalence such that*

(1) $p(G_r) = G'_s$, $p(G_{r-1}) = G'_{s-1}$ *and $p$ induces a bijection between the edges of $H_r$ and the edges of $H'_s$;*

(2) $p_\# f_\#(\sigma) = f'_\# p_\#(\sigma)$ *for all paths $\sigma \subset G_r$ with endpoints at vertices.*

*Then $p_\#$ induces a bijection between the indivisible periodic Nielsen paths in $G$ with height $r$ and the indivisible periodic Nielsen paths in $G'$ with height $s$.*

*Proof.* Let $\sigma \subset G$ be a height $r$ path with endpoints at vertices and let $\sigma' = p_\#(\sigma) \subset G'$.

We first observe that no edges in $H'_s$ are cancelled when $p(\sigma)$ is tightened to $p_\#(\sigma)$. Indeed, if this fails then (1) implies that $\sigma$ has a subpath $\sigma_0 = E\tau\overline{E}$ where $E$ is an edge in $H_r$ and where $p_\#(\sigma_0)$ is trivial. This contradicts the assumption that $p$ is a homotopy equivalence and the fact that the closed path $\sigma_0$ determines a non-trivial element of $\pi_1(G)$. As consequence of (1) and this observation we have

(3) the number of $H_r$-edges in $\sigma$ equals the number of $H'_s$-edges in $\sigma' = p_\#(\sigma)$.

Let $E$ be an edge in $H_r$ and let $E' = p(E)$. By (2) applied to $E$ and by (3) applied to $f(E)$, we see that the number of $H_r$-edges in $f(E)$ equals the number of $H'_s$-edges in $f'(E')$. It follows that the number of $H_r$-edges in (untightened) $f(\sigma)$ equals the number of $H'_s$-edges in (untightened) $f'(\sigma')$ for all $\sigma$. In conjunction with (3) applied to $\sigma$ and $f_\#(\sigma)$ this implies that $f(\sigma)$ and $f_\#(\sigma)$ have the same number of $H_r$-edges if and only if $f'(\sigma)$ and $f'_\#(\sigma)$ have the same number of $H'_s$-edges. In other words, $\sigma$ is $r$-legal if and only if $\sigma'$ is $s$-legal. It follows that the number of illegal turns of $\sigma$ in $H_r$ equals the number of illegal turns of $\sigma'$ in $H'_s$.

Assume now that $\sigma$ is an indivisible periodic Nielsen path with height $r$ and period $k$. Item (2) and the obvious induction argument shows that $(f')^k_\#(\sigma') = (f')^k_\#(p_\#(\sigma)) = p_\# f^k_\#(\sigma) = p_\#(\sigma) = \sigma'$ and hence that $\sigma'$ is a periodic Nielsen path. If $\sigma$ is indivisible then $\sigma$ begins and ends in $H_r$ and has exactly one illegal turn in $H_r$ by Lemma 2.11. Since no edges in $H'_s$ are cancelled when $p(\sigma)$ is tightened to $\sigma'$, the first and last edge of $\sigma'$ are in $H'_s$. Since $p_\#$ preserves the number of maximal height illegal turns, $\sigma'$ has exactly one illegal turn in $H'_s$ and so is indivisible. For the converse note that if $E_i$ and $E_j$ are edges in $H_r$ and $\sigma' \subset G'_s$ is a path that begins with $p(E_i)$ and ends with $p(E_j)$ then there is a unique path $\sigma$ that begins with $E_i$, ends with $E_j$ and that satisfies $p_\#(\sigma) = \sigma'$. If $\sigma'$ is a periodic Nielsen path of period $k$ then the uniqueness of $\sigma$ implies that $\sigma$ is a periodic Nielsen path of period $k$. As above $p_\#$ preserves indivisibility because it preserves the number of maximal height illegal turns. $\square$

We next recall the *sliding* operation from [2]. Suppose that $H_s$ is a non-periodic NEG stratum with edges $\{E_1, \dots, E_m\}$ satisfying $f(E_l) = E_{l+1}u_l$ for paths $u_l \subset G_{s-1}$ where $1 \le l \le m$ and indices are taken mod $m$. Choose $1 \le i \le m$ and let $\tau$ be a path in $G_{s-1}$ from the terminal endpoint $v_i$ of $E_i$ to some vertex $w_i$. Roughly speaking, we use $\tau$ to continuously change the terminal endpoint of $E_i$ from $v_i$ to $w_i$ and to mark the new graph.

More precisely, define a new graph $G'$ from $G$ by replacing $E_i$ with an edge $E'_i$ that has terminal vertex $w_i$ and that has the same initial vertex as $E_i$. There are homotopy equivalences $p\colon G \to G'$ and $p'\colon G' \to G$ that are the identity on the common edges of $G$ and $G'$ and that satisfy $p(E_i) = E'_i\bar\tau$ and $p'(E'_i) = E_i\tau$. Use $p$ to define the marking on $G'$ and define $f'\colon G' \to G'$ on edges by tightening $pfp'$. Complete details can be found in Section 5.4 of [2].

**Lemma 2.17.** *Suppose that $f\colon G \to G$ is a relative train track map and assume notation as above.*

- *$f'\colon G' \to G'$ is a relative train track map.*
- *$f'|G_{s-1} = f|G_{s-1}$.*
- *If $m = 1$ then $f'(E'_1) = E'_1[\bar\tau u_1 f(\tau)]$.*
- *If $m \neq 1$ then $f'(E_{i-1}) = E'_i[\bar\tau u_{i-1}]$, $f'(E'_i) = E_{i+1}[u_i f(\tau)]$ and $f'(E_j) = E_j u_j$ for $j \neq i-1, i$.*
- *For each EG stratum $H_r$, $p_\#$ defines a bijection between the set of the indivisible periodic Nielsen paths in $G$ with height $r$ and the indivisible periodic Nielsen paths in $G'$ with height $r$.*

*Proof.* If $m = 1$ then this is contained in Lemma 5.4.1 of [2]. The argument for $m > 1$ is a straightforward extension of $m = 1$ case and we leave the details to the reader. $\square$

**Definition 2.18.** Suppose that $u < r$ and that

(1) $H_u$ is irreducible;
(2) $H_r$ is EG and each component of $G_r$ is non-contractible;
(3) for each $u < i < r$, $H_i$ is a zero stratum that is a component of $G_{r-1}$ and each vertex of $H_i$ has valence at least two in $G_r$.

We say that each $H_i$ is *enveloped by* $H_r$ and write $H_r^z = \bigcup_{k=u+1}^r H_k$. It is often convenient to treat $H_r^z$ as a single unit.

**Theorem 2.19.** *For every $\phi \in \mathrm{Out}(F_n)$ there is a relative train track map $f : G \to G$ and filtration that represents $\phi$ and satisfies the following properties.*

(V) *The endpoints of all indivisible periodic Nielsen paths are vertices.*

(P) *If a stratum $H_m \subset \mathrm{Per}(f)$ is a forest then there exists a filtration element $G_j$ such $\mathcal{F}(G_j) \neq \mathcal{F}(G_l \cup H_m)$ for any $G_l$. (See also items* (1) *and* (5) *of Lemma* 2.20.)

(Z) *Each zero stratum $H_i$ is enveloped by an EG stratum $H_r$. Each vertex in $H_i$ is contained in $H_r$ and has link contained in $H_i \cup H_r$.*

(NEG) *The terminal endpoint of an edge in a non-periodic NEG stratum $H_i$ is periodic and is contained in a filtration element of height less than $i$ that is its own core.*

(F) *The core of each filtration element is a filtration element.*

*Moreover, if $\mathcal{C}$ is a nested sequence of non-trivial $\phi$-invariant free factor systems then we may assume that $f : G \to G$ realizes $\mathcal{C}$.*

Before proving Theorem 2.19 we record some useful observations.

**Lemma 2.20.** *Suppose that $f : G \to G$ is a relative train track map representing $\phi$ with filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_N = G$.*

(1) *Suppose that $f : G \to G$ satisfies* (P)*, that $H_m \subset \mathrm{Per}(f)$ is a forest and that $v$ has valence one in $H_m$. Then $v \in G_j$ for some $j < m$. If in addition $f : G \to G$ satisfies* (F) *then we may choose $G_j$ to be its own core.*

(2) *If $f : G \to G$ satisfies* (F) *and $H_r$ is EG then $G_r$ is its own core.*

(3) *If $f : G \to G$ satisfies* (P) *and* (NEG) *then every edge in each contractible component of a filtration element is contained in a zero stratum.*

(4) *If $f : G \to G$ satisfies* (Z) *and* (NEG) *and if $G_k$ is a filtration element that is its own core, then every vertex in $G_k$ has at least two gates in $G_k$.*

(5) *If $f : G \to G$ satisfies* (P) *then no component of a filtration element $G_m$ is a tree in* $\mathrm{Per}(f)$.

*Proof.* Suppose that $f : G \to G$ satisfies (P) and that $H_m \subset \mathrm{Per}(f)$ is a forest. Thus, the restriction of $f$ to $H_m$ acts transitively on the components, and either every component of $H_m$ is an edge, or $f$ acts transitively on the valence 1 vertices of $H_m$. By (P) there exists $j$ so that $\mathcal{F}(G_j) \neq \mathcal{F}(G_l \cup H_m)$ for any $l$. If $f : G \to G$ also satisfies (F) then we may assume that $G_j$ is its own core. Applying this property with $l = j$ we have that $\mathcal{F}(G_j \cup H_m) \neq \mathcal{F}(G_j)$. At least one vertex of some, and hence every, edge in $H_m$ is contained in $G_j$. If a vertex of some edge of $H_m$ has valence one in $G_j \cup H_m$ then every edge in $H_m$ has such a vertex in contradiction to the assumption that $\mathcal{F}(G_j \cup H_m) \neq \mathcal{F}(G_j)$. This proves (1).

Item (5) follows from the first part of (1).

If $H_r$ is EG then $G_r$ is the smallest filtration element that contains the attracting lamination associated to $H_r$. This proves (2).

For (3), suppose that $C$ is a contractible component of some $G_i$. If $C$ contains an edge in an irreducible stratum then it is non-wandering and by (NEG) the lowest stratum $H_j$ that has an edge in $C$ is either periodic or EG. But $H_j$ can not be periodic by item (1) of this lemma and cannot be EG by Lemma 2.10. Thus every edge in $C$ is contained in a zero stratum.

The proof of (4) is by induction on $k$. Suppose that $G_k$ is a filtration element that is its own core and that $v$ is a vertex in $G_k$. By Lemma 2.10 and (Z), we may assume that $v$ is not the endpoint of an edge in either an EG or zero stratum. If no illegal turns in $G_k$ are based at $v$ then the number of gates in $G_k$ based at $v$ is at least the valence of $v$ in $G_k$ and so is at least two. We may therefore assume that there is an illegal turn $(d_1, d_2)$ in $G_k$ based at $v$. At least one of the $d_i$'s is the terminal end of a non-fixed NEG edge so (NEG) and the inductive hypothesis imply that $v$ has at least two gates in $G_k$. □

*Proof of Theorem* 2.19. By Lemma 2.6.7 of [2], there is a relative train track map $f : G \to G$ and filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_N = G$ that represents $\phi$ and realizes $\mathcal{C}$. (In the statement of Lemma 2.6.7, $\mathcal{C}$ is replaced by a single invariant free factor system $\mathcal{F}$. The more general case that we use is explicitly included in the proof of that lemma.) Lemma 2.12 implies that (V) can be arranged via a finite subdivision. For convenient reference we divide the rest of the proof into steps. Changes are made to $f : G \to G$ in these steps but we start each step by referring to the current relative train track map as $f : G \to G$.

If $\mathcal{C}$ has not already been specified, let $\mathcal{C}$ be the nested sequence of free factor systems determined by the $\mathcal{F}(G_r)$'s. For possible future application we will prove the following statement in place of (P).

(P$_\mathcal{C}$) If a stratum $H_m \subset \mathrm{Per}(f)$ is a forest then there exists $\mathcal{F}^j \in \mathcal{C}$ that is not realized by $G_l \cup H_m$ for any $G_l$.

If $\mathcal{C}$ is determined by the $\mathcal{F}(G_r)$'s then (P) and (P$_\mathcal{C}$) are the same but otherwise the latter is stronger than the former. For example, if $\mathcal{C}$ is empty then (P$_\mathcal{C}$) is the statement that no filtration element is a periodic forest.

**Remark 2.21.** For reference in the proof of Theorem 4.28 we record the following property of the remainder of our construction. If $f : G \to G$ is the relative train track map as it is now and $g : G' \to G'$ is the ultimate modified relative train track map produced by the six steps listed below, then there is a bijection $H_r \to H'_s$ between the EG strata of $f : G \to G$ and the EG strata of $g : G' \to G'$ such that

(a) $H_r$ and $H'_s$ have the same number of edges,

(b) $N_r(f) = N_s(g)$.

The five moves used in our construction are valence two homotopies away from EG strata, sliding (which is defined following Lemma 2.16), reordering of strata, tree replacements (see Step 2) and collapsing forests in NEG and zero strata. Item (a) will be obvious as will (b) for the valence two homotopies and the reordering of strata. For the remaining three moves (b) will follow from Lemmas 2.17 and 2.16.

**Step 1: A weak form of (NEG).** Suppose that $H_s$ is a non-periodic NEG stratum with edges $\{E_1, \dots, E_m\}$ satisfying $f(E_i) = E_{i+1}u_i$ for paths $u_i \subset G_{s-1}$ where indices are taken mod $m$. Our goal in this step is to arrange

(1) the endpoints of an NEG edge are periodic.

Care is taken that no vertices with valence one in $G$ are created.

We first arrange that the terminal endpoint $v_1$ of $E_1$ is either periodic or has valence at least three in $G$. If this is not already the case, let $E$ be the unique oriented edge of $G$, other than $\bar{E}_1$, whose initial endpoint is $v_1$. Since $H_s$ is not a periodic stratum, $E \subset G_{s-1}$. Lemma 2.10 implies that $E$ does not belong to an EG stratum. Perform a valence two homotopy as defined on page 13 of [4].

There are three steps to a valence two homotopy. The first is to postcompose $f$ with a homotopy supported in a small neighborhood of $E$ to arrange that no vertex is mapped to $v_1$ and then tighten the map. The second is to amalgamate $E_1 E$ into a single edge named $E_1$ by removing $v_1$ from the list of vertices. Before discussing the third step we consider the effect of the first two on $f_\#$. For any path $\sigma$ with endpoints at vertices (in the new simplicial structure), the new edge path for $f_\#(\sigma)$ is obtained from the original by removing all occurrences of $E$. If $v$ is a vertex in an EG stratum $H_s$ then $f(v) \neq v_1$. This has three useful consequences. First, the restriction of $Df$ to edges incident to $v$ is unchanged. Second, if $\sigma$ is a non-trivial path in $G_{s-1}$ with endpoints in $H_s$ then the new edge path for $f_\#(\sigma)$ is non-trivial. Third, if $\sigma$ originally has height $s$ and is $s$-legal then it still has height $s$ and is $s$-legal. Together these show that the relative train track properties are still valid.

There may be some edges $E_j$ that are now mapped to points by some iterate of $f$. Each such edge was a zero stratum originally and their union is a forest. The third step in the valence two homotopy is to collapse each component of this pre-trivial forest to a point. For any path $\sigma$ with endpoints at vertices (in the new simplicial structure), the new edge path for $f_\#(\sigma)$ is obtained from the previous one by removing all occurrences of edges in the pre-trivial forest. No vertex in an EG stratum is incident to this forest. Arguing as above, it follows that the new map (still called $f : G \to G$) is a relative train track map. It is clear from the construction that (V) still holds. Collapsing pre-trivial forests does not change the free factor system determined by any filtration element so $\mathcal{C}$ is still realized. Note also that the valence two homotopy does not change the number of edges in any EG stratum and does not change the number of indivisible Nielsen paths with height corresponding to any EG stratum. After finitely many valence two homotopies we may assume that the terminal endpoint $v_1$ of $E_1$ is either periodic or has valence at least three in $G$.

The component $C_1$ of $G_{s-1}$ that contains $v_1$ does not wander and so contains a periodic vertex $w_1$. Choose a path $\tau \subset G_{s-1}$ from $v_1$ to $w_1$ in $C_1$ and slide to change the terminal endpoint of $E_1$ to $w_1$. Because of our previous move, $v_1$ still has valence at least two in $G$.

After repeating these operations finitely many times we have arranged (1) and not created any valence one vertices.

**Step 2: A weak form of (Z).** In this step we prove most of (Z). The one missing item is that we only show that the components of $G_r$ are non-wandering instead of showing that they are non-contractible as required by item (2) of Definition 2.18.

The union $NW$ of the non-wandering components of a filtration element $G_i$ is $f$-invariant and so is a union of strata. The union $W$ of the wandering components of $G_i$ is therefore also a union of strata. If $H_j \subset NW$ and $H_{j-1} \subset W$ then $f(H_j) \subset G_{j-2}$ and there is no loss in interchanging the order of $H_j$ and $H_{j-1}$. Thus the strata, if any, that are contained in $W$ can be moved up the filtration to be above the strata contained in $NW$. After finitely many such changes we may assume that the strata of $NW$ precede those of $W$. The edges in $W$ all belong to zero strata. Choose an ordering on the components of $W$ so that $C_a < C_b$ implies that $C_b \cap f^k(C_a) = \emptyset$ for all $k \geq 0$. Define a new filtration of $G_i$ by declaring $C_1$ to be the first stratum above $NW$, $C_2$ to be the second stratum above $NW$ and so on. At the end of this process we have arranged

(2) if $G_i$ has wandering components then $H_i$ is a wandering component of $G_i$.

Suppose that $K$ is a component of the union of all zero strata, that $H_i$ is the highest stratum that contains an edge in $K$ and that $H_u$ is the highest irreducible stratum below $H_i$. We prove that $K \cap G_u = \emptyset$ by assuming that $K \cap G_u \neq \emptyset$ and arguing to a contradiction. Since each component of $G_u$ is non-wandering and since some iterate of $f$ maps $K$ into $G_u$, there is a unique component $C$ of $G_u$ that intersects $K$. If each vertex $v \in K$ has valence at least two in $C \cup K$ then each edge of $K$ is contained in a path in $K$ with endpoints in $C$, and so by the connectivity of $C$, is contained in a circuit in $K \cup C$. This contradicts the fact that some iterate of $f$ maps $K \cup C$ into $C$, and we conclude that some vertex $v$ of $K$ has valence one in $K \cup C$. In particular, $v$ is non-periodic and so is not the endpoint of an NEG edge by (1) and is not the endpoint of an EG edge in a stratum $H_r$ above $G_i$ because otherwise $K \cup C$ would be contained in a non-contractible component of $G_{r-1}$ contradicting Remark 2.8. By construction, $v$ is not the endpoint of an edge in a zero stratum above $G_i$. But then $v$ has valence one in $G$. This contradiction proves that $K \cap G_u = \emptyset$. After reorganizing the edges in zero strata, we may assume that $H_i = K$. Note in particular that no vertex in $H_i$ is periodic.

Let $H_r$ be the first irreducible stratum above $H_i$. The component of $G_r$ that contains $H_i$ is non-wandering by (2) and so must intersect $H_r$. Since no vertex of $H_i$ is periodic, (1) implies that $H_r$ is EG. Moreover, the argument used in the previous paragraph proves that the link in $G$ of each vertex in $H_i$ is contained in $H_i \cup H_r \subset G_r$

and that $H_i$ is contained in the core of $G_r$.

We arrange that each vertex in $H_i$ is contained in $H_r$ by the following *tree replacement* move. Replace $H_i$ with a tree $H_i'$ whose vertex set is exactly $H_i \cap H_r$. Do this for each zero stratum $H_i$ and call the resulting graph $G'$. We view the union $X$ of the irreducible strata as a subgraph of both $G$ and $G'$. The set of vertices in $X$ is $f$-invariant by (RTT-i) and (1). There is a homotopy equivalence $p' \colon G' \to G$ that is the identity on $X$ and that sends an edge $E'$ of $H_i'$ to the unique path in $H_i$ connecting the endpoints of $E'$. The homotopy inverse $p \colon G \to G'$ is the identity on $X$ and sends $H_i$ to $H_i'$; the exact definition depends on choosing $p(w) \in H_i'$ for those vertices $w \in H_i$, if any, that are not contained in $X$. Note that $p_\#$ defines a bijection, with inverse $p'_\#$, between paths in $G$ with endpoints at vertices in $X$ and paths in $G'$ with endpoints at vertices in $X$. The homotopy equivalence $f' \colon G' \to G'$ defined on edges by $f'(E') = (pfp')_\#(E')$ is independent of the choices made in defining $p$ and represents $\phi$. To verify (RTT-ii) for $f'$ it suffices to show that if $\alpha' \subset H_i'$ is non-trivial then $f'_\#(\alpha') = p_\# f_\#(p'_\#(\alpha'))$ is non-trivial. This follows from (RTT-ii) for $f$ and the fact that both $p_\#$ and $p'_\#$ preserve non-triviality for paths with endpoints in $X$. It is easy to see that (RTT-i) for $f$ implies (RTT-i) for $f'$ and that $\mathrm{PF}(f) = \mathrm{PF}(f')$. Remark 2.8 implies that $f' \colon G' \to G'$ is a relative train track map. None of the moves in this step change the free factor systems represented by filtration elements so $f' \colon G' \to G'$ still realizes $\mathcal{C}$.

**Step 3: Property (P$_{\mathcal{C}}$).** If (P$_{\mathcal{C}}$) fails then there is a stratum $H_m \subset \mathrm{Per}(f)$ that is a forest with the property that for each $\mathcal{F}^j \in \mathcal{C}$ there is a filtration element $G_{l(j)}$ such that $G_{l(j)} \cup H_m$ realizes $\mathcal{F}^j$. We will construct a new relative train track map $f' \colon G' \to G'$ with one fewer NEG stratum that still realizes $\mathcal{C}$ and satisfies (1) and the weak form of (Z). After repeating this finitely many times we will have achieved (P$_{\mathcal{C}}$).

Let $Y$ be the set of edges in $G \setminus H_m$ that are mapped entirely into $H_m$ by some iterate of $f$. Then each edge of $Y$ is contained in a zero stratum and $H_m \cup Y$ is a forest that is mapped into itself by $f$ and into $H_m$ by some iterate of $f$. We next arrange that

($*$) if $\alpha$ is a path in a zero stratum with endpoints at vertices and if $\alpha$ is not contained in $Y$ then $f_\#(\alpha)$ is not contained in $H_m \cup Y$.

Suppose to the contrary that $\alpha \subset H_k$ violates ($*$). Choose an edge $E_i \subset H_k$ that is crossed by $\alpha$ and is not contained in $Y$. Perform a tree replacement move on $H_k$ as in Step 2, replacing $E_i$ by an edge connecting the endpoints of $\alpha$. The new edge is mapped entirely into $H_m$ by some iterate of $f$ and we add it to $Y$. After finitely many such moves ($*$) is satisfied.

Let $G'$ be the marked graph obtained by collapsing each component of $H_m \cup Y$ to a point and let $p \colon G \to G'$ be the corresponding quotient map. Identify the edges of $G'$ with those of $G \setminus (H_m \cup Y)$ and define $f' \colon G' \to G'$ by $f'(E) = [pf(E)]$. As an edge path, $f'(E)$ is obtained from $f(E)$ by removing all occurrences of edges in $H_m \cup Y$. It follows that the strata $H_r$ and $p(H_r)$ (if the latter is non-empty) have the same type (zero, EG, NEG), that $f' \colon G' \to G'$ has one fewer NEG stratum than $f : G \to G$, that $f' \colon G' \to G'$ satisfies (1) and the weak form of (Z), that $\mathrm{PF}(f') = \mathrm{PF}(f)$ and that $f' \colon G' \to G'$ satisfies (RTT-i). Lemma 5.9 of [4] implies that $f' \colon G' \to G'$ satisfies (RTT-iii).

To verify (RTT-ii), suppose that $H'_s = p(H_r)$ is EG and that $\alpha' \subset G'_{s-1}$ is a connecting path for $H'_s$. If $\alpha'$ is contained in a zero stratum then $f'_\#(\alpha')$ is non-trivial by $(*)$. We may therefore assume that the component $C'$ of $G'_{s-1}$ that contains $\alpha'$ is not a zero stratum and hence is non-wandering. To prove that $f'_\#(\alpha')$ is non-trivial, it suffices, by Remark 2.8, to show that each $v' \in H'_s \cap C'$ is a periodic point.

Since $v'$ is incident to an edge in $H'_s$, there is a vertex $v \in H_r$ such that $p(v) = v'$. If $v$ is periodic, we are done. We may therefore assume that $v \not\in H_m$. If $v \in Y$ then the component of $G_{r-1}$ that contains $v$ is a zero stratum by the weak form of (Z) contradicting the assumption that $v'$ is contained in a non-wandering component of $G'_{s-1}$. It follows that $v = p^{-1}(v')$. By the same reasoning, $v$ is contained in a non-wandering component of $G_{r-1}$ and so is periodic by Remark 2.8. Thus $v'$ is periodic and we have verified (RTT-ii) for $f'$. This completes the proof that $f' \colon G' \to G'$ is a relative train track map.

There exists $k > 0$ so that each non-contractible component of $G_{l(j)} \cup H_m$ is $f^k$-invariant and so that $f^k$ induces a rank preserving bijection between the non-contractible components of $G_{l(j)} \cup H_m \cup Y$ and the non-contractible components of $G_{l(j)} \cup H_m$. Thus $G_{l(j)} \cup H_m \cup Y$ and hence $p(G_{l(j)})$ realizes $\mathcal{F}^j$, proving that $f' \colon G' \to G'$ and its filtration realize $\mathcal{C}$.

**Step 4: Property (Z).** If $C$ is a non-wandering component of a filtration element then the lowest stratum containing an edge of $C$ is either EG or periodic. Lemma 2.10 and item (5) of Lemma 2.20 imply that $C$ is not contractible. (Z) therefore follows from the weak form of (Z).

**Step 5: Property (NEG).** Suppose that $H_s$ is a non-periodic NEG stratum with edges $\{E_1, \dots, E_m\}$ satisfying $f(E_i) = E_{i+1}u_i$ for paths $u_i \subset G_{s-1}$ where indices are taken mod $m$. The component $C_i$ of $G_{s-1}$ that contains the terminal endpoint $v_i$ of $E_i$ does not wander. The lowest stratum $H_t$ that contains an edge in $C_i$ is either EG or periodic. In the former case, every vertex in $H_t$ has at least two gates in $H_t$ by Lemma 2.10 and so $H_t$ is its own core. In the latter case, the same result follows from ($\mathrm{P}_{\mathcal{C}}$) and item (1) of Lemma 2.20 which imply that no vertex of $H_t$ has valence one in $H_t$.

Choose a path $\tau \subset G_{s-1}$ from $v_i$ to a periodic vertex $w_i$ in $H_t$ and slide to change the terminal endpoint of $E_i$ to $w_i$. After performing this sliding operation finitely many times, working up through the filtration, (NEG) is satisfied. The resulting homotopy equivalence is a relative train track map by Lemma 2.17, still realizes $\mathcal{C}$ and still satisfies (Z).

Sliding may have introduced valence one vertices to $G$. But no such vertex is the image of a vertex with valence greater than one by (NEG), (Z) and Lemma 2.10. We may therefore remove all vertices of valence one and the edges that are incident to them. After repeating this finitely many times $G$ has no valence one vertices.

If (P$_{\mathcal{E}}$) is no longer satisfied then return to Step 3. Since this reduces the number of NEG strata the process stops.

**Step 6: Property (F).** If $H_l$ is a zero stratum then $G_l$ and $G_{l-1}$ realize the same free factor system. We may therefore assume that $H_l$ is not a zero stratum and hence that every component of $G_l$ is non-contractible. If $w$ is a valence one vertex of $G_l$ then by item (1) of Lemma 2.20, (NEG) and Lemma 2.10, $w$ must be the initial endpoint of a non-periodic NEG edge in some $H_k$ with $k \leq l$ and no vertex with valence at least two in $G_l$ maps to $w$. The initial endpoint of each edge in $H_k$ has valence one in $G_l$ and $G_l \setminus H_k$ is $f$-invariant. We may therefore reorder the strata to move $H_k$ above $G_l \setminus H_k$. After finitely many such moves $\mathcal{F}(G_l)$ is realized by a core filtration element. Working our way up the filtration we arrange that (F) is satisfied. □

We conclude this section by recalling an operation from page 46 of [4] and Definitions 5.3.2 of [2].

Suppose that $H_r$ is an EG stratum of a relative train track map $f: G \to G$ that satisfies item (Z) of Theorem 2.19 and that $\rho$ is an indivisible Nielsen path of height $r$. Decompose $\rho = \alpha\beta$ into a concatenation of maximal $r$-legal subpaths as in Lemma 2.11 and let $E_1 \subset H_r$ and $E_2 \subset H_r$ be the initial edges of $\bar\alpha$ and $\beta$ respectively. If one of the edge paths $f(E_i)$, i=1 or 2, is an initial subpath of the other then we say that *the fold at the illegal turn of $\rho$ is a full fold*; otherwise it is a *partial fold*. There are two kinds of full folds. If $f(E_1) \neq f(E_2)$ then the full fold is *proper*; otherwise it is *improper*.

Suppose that the fold at the illegal turn of $\rho$ is proper, say that $f(E_1)$ is a proper initial subpath of $f(E_2)$. Write $\bar\alpha = E_1 b E_3 \ldots$ where $b$ is a (possibly trivial) subpath of $G_{r-1}$ and $E_3$ is an edge in $H_r$. The initial edge of $f(E_3)$ and the first edge of $f(\beta)$ that is not cancelled when $f(\alpha)f(\beta)$ is tightened to $\alpha\beta$ belong to $H_r$. We may therefore decompose $E_2 = E_2'' E_2'$ into subpaths such that $f(E_2'') = f_\#(E_1 b)$ and such that the first edge in $f(E_2')$ is contained in $H_r$. Form a new graph $G'$ by identifying $E_2''$ with $E_1 b$. The quotient map $F: G \to G'$ is called the *extended fold determined by $\rho_r$*.

We think of $G \setminus E_2$ as a subgraph of $G'$ on which $F$ is the identity. By construction $F(E_2) = E_1 b E_2'$. The filtration on $G'$ is defined by $H_i' = H_i$ for $i \neq r$ and $H_r' = (H_r \setminus E_2) \cup E_2'$. There is a map $g: G' \to G$ such that $gF = f$. We refer to $g: G' \to G$ as *map induced by the extended fold.*

The following lemma states that the map $f': G' \to G'$ obtained from $Fg: G' \to G'$ by tightening the images of edges is a relative train track map that satisfies item (Z) of Theorem 2.19. We say that $f': G' \to G'$ is obtained from $f: G \to G$ by *folding $\rho_r$* and that $\rho_r' = F_\#(\rho_r)$ is *the indivisible Nielsen path determined by $\rho_r$*. If the fold at the illegal turn of $\rho_r'$ is proper then this process can be repeated. This is referred to as *iteratively folding $\rho$*.

**Lemma 2.22.** *Assuming notation as above, $f'\colon G' \to G'$ is a relative train track map that satisfies item* (Z) *of Theorem* 2.19.

*Proof.* By construction, $f'|G_{r-1} = f|G_{r-1}$. If $E$ is an edge in $H_r$ then $f(E)$ does not cross the illegal turn in $\rho_r$. If $E \neq E_2$ then $f'(E)$ is obtained from $f(E)$ by replacing each occurrence of $E_2$ with $E_1 b E_2''$. Similarly, $f'(E_2')$ is obtained from $f(E_2')$ by replacing each occurrence of $E_2$ with $E_1 b E_2''$. It follows that $H_r'$ satisfies (RTT-i)–(RTT-iii).

If $H_k$ is a zero stratum above $H_r$ then each edge $E_k$ in $H_k$ is a connecting path for some EG stratum $H_s$ above $H_k$ by item (Z) of Theorem 2.19. Thus $f_\#^i(E_k)$ is non-trivial for all $i \geq 0$. Since $F$ does not identify points that are not identified by $f$, and since $F|E_k$ is the identity, $(Fg)_\#(E_k) = (Ff)_\#(E_k)$ is non-trivial. This shows that no edges are collapsed when $Fg$ is tightened to $f'$. The same argument shows that if $\sigma \subset H_k$ is any path with endpoints at vertices then $f_\#'(\sigma)$ is non-trivial.

If $H_l$ is NEG then $H_l'$ is NEG.

Suppose that $E_m$ is an edge in an EG stratum $H_m$ above $H_r$ and that $f(E_m) = \mu_1 \nu_1 \mu_2 \ldots \nu_l \mu_{l+1}$ is the decomposition into subpaths $\mu_j \subset H_m$ and subpaths $\nu_j \subset G_{m-1}$. Then $f'(E_m) = (Fg)_\#(E_m) = (Ff)_\#(E_m) = \mu_1 \nu_1' \mu_2 \ldots \nu_l' \mu_{l+1}$ where $\nu_j' = F_\#(\nu_j)$ is non-trivial because $f_\#(\nu_j)$ is non-trivial. This proves that $H_m'$ satisfies (RTT-i) and (RTT-iii).

To verify (RTT-ii) for $H_m$ suppose that $\sigma'$ is a connecting path for $H_m'$. If $\sigma'$ is contained in a zero stratum $H_k'$ then it is disjoint from $G_r$ and so is identified with a connecting path $\sigma \subset H_k$. By our previous argument, $f_\#'(\sigma')$ is non-trivial. If $\sigma'$ is contained in non-contractible component of $G_{m-1}'$ then there is a connecting path $\sigma$ for $H_m$ in a non-contractible component of $G_{m-1}$ such that $F_\#(\sigma) = \sigma'$. The endpoints of $\sigma$ are periodic for $f$ by Remark 2.8. It follows that the endpoints of $\sigma'$ are periodic for $f'$ and another application of Remark 2.8 proves that $H_m'$ satisfies (RTT-ii). This completes the proof that $f'\colon G' \to G'$ is a relative train track map.

Item (Z) of Theorem 2.19 for $f'$ therefore follows from item (Z) of Theorem 2.19 for $f$. □

## 3. Forward rotationless outer automorphisms

To avoid issues raised by finite order phenomenon, one often replaces $\phi \in \mathrm{Out}(F_n)$ with an iterate $\phi^k$. In this section we explain how this can be done canonically by exhibiting the natural class of outer automorphisms that require no iteration. We also define principal automorphisms in the context of $\mathrm{Out}(F_n)$. These automorphisms play a central role in both the definition of forward rotationless outer automorphisms (Definition 3.13) and in the formulation of the Recognition Theorem (Theorem 5.3).

In Section 3.1, we recall how principal automorphisms occur in the context of the mapping class group. Examples and definitions for $\mathrm{Out}(F_n)$ are given in Section 3.2. An equivalent definition is then given in terms of relative train track maps and the

Nielsen classes of their fixed points. Finally, in Section 3.5 we record some properties of forward rotationless outer automorphisms that justify their name; for example, we show that a $\phi$-periodic free factor is $\phi$-invariant.

**3.1. The Nielsen approach to the mapping class group.** To provide historical context and motivation for our techniques and results, we briefly recall Nielsen's point of view on the mapping class group. Further details and proofs can be found, for example, in [13].

Let $S$ be a closed orientable surface of negative Euler characteristic and let $h\colon S \to S$ be a homeomorphism representing an element $\mu \in \mathrm{MCG}(S)$. A choice of complete hyperbolic structure on $S$ identifies the universal cover $\widetilde{S}$ of $S$ with the hyperbolic plane $\mathbb{H}$. Using the Poincaré disk model for $\mathbb{H}$, there is an induced compactification of $\widetilde{S}$ by adding a topological circle $S_\infty$.

To avoid cumbersome superscripts we use $g$ to denote a positive iterate $g := h^k$ of $h$. Any lift $\tilde{g}\colon \widetilde{S} \to \widetilde{S}$ of $g$ extends to a homeomorphism of the compactification. The restriction of this extension to $S_\infty$, denoted $\hat{g}\colon S_\infty \to S_\infty$, depends only on $k$, the isotopy class of $h$ and the choice of lift. More precisely, $h$ induces an outer automorphism of $\pi_1(S)$ and $\hat{g} = \widehat{\Phi_{\tilde{g}}}$ where $\Phi_{\tilde{g}}$ is the automorphism of $\pi_1(S)$ corresponding to $\tilde{g}$ and $\widehat{\Phi_{\tilde{g}}}\colon S_\infty \to S_\infty$ is the homeomorphism determined by the identification of $S_\infty$ with $\partial\pi_1(S)$ [11].

Denote the set of non-repelling fixed points of $\hat{g}$ by $\mathrm{Fix}_N(\hat{g})$. If $\mathrm{Fix}_N(\hat{g})$ contains at least three points then we say that $\tilde{g}$ is a *principal lift of* $g$ and that $\Phi_{\tilde{g}}$ is a principal automorphism representing $\mu^k$. The sets $\mathrm{Fix}_N(\hat{g})$ determined by the principal lifts of iterates of $h$ are central to Nielsen's investigations; see for example [17] (or its translation into English by John Stillwell which appears on pages 348–400 of [18]).

The mapping class $\mu$ determined by $h$ is rotationless as defined in the Section 1 if and only for all $k$, each principal lift $\tilde{g}$ of $g = h^k$ has the form $\tilde{h}^k$ where $\tilde{h}$ is a principal lift of $h$ and where $\mathrm{Fix}_N(\hat{g}) = \mathrm{Fix}_N(\hat{h})$. Thus from the point of view of principal lifts and their $\mathrm{Fix}_N$ sets, nothing changes if $\mu$ is replaced by an iterate. For the remainder of this discussion we assume that $\mu$ is rotationless and that $k = 1$.

The intersection $\tilde{\Delta}(\tilde{g})$ of the convex hull of $\mathrm{Fix}_N(\hat{g})$ with $\widetilde{S} = \mathbb{H}$ is called the *principal region* for $\tilde{g}$ and its image in $S$ is denoted $\Delta(\tilde{g})$. Thus $\tilde{g}$ is principal if and only if $\Delta(\tilde{g})$ has non-empty interior.

Assume that $\tilde{g}$ is principal. If no point in $\mathrm{Fix}_N(\hat{g})$ is isolated then $\Delta(\tilde{g})$ is a compact subsurface and there is a homeomorphism $f\colon S \to S$ representing $\mu$ whose restriction to $\Delta(\tilde{g})$ is the identity. If $\mathrm{Fix}_N(\hat{g})$ is finite, or more generally, is finite up to the action of a single covering translation that commutes with $\tilde{g}$, then the interior of $\Delta(\tilde{g})$ is a component of the complement in $S$ of one of the pseudo-Anosov laminations $\Lambda$ associated to $\mu$. The boundary of $\Delta(\tilde{g})$ is a finite union of leaves of $\Lambda$ and perhaps one reducing curve. These are the only cases that occur if there is non-trivial twisting along each reducing curve in the Thurston normal form for $\mu$. In the general case, $\Delta(\tilde{g})$ is a finite union of the two types.

A proof of (most of) the Thurston classification theorem from this point of view is contained in [13] and [16].

**3.2. Principal automorphisms.** Suppose that $f: G \to G$ is a relative train track map representing $\phi \in \mathrm{Out}(F_n)$. Recall from Section 2.3 that there is a bijection between lifts $\tilde{f}: \Gamma \to \Gamma$ to the universal cover and automorphisms $\Phi \in \mathrm{Aut}(F_n)$ representing $\phi$.

**Definition 3.1.** For $\Phi \in \mathrm{Aut}(F_n)$ representing $\phi$, let $\mathrm{Fix}_N(\hat{\Phi}) \subset \mathrm{Fix}(\hat{\Phi})$ be the set of non-repelling fixed points of $\hat{\Phi}$. We say that $\Phi$ is a *principal automorphism* and write $\Phi \in \mathrm{P}(\phi)$ if either of the following hold.

- $\mathrm{Fix}_N(\hat{\Phi})$ contains at least three points.
- $\mathrm{Fix}_N(\hat{\Phi})$ is a two point set that is neither the set of endpoints of an axis $A_c$ nor the set of endpoints of a lift $\tilde{\lambda}$ of a generic leaf of an element of $\mathcal{L}(\phi)$.

The corresponding lift $\tilde{f}: \Gamma \to \Gamma$ is a *principal lift*.

**Remark 3.2.** For all $\phi \in \mathrm{Out}(F_n)$ there exists, by Lemma 5.2 of [3] or Proposition 9.4 of [15], $k \geq 1$ such that $\mathrm{P}(\phi^k) \neq \emptyset$. Moreover, if the conjugacy class of $a \in F_n$ is invariant under $\phi^k$, then one may choose $\Phi \in \mathrm{P}(\phi^k)$ to fix $a$.

**Remark 3.3.** If $\mathrm{Fix}(\Phi)$ has rank at least two then $\Phi$ is a principal automorphism by Lemma 2.3.

**Remark 3.4.** If $\Phi_1$ and $\Phi_2$ are distinct representatives of $\phi$ then $\mathrm{Fix}_N(\hat{\Phi}_1) \cap \mathrm{Fix}_N(\hat{\Phi}_2)$ is contained in $\mathrm{Fix}(\hat{\Phi}_1^{-1}\hat{\Phi}_2) = \{T_c^{\pm}\}$ for some non-trivial covering translation $T_c$. It follows that if $\Phi_1$ and $\Phi_2$ are principal then $\mathrm{Fix}_N(\hat{\Phi}_1) \neq \mathrm{Fix}_N(\hat{\Phi}_2)$.

**Remark 3.5.** The second item in our definition of principal automorphism does not occur in the context of mapping class groups. It arises in $\mathrm{Out}(F_n)$ to account for nonlinear NEG strata. In Example 3.10 below both attracting fixed points correspond to nonlinear NEG strata. One can also construct examples in which one attracting fixed point corresponds to a nonlinear NEG stratum and the other to an EG stratum.

**Remark 3.6.** Each $\Lambda \in \mathcal{L}(\phi)$ has infinitely many generic leaves that are invariant by an iterate of $\phi_\#$. If $\{P, Q\}$ is the endpoint set of a lift of such a leaf then (Lemma 4.38) there exists $\Phi$ representing an iterate of $\phi$ such that $P$ and $Q$ are attracting fixed points for $\hat{\Phi}$. Remark 3.9 and Lemma 2.3 imply that for all but finitely many such leaves, $\mathrm{Fix}_N(\hat{\Phi}) = \{P, Q\}$ and $\Phi$ is not principal.

**Remark 3.7.** If $\Phi$ has positive index in the sense of [12], then $\Phi$ is a principal automorphism. The converse fails for the principal automorphism $\Phi_2$ of Example 3.10.

We say that $x, y \in \mathrm{Fix}(f)$ are *Nielsen equivalent* or belong to the same *Nielsen class* if they are the endpoints of a Nielsen path for $f$. Each Nielsen class is an open subset of $\mathrm{Fix}(f)$ because every sufficiently short path with endpoints in $\mathrm{Fix}(f)$ is a Nielsen path. In particular, there are only finitely many Nielsen classes.

If $\tilde{f} : \Gamma \to \Gamma$ is a lift of $f : G \to G$, then any path $\tilde{\alpha} \subset \Gamma$ with endpoints in $\mathrm{Fix}(\tilde{f})$ projects to a Nielsen path $\alpha \subset G$ for $f$. Conversely, if $\alpha$ is a Nielsen path for $f$ and $\tilde{f}$ fixes one endpoint of a lift $\tilde{\alpha}$ of $\alpha$ then $\tilde{f}$ also fixes the other endpoint of $\tilde{\alpha}$. Thus $\mathrm{Fix}(\tilde{f})$ is either empty or projects onto a single Nielsen class in $\mathrm{Fix}(f)$.

A pair of automorphisms $\Phi_1$ and $\Phi_2$ *are equivalent* if there exists $c \in F_n$ such that $\Phi_2 = i_c \Phi_1 i_c^{-1}$. Translating this into the language of lifts, $\tilde{f}_1$ is equivalent to $\tilde{f}_2$ if $\tilde{f}_2 = T_c \tilde{f}_1 T_c^{-1}$. This equivalence relation is called *isogredience*.

**Lemma 3.8.** *Suppose that $f : G \to G$ represents $\phi \in \mathrm{Out}(F_n)$ and that $\tilde{f}_1$ and $\tilde{f}_2$ are lifts of $f$ with non-empty fixed point sets. Then $\tilde{f}_1$ and $\tilde{f}_2$ belong to the same isogredience class if and only if $\mathrm{Fix}(\tilde{f}_1)$ and $\mathrm{Fix}(\tilde{f}_2)$ project to the same Nielsen class in $\mathrm{Fix}(f)$.*

*Proof.* If $\tilde{f}_2 = T_c \tilde{f}_1 T_c^{-1}$ then $\mathrm{Fix}(\tilde{f}_2) = T_c\,\mathrm{Fix}(\tilde{f}_1)$ and $\mathrm{Fix}(\tilde{f}_2)$ and $\mathrm{Fix}(\tilde{f}_1)$ project to the same Nielsen class in $\mathrm{Fix}(f)$. Conversely, if $\mathrm{Fix}(\tilde{f}_2)$ and $\mathrm{Fix}(\tilde{f}_1)$ have the same non-trivial projection then there exists $\tilde{x} \in \mathrm{Fix}(\tilde{f}_2)$ and a covering translation $T_c$ such that $T_c(\tilde{x}) \in \mathrm{Fix}(\tilde{f}_1)$ which implies that $\tilde{f}_2$ and $T_c \tilde{f}_1 T_c^{-1}$ agree on a point and hence are equal. □

**Remark 3.9.** We show below (Corollary 3.17) that principal lifts have non-trivial fixed point sets in $\Gamma$. Since there are only finitely many Nielsen classes in $\mathrm{Fix}(f)$, it follows that there are only finitely many isogredience classes of principal lifts for $\phi$.

In the following examples, $G$ is the rose $R_3$ with basepoint $v$ at the unique vertex. We use $A, B$ and $C$ to denote both the oriented edges of $G$ and the corresponding generators of $F_3$. Our examples are all positive automorphisms $\Phi$, meaning that they are defined by $A \mapsto w_A$, $B \mapsto w_B$ and $C \mapsto w_C$ where $w_A, w_B$ and $w_C$ are words in the letters $A, B$ and $C$ (and not the inverses $\bar{A}, \bar{B}$ and $\bar{C}$). These words also define a homotopy equivalence $f : G \to G$. Since $w_A, w_B$ and $w_C$ use only $A, B$ and $C$, and not $\bar{A}, \bar{B}$ and $\bar{C}$, the homotopy equivalence is a relative train track map for $\phi$.

The universal cover of $\tilde{G}$ is denoted $\Gamma$ and we assume that a basepoint $\tilde{v}$ has been chosen. Some statements in the examples are left for the reader to verify or follow from results we establish later in this section; none of these statements are ever quoted.

**Example 3.10.** Let $\Phi_1 \in \mathrm{Aut}(F_3)$ be determined by $w_A = A, w_B = BA$ and $w_C = BCB^2$. Then $\mathrm{Fix}(\Phi_1) = \langle A, BA\bar{B} \rangle$ and $\mathrm{Fix}(\hat{\Phi}_1) = \partial(\mathrm{Fix}(\Phi_1))$. The lift $\tilde{f}_1$ that fixes $\tilde{v}$ is the principal lift corresponding to $\Phi_1$.

The unique fixed point $x$ of $f$ in the interior of $C$ is not Nielsen equivalent to $v$. Let $\widetilde{C}$ be the lift of $C$ whose initial endpoint is $\tilde{v}$ and let $\tilde{f}_2$ be the lift that fixes the unique lift of $\tilde{x}$ of $x$ in $\widetilde{C}$. Then $\tilde{f}_2$ is principal and $\mathrm{Fix}_N(\hat{f}_2)$ is a pair of attractors which bound the line that is the union of the increasing sequence $\widetilde{C} \subset (\tilde{f}_2)_\#(\widetilde{C}) \subset (\tilde{f}_2)^2_\#(\widetilde{C}) \subset \cdots$. If $\Phi_2$ is the principal automorphism corresponding to $\tilde{f}_2$ then $\Phi_2 = i_B^{-1}\Phi_1$.

**Example 3.11.** Let $\Phi_1 \in \mathrm{Aut}(F_3)$ be determined b $w_A = A, w_B = BA$ and $w_C = CB^2$. Then $\mathrm{Fix}(\Phi) = \langle A, BA\bar{B}\rangle$ and $\mathrm{Fix}(\hat{\Phi})$ is the union of $\partial(\mathrm{Fix}(\Phi))$ with the $\mathrm{Fix}(\Phi)$-orbit of a single attractor $P$. The lift $\tilde{f}$ that fixes $\tilde{v}$ is the principal lift corresponding to $\Phi$ and $P$ is the endpoint of the ray that is the union of the increasing sequence $\widetilde{C} \subset (\tilde{f})_\#(\widetilde{C}) \subset (\tilde{f})^2_\#(\widetilde{C}) \subset \cdots$.

**Example 3.12.** Let $\Phi \in \mathrm{Aut}(F_3)$ be determined by $w_A = ACBA, w_B = BA$ and $w_C = CBA$, let $\tilde{f}$ be the lift that fixes $\tilde{v}$ and let $\tilde{A}, \widetilde{B}, \widetilde{C}$ and $\tilde{A}^{-1}$ be the lifts of the oriented edges $A, B, C$ and $\bar{A}$ with $\tilde{v}$ as initial vertex. The directions determined by the initial edges of $\tilde{A}, \widetilde{B}, \widetilde{C}$ and $\tilde{A}^{-1}$ are fixed by $D\tilde{f}$. Lemma 2.13 produces attractors $P_A, P_B, P_C$ and $P_{\bar{A}}$ in $\mathrm{Fix}_N(\hat{f})$ such that lines connecting $P_{\bar{A}}$ to the other three points are generic leaves of an attracting lamination. Lemma 3.21 implies that $\mathrm{Fix}_N(\hat{f}) = \{P_A, P_B, P_C, P_{\bar{A}}\}$.

We now come to the second main definition of this section. Note that if $\Phi$ is a principal lift of $\phi$ then $\Phi^k$ is a principal lift for $\phi^k$ and $\mathrm{Fix}_N(\hat{\Phi}) \subset \mathrm{Fix}_N(\hat{\Phi}^k)$ for all $k \geq 1$. The set of non-repelling periodic points in $\mathrm{Per}(\hat{\Phi})$ is denoted $\mathrm{Per}_N(\hat{\Phi})$. By iterating $\phi$ we might pick up more principal lifts and principal lifts might pick up more non-repelling fixed points. If this doesn't happen, then we say that $\phi$ is *forward rotationless*. Here is the precise definition.

**Definition 3.13.** An outer automorphism $\phi$ is *forward rotationless* if $\mathrm{Fix}_N(\hat{\Phi}) = \mathrm{Per}_N(\hat{\Phi})$ for all $\Phi \in \mathrm{P}(\phi)$ and if for each $k \geq 1$, $\Phi \mapsto \Phi^k$ defines a bijection (see Remark 3.14) between $\mathrm{P}(\phi)$ and $\mathrm{P}(\phi^k)$. Our standing assumption is that $n \geq 2$. For notational convenience we also say that the identity element of $\mathrm{Out}(F_1)$ is forward rotationless.

**Remark 3.14.** By Remark 3.4 there is no loss in replacing the assumption that $\Phi \mapsto \Phi^k$ defines a bijection with the a priori weaker assumption that $\Phi \mapsto \Phi^k$ defines a surjection.

**3.3. Rotationless relative train track maps and principal periodic points.** We now want to characterize those relative train track maps $f : G \to G$ that represent forward rotationless $\phi \in \mathrm{Out}(F_n)$ and to determine which lifts of such $f$ are principal. We precede our main definitions by showing that principal lifts have fixed points.

Suppose that $f : G \to G$ represents $\phi$ and that $\tilde{f} : \Gamma \to \Gamma$ is a lift of $f$. We say that $\tilde{z} \in \Gamma$ *moves toward* $P \in \mathrm{Fix}(\hat{f})$ under the action of $\tilde{f}$ if the ray from

$\tilde{f}(\tilde{z})$ to $P$ does not contain $\tilde{z}$. Similarly, we say that *$\tilde{f}$ moves $\tilde{y}_1$ and $\tilde{y}_2$ away from each other* if the path in $\Gamma$ connecting $\tilde{f}(\tilde{y}_1)$ to $\tilde{f}(\tilde{y}_2)$ contains $\tilde{y}_1$ and $\tilde{y}_2$ and if $\tilde{f}(\tilde{y}_1) < \tilde{y}_1 < \tilde{y}_2 < \tilde{f}(\tilde{y}_2)$ in the order induced by the orientation on that path.

The following lemma relates the action of $\hat{f}$ to the action of $\tilde{f}$ and gives a criterion for elements of $\mathrm{Fix}(\hat{f})$ to be contained in $\mathrm{Fix}_N(\hat{f})$. Recall that $\partial F_n$ is identified with the set of ends of $\Gamma$. It therefore makes sense to say that points in $\Gamma$ are close to $P \in \partial F_n$ or that $P$ is the limit of points in $\Gamma$.

**Lemma 3.15.** *Suppose that $P \in \mathrm{Fix}(\hat{f})$ and that there does not exist $c \in F_n$ such that $\mathrm{Fix}(\hat{f}) = \{T_c^{\pm}\}$.*

(1) *If $P$ is an attractor for the action of $\hat{f}$ on $\partial\Gamma$ then $\tilde{z}$ moves toward $P$ under the action of $\tilde{f}$ for all $\tilde{z} \in \Gamma$ that are sufficiently close to $P$.*
(2) *If $P$ is an endpoint of an axis $A_c$ or if $P$ is the limit of points in $\Gamma$ that are either fixed by $\tilde{f}$ or that move toward $P$ under the action of $\tilde{f}$, then $P \in \mathrm{Fix}_N(\hat{f})$.*

*Proof.* The lemma is an immediate consequence of Proposition 1.1 of [12] if $P$ is not the endpoint $T_c^{\pm}$ of an axis $A_c$. If $P$ is $T_c^{+}$ or $T_c^{-}$, then $\mathrm{Fix}(\hat{f})$ contains $\{T_c^{\pm}\}$ and at least one other point. Lemma 2.3 implies that $P$ is not isolated in $\mathrm{Fix}(\hat{f})$ and is therefore neither an attractor nor a repeller for the action of $\hat{f}$. □

The next lemma is based on Lemma 2.1 of [4].

**Lemma 3.16.** *If $\tilde{f}$ moves $\tilde{y}_1$ and $\tilde{y}_2$ away from each other, then $\tilde{f}$ fixes a point in the interval bounded by $\tilde{y}_1$ and $\tilde{y}_2$.*

*Proof.* Denote the oriented paths connecting $\tilde{y}_1$ to $\tilde{y}_2$ and $\tilde{f}(\tilde{y}_1)$ to $\tilde{f}(\tilde{y}_2)$ by $\tilde{\alpha}_0$ and $\tilde{\alpha}_1$ respectively. Let $r\colon \Gamma \to \tilde{\alpha}_1$ be retraction onto the nearest point in $\tilde{\alpha}_1$ and let $g = r\tilde{f} : \tilde{\alpha}_0 \to \tilde{\alpha}_1$. By hypothesis, $\tilde{\alpha}_0$ is a proper subpath of $\tilde{\alpha}_1$ and $g$ is a surjection. If $\tilde{y}$ is the first point in $\tilde{\alpha}_0$ such that $\tilde{g}(\tilde{y}) = \tilde{y}$ then $\tilde{y}_1 < \tilde{y} < \tilde{y}_2$ and $\tilde{g}(\tilde{z}) < \tilde{g}(\tilde{y})$ for $y_1 < z < y$. It follows that $\tilde{f}(\tilde{y}) \in \tilde{\alpha}_1$ and hence that $\tilde{y}$ is fixed by $\tilde{f}$. □

**Corollary 3.17.** *If $\tilde{f}$ is a principal lift then* $\mathrm{Fix}(\tilde{f}) \neq \emptyset$.

*Proof.* Suppose that there is a non-trivial covering translation $T_c$ that has its endpoints in $\mathrm{Fix}_N(\hat{f})$ and so commutes with $\tilde{f}$. Assuming without loss that $A_c$ is fixed point free, there is a point in $A_c$ that moves toward one of the endpoints of $A_c$, say $P$. Since $\tilde{f}$ commutes with $T_c$, there are points in $\Gamma$ that are arbitrarily close to $P$ and that move toward $P$. The same property holds for an attractor $P \in \mathrm{Fix}(\hat{f})$ by Lemma 3.15. One may therefore choose distinct $P_1, P_2 \in \mathrm{Fix}_N(\hat{f})$ and $\tilde{x}_1, \tilde{x}_2 \in \tilde{\Gamma}$ such that $\tilde{x}_i$ is close to and moves toward $P_i$. It follows that $\tilde{x}_1$ and $\tilde{x}_2$ move away from each other. Lemma 3.16 produces the desired fixed point. □

There are two cases in which a lift $\tilde{f}$ corresponding to a Nielsen class in $\mathrm{Fix}(f)$ is not principal. The first arises from a 'non-singular' leaf of an attracting lamination as noted in Remark 3.6; in this case $\mathrm{Fix}(\tilde{f})$ is a single point. In the second case, there is a circle component of $\mathrm{Fix}(f)$ with no outward pointing periodic directions and $\mathrm{Fix}(\tilde{f})$ is an axis $A_c$. The second type could be eliminated by adding properties to Theorem 2.19. We allow the circle components for now and defer the additional properties until Section 4.

Periodic points for $f$ are *Nielsen equivalent* if they are Nielsen equivalent as fixed points for some iterate of $f$.

**Definition 3.18.** We say that $x \in \mathrm{Per}(f)$ is *principal* if neither of the following conditions are satisfied.

- $x$ is the only element of $\mathrm{Per}(f)$ in its Nielsen class and there are exactly two periodic directions at $x$, both of which are contained in the same EG stratum.
- $x$ is contained in a component $C$ of $\mathrm{Per}(f)$ that is topologically a circle and each point in $C$ has exactly two periodic directions.

Lifts to $\Gamma$ of principal periodic points in $G$ are said to be *principal*. If each principal vertex and each periodic direction at a principal vertex has period one then we say that $f : G \to G$ is *rotationless*.

In practice, we only apply these definitions to $f : G \to G$ that satisfy the conclusions of Theorem 2.19. In particular, by item (4) of Lemma 2.20, there are at least two periodic directions at each $x \in \mathrm{Per}(f)$.

Principal periodic points are either contained in periodic edges or are vertices. Thus every $f : G \to G$ has a rotationless iterate. Any endpoint of an indivisible periodic Nielsen path is principal as is the initial endpoint of any non-periodic NEG edge. The latter implies that each NEG stratum in a rotationless relative train track map is a single edge.

The following lemma shows that an $EG$ stratum has at least one principal vertex.

**Lemma 3.19.** *Assume that $f : G \to G$ satisfies the conclusions of Theorem 2.19. For every EG stratum $H_r$ there is a principal vertex whose link contains a periodic direction in $H_r$.*

*Proof.* If some vertex $v \in H_r$ belongs to a non-contractible component of $G_{r-1}$ then $v$ is periodic and there is at least one periodic direction in $H_r$ by Lemma 2.9. There is also at least one periodic direction at $v$ determined by an edge of $G_{r-1}$ so $v$ is principal. If there is no such vertex, then $H_r$ is a union of components of $G_r$. Lemma 5.2 of [3] states there is a principal lift of some iterate of $f|H_r$. In the course of proving this lemma, it is shown that either there is a vertex with three periodic directions or there is an indivisible periodic Nielsen path in $H_r$. In either case there is a vertex that is principal for $f|H_r$ and hence also for $f$. $\square$

**Remark 3.20.** Lemma 3.19 implies that the transition matrix $M_r$ of an EG stratum of a rotationless $f : G \to G$ satisfying the conclusions of Theorem 2.19 has at least one non-zero diagonal entry and so is aperiodic. For each $\Lambda \in \mathcal{L}(\phi)$ there is an EG stratum $H_r$ such that $\Lambda$ has height $r$ and this defines a bijection (see Definition 3.1.12 of [2]) between $\mathcal{L}(\phi)$ and the set of EG strata.

The next lemma relates an attractor in $\mathrm{Fix}_N(\hat{f})$ to a fixed direction of $D\tilde{f}$.

**Lemma 3.21.** *Suppose that $\tilde{f}$ is a principal lift of a relative train track map $f : G \to G$.*

1. *For each attractor $P \in \mathrm{Fix}_N(\hat{f})$ there is a (not necessarily unique) $\tilde{x} \in \mathrm{Fix}(\tilde{f})$ such that the interior of the ray $\widetilde{R}_{\tilde{x},P}$ that starts at $\tilde{x}$ and that converges to $P$ is fixed point free.*
2. *If $P \in \mathrm{Fix}_N(\hat{f})$ is an attractor, if $\tilde{x} \in \mathrm{Fix}(\tilde{f})$ and if the interior of $\widetilde{R}_{\tilde{x},P}$ is fixed point free then no point in the interior of $\widetilde{R}_{\tilde{x},P}$ is mapped by $\tilde{f}$ to $\tilde{x}$; in particular, the initial direction determined by $\widetilde{R}_{\tilde{x},P}$ is fixed.*
3. *If $P$ and $Q$ are distinct attractors in $\mathrm{Fix}_N(\hat{f})$, if $\tilde{x} \in \mathrm{Fix}(\tilde{f})$ and if the interiors of both $\widetilde{R}_{\tilde{x},P}$ and $\widetilde{R}_{\tilde{x},Q}$ are fixed point free then the directions determined by $\widetilde{R}_{\tilde{x},P}$ and $\widetilde{R}_{\tilde{x},Q}$ are distinct.*

*Proof.* To find $\tilde{x} \in \mathrm{Fix}(\tilde{f})$ and $\widetilde{R}_{\tilde{x},P}$ as in (1), start with any ray $\widetilde{R}'$ whose initial point is in $\mathrm{Fix}(\tilde{f})$ and that converges to $P$ and let $\widetilde{R}_{\tilde{x},P}$ be the subray of $\widetilde{R}'$ that begins at the last point $\tilde{x}$ of $\mathrm{Fix}(\tilde{f})$ in $\widetilde{R}'$. If $\widetilde{R}_{\tilde{x},P}$ and $\widetilde{R}_{\tilde{x},Q}$ are as in (3) and have the same initial edge then their 'difference' would be a fixed point free line whose ends converge to attractors in contradiction to Lemma 3.15 and Lemma 3.16. This verifies (3). By the same reasoning, no points in the interior of $\widetilde{R}_{\tilde{x},P}$ can map to $\tilde{x}$, which implies that the initial edge of $\widetilde{R}_{\tilde{x},P}$ determines a fixed direction at $\tilde{x}$. This proves (2). □

**Corollary 3.22.** *Assume that $f : G \to G$ satisfies the conclusions of Theorem 2.19. If $\tilde{f}$ is a principal lift then each element of $\mathrm{Fix}(\tilde{f})$ is principal.*

*Proof.* Let $\Phi$ be the automorphism corresponding to $\tilde{f}$. If $\mathrm{Fix}(\Phi)$ has rank at least two then $\mathrm{Fix}(\tilde{f})$ is neither a single point nor a single axis and we are done (see Corollary 3.17 and Lemma 2.1). If $\mathrm{Fix}(\Phi)$ has rank one then $\mathrm{Fix}(\tilde{f})$ is infinite and $\mathrm{Fix}_N(\hat{f})$ contains an attractor by the definition of principal lift and by Lemma 2.3. Lemma 3.21 implies that some $\tilde{x} \in \mathrm{Fix}(\tilde{f})$ has a fixed direction that does not come from a fixed edge and again we are done. In the remaining case, $\mathrm{Fix}(\hat{f})$ is a finite set of attractors and does not contain the endpoints of any axis. Obviously $\mathrm{Fix}(\tilde{f})$ is not an axis. Suppose that $\mathrm{Fix}(\tilde{f})$ is a single point $\tilde{x}$, that there are only two periodic directions at $\tilde{x}$ and these two directions are determined by lifts $\widetilde{E}_1$ and $\widetilde{E}_2$ of oriented edges of the same EG stratum $H_r$. Lemma 3.21 and Lemma 2.13 imply that $\mathrm{Fix}(\hat{f})$ is the endpoint set of a generic leaf of an element of $\mathcal{L}(\phi)$ in contradiction to the assumption that $\tilde{f}$ is principal. We conclude that $\tilde{x}$ is principal as desired. □

To prove the converse we use fixed directions of $\tilde{f}$ to find elements of $\mathrm{Fix}_N(\hat{f})$. The following lemma is from [2]; the proof is short and is repeated for the readers convenience.

**Lemma 3.23.** *If* $\mathrm{Fix}(\tilde{f}) = \emptyset$ *then there is a ray* $\widetilde{R} \subset \Gamma$ *converging to an element* $P \in \mathrm{Fix}(\hat{f})$ *and there are points in* $\widetilde{R}$ *arbitrarily close to* $P$ *that move toward* $P$.

*Proof.* For each vertex $\tilde{y}$ of $\Gamma$, we say that the initial edge of the path from $\tilde{y}$ to $\tilde{f}(\tilde{y})$ is *preferred* by $\tilde{y}$. Starting with any vertex $\tilde{y}_0$, inductively define $\tilde{y}_{i+1}$ to be the other endpoint of the edge preferred by $\tilde{y}_i$. If $\widetilde{E}$ is preferred by both of its endpoints then $\tilde{f}$ maps a proper subinterval of $\widetilde{E}$ over all of $\widetilde{E}$ (reversing orientation) in contradiction to the assumption that $\mathrm{Fix}(\tilde{f}) = \emptyset$. It follows that the $\tilde{y}_i$'s are contained in a ray that converges to some $P \in \mathrm{Fix}(\hat{f})$ and that $\tilde{y}_i$ moves toward $P$. □

We isolate the following notation and lemma for reference throughout the paper.

**Notation 3.24.** Suppose that $f: G \to G$ satisfies the conclusions of Theorem 2.19, that $H_r$ is a single edge $E_r$ and that $f(E_r) = E_r u$ for some non-trivial path $u \subset G_{r-1}$. Let $\widetilde{E}_r$ be a lift of $E_r$ and let $\tilde{f}: \Gamma \to \Gamma$ be the lift of $f$ that fixes the initial endpoint of $\widetilde{E}_r$. By (NEG), the component $C$ of $G_{r-1}$ that contains the terminal endpoint $w$ of $E_r$ is not contractible. Denote the copy of the universal cover of $C$ that contains the terminal endpoint of $\widetilde{E}_r$ by $\Gamma_{r-1}$ and the restriction $\tilde{f}|\Gamma_{r-1}$ by $h\colon \Gamma_{r-1} \to \Gamma_{r-1}$.

The covering translations that preserve $\Gamma_{r-1}$ define a free factor $F(C)$ of $F_n$ such that $[[F(C)]] = [[\pi_1(C)]]$. The closure in $\partial F_n$ of $\{T_c^\pm : c \in F(C)\}$ is naturally identified with $\partial F(C)$ and with the space of ends of $\Gamma_{r-1}$. Moreover, $\hat{h} = \hat{f}|\partial F(C)\colon \partial F(C) \to \partial F(C)$.

**Lemma 3.25.** *Assume that* $\tilde{f}$ *and* $h$ *are as in Notation 3.24. If* $\mathrm{Fix}(h) = \emptyset$ *then there is a ray* $\widetilde{R} \subset \Gamma_{r-1}$ *converging to an element* $P \in \mathrm{Fix}(\hat{h})$ *and there are points in* $\widetilde{R}$ *arbitrarily close to* $P$ *that move toward* $P$.

*Proof.* This follows from Lemma 3.23 applied to $h\colon \Gamma_{r-1} \to \Gamma_{r-1}$. □

Our next result is an extension of Lemma 2.13.

**Lemma 3.26.** *Suppose that* $f: G \to G$ *satisfies the conclusions of Theorem 2.19 and is rotationless, that* $\tilde{f}: \Gamma \to \Gamma$ *is a lift of* $f$*, that* $\tilde{v} \in \mathrm{Fix}(\tilde{f})$ *and that* $D\tilde{f}$ *fixes the direction at* $\tilde{v}$ *determined by a lift* $\widetilde{E}$ *of an edge* $E \subset H_r$*. Then there exists* $P \in \mathrm{Fix}(\hat{f})$ *so that the ray* $\widetilde{R}$ *from the initial endpoint of* $\widetilde{E}$ *to* $P$ *contains* $\widetilde{E}$ *and satisfies the following properties.*

(1) *There are points in $\widetilde{R}$ arbitrarily close to $P$ that are either fixed or move toward $P$. If there does not exist $c \in F_n$ such that* $\mathrm{Fix}(\hat{f}) = \{T_c^{\pm}\}$ *then* $P \in \mathrm{Fix}_N(\hat{f})$.

(2) *If $H_r$ is EG then $P$ is an attractor whose accumulation set is the unique attracting lamination of height $r$, the interior of $\widetilde{R}$ is fixed point free and $\widetilde{R}$ projects to an $r$-legal ray in $G_r$.*

(3) *If $H_r$ is NEG and non-fixed then $\widetilde{R} \setminus \widetilde{E}$ projects into $G_{r-1}$.*

(4) *No point in the interior of $\widetilde{R}$ is mapped to $\tilde{v}$ by any iterate of $\tilde{f}$.*

*Proof.* The second part of (1) follows from the first part of (1) and Lemma 3.15.

The proof is by induction on $r$, starting with $r = 1$. If $G_1 \subset \mathrm{Fix}(f)$ then we may choose $P$ to be the endpoint of any ray $\widetilde{R}$ that begins with $\widetilde{E}$ and projects into $G_1$; the existence of such a ray follows from the fact (Theorem 2.19 (F)) that $G_1$ is its own core. If $G_1$ is EG then the existence of $P$ follows from Lemma 2.13 and Lemma 3.21. This completes the $r = 1$ case so we may now assume that the lemma holds for edges with height less than $r$.

If $H_r$ is EG then the existence of $P$ follows from Lemma 2.13 and Lemma 3.21. We may therefore assume that $H_r$ is NEG. Let $h\colon \Gamma_{r-1} \to \Gamma_{r-1}$ be as in Notation 3.24. If $\mathrm{Fix}(h) \neq \emptyset$, then the initial endpoint of $\widetilde{E}$ and some $\tilde{x} \in \mathrm{Fix}(h)$ cobound an indivisible Nielsen path. Thus $\tilde{x}$ is principal, there is a fixed direction in $\Gamma_{r-1}$ at $\tilde{x}$ and the existence of an appropriate $P \in \mathrm{Fix}(\hat{h})$ follows from the inductive hypothesis. The case that $\mathrm{Fix}(h) = \emptyset$ follows from Lemma 3.25. □

We now can prove the converse to Corollary 3.22 under the assumption that $f : G \to G$ is rotationless.

**Corollary 3.27.** *Suppose that $f : G \to G$ satisfies the conclusions of Theorem 2.19 and is rotationless. If some, and hence every, $\tilde{x} \in \mathrm{Fix}(\tilde{f})$ is principal then $\tilde{f}$ is principal.*

*Proof.* Assume that $\mathrm{Fix}(\tilde{f})$ consists of principal points. Since $f : G \to G$ is rotationless, periodic directions based in $\mathrm{Fix}(\tilde{f})$ are fixed. Lemma 2.20 (4) implies that each $\tilde{x} \in \mathrm{Fix}(\tilde{f})$ has at least two fixed directions. If some $\tilde{x} \in \mathrm{Fix}(\tilde{f})$ has at least three fixed directions, then Lemma 3.26 produces at least three points in $\mathrm{Fix}_N(\hat{f})$ and we are done. We may therefore assume that there are exactly two fixed, and hence exactly two periodic, directions at each $\tilde{x} \in \mathrm{Fix}(\tilde{f})$. If $\mathrm{Fix}(\tilde{f})$ contains an edge, then by Definition 3.18 there must be such an edge with a valence one vertex in $\mathrm{Fix}(\tilde{f})$. This contradicts items (1) and (4) of Lemma 2.20 and we conclude that that there are no fixed edges. Choose an edge $E \subset H_r$ and a lift $\widetilde{E}$ whose initial direction is fixed and based at some $\tilde{x} \in \mathrm{Fix}(\tilde{f})$. Let $\widetilde{R}$ be the ray that begins with $\widetilde{E}$ and ends at some $P \in \mathrm{Fix}(\hat{f})$ as in Lemma 3.26.

If $H_r$ is EG then the accumulation set of $P$ is an attracting lamination which implies by Lemma 3.1.16 of [2] that $P$ is not the endpoint of an axis. If $H_r$ is NEG then the accumulation set of $P$ is contained in $G_{r-1}$ which implies that $P$ is not the endpoint of an axis that contains $\widetilde{E}$. It follows that the line composed of $\widetilde{R}$ and the ray determined by the second fixed direction at $\tilde{x}$ is not an axis. We have now shown that $\mathrm{Fix}(\hat{f})$ is not the endpoint set of an axis and hence that every point in $\mathrm{Fix}(\hat{f})$ produced by Lemma 3.26 is contained in $\mathrm{Fix}_N(\hat{f})$. Thus $\mathrm{Fix}_N(\hat{f})$ contains at least two points and is not the endpoint set of an axis.

To complete the proof we assume that $\mathrm{Fix}_N(\hat{f})$ is the endpoint set of a lift $\tilde{\ell}$ of a generic leaf of an attracting lamination and argue to a contradiction. Since $\ell$ is birecurrent and contains $E$, $H_r$ is EG and the second fixed direction based at $\tilde{x}$ comes from an edge in $H_r$. Lemma 3.26 (2) implies that $\ell \subset G_r$ is $r$-legal and hence does not contain any indivisible Nielsen paths of height $r$. But then $\tilde{x}$ must be the only fixed point in $\tilde{\ell}$. Since $\mathrm{Fix}(\tilde{f})$ is principal it must contain a point other than $\tilde{x}$ and that point would have a fixed direction that does not come from the initial edge of a ray converging to an endpoint of $\tilde{\ell}$. This contradiction completes the proof. □

**3.4. Rotationless is rotationless.** We prove in this section that rotationless relative train track maps represent forward rotationless outer automorphisms and vice-versa.

**Lemma 3.28.** *Suppose that $f: G \to G$ satisfies the conclusions of Theorem 2.19 and is rotationless. Every periodic Nielsen path $\sigma$ with principal endpoints has period one.*

*Proof.* There is no loss in assuming that $\sigma$ is either a single edge or an indivisible periodic Nielsen path. In the former case, $\sigma$ is a periodic edge with a principal endpoint and so is fixed. We may therefore assume that $\sigma$ is indivisible.

The proof is by induction on the height $r$ of $\sigma$ with the $r = 0$ case being vacuously true. Let $p$ be the period of $\sigma$ and let $v \in \mathrm{Fix}(f)$ be an endpoint of $\sigma$. The case that $H_r$ is EG follows from Lemma 2.11 (3).

We may therefore assume that $H_r$ is a single non-fixed NEG edge $E_r$. Lemma 4.1.4 of [2] implies that after reversing the orientation on $\sigma$ if necessary, $\sigma = E_r\mu$ or $\sigma = E_r\mu\overline{E}_r$ for some path $\mu \subset G_{r-1}$. Let $\widetilde{E}_r$ be a lift of $E_r$ with initial endpoint $\tilde{v}$, let $\tilde{f}$ be the lift that fixes $\tilde{v}$ and let $h: \Gamma_{r-1} \to \Gamma_{r-1}$ be as in Notation 3.24. By Lemma 3.27, $\tilde{f}$ is principal. Denote the terminal endpoint of the lift $\tilde{\sigma}$ that begins at $\tilde{v}$ by $\tilde{w}$.

If $\sigma = E_r\mu$ then $\tilde{w} \in \Gamma_{r-1}$. If $p \neq 1$ then the path $\tilde{\tau}$ connecting $\tilde{w}$ to $h(\tilde{w})$ projects to a non-trivial periodic Nielsen path $\tau \subset G_{r-1}$ that is closed because $\tilde{w}$ projects to $w \in \mathrm{Fix}(f)$. Since $\tilde{w}$ is principal, the inductive hypothesis implies that $\tau$ has period one and hence that the projection of the closed path $\tilde{\tau}h(\tilde{\tau})\ldots h^{p-1}(\tilde{\tau})$ to $G_{r-1}$ is homotopic to $\tau^p$. This contradicts the fact that $\tau$ and hence $\tau^p$ determines a non-trivial conjugacy class in $F_n$. Thus $p = 1$ in the case that $\sigma = E_r\mu$.

Suppose now that $\sigma = E_r\mu\overline{E}_r$. If $\mathrm{Fix}(h^p) \neq \emptyset$ then the path $\tilde{\sigma}_1$ connecting $\tilde{v}$ to $\tilde{x} \in \mathrm{Fix}(h^p)$ and the path $\tilde{\sigma}_2$ connecting $\tilde{x}$ to $\tilde{w}$ are periodic Nielsen paths. By the preceding case $\sigma_1$ and $\sigma_2$, and hence $\sigma$, has period one. We may therefore assume that $\mathrm{Fix}(h^p) = \emptyset$.

Let $T_c : \Gamma \to \Gamma$ be the covering translation satisfying $T_c(\tilde{v}) = \tilde{w}$. Then $T_c$ commutes with $\tilde{f}^p$ and the axis $A_c$ is contained in $\Gamma_{r-1}$. Lemma 2.1 implies that $T_c^{\pm} \in \mathrm{Fix}(\hat{h}^p)$. If $\Phi$ is the principal automorphism corresponding to $\tilde{f}$ then $\hat{T}_{\Phi(c)} = \hat{f}\hat{T}_c\hat{f}^{-1}$, which implies that $T_{\Phi(c)}^{\pm} = \hat{h}(T_c^{\pm}) \in \mathrm{Fix}(\hat{h}^p)$ and that $A_{\Phi(c)} \subset \Gamma_{r-1}$. If $\{T_c^{\pm}\}$ is not $\hat{h}$-invariant, then $\mathrm{Fix}_N(\hat{h}^p)$ contains the four points $\{T_c^{\pm}\} \cup \{\hat{h}(T_c^{\pm})\}$ and $h^p$ is a principal lift of $f|C$ where $C$ is the component of $G_{r-1}$ that contains the terminal endpoint of $E_r$. This contradicts Corollary 3.17 and the assumption that $\mathrm{Fix}(h^p) = \emptyset$. Thus $\{T_c^{\pm}\}$ is $\hat{h}$-invariant. If $\hat{h}$ interchanges $T_c^{\pm}$ then $\mathrm{Fix}_N(\hat{h}^p)$ contains $\{T_c^{\pm}\}$ and at least one point in $\mathrm{Fix}(\hat{h})$ by Lemma 3.25. This contradicts Corollary 3.17 and we conclude that $T_c^{\pm} \in \mathrm{Fix}(\hat{h})$. It follows that $\tilde{f}$ commutes with $T_c$ and hence that $\tilde{w} \in \mathrm{Fix}(\tilde{f})$. This proves that $p = 1$ and so completes the inductive step. □

**Proposition 3.29.** *Suppose that $f : G \to G$ represents $\phi$ and satisfies the conclusions of Theorem 2.19. Then $f : G \to G$ is rotationless if and only if $\phi$ is forward rotationless.*

*Proof.* Suppose that $f : G \to G$ is rotationless, that $k \geq 1$ and that $\tilde{g} : \Gamma \to \Gamma$ is a principal lift of $g := f^k$. Corollary 3.17 and Corollary 3.22 imply that $\mathrm{Fix}(\tilde{g})$ is a non-empty set of principal fixed points. Since $f$ is rotationless, for each $\tilde{v} \in \mathrm{Fix}(\tilde{g})$ there is a lift $\tilde{f} : \Gamma \to \Gamma$ that fixes $\tilde{v}$ and all periodic directions at $\tilde{v}$. To prove that $\phi$ is forward rotationless it suffices by Remark 3.14 to show that $\mathrm{Fix}_N(\hat{f}) = \mathrm{Fix}_N(\hat{g})$ and hence (Remark 3.4) that $\tilde{f}^k = \tilde{g}$.

The path connecting $\tilde{v}$ to another point in $\mathrm{Fix}(\tilde{g})$ projects to a Nielsen path for $g$ and hence by Lemma 3.28, a Nielsen path for $f$. Thus $\mathrm{Fix}(\tilde{f}) = \mathrm{Fix}(\tilde{g})$. It follows that $\tilde{g}$ and $\tilde{f}$ commute with the same covering translations and Lemma 2.3 implies that $\mathrm{Fix}_N(\hat{f})$ and $\mathrm{Fix}_N(\hat{g})$ have the same non-isolated points.

Each isolated point $P \in \mathrm{Fix}_N(\hat{g})$ is an attractor for $\hat{g}$. It suffices to show that $P \in \mathrm{Fix}_N(\hat{f})$. By Lemma 3.21 there is a ray $\tilde{R}$ that terminates at $P$, that intersects $\mathrm{Fix}(\tilde{g})$ only in its initial endpoint and whose initial direction is fixed by $D\tilde{g}$, and hence also by $D\tilde{f}$. We may assume that the height $r$ of the initial edge $\tilde{E}$ of $\tilde{R}$ is minimal among all choices of $\tilde{R}$. By Lemma 3.26, $\tilde{E}$ extends to a ray $\tilde{R}'$ that converges to some $P' \in \mathrm{Fix}(\hat{f})$. It suffices to show that $P' = P$ since a repeller for $\hat{f}$ could not be an attractor for $\hat{g}$. If $H_r$ is EG this follows from Lemma 3.26 (2) and Lemma 3.21 (3) applied to $g$. We may therefore assume that $H_r$ is NEG. If there exists $\tilde{x} \in \mathrm{Fix}(\tilde{g}) \cap \tilde{R}'$ then $\tilde{x} \in \tilde{R}' \setminus \tilde{E}$ and the ray connecting $\tilde{x}$ to $P$ is contained in $G_{r-1}$ in contradiction to our choice of $r$. We therefore assume that $\mathrm{Fix}(\tilde{g}) \cap \tilde{R}' = \emptyset$. Lemma 3.26 (1) implies that there exists $\tilde{x} \in \tilde{R}'$ that is moved toward $P'$ by $\tilde{f}$ and Lemma 3.16 then implies that $P = P'$. This completes the proof of the only if direction of the proposition.

For the if direction, assume that $\phi$ is forward rotationless and choose $k > 0$ so that $g := f^k$ is rotationless. For each principal $v \in \mathrm{Fix}(g)$, there exist a lift $\tilde{v}$ of $v$

and a principal lift $\tilde{g}$ of $g$ that fixes $\tilde{v}$. Since $\phi$ is forward rotationless, there is a lift $\tilde{f}$ of $f$ such that $\tilde{f}^k = \tilde{g}$ and such that $\mathrm{Fix}_N(\hat{f}) = \mathrm{Fix}_N(\hat{g})$. It suffices to show that $\tilde{v} \in \mathrm{Fix}(\tilde{f})$ and that each $D\tilde{g}$-fixed direction $\tilde{d}_1$ at $\tilde{v}$ is $D\tilde{f}$-fixed.

The edge determined by $\tilde{d}_1$ extends to a ray $\tilde{R}_1$ that converges to some $P_1 \in \mathrm{Fix}_N(\hat{g}) = \mathrm{Fix}_N(\hat{f})$. Define $P_2$ and $\tilde{R}_2$ similarly using a second $D\tilde{g}$-fixed direction $\tilde{d}_2$ based at $\tilde{v}$ and denote the line connecting $P_1$ to $P_2$ by $\tilde{\gamma}$. Thus $\tilde{f}_\#(\tilde{\gamma}) = \tilde{\gamma}$ and the turn $(\tilde{d}_1, \tilde{d}_2)$ is legal for $\tilde{g}$ and hence for $\tilde{f}$. If $\tilde{f}(\tilde{v}) \notin \tilde{\gamma}$ then there exists $\tilde{y} \in \tilde{\gamma}$ not equal to $\tilde{v}$ such that $\tilde{f}(\tilde{y}) = \tilde{f}(\tilde{v})$. But then $\tilde{f}^k(\tilde{y}) = \tilde{f}^k(\tilde{v}) = \tilde{v}$ which contradicts Lemma 3.26 (4) applied to $\tilde{g}$. This proves that $\tilde{f}(\tilde{v}) \in \tilde{\gamma}$. Suppose that $\tilde{f}(\tilde{v}) \neq \tilde{v}$. Denote $\tilde{v}$ by $\tilde{v}_0$ and orient $\tilde{\gamma}$ so that $\tilde{v} < \tilde{f}(\tilde{v})$ in the order induced from the orientation and so that there exist $\tilde{v}_i \in \tilde{\gamma}$ for $1 \le i \le k$ such that $\tilde{v}_i < \tilde{v}_{i-1}$ and such that $\tilde{f}(\tilde{v}_i) = \tilde{v}_{i-1}$. But then $\tilde{f}^k(\tilde{v}_k) = \tilde{v}$ in contradiction to Lemma 3.26 (4). We conclude that $\tilde{f}(\tilde{v}) = \tilde{v}$. A third application of Lemma 3.26 (4) implies that the directions $\tilde{d}_i$ are fixed by $D\tilde{f}$. □

### 3.5. Properties of forward rotationless $\phi$

**Lemma 3.30.** *The following hold for each forward rotationless $\phi \in \mathrm{Out}(F_n)$.*

(1) *Each periodic conjugacy class is fixed and each representative of that conjugacy class is fixed by some principal automorphism representing $\phi$.*

(2) *Each $\Lambda \in \mathcal{L}(\phi)$ is $\phi$-invariant.*

(3) *A free factor that is invariant under an iterate of $\phi$ is $\phi$-invariant.*

*Proof.* If the conjugacy class of $c$ is fixed by $\phi^k$ for $k \ge 1$ then by Remark 3.2 there exists a principal automorphism $\Phi_k \in \mathrm{P}(\phi^k)$ that fixes $c$. By Lemma 2.1 this is equivalent to $T_c^{\pm} \in \mathrm{Fix}_N(\hat{\Phi}_k)$. Since $\phi$ is forward rotationless, we may assume that $k = 1$. This completes the proof of the first item.

Item (2) follows from Remark 3.20 and Lemma 3.1.14 of [2].

For the third item, suppose that the free factor $F$ is $\phi^k$-invariant for some $k \ge 1$. If $F$ has rank one then it is $\phi$-invariant by the first item of this lemma. We may therefore assume that $F$ has rank at least two. Let $\mathcal{C}$ be the set of bi-infinite lines $\gamma$ that are carried by $F$ and for which there exist a principal lift $\Phi$ of an iterate of $\phi$ and a lift $\tilde{\gamma}$ of $\gamma$ whose endpoints are contained in $\mathrm{Fix}_N(\hat{\Phi})$. Since $\phi$ is forward rotationless, each $\gamma$ is $\phi$-invariant, so $\mathcal{C}$ is $\phi$-invariant. Obviously $\mathcal{C}$ is carried by $F$ so to prove that $F$ is $\phi$-invariant it suffices, by Corollary 2.5, to show that no proper $\phi$-invariant free factor system $\mathcal{F}$ of $F$ carries $\mathcal{C}$.

Suppose to the contrary that such an $\mathcal{F}$ exists. By Theorem 2.19 there is a relative train track map $g\colon G' \to G'$ representing $\phi^k|F$ in which $\mathcal{F}$ is represented by a proper filtration element $G'_r \subset G'$. After replacing $\phi^k|F$ and $g$ by iterates we may assume that they are (forward) rotationless. There is an principal vertex $v \in G'$ whose link contains an edge $E$ of $G' \setminus G'_r$ that determines a fixed direction. This follows from Lemma 3.19 if there is an EG stratum in $G' \setminus G'_r$ and from the definition of principal otherwise. Lemma 3.26 and the fact that there are at least two periodic directions based at $v$ imply that there is a principal lift $\tilde{g}\colon \Gamma' \to \Gamma'$ and a line $\tilde{\gamma}$ whose endpoints are contained in $\mathrm{Fix}_N(\hat{g})$ and whose projected image $\gamma$ crosses $E$ and so is not carried by $G'_r$. The automorphism $\Phi' \in \mathrm{P}(\phi^k|F)$ determined by $\tilde{g}$ extends to an element $\Phi \in \mathrm{P}(\phi^k)$ with $\mathrm{Fix}_N(\hat{\Phi}') \subset \mathrm{Fix}_N(\hat{\Phi})$. Thus $\gamma \in \mathcal{C}$ in contradiction to our choice of $\mathcal{F}$ and $G'_r$. $\square$

**Corollary 3.31.** *If $\phi$ is forward rotationless and $F$ is a $\phi$-invariant free factor, then $\theta := \phi|F \in \mathrm{Out}(F)$ is forward rotationless.*

*Proof.* Lemma 3.30 (1) handles the case that $F$ has rank one so we may assume that $F$ has rank at least two. Choose a relative train track map $f\colon G \to G$ and filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_N = G$ satisfying the conclusions of Theorem 2.19 and representing $\phi$ such that the conjugacy class of $F$ is represented by $G_l$ for some $l$. Proposition 3.29 implies that $f\colon G \to G$ is rotationless. The restriction of $f\colon G \to G$ to $G_l$ is a rotationless relative train track map representing $\theta$ and satisfying the conclusions of Theorem 2.19. A second application of Proposition 3.29 implies that $\theta$ is forward rotationless. $\square$

## 4. Completely split relative train track maps

For every $\phi \in \mathrm{Out}(F_n)$ there exists $k > 0$ such that $\phi^k$ is represented by an improved relative train track map (IRT) $f\colon G \to G$ as defined in Theorem 5.1.5 of [2]. In this section we update this theorem, replacing IRTs with CTs, by controlling the iteration index $k$, adding a very useful property called complete splitting, and by making small changes to previous definitions. Section 4.1 contains all the necessary definitions. In Section 4.2 we show that complete splittings in a CT are hard splittings in the sense of [5]. A detailed comparison of IRTs and CTs is given in Section 4.3. There is one new move needed for the construction of CTs. It is defined in Section 4.4 and the existence theorem is stated and proved in Section 4.5. A few additional properties of CTs are presented in Section 4.6

**4.1. Definitions and Notation.** For $a \in F_n$, we let $[a]_u$ be the *unoriented conjugacy class determined by $a$*. Thus, $[a]_u = [b]_u$ if and only if $b$ is conjugate to either $a$ or $\bar{a}$. If $\sigma$ is a closed path then we let $[\sigma]_u$ be the *unoriented conjugacy class determined by $\sigma$*, thought of as a circuit.

Suppose that $f\colon G \to G$ is a rotationless relative train track map with filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_N = G$ and that $f\colon G \to G$ satisfies the conclusions of Theorem 2.19. Each NEG stratum $H_i$ is a single edge $E_i$ satisfying $f(E_i) = E_i u_i$ for some (necessarily closed by (NEG)) path $u_i \subset G_{i-1}$ that is sometimes called the *suffix* for $E_i$. If $u_i$ is a non-trivial Nielsen path, then we say that $E_i$ is a *linear edge*.

In the linear case, we define the *axis* for $E_i$ to be $[w_i]_u$ where $w_i$ is root-free and $u_i = w_i^{d_i}$ for some $d_i \neq 0$.

**Definition 4.1.** If $E_i$ and $E_j$ are linear edges and if there are $m_i, m_j > 0$ and a closed root-free Nielsen path $w$ such that $u_i = w^{m_i}$ and $u_j = w^{m_j}$ then a path of the form $E_i w^p \overline{E}_j$ with $p \in \mathbb{Z}$ is called an *exceptional path*.

**Remark 4.2.** If $E_i w^p \overline{E}_j$ is an exceptional path then

$$f_{\#}^k(E_i w^p \overline{E}_j) = E_i w^{p+k(m_i - m_j)} \overline{E}_j$$

for all $k \geq 0$. It follows that $E_i w^p \overline{E}_j$ is a Nielsen path if and only if $m_i = m_j$, that $f_{\#}$ induces a height preserving bijection on the set of exceptional paths and that the interior of $E_i w^p \overline{E}_j$ is an increasing union of pre-trivial paths.

**Definition 4.3.** A filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_N = G$ that satisfies the following property is said to be *reduced* (*with respect to* $\phi$): if a free factor system $\mathcal{F}'$ is $\phi^k$-invariant for some $k > 0$ and if $\mathcal{F}(G_{r-1}) \sqsubset \mathcal{F}' \sqsubset \mathcal{F}(G_r)$ then either $\mathcal{F}' = \mathcal{F}(G_{r-1})$ or $\mathcal{F}' = \mathcal{F}(G_r)$.

**Definition 4.4.** If $E$ in an edge in an irreducible stratum $H_r$ and $k > 0$ then a maximal subpath $\sigma$ of $f_{\#}^k(E)$ in a zero stratum $H_i$ is said to be *$r$-taken* or just *taken* if $r$ is irrelevant. Note that if $H_i$ is enveloped by an EG stratum $H_s$ then $\sigma$ has endpoints in $H_s$ and so is a connecting path. A non-trivial path or circuit $\sigma$ is *completely split* if it has a splitting, called a *complete splitting*, into subpaths, each of which is either a single edge in an irreducible stratum, an indivisible Nielsen path, an exceptional path or a connecting path in a zero stratum $H_i$ that is both maximal (meaning that it is not contained in a larger subpath of $\sigma$ in $H_i$) and taken.

**Definition 4.5.** A relative train track map is *completely split* if

(1) $f(E)$ is completely split for each edge $E$ in each irreducible stratum.
(2) If $\sigma$ is a taken connecting path in a zero stratum then $f_{\#}(\sigma)$ is completely split.

The next lemma states that if $f: G \to G$ is completely split then $f_{\#}$ maps completely split paths to completely split paths.

**Lemma 4.6.** *If $f: G \to G$ is completely split and $\sigma$ is a completely split path or circuit then $f_{\#}(\sigma)$ is completely split. Moreover if $\sigma = \sigma_1 \dots \sigma_k$ is a complete splitting then $f_{\#}(\sigma)$ has a complete splitting that refines $f_{\#}(\sigma) = f_{\#}(\sigma_1) \dots f_{\#}(\sigma_s)$.*

*Proof.* This is immediate from the definitions, the fact that $f_{\#}$ carries indivisible Nielsen paths to indivisible Nielsen paths and exceptional paths to exceptional paths and the fact that each maximal subpath of $f_{\#}(\sigma)$ in a zero stratum is contained in a single $f_{\#}(\sigma_i)$. □

We now come to our main definition. When equivalent descriptions of a property are available, for example in (EG Nielsen Paths), we have chosen the one that is easiest to check.

**Definition 4.7.** A relative train track map $f: G \to G$ and filtration $\mathcal{F}$ given by $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_N = G$ is said to be a *CT* (for completely split improved relative train track map) if it satisfies the following properties.

(1) **(Rotationless)** $f: G \to G$ is rotationless. (See Remark 4.8.)

(2) **(Completely Split)** $f: G \to G$ is completely split.

(3) **(Filtration)** $\mathcal{F}$ is reduced. The core of each filtration element is a filtration element.

(4) **(Vertices)** The endpoints of all indivisible periodic (necessarily fixed) Nielsen paths are (necessarily principal) vertices. The terminal endpoint of each non-fixed NEG edge is principal (and hence fixed). (See Remark 4.9 and Lemma 4.21.)

(5) **(Periodic Edges)** Each periodic edge is fixed and each endpoint of a fixed edge is principal. If the unique edge $E_r$ in a fixed stratum $H_r$ is not a loop then $G_{r-1}$ is a core graph and both ends of $E_r$ are contained in $G_{r-1}$.

(6) **(Zero Strata)** If $H_i$ is a zero stratum, then $H_i$ is enveloped by an EG stratum $H_r$, each edge in $H_i$ is $r$-taken and each vertex in $H_i$ is contained in $H_r$ and has link contained in $H_i \cup H_r$.

(7) **(Linear Edges)** For each linear $E_i$ there is a closed root-free Nielsen path $w_i$ such that $f(E_i) = E_i w_i^{d_i}$ for some $d_i \neq 0$. If $E_i$ and $E_j$ are distinct linear edges with the same axes then $w_i = w_j$ and $d_i \neq d_j$. (See Remark 4.10.)

(8) **(NEG Nielsen Paths)** If the highest edges in an indivisible Nielsen path $\sigma$ belong to an NEG stratum then there is a linear edge $E_i$ with $w_i$ as in (Linear Edges) and there exists $k \neq 0$ such that $\sigma = E_i w_i^k \bar{E}_i$.

(9) **(EG Nielsen Paths)** (See also Lemmas 4.17 and 4.18 and Corollaries 4.19 and 4.33) If $H_r$ is EG and $\rho$ is an indivisible Nielsen path of height $r$, then $f|G_r = \theta \circ f_{r-1} \circ f_r$ where:
   - (a) $f_r: G_r \to G^1$ is a composition of proper extended folds defined by iteratively folding $\rho$;
   - (b) $f_{r-1}: G^1 \to G^2$ is a composition of folds involving edges in $G_{r-1}$;
   - (c) $\theta: G^2 \to G_r$ is a homeomorphism.

**Remark 4.8.** A CT satisfies the conclusions of Theorem 2.19. This is immediate from the definitions and from Lemma 4.21.

**Remark 4.9.** It is an immediate consequence of (Vertices), Remark 2.8 and the definitions that a vertex whose link contains edges in more than one irreducible stratum is principal.

**Remark 4.10.** If $E_i$ and $E_j$ are linear edges with the same axis then, assuming the notation of (Linear Edges), paths of the form $E_i w^p \overline{E}_j$ where $w = w_i = w_j$ and $p \in \mathbb{Z}$ are exceptional if and only if $d_i$ and $d_j$ have the same sign.

**4.2. Hard splittings.** The first item in the following lemma establishes the uniqueness of complete splittings; the second item (see also Corollary 4.12) shows that a complete splitting is a hard splitting as defined in [5].

**Lemma 4.11.** *Suppose that $f : G \to G$ is a CT, that $\sigma$ is a circuit or path and that $\sigma = \sigma_1 \dots \sigma_m$ is a decomposition into subpaths, each of which is either a single edge in an irreducible stratum, an indivisible Nielsen path, an exceptional path or a connecting path in a zero stratum that is both maximal and taken. Suppose also that each turn $(\bar{\sigma}_i, \sigma_{i+1})$ is legal. Then*

(1) *$\sigma = \sigma_1 \dots \sigma_m$ is the unique complete splitting of $\sigma$;*
(2) *each pre-trivial subpath $\tau$ of $\sigma$ is contained in a single $\sigma_i$;*
(3) *a subpath of $\sigma$ that has the same height as $\sigma$ and is either a fixed edge or an indivisible Nielsen path equals $\sigma_i$ for some $i$.*

*Proof.* Let $\tilde{\sigma} = \tilde{\sigma}_1 \dots \tilde{\sigma}_m$ be a lift of $\sigma$ and let $\tilde{f} : \Gamma \to \Gamma$ be a lift of $f : G \to G$. The main step in the proof is to establish the following property.

(4) If $\sigma_i$ is not a taken maximal connecting path in a zero stratum then for each $k \geq 0$ there exist non-trivial initial and terminal subpaths $\tilde{\alpha}_{i,k}$ and $\tilde{\beta}_{i,k}$ of $\tilde{\sigma}_i$ such that $\tilde{f}^{-k}(\tilde{f}^k(\tilde{x})) \cap \tilde{\sigma} = \{\tilde{x}\}$ for all $\tilde{x} \in \operatorname{int}(\tilde{\alpha}_{i,k}) \cup \operatorname{int}(\tilde{\beta}_{i,k})$.

The proof of (4) is by double induction, first on $k$ and then on $m$. The $k = 0$ case is obvious so we may assume that (4) holds for any iterate less than $k$.

To establish the second base case, assume that $m = 1$ or equivalently that $\sigma = \sigma_i$. If $\sigma_i$ is exceptional then (4) is clear (cf. Lemma 4.1.4 of [2]). If $\sigma_i$ is an indivisible Nielsen path then (4) follows from (NEG Nielsen Paths) and Lemma 2.11 (2). The remaining possibility is that $\sigma_i$ is an edge $E$ in an irreducible stratum and in this case we make use of the inductive hypothesis that (4) holds in general for any iterate less than $k$. The first and last terms in the complete splitting of $f(E)$ are not connecting paths in zero strata. By the inductive hypothesis there exist initial and terminal subpaths $\tilde{\alpha}'$ and $\tilde{\beta}'$ of $\tilde{f}(\tilde{E})$ such that $\tilde{f}^{-(k-1)}(\tilde{f}^{k-1}(\tilde{x})) \cap \tilde{f}(\tilde{E}) = \{\tilde{x}\}$ for all $\tilde{x} \in \operatorname{int}(\tilde{\alpha}') \cup \operatorname{int}(\tilde{\beta}')$. Since $\tilde{f}|\tilde{E}$ is an embedding, we can pull $\tilde{\alpha}'$ and $\tilde{\beta}'$ back to initial and terminal subpaths $\tilde{\alpha}_{i,k}$ and $\tilde{\beta}_{i,k}$ of $\tilde{E}$ that satisfy (4). This completes the $m = 1$ case.

Suppose now that (4) holds for $k$ and $\sigma$ if the decomposition of $\sigma$ given in (1) has fewer than $m \geq 2$ terms. As a first case suppose that $\sigma_1$ is a taken maximal connecting path in a zero stratum $H_p$. By (Zero Strata), $\sigma_2$ is an edge in a EG stratum $H_r$ with $r > p$. Define $\tilde{\alpha}_{i,k}$ and $\tilde{\beta}_{i,k}$ using $\tilde{\sigma}_2 \dots \tilde{\sigma}_m$ in place of $\tilde{\sigma}$. Then $\tilde{f}^k(\tilde{\sigma}_1) \cap \operatorname{int}(\tilde{f}^k(\tilde{\alpha}_{2,k})) = \emptyset$ because $\tilde{f}^k(\tilde{\sigma}_1)$ has height $< r$ and $\tilde{f}^k(\tilde{\alpha}_{2,k})$ is an embedded path whose initial direction has height $r$. Since $\mathrm{int}(\tilde f^k(\tilde\alpha_{2,k}))$ separates $\tilde f^k(\tilde\sigma_1)$ from each $\tilde f^k(\tilde\beta_{i,k})$ and from $\tilde f^k(\tilde\alpha_{i,k})$ for $i > 2$, $\tilde f^k(\tilde\sigma_1)$ is disjoint from each of these sets. This proves that each $\tilde\alpha_{i,k}$ and $\tilde\beta_{i,k}$ satisfies (4) with respect to $\sigma$.

As a second case, suppose that $\sigma_2$ is a taken maximal connecting path in a zero stratum $H_p$. By (Zero Strata), $\sigma_1$ and $\sigma_3$ (if $m \geq 3$) are edges in an EG stratum $H_r$ with $r > p$. Define $\tilde\alpha_{1,k}$ and $\tilde\beta_{1,k}$ using $\tilde\sigma_1$ in place of $\tilde\sigma$. For $i > 2$, define $\tilde\alpha_{i,k}$ and $\tilde\beta_{i,k}$ using $\tilde\sigma_2 \dots \tilde\sigma_m$ in place of $\tilde\sigma$. As in the previous case, $\mathrm{int}(\tilde f^k(\beta_{1,k})) \cap \tilde f^k(\tilde\sigma_2) = \emptyset$ and $\mathrm{int}(\tilde f^k(\alpha_{3,k})) \cap \tilde f^k(\tilde\sigma_2) = \emptyset$. Also as in the previous case, this implies that each $\tilde\alpha_{i,k}$ and $\tilde\beta_{i,k}$ satisfies (4) with respect to $\sigma$.

The final case is that neither $\sigma_1$ nor $\sigma_2$ is a taken maximal connecting path in a zero stratum. Define $\tilde\alpha_{1,k}$ and $\tilde\beta_{1,k}$ using $\tilde\sigma_1$ in place of $\tilde\sigma$. For $i > 2$, define $\tilde\alpha_{i,k}$ and $\tilde\beta_{i,k}$ using $\tilde\sigma_2 \dots \tilde\sigma_m$ in place of $\tilde\sigma$. Since the turn $(\bar\sigma_1, \sigma_2)$ is legal, the interiors of $\tilde f^k(\tilde\alpha_{1,k})$ and $\tilde f^k(\tilde\beta_{2,k})$ are disjoint. The proof now concludes as in the previous two cases. This completes the induction step and so the proof of (4).

If $\tau$ is a pre-trivial path then there exists $k > 0$ such that $f^k_\#(\tau)$ is trivial. For each $\tilde x \in \tilde\tau$ there exists $\tilde y \neq \tilde x$ in $\tilde\tau$ such that $\tilde f^k(\tilde x) = \tilde f^k(\tilde y)$. If $\sigma_i$ is not a taken maximal connecting path in a zero stratum and if $\tau$ intersects $\mathrm{int}(\sigma_i)$ then $\tau \subset \mathrm{int}(\sigma_i)$ by (4). Since this applies to at least one of any pair of consecutive $\sigma_i$'s we have proved (2). It follows that $\sigma = \sigma_1 \dots \sigma_m$ is a splitting, and hence a complete splitting, of $\sigma$.

Suppose that $\sigma = \sigma'_1 \dots \sigma'_q$ is also a complete splitting. If $\sigma'_i$ is an exceptional path or an indivisible Nielsen path then (by Remark 4.2 in the NEG case and by Lemma 2.11 (2) in the EG case) the interior of $\sigma'_i$ is the increasing union of pre-trivial subpaths. Item (2) implies that $\sigma'_i$ is contained in some $\sigma_j$. Since $\sigma_j$ is not a single edge and is not contained in a zero stratum, it must be an indivisible Nielsen path or an exceptional path. By symmetry, $\sigma'_i = \sigma_j$. The terms that are taken maximal connecting paths in zero strata are then characterized as the maximal subpaths, in the complement of the indivisible Nielsen paths and exceptional paths, that are contained in zero strata. All remaining edges are terms in the complete splitting. This proves that complete splittings are unique and so completes the proof of (1).

A fixed edge of maximal height in $\sigma$ is not contained in a taken maximal connecting path of a zero stratum, an indivisible Nielsen path or an exceptional path in $\sigma$ and so must be a term in the complete splitting of $\sigma$. An indivisible Nielsen path in $\sigma$ must be contained in a single $\sigma_i$ by (2). If it has maximal height then, by inspection of the four possibilities for $\sigma_i$ it must be all of $\sigma_i$. This proves (3). □

**Corollary 4.12.** *Assume that $f : G \to G$ is a CT and that $\sigma = \sigma_1 \dots \sigma_s$ is the complete splitting of a path $\sigma \subset G$. If $\tau$ is an initial segment of $\sigma$ with terminal endpoint in $\sigma_j$ then $\tau = \sigma_1 \dots \sigma_{j-1} \cdot \mu_j$ is a splitting where $\mu_j$ is the initial segment of $\sigma_j$ that is contained in $\tau$. In particular if $\tau$ is a non-trivial Nielsen path then $\sigma_i$ is a Nielsen path for all $i \leq j$ and if $\sigma_j$ is not a single fixed edge then $\mu_j = \sigma_j$.*

*Proof.* The main statement follows immediately from Lemma 4.11 (2). The statement about Nielsen paths then follows from the fact that no proper non-trivial initial segment of a non-fixed term in a complete splitting of any path is a Nielsen path. □

**4.3. CT versus IRT.** Theorem 5.1.5 of [2] is both the definition of, and the existence theorem for, an improved relative train track map. There are eight bulleted items in the statement of the theorem, the last seven of which should be considered the definition. For notational convenience, we refer to these as (IRT-1) through (IRT-7). In this section we discuss the extent to which a CT $f : G \to G$ satisfies these seven items. By the end of this section we will have verified that CTs satisfy all of the important properties of IRTs.

(IRT-1) is that $\mathcal{F}$ is reduced, which is part of (Filtration). The following lemma states that every CT satisfies (IRT-2).

**Lemma 4.13.** *If $f : G \to G$ is a CT then every periodic Nielsen path has period one.*

*Proof.* Each periodic Nielsen path is a concatenation of periodic edges and indivisible periodic Nielsen paths. The former have period one by (Periodic Edges) and the latter have period one by (Vertices) and Lemma 3.28. □

The next lemma shows that a CT satisfies most of (IRT-3). The exception is that there may be some vertices $v$ for which $f(v)$ is not fixed.

**Lemma 4.14.** *If $f : G \to G$ is a CT then every vertex $v \in G$ has at least two gates. If the link of $v$ is not contained in $H_r^z$ for some EG stratum $H_r$ then $v$ is principal and hence fixed.*

*Proof.* If $(d_1, d_2)$ is an illegal turn then either one of the $d_i$'s is the terminal end of a non-fixed NEG edge or both $d_1$ and $d_2$ belong to $H_r^z$ for some EG stratum $H_r$. (Vertices), (Zero Strata) and Lemma 2.10 imply that the vertex in both of these cases has two gates. At any other vertex the number of gates equals the valence. This proves the first statement of the lemma.

It follows from (Periodic Edges) and the definition of principal vertex that if $v$ is periodic and the link of $v$ is not contained in a single EG stratum then $v$ is principal. If $v$ is not periodic then its link is contained in some $H_r^z$ by (Vertices), Remark 4.9 and (Zero Strata). □

The difference between (IRT-4) and the conclusion of the next lemma is that a zero stratum in a IRT can be the union of contractible components.

**Lemma 4.15.** *Assume that $f : G \to G$ is a CT. Then $G_i$ has a contractible component if and only if $H_i$ is a zero stratum.*

*Proof.* The if direction follows from (Zero Strata). For the only if direction we assume that $H_i$ is not a zero stratum and prove, by induction up the filtration, that every component of $G_i$ is non-contractible. For the base case, $H_1 = G_1$ is either EG or periodic and so is connected and not contractible, by Lemma 2.10 in the former case and (Periodic Edges) in the latter. We now consider the inductive step. If some component of $G_{i-1}$ is contractible then $H_{i-1}$ is a zero stratum and (Zero Strata) implies that every component of $G_i$ is non-contractible. If every component of $G_{i-1}$ is non-contractible then (Periodic Edges) and Lemma 2.10 complete the proof. □

There are two differences between (IRT-5) and (Zero Strata). The first is that an IRT can have a vertex whose link is contained in a zero stratum but a CT cannot. The second is that the restriction of an IRT to a zero stratum is always an immersion but this need not be true for a CT. We have replaced the immersion condition with Definition 4.5 (2) and the assumption that every edge in a zero stratum is $r$-taken. The primary motivation for removing the immersion condition is that it lacks robustness. For example, it need not hold for $f^2|\sigma$. Also, since the main application of relative train track maps is in analyzing the action of the induced map $f_\#$ on paths with endpoints at vertices, it makes sense to make definitions that focus on $f_\#$ and not on $f$.

Corollary 4.19 below implies that a CT satisfies (IRT-7). In the definition of a CT, we have replaced a list of properties satisfied by indivisible Nielsen paths corresponding to EG strata (see the statement of Corollary 4.19) with the underlying property (EG Nielsen Paths) from which these properties were derived. One advantage of this is that it is easier to deduce additional properties as in Lemma 4.24.

**Lemma 4.16.** *Suppose that $H_r$ is an aperiodic EG stratum of a relative train track map $f : G \to G$, that $\rho$ is an indivisible Nielsen path of height $r$ and that $\rho$ and $H_r$ satisfy the conclusions of* (*EG Nielsen Paths*)*. Then the fold at the illegal turn at each indivisible Nielsen path obtained by iteratively folding $\rho$ is proper.*

*Proof.* We assume without loss that $G = G_r$.

Define the *data set $S$ for $f$ and $\rho$* to be the ordered sequences of $H_r$-edges in $\rho$ and in $f(E)$ for each edge $E$ of $H_r$. Then $S$ determines the type (partial, proper, improper) of the fold at the illegal turn of $\rho$. Furthermore, assuming that the fold is proper so that the extended fold is defined, $S$ also determines the data set for the relative train track map and indivisible Nielsen path obtained by folding $\rho$. Let $S_k$ be the data set for the relative train track map and indivisible Nielsen path obtained by iteratively folding $\rho$ $k$ times, assuming that such folds are defined.

Assume the notation of (EG Nielsen Paths). In particular, $f_r : G \to G^1$ is a composition of a finite number, say $K$, of proper extended folds defined by iteratively folding $\rho$. Thus $S_K$ is defined. Since $f|G_r = \theta \circ f_{r-1} \circ f_r$, the homeomorphism $\theta$ defines a bijection between the edges in the top stratum of $G^1$ and the edges of $H_r$ that conjugates $S_K$ to $S_0$. In other words, up to relabeling, $S_K = S_0$. It follows that, up to relabeling, the sequence of $S_k$'s is periodic with period $K$ and hence that the fold at the illegal turn of $\rho(k)$ is proper for all $k$. □

**Lemma 4.17.** *Theorem* 5.15 *of* [4] *remains true if the hypothesis that* $f : G \to G$ *is stable is replaced by the hypothesis that for each EG stratum* $H_r$ *there is an indivisible Nielsen path* $\rho$ *of height* $r$ *such that* $\rho$ *and* $H_r$ *satisfy the conclusions of* (*EG Nielsen Paths*).

*Proof.* The proof of Theorem 5.15 has two parts. The first is a reduction to the case that the illegal turn at each indivisible Nielsen path obtained by iteratively folding $\rho$ is proper. The second is the observation that in this case the proof for the special case that $f : G \to G$ is irreducible given in Lemma 3.9 of [4] applies to the general case as well. This lemma therefore follows from Lemma 4.16. □

**Lemma 4.18.** *Proposition* 5.3.1 *of* [2] *remains true if the hypothesis* $f : G \to G$ *is* $\mathcal{F}$*-Nielsen minimized is replaced by the hypothesis that* $H_r$ *satisfies* (*EG Nielsen Paths*).

*Proof.* The proof of Proposition 5.3.1 of [2] makes use of Lemmas 5.3.6, 5.3.7, 5.3.9 and Corollary 5.3.8 of that paper. Lemma 5.3.6 states that if $f : G \to G$ is $\mathcal{F}$-Nielsen minimized and if $\rho_r$ crosses every edge of $H_r$ exactly twice then $H_r$ satisfies (EG Nielsen Paths). The remaining three lemmas use (EG Nielsen Paths) but do not refer directly to being $\mathcal{F}$-Nielsen minimized. □

The next corollary refers to *geometric strata*; complete details can be found in Definition 5.1.4 of [2].

**Corollary 4.19.** *Suppose that* $f : G \to G$ *is a relative train track map and that* (*EG Nielsen Paths*) *holds for the EG stratum* $H_r$. *Then the following properties are satisfied.*

- eg-(i) *There is at most one indivisible Nielsen path* $\rho_r \subset G_r$ *that intersects* $H_r$ *non-trivially. The initial edges of* $\rho_r$ *and* $\bar\rho_r$ *are distinct edges in* $H_r$.
- eg-(ii) *If* $\rho_r \subset G_r$ *is an indivisible Nielsen path that intersects* $H_r$ *non-trivially and if* $H_r$ *is not geometric, then there is an edge* $E$ *of* $H_r$ *that* $\rho_r$ *crosses exactly once.*
- eg-(iii) *If* $H_r$ *is geometric then there is an indivisible Nielsen path* $\rho_r \subset G_r$ *that intersects* $H_r$ *non-trivially and satisfies the following properties:* (i) $\rho_r$ *is a closed path with basepoint not contained in* $G_{r-1}$*;* (ii) *the circuit determined by* $\rho_r$ *corresponds to the unattached peripheral curve* $\rho^*$ *of* $S$*; and* (iii) *the surface* $S$ *is connected.*

*In particular,* $H_r$ *satisfies the EG properties of an improved relative train track.*

*Proof.* Theorem 5.15 of [4], which applies here by Lemma 4.17, implies that there is at most one indivisible Nielsen path $\rho_r \subset G_i$ that intersects $H_r$ non-trivially and if such a $\rho_r$ exists then it either crosses every edge in $H_r$ exactly twice or crosses some edge of $H_r$ exactly once. Lemma 5.1.7 of [2] implies that if $\rho_r$ crosses some edge of $H_r$ exactly once then $\rho_r$ is not a closed path; in particular eg-(i) holds. The rest of eg-(i) and the remaining two items follow from Proposition 5.3.1 of [2] which applies here by Lemma 4.18. □

**Remark 4.20.** Item eg-(ii) of Corollary 4.19 and Lemma 5.1.7 in [2] imply that if $H_r$ is an EG stratum of a CT that is not geometric and if $\rho$ is an indivisible Nielsen path of height $r$ then $\rho$ has distinct endpoints.

The remaining item (IRT-6) concerns NEG strata and has three parts. The first two statements of the next lemma shows that a CT satisfies the first two parts of (IRT-6). Corollary 4.23 shows that a CT satisfies the third part of (IRT-6).

**Lemma 4.21.** *If $f: G \to G$ is a CT and $H_i$ is NEG then $H_i$ is a single edge $E_i$. If $E_i$ is not contained in* $\mathrm{Fix}(f)$ *then there is a non-trivial closed path $u_i \subset G_{i-1}$ such that $f(E_i) = E_i \cdot u_i$. Moreover $u_i$ forms a circuit and the turn $(u_i, \bar{u}_i)$ is legal.*

*Proof.* If $H_i$ consists of periodic edges then the lemma follows from (Periodic Edges). Otherwise (Rotationless), (Completely Split) and (Vertices) imply that $H_i$ is a single edge $E_i$ and that there is a non-trivial closed path $u_i$ such that $f(E_i) = E_i u_i$ is completely split. To prove that $f(E_i) = E_i \cdot u_i$ we must show that the first term $\sigma_1$ in the complete splitting of $E_i u_i$ is the single edge $E_i$. It is obviously not contained in a zero stratum and is not a Nielsen path by (NEG Nielsen Paths). It remains to show that $\sigma_1$ is not an exceptional path and for this there is no loss in assuming that $E_i$ is linear. In the notation of (Linear Edges), $f(E_i) = E_i w_i^{d_i}$, no initial segment of which is an exceptional path by Remark 4.10. This completes the proof that $f(E_i) = E_i \cdot u_i$.

The turn $(Df^{k-1}(\bar{u}_i), Df^k(u_i))$ is the $Df^k$ image of the legal turn $(\bar{E}_i, u_i)$ and is therefore legal for all $k \geq 1$. Since $f$ is rotationless and since the terminal endpoint $v$ of $E_i$ is principal by (Vertices), $Df^k(d)$ is independent of $k$ for all directions $d$ based at $v$ and all sufficiently large $k$. It follows that $(Df^k(\bar{u}_i), Df^k(u_i))$ is legal for all sufficiently large $k$ and hence that $(u_i, \bar{u}_i)$ is legal. In particular, $(u_i, \bar{u}_i)$ is non-degenerate which implies that $u_i$ forms a circuit. □

**Lemma 4.22.** *Suppose that $E_i$ is the unique edge of height $i$ in a rotationless relative train track map $f: G \to G$, that $f(E_i) = E_i \cdot u_i$ for some non-trivial closed path $u_i \subset G_{i-1}$ and that every periodic Nielsen path with height less than $i$ has period one. Suppose further that either there are no Nielsen paths of height $i$ or $E_i$ is a linear edge and all Nielsen paths of height $i$ have the form $\sigma = E_i w_i^k \bar{E}_i$ where $k \neq 0$ and where $w_i$ is root-free and $u_i = w_i^{d_i}$ for some $d_i \neq 0$. Let $h: \Gamma_{i-1} \to \Gamma_{i-1}$ be the lift of $f|G_{i-1}$ as in Notation 3.24. Then*

- $\mathrm{Fix}(h) = \emptyset$;
- *$E_i$ is a linear edge if and only if there is a covering translation $T\colon \Gamma_{i-1} \to \Gamma_{i-1}$ that commutes with $h$ and whose axis covers $u_i$.*

*Proof.* Let $\tilde f\colon \Gamma \to \Gamma$ and $\widetilde{E}_i$ be as in Notation 3.24. Thus $\tilde f(\widetilde{E}_i) = \widetilde{E}_i \cdot \tilde u_i$ where $\tilde u_i \subset \Gamma_{i-1}$ is a lift of $u_i$ and $h$ maps the initial endpoint $\tilde x_1$ of $\tilde u_i$ to the terminal endpoint $\tilde x_2$ of $\tilde u_i$. If $\tilde v \in \mathrm{Fix}(h)$ and $\tilde\gamma$ is the path from $\tilde x_1$ to $\tilde v$ then $\widetilde{E}_i\tilde\gamma$ is a Nielsen path for $\tilde f$. But then $E_i\gamma$ is a Nielsen path of height $i$ for $f$ that is not of the form $E_i w_i^k \overline{E}_i$. This contradiction verifies the first item.

If $u_i$ is a Nielsen path and $T\colon \Gamma_{i-1} \to \Gamma_{i-1}$ is the covering translation that maps $\tilde x_1$ to $\tilde x_2$, then $Th(\tilde x_1) = hT(\tilde x_1)$ is the terminal endpoint of the lift of $u_i$ that begins at $\tilde x_2$. Thus $T$ commutes with $h$. For the converse suppose that $h$ commutes with some covering translation $T\colon \Gamma_{i-1} \to \Gamma_{i-1}$. Corollary 3.17 implies that $h$ is not a principal lift of $f|G_{i-1}$ and hence that the endpoints of the axis of $T$ are the only fixed points in $\partial\Gamma_{i-1}$. On the other hand, the ray $\tilde u_i \cdot h_\#(\tilde u_i)\cdot h^2_\#(\tilde u_i)\dots$ converges to a fixed point in $\partial\Gamma_{i-1}$. The end of this ray is therefore contained in the axis of $T$. It follows that $u_i$ is a periodic Nielsen path and hence a Nielsen path and that the axis of $T$ covers $u_i$. □

Recall (Definition 4.1.3 of [2]) that if $H_i$ is an NEG strata with unique edge $E_i$ then paths of the form $E_i\gamma\overline{E}_i$, $E_i\gamma$ or $\gamma\overline{E}_i$ where $\gamma \subset G_{i-1}$ are called *basic paths of height $i$*.

**Corollary 4.23.** *Suppose that $f\colon G \to G$ is a CT and that $H_i$ is an NEG strata with unique edge $E_i$. If $\sigma \subset G_i$ is a basic path of height $i$ that does not split as a concatenation of two basic paths of height $i$ or as a concatenation of a basic path of height $i$ with a path contained in $G_{i-1}$, then either* (i) *some $f^k_\#(\sigma)$ splits into pieces, one of which equals $E_i$ or $\overline{E}_i$, or* (ii) *$u_i$ is a Nielsen path and some $f^k_\#(\sigma)$ is an exceptional path of height $i$.*

*Proof.* Lemma 4.22 and Corollary 4.12 imply that $f\colon G \to G$ satisfies the hypotheses and hence the conclusions of Proposition 5.4.3 of [2]. In conjunction with Corollary 4.19, Lemma 4.21 and Lemma 3.28 we see that $f\colon G \to G$ satisfies the hypotheses of Lemma 5.5.1 of [2], from which the corollary follows. □

We conclude this subsection with two additional properties of CTs.

**Lemma 4.24.** *Suppose that $f\colon G \to G$ is a rotationless relative train track map, that $H_r$ is an EG stratum satisfying (EG Nielsen Paths), and that $\rho$ is an indivisible Nielsen path of height $r$. Then*

(1) *$H_r^z = H_r$;*

(2) *if $\rho = a_1b_1\dots b_la_{l+1}$ is the decomposition into subpaths $a_i$ of height $r$ and maximal subpaths $b_i$ of height less than $r$ then each $b_i$ is a Nielsen path;*

(3) *if $E$ is an edge of $H_r$ then each maximal subpath of $f(E)$ in $G_{r-1}$ is one of the $b_i$'s from* (2)*. In particular $f(E)$ splits into edges in $H_r$ and Nielsen paths in $G_{r-1}$.*

*Proof.* The maps $f_r$, $f_{r-1}$ and $\theta$ induce bijections on the set of components in the filtration element of height $r-1$. It follows that $f = \theta f_{r-1} f_r$ induces a bijection on the set of components of $G_{r-1}$ and hence that each component of $G_{r-1}$ is non-wandering. This proves (1).

For (2), let $(f_r)_\#(\rho) = a'_1 b'_1 \dots a'_m b'_m a'_{m+1}$ be the decomposition into subpaths $a'_j$ of height $r$ and maximal subpaths $b'_j$ of height less than $r$. It is an immediate consequence (see the proof of Lemma 5.3.3 of [2]) of the definition of an extended fold that the set of distinct $b'_j$'s is contained in the set of distinct $b_i$'s. Now let $(\theta f_{r-1})_\#(a'_1 b'_1 \dots a'_m b'_{m+1}) = c_1 d_1 \dots c_p d_p c_{p+1}$ be the decomposition into subpaths $c_k$ of height $r$ and maximal subpaths $d_k$ of height less than $r$. Then for each $k$ there exists $j$ such that $d_k = (\theta f_{r-1})_\#(b'_j)$. Combining this with the fact that $a_1 b_1 \dots b_l a_{l+1} = c_1 d_1 \dots c_p d_p c_{p+1}$, we conclude that $f_\#$ permutes the $b_i$'s. Since $f$ is rotationless, each $b_i$ is a Nielsen path.

If $E$ is an edge of $H_r$ then, by construction, each maximal subpath of $f_r(E)$ in $G_{r-1}$ is a $b_i$. By (2), each $b_i$ is a Nielsen path for $f$ and hence for $\theta f_{r-1}$. This completes the proof of (3). $\square$

**Lemma 4.25.** *If $f : G \to G$ is a CT and $\sigma \subset G_r$ is a path with endpoints at vertices then $f^k_\#(\sigma)$ is completely split for all sufficiently large $k$.*

*Proof.* The proof is by induction on the height of $\sigma$. The height zero case is vacuously true so suppose that $\sigma$ has height $j \geq 1$ and that the lemma holds for all paths of height less than $j$. By Lemmas 4.6, 4.11 and the inductive hypothesis, it suffices to show that some $f^k(\sigma)$ has a splitting into subpaths that are either completely split or contained in $G_{j-1}$. This is immediate if $H_j$ is a zero stratum or if $H_j$ is a single fixed edge. If $H_j$ is NEG then $\sigma$ has a splitting into basic paths of height $j$ and subpaths in $G_{j-1}$ by Lemma 4.1.4 of [2]. The desired splitting of $\sigma$ therefore follows from Lemma 4.23. If $H_j$ is EG, then Lemmas 4.2.6 and 4.2.5 of [2] imply that some $f^k(\sigma)$ splits into pieces, each of which is either $j$-legal or a Nielsen path and Lemma 4.2.1 of [2] implies that the $j$-legal paths in $G_j$ split into single edges in $H_j$ and subpaths in $G_{j-1}$. $\square$

**4.4. A new move.** We make use of a move that plays the same role for zero and EG strata that sliding (Section 5.4 of [2]) does for NEG strata. See item (7) of Lemma 4.27 below for its main application.

**Definition 4.26.** Suppose that $f : G \to G$ is a rotationless relative train track map satisfying the conclusions of Theorem 2.19 with respect to the filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_N = G$, that $1 \leq j \leq N$, that every component of $G_j$ is non-contractible and that $f$ fixes every vertex in $G_j$ whose link is not contained in $G_j$.

Define a homotopy equivalence $g\colon G \to G$ by $g|G_j = f|G_j$ and $g|(G \setminus G_j) =$ identity.

Define $G'$ from $G$ by changing the marking via $g$. More precisely, if $X$ is the underlying graph of $G$ and $\tau\colon R_n \to X$ is the marking that defines $G$, then $g\tau\colon R_n \to X$ is the marking that defines $G'$. Since $G$ and $G'$ have the same underlying graph, there is a natural identification of $G$ with $G'$ and we use this when discussing edges and strata.

Define $f'\colon G' \to G'$ by $f'|G'_j = f|G_j$ and $f'(E) = (gf)_\#(E)$ for all edges $E$ in $H_i$ with $i > j$.

We say that $f'\colon G' \to G'$ is obtained from $f\colon G \to G$ by *changing the marking on $G_j$ via $f$*.

The following lemma is the analog of Lemma 5.4.1 of [2].

**Lemma 4.27.** *Suppose that $f'\colon G' \to G'$ is obtained from $f\colon G \to G$ by changing the marking on $G_j$ via $f$. Then:*

(1) $f'|G_j = f|G_j$*;*

(2) *for every path $\sigma \subset G$ with endpoints at vertices and for every $k > 0$, $g_\# f_\#^k(\sigma) = (f')_\#^k g_\#(\sigma)$;*

(3) *$f'\colon G' \to G'$ is a homotopy equivalence that determines the same element of* $\mathrm{Out}(F_n)$ *as $f\colon G \to G$;*

(4) *there is a one-to-one correspondence between Nielsen paths for $f$ and Nielsen paths for $f'$;*

(5) *$f'\colon G' \to G'$ is a rotationless relative train track map satisfying the conclusions of Theorem 2.19 with respect to $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_N = G$.*

*Proof.* Item (1) is immediate from the definitions as is the fact that $f'$ preserves the filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_N = G$. Also immediate are:

(6) If $f(x) \neq f'(x)$ then $x \notin G_j$ and $f(x), f'(x) \in G_j$. In particular, $\mathrm{Fix}(f) = \mathrm{Fix}(f') \subset \mathrm{Fix}(g)$, $\mathrm{Per}(f) = \mathrm{Per}(f') \subset \mathrm{Per}(g)$ and $Df$ and $Df'$ have the same fixed and periodic directions.

(7) Suppose that $E$ is an edge in $H_i$ for $i > j$ and that $f(E) = \mu_1\nu_1\mu_2\ldots\nu_{k-1}\mu_k$ where the $\nu_l$'s are the maximal subpaths in $G_j$ and where $\mu_1$ and $\mu_k$ may be trivial. Then $f'(E) = \mu_1 f_\#(\nu_1)\mu_2\ldots f_\#(\nu_{k-1})\mu_k$ and all the $f_\#(\nu_l)$'s are non-trivial. (The non-triviality follows from the fact that $f$ fixes the endpoints of each $\nu_l$.)

This implies:

(8) Each stratum $H_i$ has the same type (EG, NEG, zero) for $f$ as for $f'$.

To verify (2), it suffices to assume that $k = 1$ and that $\sigma$ is a single edge $E$. If $E \subset G_j$ then $g_\# f_\#(E) = f_\#(f_\#(E)) = f'_\# g_\#(E)$. If $E \subset G_i$ for $i > j$ then $g_\# f_\#(E) = f'_\#(E) = f'_\# g_\#(E)$. This completes the proof of (2) which implies (3).

If $\rho' \subset G'$ is a path in $G$ with endpoints $P_1, P_2 \in \mathrm{Fix}(f') = \mathrm{Fix}(f)$, then there is a unique path $\rho \subset G$ with endpoints $P_1$ and $P_2$ such that $g_\#(\rho) = \rho'$. Condition (2) implies that $\rho'$ is fixed by $(f')_\#$ if and only if $\rho$ is fixed by $f_\#$. This proves (4).

To show that $f' \colon G' \to G'$ is a relative train track map it suffices by (1), (6) and (7) to prove that if $H_i$ is an EG stratum with $i > j$ and if $\sigma \subset G_{i-1}$ is a connecting path for $H_i$ then $(f')_\#(\sigma)$ is non-trivial. If $\sigma$ is contained in a non-contractible component of $G_{i-1}$ then its endpoints are $f$-fixed, and hence $f'$-fixed, by Remark 2.8. Non-triviality of $(f')_\#(\sigma)$ therefore follows from the fact that $f'$ is a homotopy equivalence. If $\sigma$ is contained in a contractible component of $G_{i-1}$ then it is contained in a zero stratum that has height greater than $j$ (because it is contained in $H_i^z$) and so $(f')_\#(\sigma) = g_\# f_\#(\sigma)$. If the connecting path $f_\#(\sigma)$ is contained in a non-contractible component of $G_{i-1}$ then $g_\# f_\#(\sigma)$ is non-trivial by the same argument used in the previous case. Otherwise, $g_\# f_\#(\sigma) = f_\#(\sigma)$ and again we are done. This completes the proof that $f' \colon G' \to G'$ is a relative train track map.

Item (5) follows from (4) and (6). □

## 4.5. Existence theorem

**Theorem 4.28.** *Suppose that $\phi \in \mathrm{Out}(F_n)$ is forward rotationless and that $\mathcal{C}$ is a nested sequence of $\phi$-invariant free factor systems. Then $\phi$ is represented by a CT $f : G \to G$ and filtration $\mathcal{F}$ that realizes $\mathcal{C}$.*

*Proof.* We assume without loss $\mathcal{C}$ is maximal with respect to $\sqsubset$. Thus any filtration that realizes $\mathcal{C}$ is reduced. By Theorem 2.19 and item (3) of Lemma 2.20 we may choose a relative train track map $f : G \to G$ that represent $\phi$ and realizes $\mathcal{C}$ and such that each contractible component of a filtration element is a union of zero strata and the endpoints of all indivisible Nielsen paths of EG height are vertices. For the remainder of the proof all relative train track maps are assumed to satisfy these properties.

**Step 1: (EG Nielsen Paths).** Let $N(f)$ be the number of indivisible Nielsen paths of EG height. In the construction of an IRT in [2] it is assumed (see Definition 5.2.1 of [2]) that $N(f)$ is as small as possible. The EG properties of an IRT are then established by contradiction: the failure of these properties allows one to reduce $N(f)$ which is impossible. In order to make our constructions more algorithmic, we drop the assumption that $N(f)$ is minimal and argue inductively: the failure of (EG Nielsen Paths) allows one to reduce $N(f)$ and since $N(f)$ is finite, this process eventually terminates in an $f : G \to G$ satisfying (EG Nielsen Paths). As we are no longer assuming that $N(f)$ is minimal we cannot quote statements of results from [2] but must instead refer to their proofs.

**Lemma 4.29.** *Suppose that $H_r$ is an EG stratum of a relative train track map $f : G \to G$ and that $\rho$ is an indivisible Nielsen path of height $r$. If the fold at the illegal turn* *of $\rho$ is partial then there is a relative train track map $f' \colon G' \to G'$ satisfying $N(f') < N(f)$.*

*Proof.* This follows from the proofs of Lemmas 5.2.3 and 5.2.4 of [2]. The latter constructs a topological representative $f'' \colon G'' \to G''$ with $N(f'') < N(f)$. The former constructs a relative train track map $f' \colon G' \to G'$ representing $\phi$ with $N(f') = N(f'')$. $\square$

**Lemma 4.30.** *Suppose that $H_r$ is an EG stratum of a relative train track map $f : G \to G$, that $\rho$ is an indivisible Nielsen path of height $r$ and that the fold at the illegal turn of $\rho$ is proper. Let $f' \colon G' \to G'$ be the relative train track map obtained from $f : G \to G$ by folding $\rho$. Then $N(f') = N(f)$ and there is a bijection $H_s \to H'_s$ between the EG strata of $f$ and the EG strata of $f'$ such that $H'_s$ and $H_s$ have the same number of edges for all $s$.*

*Proof.* This follows from the definition of $f' \colon G' \to G'$ and the proof of Lemma 5.3.3 of [2]. $\square$

**Lemma 4.31.** *Suppose that $H_r$ is an EG stratum of a relative train track map $f : G \to G$ and that $\rho$ is an indivisible Nielsen path of height $r$. If the fold at the illegal turn of $\rho$ is improper then there is a relative train track map $f' \colon G' \to G'$ and a bijection $H_s \to H'_s$ between the EG strata of $f$ and the EG strata of $f'$ with the following properties.*

(1) $N(f') = N(f)$.
(2) *$H'_r$ has fewer edges than $H_r$.*
(3) *If $s > r$ then $H'_s$ and $H_s$ have the same number of edges.*

*Proof.* This follows from Definition 5.3.4 and the proof of Lemma 5.3.5 of [2]. $\square$

**Lemma 4.32.** *If $H_r$ is an EG stratum of $f : G \to G$ and $\rho$ is an indivisible Nielsen path of height $r$ such that the fold at the illegal turn at each indivisible Nielsen path obtained by iteratively folding $\rho$ is proper then $H_r$ satisfies (EG Nielsen Paths).*

*Proof.* The conclusion of Lemma 5.3.6 of [2] is that $H_r$ satisfies (EG Nielsen Paths). The proof of that lemma uses only standard folding arguments, the hypotheses of our lemma and uniqueness of the illegal turn of height $r$, which follows from Lemma 4.17. $\square$

**Corollary 4.33.** *Suppose that $H_r$ is an EG stratum of a relative train track map $f : G \to G$ and that $\rho$ is an indivisible Nielsen path of height $r$. Then the fold at the illegal turn at each indivisible Nielsen path obtained by iteratively folding $\rho$ is proper if and only if $H_r$ satisfies (EG Nielsen Paths).*

*Proof.* This is an immediate corollary of Lemmas 4.16 and 4.32. □

Our algorithm for modifying a given $f : G \to G$ so that it satisfies (EG Nielsen Paths) is as follows. If some EG stratum does not satisfy (EG Nielsen Paths), let $H_r$ be the highest such stratum. By Lemma 4.32, there is a (possibly empty) sequence of proper folds leading to a relative train track map and an indivisible Nielsen path with either a partial fold or an improper fold. Apply Lemma 4.29 or Lemma 4.31 respectively. If the resulting relative train track map does not satisfy (EG Nielsen Paths) go back to the beginning and start again.

**Remark 4.34.** Iteratively folding any $\rho$ in $H_r$ either determines $f_r$ as in (EG Nielsen Paths) or leads to a partial or improper fold in a predictable number of steps.

Suppose that the algorithm does not terminate. Denote the relative train track maps that are produced by $f = f_0, f_1, f_2 \dots$. Since $N(f_i)$ is non-decreasing and is strictly decreasing when a partial fold occurs, there are only finitely many such occurrences and we may assume without loss that all the folds are full. We make use of the bijection $H_s(i) \to H_s(j)$ between EG strata for $f_i$ and EG strata of $f_j$ given by Lemmas 4.30 and 4.31. Let $H_r$ be the highest stratum for which (EG Nielsen Paths) is not satisfied by $f_k$ for all sufficiently large $k$. Then the number of edges of height $r$ is a non-increasing function of $k$ that strictly decreases when an improper fold of height $r$ occurs. These folds do not therefore occur for sufficiently large $k$. But this contradicts Lemma 4.32 and the choice of $r$. This proves that the algorithm terminates at a relative train track map (still called) $f : G \to G$ satisfying (EG Nielsen Paths).

In the steps that follow the number of edges in each EG strata and the number of indivisible Nielsen paths of EG height are not increased. If after some modification, (EG Nielsen Paths) fails then we can return to Step 1 and start again. By the above argument this terminates after finitely many repetitions. (In fact, it is never necessary to return to Step 1 but this requires an additional argument.)

**Step 2: (Theorem 2.19).** Apply Steps 1 through 6 of the proof of Theorem 2.19 to produce a new $f : G \to G$ satisfying the conclusions of that theorem. By Remark 2.21, the number of edges in each EG strata and the number of indivisible Nielsen paths of EG height is unchanged. As noted in the preceding paragraph, we may assume that (EG Nielsen Paths) is still satisfied.

**Step 3: ((Rotationless), (Filtration) and (Zero Strata)).** Items (Rotationless) and (Filtration) follow from Proposition 3.29 and Theorem 2.19 (F). To achieve (Zero Strata) it suffices, by item (Z) of Theorem 2.19, to arrange that every edge in a zero stratum $H_i$ is $r$-taken. Each edge $E$ in $H_i$ is contained in an $r$-taken path $\sigma \subset H_i$. If $E$ is not $r$-taken, replace $E$ by a path that has the same endpoints as $\sigma$ and is marked by $\sigma$. After finitely many such tree replacements, (Zero Strata) is satisfied.

**Step 4: (Periodic Edges).** Suppose at first that no component $C$ of $\text{Per}(f)$ is topologically a circle with each point in $C$ having exactly two periodic directions. Then the endpoints of any periodic edge are principal, each periodic edge is fixed and each periodic stratum $H_r$ has a single edge $E_r$. If $G_{r-1}$ is not a core graph that contains both endpoints of $E_r$ then one could collapse $E_r$ without changing the free factor systems realized by the filtration elements, in violation of item (P) of Theorem 2.19). Thus (Periodic Edges) is satisfied.

For the general case, it suffices to assume that some component $C$ of $\text{Per}(f)$ is topologically a circle with each point in $C$ having exactly two periodic directions and modify $f : G \to G$ to reduce the number of such components.

Lemma 3.30 (1) implies that $C$ is $f$-invariant and that $g = f|C$ is orientation preserving. By (Zero Strata) and the fact that there are no periodic directions based in $C$ and pointing out of $C$, every edge $E_j$ not in $C$ that has an endpoint in $C$ is non-periodic, NEG and intersects $C$ in exactly its terminal endpoint. Since all non-periodic vertices are contained in EG strata, no vertex in the complement of $C$ maps into $C$. Also, $C$ is a component of some $G_l$ by item (NEG) of Theorem 2.19. Let $E_m$ be the first non-periodic NEG edge $E_j$ that has terminal endpoint in $C$ and note that $f(E_m) = E_m C^d$. We modify $f : G \to G$ near $C$ in two steps as follows.

In the first step we make $C \subset \text{Fix}(f)$. Extend the rotation $g^{-1} : C \to C$ to a map $h : G \to G$ that has support on a small neighborhood of $C$, that is homotopic to the identity and such that $h(E_j) \subset E_j \cup C$ for each non-periodic NEG edge $E_j$ that has terminal endpoint in $C$. Redefine $f$ on each edge $E$ to be $h_\# f_\#(E)$. The filtration is unchanged. Edges in $C$ are now fixed. If $f(E_j) = E_j u_j$ then the new $u_j$ and the old $u_j$ agree with the possible exception of initial and terminal segments in $C$. The $f$-image of all other edges is unchanged. In fact, $f_\#(\sigma)$ is unchanged for any path $\sigma$ with the property that endpoints of $f(\sigma)$ are not in the support of $h$. It is straightforward to check that $f : G \to G$ is a relative train track map and that all of the properties that we have established to date are preserved with the possible exception of item (P) of Theorem 2.19, which fails if one or more of the $E_j$'s is now a fixed edge that should be collapsed. If there is no such edge then proceed to the next paragraph. If there is such an edge, collapse it as in Step 3 of the proof of Theorem 2.19. That step is described very explicitly and we leave it to the reader, here and later in the proof, to check that this operation does not undo previously established properties. After finitely many such collapses, we have $C \subset \text{Fix}(f)$ and all previously established properties are preserved. If $C$ now has outward pointing periodic directions we have finished our modifications of $C$. Otherwise proceed to the next paragraph.

Recall that if $E_m$ is the first non-periodic NEG edge $E_j$ that has terminal endpoint in $C$, then $f(E_m) = E_m C^d$ for some $d \in \mathbb{Z}$. In this second step we arrange that $d = 0$. Choose $h' : G \to G$ that is the identity on $C$, that satisfies $h'(E_j) = E_j C^{-d}$ for all $E_j$ and that has support in a small neighborhood of $C$. This map is homotopic to the identity since we can simply unwind the twisting on $C$. Redefine $f$ on each edge $E$ to be $h_\# f_\#(E)$ and note that $C \cup E_m \subset \text{Fix}(f)$ so the component of $\text{Fix}(f)$ containing

$C$ is no longer a topological circle. The filtration is unchanged. If necessary, collapse fixed edges with an endpoint in $C$ and repeat this second step.

**Step 5: (Induction: the NEG case).** It remains to establish (Completely Split) and the items related to non-fixed NEG edges. We do this by induction up the filtration making use of sliding and the new move described in Section 4.4.

Let NI be the number of irreducible strata in the filtration and for each $0 \le m \le \text{NI}$, let $G_{i(m)}$ be the smallest filtration element containing the first $m$ irreducible strata. We will prove by induction on $m$ that for all $0 \le m \le \text{NI}$, one can modify $f$ to arrange that $f|G_m$ (or more precisely the restriction of $f$ to each component of $G_m$) is a CT. The $m = 0$ case is vacuously true so we assume that $f|G_r$ is a CT for $r = i(m)$ and make modifications to arrange that $f|G_s$ is a CT for $s = i(m+1)$. In this step we assume that $H_s$ is NEG and is hence a single edge $E_s$ satisfying $f(E_s) = E_s u_s$ for some path $u_s \subset G_{s-1}$.

By (Zero Strata), $r = s - 1$. The sliding operation described in Section 2.7 (complete details in Section 5.4 of [2]) allows us to modify $E_s$ and $u_s$ by choosing a path $\tau \subset G_{s-1}$ with initial endpoint equal to the terminal vertex of $E_s$ and 'sliding' the terminal end of $E_s$ to the terminal vertex of $\tau$. As noted in Step 2, we may assume that sliding preserves (EG Nielsen Paths).

As a first case suppose that after sliding along $\tau$ we have $E_s \subset \text{Fix}(f)$. For future reference note that by Lemma 2.17 this is equivalent to $[\bar{\tau} u_s f_\#(\tau)]$ being trivial and hence equivalent to $f_\#(E_s\tau) = E_s[u_s f_\#(\tau)] = E_s\tau$; i.e., to $E_s\tau$ being a Nielsen path.

If both endpoints of $E_s$ are contained in $G_{s-1}$ then (Periodic Edges) is satisfied as are all of the conclusions of Theorem 2.19 and the three properties established in Step 3. The remaining properties of a CT follow from the inductive hypothesis.

If either endpoint of $E_s$ is not contained in $G_{s-1}$ then collapse $E_s$ to a point as in Step 4. None of the previously achieved properties are lost and the remaining properties of a CT follow from the inductive hypothesis. This completes the inductive step in the case that $E_s \subset \text{Fix}(f)$ is trivial after sliding.

We assume now that there is no choice of $\tau$ such that $E_s\tau$ is a Nielsen path. The following proposition is a combination of Proposition 5.4.3 and Lemma 5.5.1 of [2].

**Proposition 4.35.** *Suppose that*

(i) $f : G \to G$ *is a relative train track map that satisfies* (*EG Nielsen Paths*),

(ii) $f|G_{s-1}$ *is a CT,*

(iii) $H_s$ *is an NEG stratum with single edge* $E_s$ *for which there does not exists a path* $\mu \subset G_{s-1}$ *such that* $E_s\mu$ *is a Nielsen path.*

*Then there exists a path* $\tau \subset G_{s-1}$ *with initial endpoint equal to the terminal endpoint of* $E_s$ *such that after performing the slide associated to* $\tau$ *the following conditions are satisfied.*

(1) $f(E_s) = E_s \cdot u_s$ *is a non-trivial splitting.*

(2) *If $\sigma$ is a circuit or path with endpoints at vertices and if $\sigma$ has height $s$ then there exists $k \geq 0$ such that $f_\#^k(\sigma)$ splits into subpaths of the following type.*
   (a) *$E_s$ or $\overline{E}_s$,*
   (b) *an exceptional path of height $s$,*
   (c) *a subpath of $G_{s-1}$.*

(3) *$u_s$ is completely split and its initial vertex is principal.*

(4) *$f|G_s$ satisfies (Linear Edges).*

*Proof.* The construction of a path $\tau$ along which to slide is carried out in the proof of Proposition 5.4.3 of [2]. We assume that $\tau$ has been chosen to satisfy the conclusions of that proposition. In particular, $f(E_s) = E_s \cdot u_s$ is a splitting that is non-trivial by (iii). Thus (1) is satisfied. (The statement of Proposition 5.4.3 of [2] allows the possibility that $G$ is subdivided at a periodic point and that the terminal endpoint of $E_s$ is one of the new periodic vertices. By the end of the construction, we will have shown that the terminal endpoint of $E_s$ is principal and hence fixed. At that point we can undo the subdivision.)

For (3) we must make use of facts that are explicitly stated and used in the proof of Proposition 5.4.3 of [2] but are not contained in its statement. The first is that by a further slide one can replace $u_s$ with $f_\#^k(u_s)$ for any $k \geq 1$. Since $f|G_{s-1}$ satisfies (Completely Split) we may assume by Lemma 4.25 that $u_s$ is completely split. The second is that if $u_s = \alpha \cdot \beta$ is a coarsening of the complete splitting of $u_s$, then by a further slide we may assume that the terminal endpoint of (the new) $E_s$ is the terminal endpoint of $\alpha$. Thus to complete the proof of (3) we need only show that the endpoint of some term in the complete splitting of $u_s$ is principal. The only way that this could fail would be if $u_s$ has height $r'$ where $H_{r'}$ is EG and if each height $r'$ term in the complete splitting of $u_s$ is a single edge. After replacing $u_s$ with a sufficiently high iterate, we may assume that $u_s$ has such a long $r'$-legal segment that every edge in $H_{r'}$ occurs as a term in the complete splitting of $u_s$. Lemma 3.19 then completes the proof of (3).

If $E_s$ is a linear edge, choose a root-free Nielsen path $w_s$ and $d_s \neq 0$ so that $u_s = w_s^{d_s}$. If $E_t \subset G_{s-1}$ is a linear edge with the same axis as $E_s$ then after reversing the orientation on $w_s$ we may assume that $w_t$ and $w_s$ agree as oriented loops. After a further slide as in the proof of (3) we may assume that $w_s = w_t$. Item (iii) implies that $E_s \overline{E}_t$ is not a Nielsen path and hence that $d_s \neq d_t$. This completes the proof of item (4).

Lemma 4.1.4 of [2] states that if $\sigma$ is a height $s$ circuit or path with endpoints at vertices then $\sigma$ splits into subpaths that are either contained in $G_{s-1}$ or are basic paths of height $s$ meaning that they and their inverses have the form $E_s\gamma$ or $E_s\gamma\overline{E}_s$ for some $\gamma \subset G_{s-1}$. It therefore suffices, in proving (2), to assume that $\sigma$ is a basic path of height $s$. Lemma 4.22 and Corollary 4.12 imply that $f : G \to G$ satisfies the conclusions of Proposition 5.4.3 of [2]. Corollary 4.19 therefore implies that $f : G \to G$ satisfies the hypotheses, and hence the conclusions, of

Lemma 5.5.1 of [2]. These conclusions address both types of basic paths of height $s$ and verify (2). □

We assume now that we have performed the slide move of Proposition 4.35. Since $u_s$ is non-trivial, $f|G_s$ satisfies (Periodic Edges) and all of the properties achieved in the first four steps of our construction. Items (Completely Split), (Vertices), (NEG Strata), (Linear Edges) and (Nielsen Paths) for $f|G_s$ follows from Proposition 4.35 and these properties for $f|G_{s-1}$. This completes the proof of the inductive step in the case that $H_s$ is NEG.

**Step 6: (Induction: the EG case).** Suppose now that $H_s$ is EG. Items (Vertices), (NEG Strata), (Linear Edges) and (Nielsen Paths) for $f|G_s$ follow from these properties for $f|G_{s-1}$.

For each edge $E \subset H_s$, there is a decomposition $f(E) = \mu_1 \cdot \nu_1 \cdot \mu_2 \dots \nu_{m-1} \cdot \mu_m$ where the $\nu_l$'s are the maximal subpaths in $G_r$. Let $\{\nu_l\}$ be the collection of all such paths that occur as $E$ varies over the edges of $H_s$. By (RTT-ii), $f_\#^k(\nu_l)$ is non-trivial for each $k$ and $l$. By Lemma 4.25 we may choose $k$ so large that each $f_\#^k(\nu_l)$ is completely split. We may also assume that the endpoints of $f_\#^k(\nu_l)$ are periodic and hence principal. There are finitely many connecting paths $\sigma$ contained in the strata (if any) between $G_r$ and $H_s$. Each $f(\sigma)$ is either a connecting path or a non-trivial path in $G_r$ with fixed endpoints. We may therefore assume that $f_\#^k(\sigma)$ is completely split for each such $\sigma$. After $k$ applications of Lemma 4.27 with $j = r$ (see in particular item (7) of that lemma) we have that $f|G_s$ is completely split. This completes the induction step and so also the proof of the theorem. □

**4.6. Further properties of a CT.** The next lemma is an extension of Lemma 3.26.

**Lemma 4.36.** *Assume that $f : G \to G$ is a CT. The following properties hold for every principal lift $\tilde{f} : \Gamma \to \Gamma$.*

(1) *If $\tilde{v} \in \mathrm{Fix}(\tilde{f})$ and a non-fixed edge $\tilde{E}$ determines a fixed direction at $\tilde{v}$, then $\tilde{E} \subset \tilde{f}_\#(\tilde{E}) \subset \tilde{f}_\#^2(\tilde{E}) \subset \cdots$ is an increasing sequence of paths whose union is a ray $\tilde{R}$ that converges to some $P \in \mathrm{Fix}_N(\hat{f})$ and whose interior is fixed point free.*

(2) *For every isolated $P \in \mathrm{Fix}_N(\hat{f})$ there exists $\tilde{E}$ and $\tilde{R}$ as in (1) that converges to $P$. The edge $E$ is non-linear.*

*Proof.* For $\tilde{E}$ as in (1) and for each $m > 0$, Lemma 4.6 implies that $\tilde{E} \subset \tilde{f}_\#(\tilde{E}) \subset \tilde{f}_\#^2(\tilde{E}) \subset \cdots$ is a nested sequence of completely split paths. This increasing sequence of paths defines a ray $\tilde{R}'$ that converges to some non-repelling fixed point $P \in \mathrm{Fix}_N(\hat{f})$ and that, by Corollary 4.12, intersects $\mathrm{Fix}(\tilde{f})$ only in its initial endpoint. This completes the proof of (1).

If $P \in \mathrm{Fix}_N(\hat{f})$ is isolated then $\tilde{f}$ moves points that are sufficiently close to $P$ toward $P$ by Lemma 2.3. We may therefore choose a ray $\tilde{R}$ that converges to $P$ and that intersects $\text{Fix}(\tilde{f})$ only in its initial endpoint. Moreover, the initial edge $\widetilde{E}$ of $\widetilde{R}$ determines a fixed direction by Lemma 3.16 and so extends to a fixed point free ray $\widetilde{R}$ converging to some $Q \in \text{Fix}_N(\hat{f})$ by (1). Lemma 3.16 implies that $P = Q$. Since $P$ is isolated, Lemma 2.1 and Lemma 2.3 (i) imply that $P$ is not an endpoint of the axis of a covering translation; in particular, $E$ is not a linear edge. □

**Notation 4.37.** If $\widetilde{E}$ and $P$ are as in Lemma 4.36 (1) then we say that $\widetilde{E}$ *iterates to* $P$ and that $P$ is *associated to* $\widetilde{E}$.

The following lemma is used in Section 5 and also in [10]. It is related to the fact (Proposition 3.3.3 (3) of [2]) that if $\Lambda$ is an attracting lamination for some element of $\text{Out}(F_n)$ then $\Lambda \in \mathcal{L}(\psi)$ if and only if $\text{PF}_\Lambda(\psi) > 0$.

**Lemma 4.38.** *Suppose that* $\psi \in \text{Out}(F_n)$ *is forward rotationless and that* $P \in \text{Fix}_N(\hat{\Psi})$ *for some* $\Psi \in \text{P}(\psi)$*. Suppose further that* $\Lambda$ *is an attracting lamination for some element of* $\text{Out}(F_n)$*, that* $\Lambda$ *is* $\psi$*-invariant and that* $\Lambda$ *is contained in the accumulation set of* $P$*. Then* $\text{PF}_\Lambda(\psi) \geq 0$ *and* $\text{PF}_\Lambda(\psi) > 0$ *if and only if* $P$ *is isolated in* $\text{Fix}_N(\hat{\Psi})$.

*Proof.* Let $g\colon G \to G$ be a CT representing $\psi$, let $\tilde{g}\colon \Gamma \to \Gamma$ be the lift corresponding to $\Psi$ and let $\widetilde{R} \subset \Gamma$ be a ray converging to $P$. Choose a generic leaf $\gamma \subset G$ of the realization of $\Lambda$ in $G$. Then every finite subpath of $\gamma$ lifts to a subpath of every subray of $\widetilde{R}$. If $P$ is not isolated in $\text{Fix}_N(\hat{\Psi})$ then Lemma 2.3 implies that there are points in $\text{Fix}(\tilde{g})$ whose nearest points in $\widetilde{R}$ converge to $P$. It follows that there is a subray of $\widetilde{R}$ each of whose finite paths is contained in a Nielsen path for $\tilde{g}$. In particular, every finite subpath of $\gamma$ extends to a Nielsen path for $g$. By (NEG Nielsen Paths), (EG Nielsen Paths) and Corollary 4.19, this extension can be done with a uniformly bounded number of edges. It is an immediate consequence of the definition of the expansion factor (Definition 3.3.2 of [2]) that $\text{PF}_\Lambda(\psi) = 0$.

Assume now that $P$ is isolated in $\text{Fix}_N(\hat{\Psi})$ and let $\Sigma$ be the set of finite paths $\sigma \subset G$ with endpoints at vertices and with the property that every finite subpath of $\gamma$ is contained in $g^m_\#(\sigma)$ for some $m > 0$. Lemma 4.36 and the assumption that $\Lambda$ is contained in the accumulation set of $P$ imply that $\Sigma$ contains a path that is a single edge and in particular is non-empty.

Let $\sigma \in \Sigma$ be an element of minimal height, say $k$. Then $\sigma$ decomposes as a concatenation of edges $\mu_i \subset H_k$ and subpaths $\nu_i \subset G_{k-1}$ and we let $K$ be the number of elements in this decomposition. Choose a nested sequence of subpaths $\gamma_j$ of $\gamma$ whose union equals $\gamma$. Since the $\nu_i$'s are not in $\Sigma$, there exists $J > 0$ so that $\gamma_j$ is not contained in any $g^m_\#(\nu_i)$ for $j > J$. Since $\gamma$ is generic, it is birecurrent. Choose $j_0 > J$ and $j > j_0$ so that $\gamma_j$ contains at least $3K$ disjoint copies of $\gamma_{j_0}$. There exists $m > 0$ such that $\gamma_j \subset g^m_\#(\sigma)$. It follows that $\gamma_{j_0} \subset g^m_\#(\mu_i)$ for some $i$. There is a choice of $i$ that works for all choices of $j_0$ and this proves that $\mu_i \in \Sigma$.

Let $E$ be the single edge in $\mu_i$. We assume that $E$ is NEG and argue to a contradiction. There is a path $u \subset G_{k-1}$ such that $g^m_\#(E) = E \cdot u \dots g^{m-1}_\#(u)$ for all $m > 0$. Lemma 3.1.16 of [2] states that $\gamma$ is not a circuit. It follows that $u$ is not a Nielsen path and hence that the length of $g_\#^l(u)$ goes to infinity with $l$. The birecurrence of $\gamma$ and the fact that $E \in \Sigma$ imply that for every $\gamma_j$ there exits $p > 0$ such that $\gamma_j \subset g_\#^{p-1}(u)g_\#^p(u) = g_\#^{p-1}(ug_\#(u))$ in contradiction to the assumption that no element of $\Sigma$ has height less than $k$.

We now know that $H_k$ is EG. Since $E \in \Sigma$, $\gamma$ is a leaf in the attracting lamination $\Lambda_k$ associated to $H_k$. There is a splitting of $g(E)$ into subpaths in $H_k$ and subpaths in $G_{k-1}$. If $\gamma$ were contained in $G_{k-1}$ then one of the subpaths in $G_{k-1}$ would be contained in $\Sigma$ in contradiction to our choice of $k$. Thus $\gamma$ is not entirely contained in $G_{k-1}$ and Lemma 3.1.15 of [2] implies that $\gamma$ is a generic leaf of $\Lambda_k$. In other words, $\Lambda = \Lambda_k$. It follows that $\mathrm{PF}_\Lambda(\psi) > 0$ which completes the proof. □

Assume that $\phi$ is forward rotationless and that $f : G \to G$ is a CT representing $\phi$. Following the notation of [3] we say that an unoriented conjugacy class $\mu$ of a root-free element of $F_n$ is an *axis for* $\phi$ if for some (and hence any) representative $c \in F_n$ of $\mu$ there exist distinct $\Phi_1, \Phi_2 \in \mathrm{P}(\phi)$ satisfying $\Phi_1(c) = \Phi_2(c) = c$, which by Remark 3.4 is equivalent to $\mathrm{Fix}_N(\hat{\Phi}_1) \cap \mathrm{Fix}_N(\hat{\Phi}_2) = \partial A_c$. It is a consequence of Lemma 4.40 below that an unoriented conjugacy class $\mu$ is an axis for $\phi$ if and only if it is an axis for a linear edge in some (every) CT representing $\phi$.

**Remark 4.39.** In the context of the mapping class group, a conjugacy class is an axis if and only if it is represented by a reducing curve in the minimal Thurston normal form.

Lemma 3.30 implies that the oriented conjugacy class of $c$ is $\phi$-invariant. By Lemmas 4.1.4 and 4.2.6 of [2], the circuit $\gamma$ representing $c$ splits into a concatenation of subpaths $\alpha_i$, each of which is either a fixed edge or an indivisible Nielsen path. (NEG Nielsen Paths) and Corollary 4.19 imply that each turn $(\bar{\alpha}_i, \alpha_{i+1})$ is legal. Item 1 of Lemma 4.11 therefore implies that this splitting is the complete splitting of $\gamma$.

There is an induced complete splitting of $A_c$ into subpaths $\tilde{\alpha}_i$ that project to either fixed edges or indivisible Nielsen paths. The lift $\tilde{f}_0 : \Gamma \to \Gamma$ that fixes the endpoints of each $\tilde{\alpha}_i$ is a principal lift by Corollary 3.27 and commutes with $T_c$. We say that $\tilde{f}_0$ and the corresponding $\Phi_0 \in \mathrm{P}(\phi)$ are the *base lift* and *base principal automorphism* associated to $\mu$ and the choices of $T_c$ and $f : G \to G$. (If $\mu$ is not represented by a basis element then $\Phi_0$ is independent of the choice of $f : G \to G$. To see this, let $F$ be the smallest free factor that carries $\mu$ and note that there can not be a linear edge with axis $\mu$ in any CT representing $\phi|F$. Lemma 4.40 below implies that $\mu$ is not an axis for $\phi|F$ and so there is a unique principal automorphism $\Phi|F$ that fixes $c$. The automorphism $\Phi_0$ is the unique (because $F$ has rank greater than one) extension of $\Phi|F$. It is not hard to show that if $\mu$ is represented by a basis element then $\Phi_0$ is not independent of the choice of $f : G \to G$; see, for example, the proof of Proposition 8.9 of [10].) Item 2 of Lemma 4.11 implies that, for each $\tilde{\alpha}_i$ and for each $\tilde{x} \in \tilde{\alpha}_i$, the nearest point to $\tilde{f}_0(\tilde{x})$ in $A_c$ is contained in $\tilde{\alpha}_i$. It follows that $\mathrm{Fix}(T_c^j \tilde{f}_0) = \emptyset$ for all $j \neq 0$ and hence that $\tilde{f}_0$ is the only lift that commutes with $T_c$ and has fixed points in $A_c$.

**Lemma 4.40.** *Suppose that $\phi$ is forward rotationless and that the unoriented conjugacy class $\mu$ is an axis for $\phi$. Assume notation as above. There is a bijection between the set of principal lifts [principal automorphisms] $\tilde{f}_j \neq \tilde{f}_0$ [respectively $\Phi_j \neq \Phi_0 \in \mathrm{P}(\phi)$] that commute with $T_c$ [fix c] and the set of linear edges $\{E_j\}$ with axis equal to $\mu$. Moreover, if $f(E_j) = E_j w_j^{d_j}$ then $\tilde{f}_j = T_c^{d_j} \tilde{f}_0$ $[\Phi_j = i_c^{d_j} \Phi_0]$.*

*Proof.* The $w_j$'s in question are equal by (Linear Edges) and we label this path $w$. There is a lift $\tilde{w}$ that is a fundamental domain of $A_c$ and that is a Nielsen path for $\tilde{f}_0$. Let $\tilde{E}_j$ be the lift of $E_j$ that terminates at the initial endpoint $\tilde{v}$ of $\tilde{w}$ and let $\tilde{f}_j$ be the lift that fixes the initial endpoint of $\tilde{E}_j$. Then $\tilde{f}_j$ is a principal lift that commutes with $T_c$ and satisfies $\tilde{f}_j(\tilde{v}) = T_c^{d_j}(\tilde{v}) = T_c^{d_j}(\tilde{f}_0 \tilde{v})$ which implies that $\tilde{f}_j = T_c^{d_j} \tilde{f}_0$.

Conversely, if $\tilde{f} \neq \tilde{f}_0$ is a principal lift that commutes with $T_c$ then $\tilde{f} = T_c^d \tilde{f}_0$ for some $d \neq 0$. In particular, $A_c$ is disjoint from $\mathrm{Fix}(\tilde{f})$ and there is a ray $\tilde{R}_1$ that intersects $\mathrm{Fix}(\tilde{f})$ in exactly its initial endpoint and that terminates at the endpoint $P$ of $A_c$ that is the limit of the forward $\tilde{f}$ orbit of $\tilde{v}$. Let $\tilde{E}$ be the initial edge of $\tilde{R}_1$. Lemma 3.16 implies that $\tilde{E}$ determines a fixed direction and also that $\tilde{R}_1$ must be the ray constructed from the initial edge $\tilde{E}$ of $\tilde{R}_1$ by Lemma 4.36. If $\tilde{f}(\tilde{E}) = \tilde{E} \cdot \tilde{u}$ then $\tilde{R}_1 = E \cdot \tilde{u} \cdot \tilde{f}_\#(\tilde{u}) \cdot \tilde{f}_\#^2(\tilde{u}) \ldots$. Since $\tilde{R}_1$ has a common infinite end with $A_c$, it follows that $f_\#^k(u)$ is a periodic, hence fixed, Nielsen path for sufficiently large $k$ and for $u$ equal to the projected image of $\tilde{u}$. In particular, $u$ and $f_\#(u)$ have the same $f_\#^k$-image, and since they have the same endpoints, they must be equal. In other words, $u$ is a Nielsen path. This proves that $\tilde{E}$ is the lift of a linear edge $E$ whose associated axis is $\mu$. By (Linear Edges), $E = E_j$ and $d = d_j$ for some $j$ and, after translating $\tilde{E}$ by some iterate of $T_c$ if necessary, $\tilde{v}$ is the terminal endpoint of $\tilde{E}$. □

**Remark 4.41.** Suppose that $f : G \to G$ is a CT, that $C$ is a component of some filtration element $G_s$, that $C$ has no valence one vertices and that $\phi|\mathcal{F}(C)$ is the trivial outer automorphism. Then $f|C$ is the identity. To see this, let $H_r$ be the first non-fixed stratum in $C$. It can not be EG because the identity element has no attracting laminations. If it were NEG it would have to be linear because $f|G_{r-1}$ is the identity and it cannot be linear because the identity element has no axes.

We conclude this section by showing that every element of $\mathrm{Out}(F_n)$ has a uniformly bounded iterate that is forward rotationless.

**Lemma 4.42.** *For all $n \geq 1$ there exists $K_n > 1$ so that $\phi^{K_n}$ is forward rotationless for all $\phi \in \mathrm{Out}(F_n)$.*

*Proof.* Given $\phi \in \mathrm{Out}(F_n)$, let $f : G \to G$ be a CT representing some forward rotationless iterate $\psi = \phi^N$ of $\phi$. By Corollary 3.17 and Lemma 3.8, the number of isogredience classes of principal lifts of $\psi$ is less than or equal to the number of Nielsen classes for $f : G \to G$. If $x$ is a principal vertex that has valence two and that is isolated in $\mathrm{Fix}(f)$ then $x$ is either the initial endpoint of a non-fixed NEG edge or an endpoint of an indivisible Nielsen path of EG height. By (Vertices) and Corollary 4.19, there is a uniform (i.e. depending only on $n$) upper bound to the number of isolated fixed principal vertices. By (Periodic Edges) there is also a uniform upper bound to the number of components of $\mathrm{Fix}(f)$ that contain at least one edge. It follows that there is a uniform upper bound to the number of Nielsen classes for $f : G \to G$ and to the number of edges based at principal vertices. From the former we conclude that the number of isogredience classes of principal lifts $\tilde{f} : \Gamma \to \Gamma$ of $f$ is uniformly bounded.

Since $\phi$ commutes with $\psi$, it acts on the set of isogredience classes of principal automorphisms representing $\psi$. After replacing $\phi$ with a uniformly bounded iterate, we may assume that $\phi$ fixes each isogredience class. Thus, if $\Psi$ is a principal automorphism representing $\psi$ then there exists an automorphism $\Phi$ representing $\phi$ such that $\Phi$ commutes with $\Psi$. In particular, $\mathbb{F} := \mathrm{Fix}(\Psi)$ is $\Phi$-invariant. By construction, the outer automorphism determined by $\Phi|\mathbb{F}$ has finite order and so is represented by a homeomorphism of a graph with no valence one or valence two vertices. Since the rank of $\mathbb{F}$ is uniformly bounded, the period of the outer automorphism determined by $\Phi|\mathbb{F}$ is uniformly bounded. After replacing $\phi$ with a further uniformly bounded iterate, we may assume that $\mathbb{F} \subset \mathrm{Fix}(\Phi)$. Thus $\mathrm{Fix}(\hat{\Phi})$ contains each non-isolated point of $\mathrm{Fix}(\hat{\Psi})$ by Lemma 2.3.

By Lemma 4.36 (2), the number of isolated points in $\mathrm{Fix}(\hat{\Psi})$, up to the action of $\mathbb{F}$, is bounded above by the number of edges based at principal fixed points for $f$ and so is uniformly bounded. We may therefore assume that if $P$ is an isolated point in $\mathrm{Fix}(\hat{\Psi})$ then $\hat{\Phi}(P) = \hat{T}_a(P)$ for some $a = a_P \in \mathbb{F}$, from which it follows that $\hat{\Phi}^N(P) = \hat{T}_a^N(P)$.

The proof now divides into cases. If $\mathbb{F}$ is trivial then $\mathrm{Fix}(\hat{\Psi}) \subset \mathrm{Fix}(\hat{\Phi})$. If $\mathbb{F}$ has rank at least two then $\Phi^N = \Psi$. It follows that $T_a$ is trivial and again $\mathrm{Fix}(\hat{\Psi}) \subset \mathrm{Fix}(\hat{\Phi})$. The final case is that $\mathbb{F}$ has rank one. After replacing $\Phi$ with $T_a^{-1}\Phi$ we may assume that $P \in \mathrm{Fix}(\hat{\Phi})$. Since $\mathrm{Fix}(\hat{\Phi})$ and $\mathrm{Fix}(\hat{\Psi})$ have at least three points in common, $\Phi^N = \Psi$. As in the higher rank case, it follows that $\mathrm{Fix}(\hat{\Psi}) \subset \mathrm{Fix}(\hat{\Phi})$ in this case as well. As this holds for each principal automorphism representing $\psi$, $\phi$ is forward rotationless. □

## 5. Recognition Theorem

In this section we specify invariants that uniquely determine a forward rotationless $\phi$. As a warm-up to the general theorem, we consider the special case, essentially proved in [1], that $\phi$ is irreducible, meaning that there are no non-trivial proper $\phi$-invariant free factor systems. It follows that a CT $f : G \to G$ representing $\phi$ has a single stratum and that the stratum is EG. In particular, $\phi$ has infinite order and $\mathcal{L}(\phi)$ has exactly one element. Lemma 3.30 (3) implies that all iterates of $\phi$ are irreducible.

**Lemma 5.1.** *If $\phi \in \mathrm{Out}(F_n)$ is irreducible and forward rotationless, then $\phi$ has infinite order and is determined by its unique attracting lamination $\Lambda$ and the expansion factor $\mathrm{PF}_\Lambda(\phi)$. More precisely, if $\phi$ and $\psi$ are forward rotationless and irreducible and if they have the same unique attracting lamination and the same expansion factor then $\phi = \psi$.*

*Proof.* As noted above, $\phi$ and $\psi$ have infinite order and all iterates of $\phi$ and $\psi$ are irreducible. Theorem 2.14 of [1] implies that $\psi^{-1}\phi$ has finite order and that $\psi^k = \phi^k$ for some $k \geq 1$. By Lemma 3.19 and Lemma 2.13 there exists $\Phi \in \mathrm{P}(\phi)$ such that $\mathrm{Fix}_N(\Phi)$ contains at least three points $P_1$, $P_2$ and $P_3$, each of whose accumulation set equals $\Lambda$. The $\mathrm{Fix}_N$-preserving bijections between $\mathrm{P}(\phi)$ and $\mathrm{P}(\phi^k)$ and between $\mathrm{P}(\psi)$ and $\mathrm{P}(\psi^k)$ induce a $\mathrm{Fix}_N$-preserving bijection between $\mathrm{P}(\phi)$ and $\mathrm{P}(\psi)$. Thus there exists $\Psi \in \mathrm{P}(\psi)$ such that $\mathrm{Fix}_N(\hat{\Psi}) = \mathrm{Fix}_N(\hat{\Phi})$.

Choose a finite order homeomorphism $f : G \to G$ of a marked graph $G$ representing $\psi^{-1}\phi$, let $\tilde{f} : \Gamma \to \Gamma$ be the lift corresponding to $\Psi^{-1}\Phi$ and note that $P_1, P_2, P_3 \in \mathrm{Fix}(\hat{f})$. The line with endpoints $P_1$ and $P_2$ and the line with endpoints $P_1$ and $P_3$ are $\tilde{f}_\#$-invariant and since $\tilde{f}$ is a homeomorphism they are $\tilde{f}$-invariant. The intersection of these lines is an $\tilde{f}$- invariant, and hence $\tilde{f}$-fixed, ray $\widetilde{R}$ that terminates at $P_1$. The lamination $\Lambda$ is carried by the subgraph $G_0 \subset \mathrm{Fix}(f)$ that is the image of $\widetilde{R}$. Example 2.5(1) of [1] implies that $G_0 = G$; thus $f$ is the identity and $\psi = \phi$. ☐

If $\Phi_1, \Phi_2 \in \mathrm{P}(\phi)$ and if there exists a non-trivial indivisible $a \in \mathrm{Fix}(\Phi_1) \cap \mathrm{Fix}(\Phi_2)$, then $\Phi_2\Phi_1^{-1} = i_a^d$ for some $d \neq 0$. We think of $d$ as a *twist coefficient* for the ordered pair $(\Phi_1, \Phi_2)$ relative to $a$. In our next example we show that an elementary linear outer automorphism is determined by the fixed subgroups of its principal automorphisms and by a twist coefficient.

**Example 5.2.** Let $x_1, x_2, \ldots, x_n$ be a basis for $F_n$ and let $F_{n-1} = \langle x_1, \ldots, x_{n-1}\rangle$. Define $\Phi_1$ by $\Phi_1 | F_{n-1} =$ identity and $\Phi_1(x_n) = x_n a^d$ for some non-trivial root-free $a \in F_{n-1}$ and some $d > 0$. Define

$$\Phi_2 = i_{x_n}^{-1}\Phi_1 i_{x_n} = i_{\bar{x}_n\Phi_1(x_n)}\Phi_1 = i_a^d\Phi_1.$$

Then $\mathrm{Fix}(\Phi_1) = \langle x_1, \ldots, x_{n-1}, x_n a \bar{x}_n\rangle$, $\mathrm{Fix}(\Phi_2) = i_{\bar{x}_n}\mathrm{Fix}(\Phi_1)$ and $\mathrm{Fix}(\Phi_1) \cap \mathrm{Fix}(\Phi_2) = \langle a\rangle$. Since $\mathrm{Fix}(\Phi_1)$ and $\mathrm{Fix}(\Phi_2)$ have rank greater than one, $\Phi_1, \Phi_2 \in \mathrm{P}(\phi)$.

We claim that for any $\psi \in \mathrm{Out}(F_n)$, if there exist $\Psi_1, \Psi_2 \in \mathrm{P}(\psi)$ such that $\Psi_2 = i_a^d\Psi_1$ and such that $\mathrm{Fix}(\Psi_i) = \mathrm{Fix}(\Phi_i)$ for $i = 1, 2$, then $\psi = \phi$. It is obvious that $\Psi_1 | F_{n-1}$ = identity. Moreover,

$$\begin{aligned}\mathrm{Fix}(i_{a^d}\Psi_1) = \mathrm{Fix}(i_{a^d}\Phi_1) &= \mathrm{Fix}(i_{x_n}^{-1}\Phi_1 i_{x_n}) = i_{\bar{x}_n}\,\mathrm{Fix}(\Phi_1)\\ &= i_{\bar{x}_n}\,\mathrm{Fix}(\Psi_1) = \mathrm{Fix}(i_{x_n}^{-1}\Psi_1 i_{x_n}) = \mathrm{Fix}(i_{\bar{x}_n\Psi_1(x_n)}\Psi_1).\end{aligned}$$

Since $i_{a^d}\Psi_1$ and $i_{\bar{x}_n\Psi_1(x_n)}\Psi_1$ represent the same outer automorphism and have a common fixed subgroup of rank greater than one, they are equal. Thus $a^d = \bar{x}_n\Psi_1(x_n)$ or equivalently $\Psi_1(x_n) = x_n a^d$. This proves that $\Psi_1 = \Phi_1$ and $\phi = \psi$.

We now turn to the general case.

**Theorem 5.3** (Recognition Theorem). *Suppose that $\phi, \psi \in \mathrm{Out}(F_n)$ are forward rotationless and that*

(1) $\mathrm{PF}_\Lambda(\phi) = \mathrm{PF}_\Lambda(\psi)$, *for all* $\Lambda \in \mathcal{L}(\phi) = \mathcal{L}(\psi)$.
(2) *there is bijection* $B\colon\ \mathrm{P}(\phi) \to \mathrm{P}(\psi)$ *such that:*
   (i) **(fixed sets preserved)** $\mathrm{Fix}_N(\hat{\Phi}) = \mathrm{Fix}_N(\widehat{B(\Phi)})$
   (ii) **(twist coordinates preserved)** *If* $w \in \mathrm{Fix}(\Phi)$ *and* $\Phi, i_w\Phi \in \mathrm{P}(\phi)$, *then* $B(i_w\Phi) = i_w B(\Phi)$.

*Then* $\phi = \psi$.

**Remark 5.4.** The bijection $B$ is necessarily equivariant in the sense that $B(i_c\Phi i_c^{-1}) = i_c B(\Phi) i_c^{-1}$ for all $c \in F_n$. This follows from the fact that $\mathrm{Fix}(i_c\Phi i_c^{-1}) = i_c(\mathrm{Fix}(\Phi_1))$ and from Remark 3.4. Thus $B$ is determined by its value on one representative from each of the finitely many isogredience classes in $\mathrm{P}(\phi)$ and 2 (i) can be verified by checking finitely many cases. Similarly, 2 (ii) can be verified by checking finitely many cases. The $w$'s to which 2 (ii) apply have the form $w = a^d$ where $a$ represents a common axis of $\phi$ and $\psi$. The values of $d$ can be read off from relative train track maps as in Lemma 4.40.

**Remark 5.5.** The assumption in (1) that $\mathcal{L}(\phi) = \mathcal{L}(\psi)$ is redundant. It follows from Lemma 3.26 and 2 (i). We include it in the statement of the theorem for clarity.

*Proof.* The proof is by induction on $n$. By convention, all forward rotationless outer automorphisms are the identity when $n = 1$ so we may assume that the theorem holds for all ranks less than $n$ and prove it for $n$.

The case that both $\phi$ and $\psi$ are irreducible is proved in Lemma 5.1 so we may assume that at least one of these, say $\phi$, is reducible and so admits a proper nontrivial invariant free factor system. Since this free factor system is realized by a filtration element in a relative train track map representing $\phi$, some proper free factor carries either an attracting lamination $\Lambda$ for $\phi$ or a $\phi$-periodic conjugacy class $[c]$. Lemma 3.30 implies that the elements of $\mathcal{L}(\phi) = \mathcal{L}(\psi)$ are invariant by both $\phi$ and $\psi$, that $\phi$ and $\psi$ have the same periodic conjugacy classes and that all these conjugacy classes are fixed. The smallest free factor that carries $\Lambda$ or $[c]$ is both $\phi$-invariant and $\psi$-invariant by Corollary 2.5. This proves the existence of non-trivial proper free factors that are that both $\phi$-invariant and $\psi$-invariant.

Among all proper free factor systems, each of whose elements is both $\phi$-invariant and $\psi$-invariant, choose one $\mathcal{F} = \{[[F^1]], \dots, [[F^k]]\}$ that is maximal with respect to inclusion. We claim that $\phi|F^i = \psi|F^i$ for each $i$. If $F^i$ has rank one then this follows from Lemma 3.30. If $F^i$ has rank at least two then principal automorphisms representing $\phi|F^i$ and $\psi|F^i$ extend uniquely to principal automorphisms representing $\phi$ and $\psi$. Thus $\phi|F^i$ and $\psi|F^i$, which are forward rotationless by Corollary 3.31, satisfy the hypothesis of Theorem 5.3 and the inductive hypothesis implies that $\phi|F^i = \psi|F^i$. This verifies the claim. Let $f : G \to G$ be a CT representing $\phi$ with $[\pi_1(G_r)] = \mathcal{F}$ for some filtration element $G_r$, which we may assume without loss has no valence one vertices. Then $f|G_r$ represents both $\phi|\mathcal{F}$ and $\psi|\mathcal{F}$. The proof now divides into two cases. The arguments are sufficiently elaborate that we treat the cases in separate subsections.

**5.1. The NEG case.** In this subsection we complete the proof of Theorem 5.3 in the case that there exists $s > r$ such that $G_s$ is not homotopy equivalent to $G_r$ and such that $H_i$ is NEG for all $r < i \le s$. After reordering the $H_i$'s if necessary, we may assume by Lemma 4.21 that $G_s$ is obtained from $G_r$ as a topological space by either adding a disjoint circle or by attaching an arc $E$ with both endpoints in $G_r$. In the former case, $[\pi_1(G_s)]$ is both $\phi$-invariant and $\psi$-invariant in contradiction to the assumption that $\mathcal{F}$ is maximal and the fact that $G_s$ is disconnected. Thus $G_s = G_r \cup E$ where $f(E) = \bar{u}_1 E u_2$ for some closed paths $u_1, u_2 \subset G_r$. If $E$ is a single edge of $G$ then $s = r + 1$ and at least one of $u_1$ or $u_2$ is trivial. Otherwise $E$ is made up of two edges and $s = r + 2$.

Let $\Gamma$ be the universal cover of $G$. Choose lifts $\tilde{E}, \tilde{u}_1, \tilde{u}_2 \subset \Gamma$ and $\tilde{f} : \Gamma \to \Gamma$ such that $\tilde{f}(\tilde{E}) = \tilde{u}_1^{-1} \tilde{E} \tilde{u}_2$. Denote the component of $G_r$ that contains $u_i$ by $C^i$ and the copy of the universal cover of $C^i$ that contains $\tilde{u}_i$ by $\Gamma_r^i$. If $u_1$ or $u_2$ is trivial then (Remark 4.9) at least one of the endpoints of $\tilde{E}$ is a principal vertex that is fixed by $\tilde{f}$. Otherwise $\tilde{E}$ subdivides into two NEG edges whose common initial vertex is principal and is fixed by $\tilde{f}$. Corollary 3.27 therefore implies that $\tilde{f}$ is a principal lift. Lemma 3.26 implies that there is a line $\tilde{\gamma} \subset \Gamma$ that crosses $\tilde{E}$ and has endpoints in $\mathrm{Fix}_N(\hat{f})$. The projection $\gamma$ of $\tilde{\gamma}$ is $\phi$-invariant and by (2) (i) is $\psi$-invariant as well. The smallest free factor system that carries $[\pi_1(G_r)]$ and $\gamma$ is both $\phi$-invariant and $\psi$-invariant. Since $\mathcal{F}$ is maximal, $G = G_s$.

Corollary 3.2.2 of [2] implies that $\psi$ is represented by $g : G \to G$ such that $g|G_r = f|G_r$ and such that $g(E) = \bar{w}_1 E w_2$ for some closed paths $w_1, w_2 \subset G_r$. It suffices to prove that $u_i = w_i$. The cases are symmetric so we show that $u_1 = w_1$.

Suppose at first that $C^1$ has rank one and hence is a topological circle that is contained in $\mathrm{Fix}(f)$. By (Periodic Edges), the vertices in $C^1$ are principal. Thus at least one of $E$ or $\bar{E}$ determines a direction based in $C^1$ that is fixed by $Df$. If $u_1$ is non-trivial then it must be that $\bar{E}$ determines a fixed direction based in $C^1$. In this case $C^1 \cup E$ is a component of $G$ and hence equal to $G$. We conclude that $n = 2$ and that there is a basis $\{x_1, x_2\}$ for $F_2$ and $d \neq 0$ such that $x_1 \mapsto x_1$ and $x_2 \mapsto x_2 x_1^d$ defines an automorphism representing $\phi$. This is a special case of Example 5.2 and so $u_1 = w_1$ in this case. We may therefore assume that $u_1$ is trivial. The symmetric argument with $g$ replacing $f$ reduces us to the case that $u_1$ and $w_1$ are both trivial and so equal. We may now assume that $C^1$ has rank at least two.

The principal lift $\tilde{g}\colon \Gamma \to \Gamma$ that corresponds to $\tilde{f}$ under the bijection $B$ satisfies $\mathrm{Fix}_N(\hat{g}) = \mathrm{Fix}_N(\hat{f})$. Since $\hat{g}$ fixes the endpoints of $\tilde{\gamma}$ and $\widetilde{E}$ is the only edge in $\tilde{\gamma}$ that does not project into $G_r$, it follows that $\tilde{g}(\widetilde{E}) = \tilde{w}_1 \widetilde{E} \tilde{w}_2$. Let $\tilde{v} \in \Gamma_r^1$ be the initial endpoint of $\widetilde{E}$. Then $\tilde{u}_1$ and $\tilde{w}_1$ are the paths in $\Gamma_r^1$ connecting $\tilde{v}$ to $\tilde{f}(\tilde{v})$ and $\tilde{v}$ to $\tilde{g}(\tilde{v})$ respectively. It therefore suffices to show that $\tilde{f}|\Gamma_r^1 = \tilde{g}|\Gamma_r^1$.

We know that $\tilde{f}|\Gamma_r^1$ and $\tilde{g}|\Gamma_r^1$ are both lifts of $f|C^1$ and that $\hat{f}|\partial\Gamma_r^1$ and $\hat{g}|\partial\Gamma_r^1$ have a common fixed point $P$. If $P$ is not an endpoint of the axis of some covering translation $T_c$ of $\Gamma_r^1$, then there is at most one lift of $f|C^1$ that fixes $P$ and we are done. Suppose then that $P \in \{T_c^{\pm}\}$. By Remark 3.2, there exists a principal lift of $f|C^1$ that commutes with $T_c$. This lift extends over $\Gamma$ to principal lifts $\tilde{f}'$ and $\tilde{g}'$ of $f$ and $g$ respectively. Since $\mathrm{Fix}_N(\widehat{f'}|\partial\Gamma_r^1) \subset \mathrm{Fix}_N(\widehat{f'}) \cap \mathrm{Fix}_N(\widehat{g'})$ contains at least three points, $\tilde{g}' = B(\tilde{f}')$. Condition 2 (ii) therefore implies that $\tilde{g} = T_c^d \tilde{g}'$ and $\tilde{f} = T_c^d \tilde{f}'$ for some $d \in \mathbb{Z}$. We conclude that $\tilde{f}|\Gamma_r^1 = \tilde{g}|\Gamma_r^1$ as desired.

**5.2. The EG case.** In this subsection we prove Theorem 5.3 assuming that there exists $s > r$ where $H_s$ is exponentially growing and where the union of the non-contractible components of $G_{s-1}$ is homotopy equivalent to $G_r$. In light of sub-Section 5.1 and (Zero Strata) this completes the proof of Theorem 5.3. Since $G_{s-1}$ and $G_r$ carry the same elements of $\mathcal{L}(\phi)$, all irreducible strata between $G_r$ and $G_s$ are NEG. Since $\mathcal{F}$ is maximal, $[\pi_1(G_s)]$ is the smallest free factor system carrying $[\pi_1(G_r)]$ and $\Lambda_s$. By (1), $\Lambda_s$, and hence $[\pi_1(G_s)]$, is $\psi$-invariant. It follows that $G_s = G$.

Denote $\psi^{-1}\phi$ by $\theta$. We must show that $\theta$ is trivial. By construction, $\theta|[[F^l]]$ is trivial for each $[[F^l]] \in \mathcal{F}$ and the attracting lamination $\Lambda$ associated to $H_s$ is $\theta$-invariant with expansion factor one. Moreover, for any principal lift $\Phi$ of $\phi$ there is a unique lift $\Theta$ of $\theta$ such that $\mathrm{Fix}(\hat{\Theta}) \supset \mathrm{Fix}_N(\hat{\Phi})$.

Each $F^l$ corresponds to a non-contractible component $D_l$ of $G_{s-1}$. Let $\widetilde{D}_l$ be the component of the full pre-image of $D_l$ whose accumulation set in $\partial F_n$ is $\partial F^l$. Suppose that $v \in H_s \cap D_l$ and that $\tilde{v} \in \widetilde{D}_l$ is a lift of $v$. Then $\tilde{v}$ is principal by Remark 4.9 and the lift $\tilde{f}_{\tilde{v}}$ of $f$ that fixes $\tilde{v}$ is principal by Lemma 3.27. The link of $\tilde{v}$ contains edges that project to $H_s$ and determine fixed directions for $\tilde{f}_{\tilde{v}}$. Lemma 3.26 implies that any such edge extends to a ray whose interior is fixed point free and that terminates at a point $P \in \mathrm{Fix}_N(\hat{f}_{\tilde{v}})$ whose accumulation set is $\Lambda$. Let $\mathcal{P}_l$ be the union of such $P$ for all $v \in H_s \cap D_l$ and all lifts $\tilde{v} \in \widetilde{D}_l$.

**Lemma 5.6.** *For each $l$ there is a lift $\Theta$ of $\theta$ such that $\partial F^l \cup \mathcal{P}_l \subset \mathrm{Fix}(\hat{\Theta})$.*

*Proof.* Assume at first that $F^l$ has rank at least two. Let $\Theta$ be the unique lift of $\theta$ such that $\partial F^l \subset \mathrm{Fix}(\hat{\Theta})$. If $P \in \mathcal{P}_l$ corresponds to $\tilde{v}$ as above and if $\Phi_1 \in \mathrm{P}(\phi)$ corresponds to $\tilde{f_{\tilde{v}}}$ then $\mathrm{Fix}_N(\hat{\Phi}_1)$ contains $P$ and intersects $\partial F^l$ non-trivially. There exists $\Theta_1$ such that $\mathrm{Fix}(\hat{\Theta}_1) \supset \mathrm{Fix}_N(\hat{\Phi}_1)$. If there does not exist a covering translation $T_c$ such that $\mathrm{Fix}(\hat{\Phi}_1) \cap \partial F^l = \{T_c^{\pm}\}$ then $\Theta_1 = \Theta$ and we are done. Suppose then that $T_c$ with this property exists. By Remark 3.2 there is a principal lift $\Phi_2$ such that $\mathrm{Fix}(\Phi_2)$ contains $T_c^{\pm}$ and such that $\Phi_2|F_l$ is a principal lift of $\phi|F_l$. In particular, $\Phi_2 = i_c^d\Phi_1$ for some $d \neq 0$. By hypothesis, there are principal lifts $\Psi_1$ and $\Psi_2$ such that $\mathrm{Fix}_N(\hat{\Psi}_i) = \mathrm{Fix}_N(\hat{\Phi}_i)$ and such that $\Psi_2 = i_c^d\Psi_1$. Thus $\Theta = \Psi_2^{-1}\Phi_2 = \Psi_1^{-1}\Phi_1 = \Theta_1$ where the first equality comes from the fact that $\mathrm{Fix}(\hat{\Psi}_2^{-1}\hat{\Phi}_2)$ contains at least three points in $\partial F^l$.

It remains to consider the case that $F^l$ has rank one. For each $P \in \mathcal{P}_l$, there exist $\Phi_P$ and $\Psi_P$ such that $\mathrm{Fix}_N(\hat{\Phi}_P) = \mathrm{Fix}_N(\hat{\Psi}_P)$ contains $P \cup \partial F^l$. Define $\Theta_P = \Psi_P^{-1}\Phi_P$. For any $Q \in \mathcal{P}_l$ there exists $w \in F^l$ such that $\Phi_P = i_w\Phi_Q$ and $\Psi_P = i_w\Psi_Q$. It follows that $\Theta_P = \Psi_P^{-1}\Phi_P = \Psi_Q^{-1}\Phi_Q = \Theta_Q$ as desired. □

**Corollary 5.7.** *If $\theta$ has finite order then $\theta$ is trivial.*

*Proof.* If $\theta$ has finite order then [8] there is a marked graph $X$, a subgraph $X_0$ such that $\mathcal{F}(X_0) = \mathcal{F}$ and a homeomorphism $h: X \to X$ that represents $\theta$ and is the identity on $X_0$. By Lemma 5.6 there is an $h_\#$-invariant ray $R$ whose initial endpoint is in $X_0$ and whose accumulation set contains $\Lambda$. No proper free factor system carries $\mathcal{F}$ and $\Lambda$, so $R$ crosses every edge in $X \setminus X_0$. Since $h$ is a homeomorphism $R \subset \mathrm{Fix}(h)$ and we conclude that $h$ is the identity. □

We now assume that $\theta$ has infinite order and argue to a contradiction. There is no loss in replacing $\theta$ by an iterate, so we may assume that both $\theta$ and $\theta^{-1}$ are forward rotationless. There is a CT $h: G' \to G'$ representing $\theta$ and there exists $r' < s'$ such that $G'_{s'} = G'$, such that $\mathcal{F}(G'_{r'}) = \mathcal{F}$ and such that $h|G'_{r'} = $ identity (see Remark 4.41).

**Lemma 5.8.** *Suppose that $\mathcal{P}_l$ and $\Theta$ are as in Lemma 5.6 and that $P \in \mathcal{P}_l$. Then $P$ is not isolated in* $\mathrm{Fix}(\hat{\Theta})$.

*Proof.* Suppose at first that $P$ is an attractor for $\hat{\Theta}$. Let $\tilde{h}$ be the lift of $h$ corresponding to $\Theta$. By Lemma 4.36 there is an edge $\widetilde{E}$ that iterates to $P$; let $\widetilde{R}$ be the ray connecting $\widetilde{E}$ to $P$. If $E$ belongs to an EG stratum, then $\Lambda$, which is the accumulation set of $P$, is an attracting lamination for $\theta$ by Lemma 4.38. This contradicts the fact that $\theta$ acts on $\Lambda$ with expansion factor one. If $E$ is NEG, then $\Lambda$ is carried by $G' \setminus E$ in contradiction to the fact that no proper free factor can carry $\mathcal{F}$ and $\Lambda$. This proves that $P$ is not an attractor for $\hat{\Theta}$.

The symmetric argument using a relative train track map for $\theta^{-1}$ proves that $P$ is not a repeller so Lemma 2.3 completes the proof. □

**Corollary 5.9.** *If $\gamma' \subset G'$ is a finite subpath of either:*

(1) *a leaf of the realization of $\Lambda$ in $G'$ or*

(2) *the projection of the line in the universal cover $\Gamma'$ of $G'$ connecting a pair of points $P_1, P_2 \in \mathcal{P}_l$*

*then $\gamma'$ extends to a Nielsen path for $h$.*

*Proof.* Let $\Theta$ be as in Lemma 5.6 and let $\tilde{h}\colon \Gamma' \to \Gamma'$ be the lift corresponding to $\Theta$. For case (1), let $\widetilde{R}' \subset \Gamma'$ be a ray converging to $P \in \mathcal{P}_l$. There are lifts $\tilde{\gamma}' \subset \widetilde{R}'$ of $\gamma'$ that are arbitrarily close to $P$. Lemma 5.8 and Lemma 2.3 therefore imply that $\tilde{\gamma}'$ extends to a Nielsen path for $\tilde{h}$. In case (2), $P_1, P_2 \in \mathrm{Fix}(\hat{h})$. Lemma 5.8 and Lemma 2.3 imply that any finite subpath of the line connecting $P_1$ to $P_2$ extends to a Nielsen path for $\tilde{h}$. □

It is well known that if $\phi$ acts trivially on conjugacy classes in $F_n$ then $\phi$ is the trivial element. This can be proved by induction up the strata of $f : G \to G$ representing $\phi$ or directly as in Lemma 3.3 of [9]. The following lemma therefore completes the proof of Theorem 5.3.

**Lemma 5.10.** *$\theta$ fixes each conjugacy class $[c]$ in $F_n$.*

*Proof.* If $v$ is a vertex in $G$ whose link $lk(v)$ is contained in $H_s$, then the local stable Whitehead graph $SW_v$ is defined to be the graph with one vertex for each oriented edge based at $v$ whose initial direction is fixed by $Df$ and an edge connecting the vertices corresponding to $E_1$ and $E_2$ if there is an edge $E$ of $H_s$ and $k \geq 1$ so that the path $f_\#^k(E)$ contains $\bar{E}_1E_2$ or $\bar{E}_2E_1$ as a subpath. By Lemma 2.13 this is equivalent to $\bar{E}_1E_2$ or $\bar{E}_2E_1$ being a subpath of a generic leaf of $\Lambda$. If $SW_v$ is not connected then one can blow up $v$ to an edge $E$ as in Proposition 5.4 of [4] to obtain a proper free factor that carries $\mathcal{F}$ and $\Lambda$. Since this is impossible, $SW_v$ is connected.

Choose a positive integer $M$ such that $Df^M$ maps every direction in $H_s$ to a fixed direction in $H_s$. At one point in the proof we need a way to choose partial edges and for this we subdivide the edges of $H_s$ at the full $f^M$-pre-image of the set of vertices. Edges in this subdivision will be called *edgelets*. Thus an edgelet maps by $f^M$ to an edge.

Let $g\colon G \to G'$ be a homotopy equivalence that respects the marking and that satisfies $g(G_r) = G'_{r'}$. We show below that there is a positive integer $N$ so that for all circuits $\sigma \subset G$ the conjugacy class in $F_n$ determined by $g_\# f_\#^{MN}(\sigma) \subset G'$ is fixed by $\theta$. Since every conjugacy class in $F_n$ is realized in this manner by some $\sigma$, this completes the proof of the lemma.

Since $\theta$ acts by the identity on $\mathcal{F}(G'_{r'})$ we may assume without loss that $\sigma$ crosses at least one edge in $H_s$. The proof involves choosing a closed curve that is homotopic to $f_\#^{MN}(\sigma)$ and a covering of that curve by subpaths with large overlap.

To begin, choose a cyclic ordering of the $m$ edges of $H_s$ in $\sigma$. Define $\sigma_1$ to be first edge of $H_s$ in $\sigma$, $\sigma_3$ to be second edge of $H_s$ in $\sigma$ and $\sigma_2$ to be the subpath of $\sigma$ that begins with the last edgelet in $\sigma_1$ and ends with the first edgelet in $\sigma_3$. Define $\sigma_5$ to be third edge of $H_s$ in $\sigma$ and $\sigma_4$ to be the subpath of $\sigma$ that begins with the last edgelet in $\sigma_3$ and ends with the first edgelet in the $\sigma_5$. Continue in this manner stopping with $\sigma_{2m}$ that begins with the last edgelet in $\sigma_{2m-1}$ and ends with the first edgelet in $\sigma_1$.

Let $\rho = f_\#^M(\sigma)$ and let $\rho_i = f_\#^M(\sigma_i)$. Then each $\rho_i$ is an edge path in $G$ whose initial and terminal edges are in $H_s$ and whose initial and terminal directions are fixed by $Df$. Suppose that $\rho_{2j} = \bar{E}E'$ where the link of the common initial endpoint $v$ of $E$ and $E'$ is entirely contained in $H_s$. Since $SW_v$ is connected, there are edges $E = E_1, E_2, \dots, E_l = E'$ in $lk(v)$ with initial directions fixed by $Df$ such that each $\bar{E}_p E_{p+1}$ is a subpath of a generic leaf of $\Lambda$. Replace $\rho_{2j}$ by the concatenation $(\bar{E}_1 E_2) \cdot (\bar{E}_2 E_3) \cdot \cdots \cdot (\bar{E}_{l-1} E_l)$.

After adjusting the indices, we have produced paths $\rho_1, \dots, \rho_t$ with the following properties:

(a) The initial edge $E_i^1$ of $\rho_i$ and the terminal edge $E_i^2$ of $\rho_i$ are contained in $H_s$ and $E_i^2$ equals $E_{i+1}^1$ up to a possible change of orientation.

(b) For all $k \ge 0$, $f_\#^k(\rho_i)$ is a finite subpath of either

  (1) a generic leaf of the realization of $\Lambda$ in $G$, or
  (2) the projection of the line in $\Gamma$ connecting a pair of points $P_1, P_2 \in \mathcal{P}_l$ for some $l$.

Suppose that each $E_i^1$ as been decomposed into proper subpaths $E_i^1 = a_i b_i$. The equality $E_i^2 = E_{i+1}^1$ or $\bar{E}_{i+1}^1$ determines a corresponding decomposition of $E_i^2$. Define $\tau_i$ from $\rho_i$ by deleting the initial $a_i$ segment of $E_i^1$ and by deleting the terminal segment of $E_i^2$ determined from (a) as follows. If $E_i^2 = E_{i+1}^1$ then remove the terminal $b_{i+1}$ ; if $E_i^2 = \bar{E}_{i+1}^1$ then remove the terminal $a_{i+1}$.

(c) For any $\{a_i\}$ as above, $\rho$ is homotopic to the loop determined by the concatenation of the $\tau_i$'s.

Choose $K$ greater than the number of edges with height $s'$ in any indivisible Nielsen path for $h$. By [7] there is a positive constant $C$ so that if $\beta_1 \subset \beta_2$ are finite subpaths in $G$ then $g_\#(\beta_2) \subset G'$ contains the subpath of $g_\#(\beta_1)$ obtained by removing the initial and terminal segments of edge length $C$. Since generic leaves of $\Lambda$ are birecurrent and since the realization of $\Lambda$ in $G'$ can not be contained in $G'_{r'}$, there is a subpath $\gamma'$ of a generic leaf of the realization of $\Lambda$ in $G'$ that contains $2K + 3C$ edges of $H'_{s'}$. Choose a subpath $\gamma$ of a generic leaf of the realization of $\Lambda$ in $G$ whose $g_\#$ image contains $\gamma'$. There exists $N > 0$ so that $f_\#^N(E)$ contains $\gamma$ as a subpath for each edge $E$ of $H_s$. It follows that the path $g_\#(f_\#^N(E))$ contains at least $2K + C$ edges of $H'_{s'}$ for each edge $E$ of $H_s$.

The subpath $\nu_i'$ of $g_\#(f_\#^N(\rho_i))$ defined by removing initial and terminal segments with exactly $C$ edges of $H'_{s'}$ is contained in either the realization of a leaf of $\Lambda$ in $G'$ or the projection of a line in $\Gamma'$ connecting a pair of points $P_1, P_2 \in \mathcal{P}_l$. Corollary 5.9 implies that $\nu_i'$ extends to a Nielsen path $\mu_i'$ for $h$. Let $\mu_i' = \mu'_{i,1} \cdot \mu'_{i,2} \dots \mu'_{i,m_i}$ be the complete splitting of $\mu'_i$. There is no loss in assuming that $\mu'_{i,2} \subset \nu'_i$. There are at most $K + C$ edges of $H'_{s'}$ in $g_\#(f_\#^N(\rho_i))$ that precede $\mu'_{i,2}$. Without changing this estimate we may assume that $\mu'_{i,2}$ has height $s'$. Note that $\mu'_{i,2} \subset g_\#(f_\#^N(E_i^1))$ and hence that $\mu'_{i+1,2} \subset g_\#(f_\#^N(E_i^2))$. Let $a_i$ be an initial segment of $E_i^1$ such that $g_\# f_\#^N(a_i)$ is the initial segment of $g_\#(f_\#^N(\rho_i))$ that precedes $\mu'_{i,2}$. Define the $\tau_i$'s as in (c).

Lemma 4.11 (3) implies that there exists $l \leq m_i$ such that $\mu'_{i,l} = \mu'_{i+1,2}$ up to a change of orientation. Thus $g_\# f^N(\tau_i) = \mu'_{i,2} \ldots \mu'_{i,l}$ is a Nielsen path for $h$. Property (c) implies that the conjugacy class determined by $g_\# f_\#^N(\rho) = g_\# f^{MN}(\sigma)$ is $\theta$-invariant as desired. □

**Corollary 5.11.** *If $\phi$ and $\psi$ are forward rotationless and if $\phi^m = \psi^m$ for some $m > 0$ then $\phi = \psi$.*

*Proof.* Since $\phi$ and $\psi$ are forward rotationless there are $\mathrm{Fix}_N$-preserving bijections between $\mathrm{P}(\phi^m)$ and $\mathrm{P}(\phi)$ and between $\mathrm{P}(\psi^m)$ and $\mathrm{P}(\psi)$. By assumption, $\mathrm{P}(\phi^m) = \mathrm{P}(\psi^m)$ so there is a $\mathrm{Fix}_N$-preserving bijection between $\mathrm{P}(\phi)$ and $\mathrm{P}(\psi)$. The lemma now follows from the Recognition theorem and the fact that expansion factors and twist coefficients for $\phi^m$ are $m$ times those of $\phi$ and similarly for $\psi$. □

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M. Feighn, Department of Mathematics and Computer Science, Rutgers University, Newark, NJ 07102, U.S.A.

E-mail: feighn@rutgers.edu

M. Handel, Department of Mathematics and Computer Science, Lehman College, Bronx, NY 10468, U.S.A.

E-mail: michael.handel@lehman.cuny.edu