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# The end of the curve complex

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**Abstract.** Suppose that $S$ is a surface of genus two or more, with exactly one boundary component. Then the curve complex of $S$ has one end.

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## 1. Introduction

We denote the compact, connected, orientable surface of genus $g$ with $b$ boundary components by $S_{g,b}$. The *complexity* of $S = S_{g,b}$ is $\xi(S) := 3g - 3 + b$. A simple closed curve $\alpha$ in $S$ is *essential* if $\alpha$ does not cut a disk out of $S$. Also, $\alpha$ is *non-peripheral* if it does not cut an annulus out of $S$.

When $\xi(S) \geq 2$ the *complex of curves*, $\mathcal{C}(S)$, is the simplicial complex where vertices are isotopy classes of essential non-peripheral curves. The $k$-simplices are collections of $k+1$ distinct vertices having disjoint representatives. We regard every simplex as a Euclidean simplex of side-length one. If $\alpha$ and $\beta$ are vertices of $\mathcal{C}(S)$ let $d_S(\alpha, \beta)$ denote the distance between $\alpha$ and $\beta$ in the one-skeleton $\mathcal{C}^1(S)$. It is a pleasant exercise to prove that $\mathcal{C}(S)$ is connected. It is an important theorem of H. Masur and Y. Minsky [8] that $\mathcal{C}(S)$ is Gromov hyperbolic.

Let $B(\omega, r) := \{\alpha \in \mathcal{C}^0(S) \mid d_S(\alpha, \omega) \leq r\}$ be the ball of radius $r$ about the vertex $\omega$. We will prove:

**Theorem 5.1.** *Fix $S := S_{g,1}$ for some $g \geq 2$. For any vertex $\omega \in C(S)$ and for any $r \in \mathbb{N}$: the subcomplex spanned by $\mathcal{C}^0(S) \setminus B(\omega, r)$ is connected.*

For such surfaces, Theorem 5.1 directly answers a question of Masur's. It also answers a question of G. Bell and K. Fujiwara [1] in the negative: the complex of curves need not be quasi-isometric to a tree. Theorem 5.1 is also evidence for a positive answer to a question of P. Storm:

**Question 1.1.** Is the Gromov boundary of $\mathcal{C}(S)$ connected?[1]

Note that Theorem 5.1 is only evidence for, and not an answer to, Storm's question: for example, there is a one-ended hyperbolic space where the Gromov boundary is a pair of points. Finally, as we shall see in Remark 4.2, it is not obvious how to generalize Theorem 5.1 to surfaces with more (or fewer) boundary components.

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## 2. Definitions and necessary results

An important point elided above is how to define $\mathcal{C}(S)$ when $\xi(S) = 1$. The complex as defined is disconnected in these cases. Instead we allow a $k$-simplex to be a collection of $k+1$ distinct vertices which have representatives with small intersection. For $S_{1,1}$ exactly one intersection point is allowed while $S_{0,4}$ requires two. In both cases $\mathcal{C}(S)$ is the famous *Farey tessellation*. Note that $\mathcal{C}(S_{0,3})$ is empty. We will not need to consider the other low complexity surfaces: the sphere, the disk, the annulus, and the torus.

A subsurface $X \subset S$ is *essential* if every component of $\partial X$ is essential in $S$. We will generally assume that $\xi(X) \geq 1$. A pair of curves, or a curve and a subsurface, are *tight* if they cannot be isotoped to reduce intersection. We will generally assume that all curves and subsurfaces discussed are tight with respect to each other. We say a curve $\alpha$ *cuts* $X$ if $\alpha \cap X \neq \emptyset$. If $\alpha \cap X = \emptyset$ then we say $\alpha$ *misses* $X$.

Following Masur and Minsky [9], we define the *subsurface projection* map $\pi_X$: this maps vertices of $\mathcal{C}(S)$ to collections of vertices of $\mathcal{C}(X)$. Fix a vertex $\alpha \in \mathcal{C}(S)$ and, for every component $\delta \subset \alpha \cap X$, form $N_\delta := \overline{\text{neigh}(\delta \cup \partial X)}$, a closed regular neighborhood of $\delta \cup \partial X$. Take $\pi_X(\alpha)$ to be the set of all vertices of $\mathcal{C}(X)$ which appear as a boundary component of some $N_\delta$. If $\alpha$ misses $X$ then $\pi_X(\alpha) = \emptyset$. Note if $\alpha \subset S$ is contained in $X$ after tightening then $\pi_X(\alpha) = \{\alpha\}$.

As a useful bit of notation, if $\alpha$ and $\beta$ both cut $X$, we set

$$d_X(\alpha, \beta) := \text{diam}_X(\pi_X(\alpha) \cup \pi_X(\beta))$$

with diameter computed in $\mathcal{C}^1(X)$. Masur and Minsky give a combinatorial proof (Lemma 2.2 in [9]) that:

[1]Leininger and I answered Storm's question in many cases [7]. The complete conjecture has been verified by D. Gabai [3]. The material in this note is of independent interest as it is self-contained and elementary.

**Lemma 2.1.** *If $\alpha$ and $\beta$ are disjoint and both cut $X$ then $d_X(\alpha, \beta) \leq 2$.* □

By *geodesic* in $\mathcal{C}(S)$ we will always be referring to a geodesic in the one-skeleton. Since $\mathcal{C}(S)$ is Gromov hyperbolic the exact position of the geodesic is irrelevant; we often use the notation $[\alpha, \beta]$ as if the geodesic was determined by its endpoints. We immediately deduce from Lemma 2.1:

**Lemma 2.2.** *Suppose that $\alpha, \beta$ are vertices of $\mathcal{C}(S)$, both cutting $X$. Suppose that $d_X(\alpha, \beta) > 2 \cdot d_S(\alpha, \beta)$. Then every geodesic $[\alpha, \beta] \subset \mathcal{C}(S)$ has a vertex which misses $X$.* □

This is essentially Lemma 2.3 of [9].

**Remark 2.3.** There is a useful special case of Lemma 2.2: assume all the hypotheses and in addition that $\gamma$ is the unique vertex of $\mathcal{C}(S)$ missing $X$. Then every geodesic connecting $\alpha$ to $\beta$ contains $\gamma$.

In fact, $\gamma$ is the unique vertex missing $X$ exactly when $S \setminus \text{neigh}(\gamma) = X$ or $S \setminus \text{neigh}(\gamma) = X \cup P$ with $P \cong S_{0,3}$: a *pair of pants*.

**Remark 2.4.** Note that Lemma 2.2 is a weak form of the Bounded Geodesic Image Theorem ([9], Theorem 3.1). The proof of the stronger result appears to require techniques from Teichmüller theory.

We now turn to the *mapping class group* $\mathcal{MCG}(S)$: the group of isotopy classes of homeomorphisms of $S$. Note that the natural action of $\mathcal{MCG}(S)$ on $\mathcal{C}(S)$ is via isometries. We have an important fact:

**Lemma 2.5.** *If $\psi\colon S \to S$ is a pseudo-Anosov and $\alpha$ is a vertex of $\mathcal{C}^0(S)$ then* $\text{diam}_S\{\psi^n(\alpha) \mid n \in \mathbb{Z}\}$ *is infinite.* □

It follows that the diameter of $\mathcal{C}(S)$ is infinite whenever $\xi(S) \geq 1$. A proof of Lemma 2.5, relying on Kobayashi's paper [5], may be found in the remarks following Lemma 4.6 of [9]. As a matter of fact, Masur and Minsky there prove more using train track machinery: any orbit of a pseudo-Anosov map is a quasi-geodesic. We will not need this sharper version.

Note that if $\psi\colon S \to S$ is a homeomorphism then we may restrict $\psi$ to the curve complex of a subsurface $\psi|X\colon \mathcal{C}(X) \to \mathcal{C}(\psi(X))$. This restriction behaves well with respect to subsurface projection: that is, $\pi_{\psi(X)} \circ \psi = \psi|X \circ \pi_X$.

We conclude this discussion by examining *partial maps*. Suppose that $X \subset S$ is an essential surface, not homeomorphic to $S$. If $\psi\colon S \to S$ has the property that $\psi|S \setminus X = \text{Id}\,|S \setminus X$ then we call $\psi$ a *partial map* supported on $X$. Note that if $\psi$ is supported on $X$ then the orbits of $\psi$ do not have infinite diameter in $\mathcal{C}(S)$. Since $\psi$ fixes $\partial X$ and acts on $\mathcal{C}(S)$ via isometry, every point of an orbit has the same distance to $\partial X$ in $\mathcal{C}(S)$. Nonetheless, Lemmas 2.2 and 2.5 imply:

**Lemma 2.6.** *Suppose* $\psi : S \to S$ *is supported on* $X$ *and* $\psi|X$ *is pseudo-Anosov. Fix a vertex* $\sigma \in \mathcal{C}(S)$ *and define* $\sigma_n := \psi^n(\sigma)$. *Then for any* $K \in \mathbb{N}$ *there is a power* $n \in \mathbb{Z}$ *so that* $d_X(\sigma, \sigma_n) \geq K$. *In particular, if* $K > 4 \cdot d_S(\sigma, \partial X)$ *then every geodesic* $[\sigma, \sigma_n] \subset \mathcal{C}(S)$ *contains a vertex which misses* $X$. □

## 3. No dead ends

We require a pair of tools in order to prove Theorem 5.1. The first is:

**Proposition 3.1.** *Fix* $S = S_{g,b}$. *For any vertex* $\omega \in \mathcal{C}(S)$ *and for any* $r \in \mathbb{N}$: *every component of the subcomplex spanned by* $\mathcal{C}^0(S) \setminus B(\omega, r)$ *has infinite diameter.*

A more pithy phrasing might be: the complex of curves has no *dead ends*. Proposition 3.1 allows us to push vertices away from $\omega$ while remaining inside the same component of $\mathcal{C}(S) \setminus B(\omega, r)$. The proof is a bit subtle due to the behavior of $\mathcal{C}(S)$ near a non-separating curve.

*Proof of Proposition* 3.1. If $S = S_{0,3}$ is a pants then the curve complex is empty and there is noting to prove. If $\mathcal{C}(S)$ is a copy of the Farey graph then the claim is an easy exercise. So we may suppose that $\xi(S) \geq 2$.

Now fix a vertex $\alpha \in \mathcal{C}(S) \setminus B(\omega, r)$. Set $n := d_S(\alpha, \omega)$. Thus $n > r$. Our goal is to find a curve $\delta$, connected to $\alpha$ in the complement of $B(\omega, n-1)$, with $d_S(\delta, \omega) = n + 1$. Doing this repeatedly proves the proposition. Note that finding such a vertex $\delta$ is straight-forward if $r = 0$ and $n = 1$. This is because $\mathcal{C}(S) \setminus \omega$ is connected and because, following Lemma 2.5, we know that the diameter of $\mathcal{C}(S)$ is infinite. Henceforth we will assume that $n \geq 2$; that is, $\omega$ cuts $\alpha$.

Fix attention on a component $X$ of $S \setminus \text{neigh}(\alpha)$ which is not a pair of pants. So $\xi(X) \geq 1$ and, by the comments following Lemma 2.5, $\mathcal{C}(X)$ has infinite diameter. Since $\omega$ cuts $\alpha$ we find that $\omega$ also cuts $X$. Choose a curve $\beta$ contained in $X$ with $d_X(\beta, \omega) \geq 2n + 1$. Note that $d_S(\alpha, \beta) = 1$. We may assume that $\beta$ is either non-separating or cuts a pants off of $S$. (To see this: if $\beta$ cannot be chosen to be non-separating then $X$ is planar. As $\xi(X) \geq 1$ we deduce that $X$ has at least four boundary components. At most two of these are parallel to $\alpha$.) It follows from Lemma 2.2 that any geodesic from $\beta$ to $\omega$ in $\mathcal{C}(S)$ has a vertex $\gamma$ which misses $X$.

By the triangle inequality $d_S(\gamma, \omega)$ equals $n$ or $n - 1$. In the former case we are done: simply take $\delta = \beta$ and notice that $d_S(\beta, \omega) = n + 1$. In the latter case $d_S(\beta, \omega) = n$ and we proceed as follows: replace $\alpha$ by $\beta$ and replace $X$ by $Z := S \setminus \text{neigh}(\beta)$. We may now choose $\delta$ to be a vertex of $\mathcal{C}(Z)$ with $d_Z(\delta, \omega) \geq 2n + 1$. As above, any geodesic $[\delta, \omega] \subset \mathcal{C}(S)$ has a vertex which misses $Z$. Since $\beta$ is the *unique* vertex not cutting $Z$, our Remark 2.3 implies that $\beta \in [\delta, \omega]$. Thus $d_S(\delta, \omega) = n + 1$ and we are done. □

## 4. The Birman short exact sequence

We now discuss the second tool needed in the proof of Theorem 5.1. Following I. Kra's notation in [6] let $\dot{S} := S_{g,1}$ and $S := S_g$ for a fixed $g \geq 2$. Let $\rho\colon \dot{S} \to S$ be the quotient map crushing $\partial \dot{S}$ to a point, say $x \in S$. This leads to the *Birman short exact sequence*:

$$\pi_1(S, x) \to \mathcal{MCG}(\dot{S}) \to \mathcal{MCG}(S)$$

for $g \geq 2$. The map $\rho$ gives the second arrow. The first arrow is defined by sending $\gamma \in \pi_1(S, x_0)$ to a mapping class $\psi_\gamma$. There is a representative of this class which is isotopic to the identity, in $S$, via an isotopy dragging $x$ along the path $\gamma$. See Birman's book [2] or Kra's paper [6] for further details.

Fix an essential subsurface $\dot{X} \subset \dot{S}$ and let $X := \rho(\dot{X})$. If $\gamma \in \pi_1(S, x)$ is contained in $X$ then $\psi_\gamma$ is a partial map, supported in $\dot{X}$. We say that $\gamma$ *fills* $X$ if $\gamma \subset X$ and, in addition, every representative of the free homotopy class of $\gamma$ cuts $X$ into a collection of disks and peripheral annuli. For future use we record a well-known theorem of Kra [6]:

**Theorem 4.1.** *Suppose that $\xi(\dot{X}) \geq 1$. If $\gamma$ fills $X$ then $\psi_\gamma|\dot{X}$ is pseudo-Anosov.* □

Now note that, corresponding to the Birman short exact sequence, there is a "fibre bundle" of curve complexes:

$$\mathcal{F}_\tau \to \mathcal{C}(\dot{S}) \to \mathcal{C}(S).$$

Here $\tau$ is an arbitrary vertex of $\mathcal{C}(S)$ and $\mathcal{F}_\tau := \rho^{-1}(\tau)$. The second arrow is given by $\rho$. The first is the inclusion of $\mathcal{F}_\tau$ into $\mathcal{C}(\dot{S})$.

**Remark 4.2.** If $|\partial S| \geq 2$ then collapsing one boundary component does not induce a map on the associated curve complexes. Thus, it is not clear how to generalize Theorem 5.1 to such surfaces. If $\partial S$ is empty then it appears to be very difficult to find interesting mapping class group equivariant electrifications of $\mathcal{C}(S)$.

Using the Birman short exact sequence we obtain an action of $\pi_1(S, x)$ on the curve complex $\mathcal{C}(\dot{S})$. Behrstock and Leininger observe that:

**Proposition 4.3.** *The map $\rho\colon \mathcal{C}(\dot{S}) \to \mathcal{C}(S)$ has the following properties:*

- *It is 1-Lipschitz.*
- *For any $\alpha \in \mathcal{C}(\dot{S})$, $\gamma \in \pi_1(S, x)$ we have $\rho(\alpha) = \rho(\psi_\gamma(\alpha))$.*
- *Every fibre $\mathcal{F}_\tau$ is connected.*

**Remark 4.4.** Behrstock and Leininger's interest in the fibre $\mathcal{F}_\tau$ was to give a "natural" subcomplex of $\mathcal{C}(S)$ which is not quasi-convex: this is implied by the first pair of properties.

**Remark 4.5.** More of the structure of $\mathcal{F}_\tau$ is known. For example, since $S$ is closed, the fibre $\mathcal{F}_\tau$ is either a single $\pi_1(S, x)$-orbit or the union of a pair of orbits depending on whether $\tau$ is non-separating or separating. Furthermore, $\mathcal{F}_\tau$ is a tree. See [4] for a detailed discussion.

*Proof of Proposition* 4.3. Fix an essential non-peripheral curve $\alpha$ in $\dot{S}$. Note that $\rho(\alpha)$ is essential in $S$ and so the induced map $\rho\colon \mathcal{C}(\dot{S}) \to \mathcal{C}(S)$ is well-defined. If $\alpha$ and $\beta$ are disjoint in $\dot{S}$ then so are their images in $S$. Thus $\rho$ does not increase distance between vertices and the first conclusion holds.

Now fix a curve $\alpha \subset \dot{S}$ and $\gamma \in \pi_1(S, x)$. Note that $\psi_\gamma$ is isotopic to the identity in $S$. Thus the images $\rho(\psi_\gamma(\alpha))$ and $\rho(\alpha)$ are isotopic in $S$. It follows that $\rho(\alpha) = \rho(\psi_\gamma(\alpha))$ as vertices of $\mathcal{C}(S)$, as desired.

Finally, fix $\tau \in \mathcal{C}(S)$. Let $\mathcal{F}_\tau$ be the fibre over $\tau$. Pick $\alpha, \beta \in \mathcal{F}_\tau$. It follows that $a := \rho(\alpha)$ and $b := \rho(\beta)$ are both isotopic to $\tau$ and so to each other. We induct on the intersection number $\iota(\alpha, \beta)$. Suppose the intersection number is zero. Then $\alpha$ and $\beta$ are disjoint and we are done. Suppose that the intersection number is non-zero. Since $a$ and $b$ are isotopic, yet intersect, they are *not* tight with respect to each other. It follows that there is a bigon $B \subset S \setminus (a \cup b)$. Since $\alpha$ and $\beta$ are tight in $\dot{S}$ the point $x$ must lie in $B$. Let $\dot{B} := \rho^{-1}(\bar{B})$. Now construct a curve $\beta' \subset \dot{S}$ by starting with $\beta$, deleting the arc $\beta \cap \dot{B}$, and adding the arc $\alpha \cap \dot{B}$. Isotope $\beta'$ to be tight with respect to $\alpha$. Now $\beta' \in \mathcal{F}_\tau$ because $\rho(\beta')$ is isotopic to $\rho(\beta)$ in $S$. Finally, $\iota(\alpha, \beta') \leq \iota(\alpha, \beta) - 2$. □

## 5. Proving the theorem

We are now equipped to prove:

**Theorem 5.1.** *Fix $\dot{S} := S_{g,1}$ for some $g \geq 2$. For any vertex $\omega \in C(\dot{S})$ and for any $r \in \mathbb{N}$: the subcomplex spanned by $\mathcal{C}^0(\dot{S}) \setminus B(\omega, r)$ is connected.*

As above we use the notation $\dot{S} = S_{g,1}$ and $S = S_g$ for some fixed $g \geq 2$. Also, we have defined a map $\rho\colon \mathcal{C}(\dot{S}) \to \mathcal{C}(S)$ induced by collapsing $\partial\dot{S}$ to a point, $x$. As above we use $\mathcal{F}_\tau = \rho^{-1}(\tau)$ to denote the fibre over $\tau$.

*Proof of Theorem* 5.1. Choose $\alpha'$ and $\beta'$ vertices of $\mathcal{C}(\dot{S}) \setminus B(\omega, r)$. By Proposition 3.1 we may connect $\alpha'$ and $\beta'$, by paths disjoint from $B(\omega, r)$, to vertices outside of $B(\omega, 3r)$. Call these new vertices $\alpha$ and $\beta$. We may assume that both $\alpha$ and $\beta$ are non-separating because such vertices are 1-dense in $\mathcal{C}(\dot{S})$.

Choose any vertex $\tau \in \mathcal{C}(S)$ so that $d_S(\tau, \rho(\omega)) \geq 4r$. This is always possible because $\mathcal{C}(S)$ has infinite diameter. (See the remarks after Lemma 2.5.) It follows from Proposition 4.3 that $\mathcal{F}_\tau \cap B(\omega, r) = \emptyset$. We will now connect each of $\alpha$ and $\beta$ to some point of $\mathcal{F}_\tau$ via a geodesic disjoint from $B(\omega, r)$. Since $\mathcal{F}_\tau$ is connected, by Proposition 4.3, this will complete the proof of Theorem 5.1.

Let $\dot{X} := \dot{S} \setminus \alpha$ and take $X := \rho(\dot{X})$. Fix any point $\sigma$ in $\mathcal{F}_\tau$. If $\sigma = \alpha$ then $\alpha$ is trivially connected to the fibre. So suppose that $\sigma \neq \alpha$. Since $\alpha$ is non-separating deduce that $\sigma$ cuts $\dot{X}$. Now, since $\xi(\dot{S}) \geq 4$ we have $\xi(\dot{X}) \geq 3$. Let $\gamma \in \pi_1(S, x)$ be any homotopy class so that $\psi_\gamma$ is supported in $\dot{X}$ and so that $\gamma$ fills $X$. By Kra's Theorem (4.1) $\psi_\gamma | \dot{X}$ is pseudo-Anosov.

Since $\mathcal{F}_\tau$ is left setwise invariant by $\pi_1(S, x)$ (Proposition 4.3) the curves $\sigma_n := \psi_\gamma^n(\sigma)$ all lie in $\mathcal{F}_\tau$. Since $\psi_\gamma | \dot{X}$ is pseudo-Anosov, Lemma 2.6 gives an $n \in \mathbb{Z}$ so that every geodesic $g := [\sigma, \sigma_n] \subset \mathcal{C}(\dot{S})$ has a vertex which misses $\dot{X}$. Since $\alpha$ is non-separating, as in Remark 2.3, it follows that $\alpha$ is actually a vertex of $g$.

We now claim that at least one of the two segments $[\sigma, \alpha] \subset g$ or $[\alpha, \sigma_n] \subset g$ avoids the ball $B(\omega, r)$. For suppose not: then there are vertices $\mu, \mu' \in g$ on opposite sides of $\alpha$ which both lie in $B(\omega, r)$. Thus $d_{\dot{S}}(\mu, \mu') \leq 2r$. Since $g$ is a geodesic the length along $g$ between $\mu$ and $\mu'$ is at most $2r$. Thus $d_{\dot{S}}(\omega, \alpha) \leq 2r$. This is a contradiction.

Thus we can connect $\alpha$ to a vertex of $\mathcal{F}_\tau$ (namely, $\sigma$ or $\sigma_n$) avoiding $B(\omega, r)$. Identically, we can connect $\beta$ to a vertex of $\mathcal{F}_\tau$ while avoiding $B(\omega, r)$. As noted above, this completes the proof. $\square$

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