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(Non)-completeness of $\mathbb{R}$-buildings and fixed point theorems

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Abstract. We prove two generalizations of results of Bruhat and Tits involving metrical completeness and $\mathbb{R}$-buildings. Firstly, we give a generalization of the Bruhat–Tits fixed point theorem also valid for non-complete $\mathbb{R}$-buildings with the added condition that the group is finitely generated.

Secondly, we generalize a criterion which reduces the problem of completeness to the wall trees of the $\mathbb{R}$-building. This criterion was proved by Bruhat and Tits for $\mathbb{R}$-buildings arising from root group data with valuation.

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1. Introduction

In 1986, Jacques Tits classified affine buildings of rank at least four ([13]). He also included in this classification non-discrete generalizations of affine buildings, these metric spaces are called non-discrete affine apartment systems or $\mathbb{R}$-buildings. Although the first definition for these $\mathbb{R}$-buildings only appeared in the aforementioned paper, the examples that arise from root group data with a (non-discrete) valuation were already studied in 1972 in the book [5] by Bruhat and Tits. The classification of Tits shows that an $\mathbb{R}$-building of rank at least four (or equivalently dimension at least three) necessarily arises from such a root group datum with valuation. For dimension two there exist various explicit and 'free' constructions (for example see [4] and [12]). The one-dimensional $\mathbb{R}$-buildings are also known as $\mathbb{R}$-trees.

As $\mathbb{R}$-buildings are metric spaces, one has the notion of (metrical) completeness. While affine buildings are always complete due to their discrete nature, this is rarely the case for non-discrete $\mathbb{R}$-buildings. Perhaps the simplest example of a field with non-discrete valuations which can be used to define complete $\mathbb{R}$-buildings, is coming from the Hahn–Mal'cev–Neumann series.

The current paper generalizes two results appearing in [5], both involving completeness. The first such result is the Bruhat–Tits fixed point theorem. It says that a bounded group of isometries of a complete $\mathbb{R}$-building (or more generally a complete

CAT(0)-space) has a fixed point. Our result proves the existence of a fixed point also in the non-complete case, but with the added condition that the group is finitely generated.

The second result provides a criterion of (metric) completeness of $\mathbb{R}$-buildings. In [5] it is proved that an $\mathbb{R}$-building arising from a root group datum with valuation is complete if and only if its corresponding wall trees (which are $\mathbb{R}$-trees) are complete. This reduces the question to the easier one-dimensional case. We generalize this result to all $\mathbb{R}$-buildings, not only those arising from a root group datum with valuation, by giving a geometric proof instead of an algebraic proof.

The proofs use an embedding into an ultralimit. A similar method can be found in [2], where a fixed point theorem is proved for certain weakly contracting self-maps of buildings.

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2. Preliminaries

2.1. $\mathbb{R}$-buildings

2.1.1. Definition. Let $(\overline{W}, S)$ be a spherical Coxeter system of rank n. The group $\overline{W}$ can be realized as a finite reflection group acting on an n-dimensional real affine space $\mathbb{A}$, called the *model space*. A *wall* of $\mathbb{A}$ is a hyperplane of it fixed by a conjugate of an involution in S. A *root* is a (closed) half-space of $\mathbb{A}$ bordered by a wall. The set of all walls of $\mathbb{A}$ defines a poset of simplicial cones in $\mathbb{A}$ (called *sector-faces*), which forms the simplicial complex of the Coxeter system $(\overline{W}, S)$. The maximal cones are called *sectors*, the one less than maximal the *sector-panels*. The apex of a cone formed by a sector-face in Λ is called the *base point* of that sector-face. (See [1], Chapter 1, for a detailed discussion on finite reflection groups.) Let W be the group acting on $\mathbb{A}$ generated by $\overline{W}$ and the translations of $\mathbb{A}$.

Consider a pair $(\Lambda, \mathcal{F})$ where Λ is the set of *points* forming a metric space together with a metric d, and $\mathcal{F}$ a set of isometric injections (called *charts*) from the model space $\mathbb{A}$ (equipped with the Euclidean distance) into Λ. An image of the model space is called an *apartment*, an image of a root a *half-apartment* and an image of a sector (-face/panel) is called again a *sector(-face/panel)*. The pair $(\Lambda, \mathcal{F})$ is an $\mathbb{R}$-*building* if the following five properties are satisfied:

(A1) If $w \in W$ and $f \in \mathcal{F}$, then $f \circ w \in \mathcal{F}$.

(A2) If $f, f' \in \mathcal{F}$, then $X = f^{-1}(f'(\mathbb{A}))$ is a closed and convex subset of $\mathbb{A}$, and $f|_X = f' \circ w|_X$ for some $w \in W$.

(A3) Each two points of Λ lie in a common apartment.

This last axiom implies that the metric d on Λ is defined implicitly by the isometric injections $\mathcal{F}$.

(A4) Any two sectors S_1 and S_2 contain subsectors $S_1' \subset S_1$ and $S_2' \subset S_2$ lying in a common apartment.

(A5) If three apartments intersect pairwise in half-apartments, then the intersection of all three is non-empty.

2.1.2. Global and local structure. Two sector-faces are *parallel* if the Hausdorff distance between both is finite. This relation is an equivalence relation due to the triangle inequality. The equivalence classes (named *simplices at infinity*, or the *direction* F_∞ of a sector-face F) form a spherical building Λ_∞ of type $(\overline{W}, S)$ called the *building at infinity* of the $\mathbb{R}$-building $(\Lambda, \mathcal{F})$. The chambers of this building are the equivalence classes of parallel sectors. Each apartment Σ of $(\Lambda, \mathcal{F})$ corresponds to an apartment Σ_∞ of Λ_∞ in a bijective way.

Two sector-faces are *asymptotic* if they contain a common subsector-face having the same dimension as the original two. Asymptotic sector-faces are necessarily parallel, the inverse is only true for sectors (see [10], Corollary 1.6). Asymptoticness is an equivalence relation as well.

One can also define local equivalences. Let α be a point of Λ, and F, F' two sector-faces based at α. These two sector-faces *locally coincide* if their intersection is a neighbourhood of α in both F and F'. This relation forms an equivalence relation defining *germs of sector-faces* as equivalence classes (notation $[F]_\alpha$). These germs form a (weak) spherical building Λ_α of type $(\overline{W}, S)$, called the *residue* at α. We say that a germ of some sector-face F based at α lies in a subset K of Λ if a neighbourhood of α in F is contained in K.

A detailed study of $\mathbb{R}$-buildings can be found in [10]. We list some results from that paper here for future reference.

Lemma 2.1 ([10], Proposition 1.8). *Let C be a chamber of the building at infinity Λ_∞ and S a sector based at $\alpha \in \Lambda$. Then there exists an apartment Σ containing the germ $[S]_\alpha$ and a sector with direction C.*

Corollary 2.2 ([10], Corollary 1.9). *Let α be a point of Λ and F_∞ a facet of the building at infinity. Then there is a unique sector-face $F' \in F_\infty$ based at α.*

The unique sector-face of the previous corollary will be denoted by F_α. We will often use the subscript to indicate the base point of a sector-face. An exception is when we use the symbol ∞ as subscript to denote the direction of the sector-face. To give an example, if one says S_∞, S_α and S_β; then S_∞ denotes some simplex at infinity and S_α, S_β are the unique parallel sector-faces based at respectively α and β with direction S_∞.

Lemma 2.3 ([10], Proposition 1.12). *If the germs $[S]_\alpha$ and $[S']_\alpha$ of two sectors S_α and S'_α form opposite chambers of the residue Λ_α, then there exists a unique apartment containing both sectors.*

Lemma 2.4 ([10], Proposition 1.17). *Let Σ be an apartment and S_α a sector in Σ based at some point α. There exists a retraction r of Λ on Σ such that r preserves the distance of points of Λ to α and does not increase other distances. Also each sector based at α is mapped isometrically to a sector in Σ. The only sectors based at α mapped to S_α are the sectors with the same germ as S_α.*

2.1.3. Wall and panel trees. With a wall M of an $\mathbb{R}$-building one can associate a direction at infinity (the set of all parallel classes of sector-faces it contains). This direction M_∞ at infinity will be a wall of the spherical building at infinity.

Let m (respectively π) be a wall (resp. a panel contained in the wall m) of the building at infinity. Let $T(m)$ be the set of all walls M of the $\mathbb{R}$-building with $M_\infty = m$, and $T(\pi)$ the set of all asymptotic classes of sector-panels in the parallel class of π.

One can define charts (and so also apartments) from $\mathbb{R}$ to $T(m)$ (resp. $T(\pi)$). First choose M (resp. D) a wall (resp. a sector-panel contained in M) of the model space such that there exists some chart f such that $f(M)_\infty = m$ and $f(D) \in \pi$. One can identify the model space $\mathbb{A}$ with the product $\mathbb{R} \times M$. For every chart $g \in \mathcal{F}$ of the $\mathbb{R}$-building $(\Lambda, \mathcal{F})$ such that $g(M)_\infty = m$ (resp. $f(D) \in \pi$), one defines a chart g' as follows: if $x \in \mathbb{R}$, then $g'(x)$ is the wall $g(\{x\} \times M)$ (resp. the asymptotic class containing $g(\{x\} \times D)$).

These two constructions yield $\mathbb{R}$-buildings with a rank one building at infinity, such $\mathbb{R}$-buildings are better know as $\mathbb{R}$-*trees* (or shortly trees when no confusion can arise). The following theorem shows the connection between both constructions.

Theorem 2.5 ([13], Proposition 4). *If π is a panel in some wall m at infinity, then for each asymptotic class D of sector-panels with direction π, there is a unique wall M with direction m containing an element of D. The corresponding map $D \mapsto M$ is an isometry from the $\mathbb{R}$-tree $T(\pi)$ to the $\mathbb{R}$-tree $T(m)$.*

The trees obtained from walls (resp. panels) are called *wall trees* (resp. *panel trees*).

2.2. CAT(0)-spaces. For now suppose that (X, d) is some metric space, not necessarily an $\mathbb{R}$-building. A *geodesic* is a subset of the metric space X isometric to a closed interval of real numbers. The metric space (X, d) is a *geodesic metric space* if each two points of X can be connected by a geodesic. From Condition (A3) it follows that $\mathbb{R}$-buildings are geodesic metric spaces.

Let x, y and $z \in X$ be three points in a geodesic metric space (X, d). Because of the triangle inequality we can find three points $\bar{x}$, $\bar{y}$ and $\bar{z}$ in the Euclidean plane

$\mathbb{R}^2$ such that each pair of points has the same distance as the corresponding pair in x, y, z. The triangle formed by the three points is called a *comparison triangle* of x, y and z. Consider a point a on a geodesic between x and y (note that this geodesic is not necessarily unique). We now can find a point $\bar{a}$ on the line through $\bar{x}$ and $\bar{y}$ such that the pairwise distances in $\bar{x}$, $\bar{y}$, $\bar{a}$ are the same as in x, y, a. If the distance between z and a is not larger than the distance between $\bar{z}$ and $\bar{a}$ for all possible choices of x, y, z and a, we say that the geodesic metric space (X, d) is a CAT(0)-*space*.

A metric space is *complete* if all Cauchy sequences converge. A group of isometries acting on a metric space is *bounded* if at least one orbit (and hence all orbits) is a bounded set. Finite groups of isometries are always bounded.

The metric spaces formed by $\mathbb{R}$-buildings are examples of CAT(0)-spaces. For complete CAT(0)-spaces one has the following important theorem known as the Bruhat–Tits fixed point theorem.

Theorem 2.6 ([5], Proposition 3.2.4). *If G is a bounded group of isometries of a complete* CAT(0)*-space* (X, d)*, then G fixes some point of X.*

Remark 2.7. The notion of completeness has also another meaning when used for $\mathbb{R}$-buildings, in the sense of 'the complete system of apartments'. However, there will be no confusion possible as we will not use this other notion.

2.3. Convex sets in spherical buildings. Consider a (weak) spherical building Δ of type $(\overline{W}, S)$ as a chamber complex. Between two chambers of the building one can define a $\overline{W}$-valued function δ, called the *Weyl distance* (see [1], Section 4.8). The word length with respect to the generating set S makes this Weyl distance into an integer-valued metric on the set of chambers. A chamber subcomplex K of Δ is *convex* if each minimal gallery between two chambers of K also lies in K. An equivalent definition is if C and D are chambers of K, and E is a chamber of Δ such that $\delta(C, D) = \delta(C, E)\delta(E, D)$, then E is a chamber of K as well.

The following statement is known as the center conjecture:

Conjecture 2.8. *Let Δ be a* (*weak*) *spherical building and K a convex chamber subcomplex of Δ. Then* (*at least*) *one of the following two possibilities holds.*

- *For each chamber C in K there is a chamber D in K opposite to C.*
- *The group of automorphisms of Δ* (*setwise*) *stabilizing K stabilizes a non-trivial simplex of K.*

The center conjecture is most often stated in a more general way to include subcomplexes of lower rank than Δ, but we omit this as we will not need it. Although it is called a conjecture, it has been proved except for the case where one has a direct factor of type H_4. The cases A_n, B_n, C_n and D_n have been proved by Bernhard Mühlherr and Jacques Tits ([8]). The F_4 case has been announced by Chris Parker and Katrin Tent ([9]). Bernhard Leeb and Carlos Ramos-Cuevas gave an alternative proof for the

F_4 case and also proved the E_6 case ([7]). Finally Carlos Ramos-Cuevas proved the E_7 and E_8 case in [11]. The reducible cases obey the conjecture if their irreducible components do. So all thick spherical buildings obey the center conjecture.

For weak spherical buildings a direct factor of type H_3 or H_4 is possible as well. The center conjecture for the first case follows from results in [3], the second is still open.

3. Main results

The first main result is a fixed point theorem also valid for $\mathbb{R}$-buildings which are not metrically complete.

Main Result 1. *A finitely generated bounded group G of isometries of an $\mathbb{R}$-building Λ admits a fixed point.*

The second main result characterizes metrically complete $\mathbb{R}$-buildings in terms of their wall trees.

Main Result 2. *An $\mathbb{R}$-building is metrically complete if and only if all of its wall trees are metrically complete.*

4. Useful lemmas

Let $(\Lambda, \mathcal{F})$ be an $\mathbb{R}$-building.

Lemma 4.1. *Let C_β and C_γ be two sectors having the same direction C_∞ and based at respectively β and γ . There exists a constant $k \in \mathbb{R}^+$ depending only on the type of the $\mathbb{R}$-building such that there exists a point δ, with $\mathsf{d}(\beta,\delta), \mathsf{d}(\gamma,\delta) \leq k\mathsf{d}(\beta,\gamma)$, for which the sector C_δ is a subsector of both C_β and C_γ.*

Proof. Embed the sector C_β in an apartment Σ and the sector C_γ in an apartment Σ'. Let δ be the point of $C_\beta \cap C_\gamma$ closest to β (possible because this intersection is a closed subset of Σ due to Condition (A2), and non-empty because parallel sectors are asymptotic, and hence contain a common subsector). The sector C_δ is a subsector of both C_β and C_γ.

Let F_δ and F'_δ be the minimal sector-faces based at δ lying in respectively Σ and Σ', containing respectively β and γ. Notice that the germs of F_δ and F'_δ lie in respectively C_β and C_γ. This implies that the intersection of these sector-faces is the singleton $\{\delta\}$, otherwise one could find another point which lies in both $F_\delta \cap F'_\delta$ and $C_\beta \cap C_\gamma$. Such a point would be closer to β than δ, which is a contradiction. Hence $F_\delta \cap F'_\delta = \{\delta\}$.

Choose a sector D_δ based at δ containing F_δ. Let r be the retraction on the apartment Σ centered at the germ of D_δ (see Lemma 2.4). This retraction maps the sector-face F'_δ to some sector-face F''_δ in Σ, only sharing its base point δ with the sector-face F_δ. As $r(\gamma)$ lies in $F''_\delta \subset \Sigma$, it follows that there exists some constant k (depending on the minimal angle of half-lines in two sector-faces in the same apartment only having their base point in common) such that $\mathsf{d}(\beta,\delta), \mathsf{d}(r(\gamma),\delta) \leq k\mathsf{d}(\beta, r(\gamma))$. As the retraction preserves distances to δ, and does not increase other distances, this implies the desired result. □

Corollary 4.2. *There exists a constant k' depending only on the type of the $\mathbb{R}$-building, such that for each sector C_β and $l \in \mathbb{R}^+$, there exists a point $\delta \in C_\beta$ with $\mathsf{d}(\beta,\delta) = k'l$, such that for each point γ at distance at most l from β, the sector C_δ is a subsector of C_γ.*

Proof. If γ is a point at distance at most t from β, then by the above lemma one has that the sectors C_β and C_γ contain a common subsector C_τ based at a point $\tau \in C_\beta$ with $\mathsf{d}(\beta,\tau) \leq kt$.

All the sectors C_ρ with $\mathsf{d}(\rho,\beta) < t'$, $t' \in \mathbb{R}^+$ and $\rho \in C_\beta$ contain a common point τ which lies at a distance $k''t'$ from β, where k'' is a constant depending only on the type of the $\mathbb{R}$-building. Combining both observations one sees that this corollary holds for the constant $k' := kk''$. □

Corollary 4.3. *Let C_∞ and D_∞ be two adjacent chambers at infinity. If the germs of the sectors C_β and D_β based at some point β are the same, then there exists an $l > 0$ such that for each point $\gamma \in \Lambda$ with $\mathsf{d}(\beta,\gamma) < l$, the germs of the sectors C_γ and D_γ based at γ are the same.*

Proof. Because the germs of the sectors C_β and D_β are the same, there is an $l' > 0$ such that the intersections of the closed ball with radius l' centered at β with either the sector C_β or D_β are the same. This implies that for each point δ in the intersection of C_β and the open ball with radius l' centered at β, the germs of the sectors C_δ and D_δ are the same. Applying Lemma 4.1 there exists an $l > 0$ such that for each point $\gamma \in \Lambda$ with $\mathsf{d}(\beta,\gamma) < l$, the sectors C_β and C_γ have a point ϵ in common at distance strictly less than l' from β. If the germs of the sectors C_γ and D_γ were different, then the two sectors only share a sector-panel. This would also imply that the germs of the sectors C_ϵ and D_ϵ have to be different as well, which is a contradiction. □

Lemma 4.4. *The $\mathbb{R}$-building $(\Lambda, \mathcal{F})$ can be embedded in a metrically complete $\mathbb{R}$-building $(\Lambda', \mathcal{F}')$ of the same type, such that the isometries of $(\Lambda, \mathcal{F})$ extend to isometries of $(\Lambda', \mathcal{F}')$.*

Proof. This is a (direct) consequence of [6], Theorem 5.1.1, by taking the ω-limit with respect to the constant scaling sequence (Kleiner and Leeb assume completeness in their paper, but in this theorem it is irrelevant whether the $\mathbb{R}$-building you start with is complete or not). The resulting $\mathbb{R}$-building is metrically complete by [6], Lemma 2.4.2. The choice of the base point does not matter and the construction is functorial, so isometries extend. □

From now on assume that one has an $\mathbb{R}$-building $(\Lambda', \mathcal{F}')$ as described in the above lemma. Let $\overline{\Lambda}$ be the closure of Λ in Λ'. The metric space defined on Λ is complete if and only if $\Lambda = \overline{\Lambda}$.

Choose α to be a point in $\overline{\Lambda} \backslash \Lambda$ (which is only possible if the metric space defined on Λ is not complete). Let K be the chamber subcomplex of the residue Λ'_α in α consisting of the germs of sectors-faces based at α with direction a simplex of Λ_∞.

Corollary 4.5. *Let C_∞ be some chamber of the building at infinity Λ_∞. Then the interior of the sector C_α lies in Λ.*

Proof. This follows from applying Corollary 4.2 to a sequence of points in Λ converging to α. □

Lemma 4.6. *Let C_α and C'_α be two sectors based at α with their directions C_∞, C'_∞ in Λ_∞. Let w be the Weyl distance from the germ of C_α to C'_α. Then there exists a sector C''_α based at α with the same germ as C'_α such that the direction C''_∞ is a chamber of Λ_∞ and that the Weyl distance from C_∞ to C''_∞ is w.*

Proof. Using Lemma 2.1 we know that there exists an apartment Σ of Λ' containing both the sector C_α and the germ of the sector C'_α. Let β be a point of Σ in the interior of C'_α such that the germ of C'_β also lies in Σ. The Weyl distance from the germ of C_β to the germ of C'_β is again w. Because both the point β and the sectors C_β and C'_β lie in Λ, one can use Lemma 2.1 again to find an apartment Σ' of Λ containing C_β and the germ of C'_β. Let C''_β be the sector in this apartment Σ', based at β, with the same germ as C'_β. The Weyl distance between C_∞ and C''_∞ is w because the sectors C_β and C''_β lie in one apartment. As the germs of the sectors C'_α and C''_α are the same, we have proved the lemma. □

Lemma 4.7. *The chamber subcomplex K of Λ'_α is convex.*

Proof. Let C_α and C'_α be two sectors with chambers of Λ_∞ at infinity. Let the Weyl distance from the germ of C_α to the germ of C'_α be w. By the previous lemma, one can assume that the Weyl distance between C_∞ and C'_∞ also is w.

Assume we have a germ of sector D_α in the convex hull of the germs of the sectors C_α and C'_α. So if the Weyl distance from C_α to D_α is v, and the Weyl distance from D_α to C'_α is v' then $w = vv'$. These distances define D_α uniquely. Because the Weyl distance from C_∞ to C'_∞ also is w, one can find a chamber E_∞ at infinity such that the Weyl distance from C_∞ to E_∞ is v and the Weyl distance from E_∞ to C'_∞ is v'.

The distances between the germs C_α, E_α and C'_α have to be smaller than or equal to these ('smaller' with respect to the word metric on $\overline{W}$ defined by the generating set S), but as the distance between C_α and C'_α stays the same, they all stay the same. By uniqueness we can conclude that the germs $[D]_\alpha$ and $[E]_\alpha$ are the same and that the chamber subcomplex K is convex. □

Corollary 4.8. *No two germs of sectors in K are opposite.*

Proof. Assume that the two germs of the sectors C_α and C'_α based at α are opposite with C_∞, C'_∞ chambers of Λ_∞. Lemma 2.3 implies that there is an unique apartment Σ containing both sectors C_α and C'_α. The apartment at infinity will be the unique apartment Σ_∞ containing the opposite chambers C_∞ and C'_∞. A consequence is that Σ_∞ is an apartment of Λ_∞, and so also that Σ is an apartment of Λ. This yields that α, being a point of the apartment Σ, lies in Λ, which is a contradiction. □

5. Proof of the first main result

Let G be a finitely generated bounded group of isometries of an $\mathbb{R}$-building $(\Lambda, \mathcal{F})$. Assume that the Main Result 1 does not hold, or equivalently, that G does not fix points in Λ (the Bruhat–Tits fixed point theorem 2.6 implies that Λ is not complete if this is the case).

Embed Λ in a complete building Λ' as described in Lemma 4.4. The closure $\overline{\Lambda}$ in Λ' is a complete CAT(0)-space, so we can apply the Bruhat–Tits fixed point theorem 2.6 and obtain a point $\alpha \in \overline{\Lambda} \subset \Lambda'$ fixed by G. Then again as in the previous section, we obtain a convex chamber subcomplex K of the residue Λ'_α. As α is fixed and Λ stabilized by G, this group also acts on the (weak) spherical building Λ'_α and stabilizes the convex chamber complex K. One can now apply the center conjecture 2.8, and as Corollary 4.8 eliminates one option, we know that G stabilizes some non-trivial simplex of K. Let A be a maximal stabilized simplex of K.

The next step is to investigate the residue (in the spherical building Λ'_α) of the simplex A, which is again a (weak) spherical building. The germs in K which contain A form a convex chamber complex of this residue, so we can again apply the center conjecture 2.8 on this new convex chamber complex. However, a stabilized non-trivial simplex in the residue is impossible due to the maximality of A. So there exist two chambers C and D in K, both containing the fixed simplex A, and such that the corresponding chambers in the residue of A are opposite.

Using Lemma 4.6, one can find two sectors S_α and S'_α lying in one apartment such that their germs equal respectively C and D. The intersection $S_\alpha \cap S'_\alpha$ is a sector-face R_α with germ A. The interior of both sectors lies in Λ due to Corollary 4.5. Because Λ is convex within Λ', it follows that the points of R_α not lying on a non-maximal face of this sector-face lie in Λ.

Let L be the barycentric closed half-line (with endpoint α) of the sector-face R_α. From the above discussion it follows that this half-line, except for the point α, lies in Λ. Parametrize this line by a map $\phi : \mathbb{R}^+ \to L$ such that for each $l \in \mathbb{R}^+$ one has that $\mathsf{d}(\phi(l), \alpha) = l$. Note that $\phi(0) = \alpha$. As the group G stabilizes the germ A, there exists for each element $g \in G$ a positive number $l_g > 0$ such that each point $\phi(l)$ with $l \in [0, l_g]$ is fixed by g. Let $\{g_1, \dots, g_k\}$ be the chosen finite generating set of G and $l' > 0$ be the minimum of the l_{g_i} with $i \in \{1, \dots, k\}$. The point $\phi(l')$, which lies in Λ, is fixed by a generating set of G, and hence by the entire group G. So we have proved that there does exist a point in Λ fixed by G.

Remark 5.1. If the Coxeter system has a direct factor of type H_4, then the center conjecture 2.8 has not been proved yet. However this does not pose a problem for our purposes. If such a case occurs one can restrict the Weyl group of both the spherical building at infinity and the residue at α (which is a retract of the building at infinity) to no longer have a direct factor of type H_4. The convex chamber complex K stays a chamber complex after this restriction of the Weyl group, so we can apply the center conjecture in this case (compare [11], p. 3).

6. Proof of the second main result

First assume that the metric space defined on Λ is complete, and let m be a wall of the spherical building at infinity. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in the wall tree $T(m)$. The union of the apartments of the $\mathbb{R}$-building which at infinity contain m forms a subset $K \subset \Lambda$ isometric to the direct product of the metric space formed by $T(m)$ and $\mathbb{R}^{k-1}$ (where k is the dimension of the apartments).

Using this subset K, we can 'lift' the Cauchy sequence $(\alpha_n)_{n\in\mathbb{N}}$ to a Cauchy sequence $(\beta_n)_{n\in\mathbb{N}}$ in $K \subset \Lambda$. As the metric space defined on Λ is complete, this sequence converges to some point $\beta \in \Lambda$. Our goal is to prove that the point β lies in K, implying that the original sequence $(\alpha_n)_{n\in\mathbb{N}}$ converges. For this we have to prove that β lies in an apartment which at infinity contains the wall m. Let S_∞ and S'_∞ be two opposite sector-panels in m; if we can prove that the germs of sector-panels $[S]_\beta$ and $[S']_\beta$ in the residue at β are still opposite, we are done (because Lemma 2.3 then implies that β lies in an apartment containing m at infinity). Equivalent with this last statement is that for a shortest gallery from a chamber C_∞ containing S_∞ to a chamber C'_∞ containing S'_∞ ('shortest' meaning minimal over all choices of C_∞ and C'_∞), the corresponding gallery in the residue Λ_β between the germs of the sectors $[C]_\beta$ to $[C']_\beta$ always is non-stammering. As this is the case for each point of K, and hence each point of the sequence $(\beta_n)_{n\in\mathbb{N}}$, Corollary 4.3 implies that this is also the case for β. So we have proved that the metric space defined by the $\mathbb{R}$-tree $T(m)$ is complete.

We now prove the other direction. Assume that all the trees corresponding to walls at infinity are complete. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in the metric space

(Λ, d). Using Lemma 4.4, we embed $(\Lambda, \mathcal{F})$ in a complete $\mathbb{R}$-building $(\Lambda', \mathcal{F}')$. The Cauchy sequence $(\alpha_n)_{n\in\mathbb{N}}$ then converges to some point $\alpha \in \overline{\Lambda}$. We need to prove that α is a point of Λ, so assume this is not the case.

Choose a sector C_α based at α such that C_∞ is a chamber of Λ_∞. The interior of this sector lies in Λ due to Corollary 4.5, and one can find a sequence of points $(\beta_n)_{n\in\mathbb{N}}$ herein which also converges to the point α.

Let P_∞ be a panel of C_∞. The sequence $(P_{\beta_n})_{n\in\mathbb{N}}$ of parallel sector-panels forms a Cauchy sequence in the panel tree $T(P_\infty)$, contained in an open half-line. Using the completeness of this tree, we can embed this open half-line into a closed half-line, and then into to an apartment (essentially using Lemma 2.1), and find a chamber C'_∞ in Λ_∞ containing P_∞ such that the germs of the sectors C_{β_n} and C'_{β_n} are not the same for all $n \in \mathbb{N}$. It follows that the germs of the sectors C_α and C'_α are not the same, but adjacent, having the germ of the sector-panel P_α in common.

Repeating this algorithm one can obtain two sectors based at α with at infinity chambers of Λ_∞ and opposite germs, but this is in contradiction with Corollary 4.8 and $\alpha \notin \Lambda$. This proves the second main result.

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