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Mod-p cohomology growth in p-adic analytic towers of 3-manifolds

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Dedicated to the memory of Fritz Grunewald

Abstract. Let M be a compact 3-manifold with infinite fundamental group Γ. Given a homomorphism from Γ to a p-adic analytic group G with dense image, we describe the possible mod-p homology growth of covers M_n of M determined by the congruence subgroups G_n. If $d = \dim(G) > 3$, this growth is always non-trivial, growing at least as fast as $\mathrm{Vol}(M_n)^{(d-1)/d}$.

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1. Introduction

Let M be a compact, orientable, connected 3-manifold with fundamental group Γ. Let p be a prime, let $n \geq 1$ be a positive integer, and let $\phi\colon \Gamma \to \mathrm{GL}_n(\mathbb{Z}_p)$ be a homomorphism. If we let G denote the closure in $\mathrm{GL}_n(\mathbb{Z}_p)$ of the image of Γ, then G is a p-adic analytic group, admitting a normal exhaustive filtration

$$G_n := G \cap \ker(\mathrm{GL}_N(\mathbb{Z}_p) \to \mathrm{GL}_N(\mathbb{Z}/p^n\mathbb{Z})).$$

This filtration gives rise to a corresponding filtration $\{\Gamma_n\}$ of Γ via the normal subgroups $\Gamma_n := \phi^{-1}(G_n)$. This filtration may or may not be exhaustive (depending on whether or not ϕ is injective).

Associated to each of the finite index subgroups Γ_n of Γ is a finite connected cover M_n of M. The main concern of this paper is to estimate the growth of $H_1(M_n, \mathbb{F}_p)$ as a function of n, in terms of the geometry of M and the dimension of the group G.

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The methods of this paper are very similar to those of [4], [5]. The point of this note, however, is to point out that no assumption on the arithmeticity (or even hyperbolicity) of M is required, and that ϕ need not be a congruence homomorphism. Indeed, our conclusions do not depend at all on the geometry of M, beyond the fact that it has dimension 3.

If H is any pro-p group, let $\delta(H) := \dim_{\mathbb{F}_p} H/\Phi(H)$ where $\Phi(H)$ is the Frattini subgroup of H. If $\delta(H)$ is finite, it is equal to the minimal number of topological generators of H. Recall that the dimension $d := \dim(G)$ of a p-adic analytic group G is equal to $\delta(H)$ for any open uniform subgroup H of G. The sequence of normal subgroups $G_n \subset G$ constructed above is an *exhaustive* sequence of subgroups, that is, $\bigcap G_n = \{1\}$. It follows that for any finite index open subgroup $H \subset G$, there is an inclusion $G_n \subset H$ for sufficiently large n. In particular, taking H to be uniform, we deduce from Theorem 3.8 of [7] that $\delta(G_n) \leq d$ for sufficiently large n.

The following result is the main theorem of this note.

Theorem 1.1. *Let* $d = \dim(G)$. *Then one of the following holds:*

(1) $\dim H_1(M_n, \mathbb{F}_p) = \lambda \cdot p^{dn} + O(p^{(d-1)n})$ *for some rational constant* $\lambda \neq 0$.
(2) $\dim H_1(M_n, \mathbb{F}_p) = \lambda \cdot p^{(d-1)n} + O(p^{(d-2)n})$ *for some rational constant* $\lambda \neq 0$.
(3) $d = 2$, *and* $\dim H_1(M_n, \mathbb{F}_p) = O(1)$.
(4) $d = 3$, *the boundary of* M *is a (possibly empty) union of spheres, and one has* $\dim H_1(M_n, \mathbb{F}_p) = \delta(G_n)$ *and* $\dim H_1(M_n, \mathbb{F}_p) \leq 3$ *for sufficiently large* n.

If $d > 3$, then Theorem 1.1 implies that the rate of growth of mod-p homology comes in two possible flavours. Since $[G : G_n]$ is of order p^{dn}, case (1) of Theorem 1.1 corresponds to "linear growth" of mod-p homology. The question of linear mod-p homology growth is considered in recent work of Lackenby [8], [9], where the existence of such growth in certain circumstances can be used to deduce that some finite index subgroup of Γ is "large" (i.e., admits a surjection onto a free group of rank ≥ 2). (We shall see in Example 5.1 that large subgroups give rise to examples where case (1) applies.) Case (2) of Theorem 1.1, however, corresponds to "sub-linear growth" of mod-p homology. Perhaps the most surprising aspect of our result is that whenever the growth of mod-p homology in a p-adic analytic tower is sub-linear (and $\dim(G) > 3$) the rate of growth is still quite fast, and, moreover, is determined (up to a non-zero scalar) by the single invariant $\dim(G)$. A natural question to consider is whether one should expect case (1) or (2) to hold. We conjecture the following:

Conjecture 1.2. *Suppose that* M *is a finite volume hyperbolic manifold and that* ϕ *is injective, or equivalently, that* $\bigcap \Gamma_n = \{1\}$. *Then case* (1) *of Theorem* 1.1 *does not occur.*

Although this conjecture is motivated by questions in the theory of automorphic forms (see, for example, [5]), it reflects the general principle that fundamental groups of finite volume hyperbolic 3-manifolds are a long way from being free, unlike the fundamental groups of surfaces. (For example, the fundamental group of any closed 3-manifold admits a balanced presentation, i.e., one with an equal number of generators and relations.)

In Section 5, we give examples illustrating our theorem. In particular, every possible case of Theorem 1.1 does occur.

2. Iwasawa theory

We maintain the notation of the introduction. We define the completed homology groups as follows:

$$\widetilde{H}_i(\mathbb{F}_p) = \varprojlim H_i(M_n, \mathbb{F}_p).$$

Example 2.1. Consider the case $i = 0$. Each $H_i(M_n, \mathbb{F}_p)$ is then canonically identified with $\mathbb{F}_p$, and the transition maps in the projective limit are just the identity map. Hence $\widetilde{H}_0(\mathbb{F}_p) = \mathbb{F}_p$.

Example 2.2. If M is not closed, then $H_3(M_n, \mathbb{F}_p) = 0$ for each n, and so certainly $\widetilde{H}_3(\mathbb{F}_p) = 0$. On the other hand, suppose that M is closed, so that $H_3(M_n, \mathbb{F}_p) = \mathbb{F}_p$ for each value of n, generated by the fundamental class $[M_n]$. If the dimension d of G is at least 1, then for any $n \geq 1$, there is $n' > n$ such that the degree of the map $M_{n'} \to M_n$ is divisible by p, and hence induces the zero map on H_3. Thus, when $d \geq 1$, we see that $\widetilde{H}_3(\mathbb{F}_p) = 0$ even when M is closed.

Each of the modules $\widetilde{H}_i(\mathbb{F}_p)$ carries a continuous action of G, and thus is a $\Lambda := \mathbb{F}_p[[G]]$-module. The modules $\widetilde{H}_i(\mathbb{F}_p)$ are, in fact, finitely generated as Λ-modules. One way to see this, which also sheds light on the nature of the groups $\widetilde{H}_i(\mathbb{F}_p)$, is as follows: Fix a triangulation of M, and pull this back to obtain a G/G_n-equivariant triangulation to M_n. We obtain a chain complex of free $\mathbb{F}_p[G/G_n]$-modules which compute $H_i(M_n, \mathbb{F}_p)$ together with the action of G/G_n. The rank of these modules is bounded by the number of cells in the triangulation of M, and is hence bounded independently of n. The projective limit of these complexes defines a chain complex of finite rank free $\Lambda = \mathbb{F}_p[[G]]$-modules

$$S_\bullet \colon S_3 \to S_2 \to S_1 \to S_0,$$

whose homology equals $\widetilde{H}_i(\mathbb{F}_p)$.

In particular, $\widetilde{H}_1(\mathbb{F}_p)$ can be thought of as an analogue of the Alexander invariant, where the group G plays the role of the covering group $\mathbb{Z}$ of a knot complement in the theory of the Alexander invariant, and the ring Λ plays the role of the ring $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$ in that theory.

A basic fact underlying the viewpoint of the paper is that the category of finitely generated modules over $\Lambda := \mathbb{F}_p[[G]]$ has good properties whenever G is p-adic analytic; e.g., for any such G, the ring Λ is (left and right) Noetherian [10], and if G is furthermore sufficiently small, then Λ is Auslander regular [14]. For example, If $G = \mathbb{Z}_p^N$, then $\Lambda \simeq \mathbb{F}_p[[T_1, \dots, T_N]]$ is a regular local ring of dimension N.

The key technical ingredient that we employ is the notion of codimension for finitely generated Λ-modules. A finitely generated Λ-module A is defined to have codimension c if $\mathrm{Ext}^i(A, \Lambda) = 0$ for all $i < c$ and is non-zero for $i = c$. A non-zero finitely generated module has codimension $\le d$; we define the codimension of the zero module to be ∞. The utility of this definition is the following result (see [1], §5):

Theorem 2.3. *Suppose that A is a finitely generated Λ-module of codimension c, and as usual, write* $\dim(G) = d$. *Then, letting $\{G_n\}$ denote the filtration of G defined above, we have that*

$$\begin{aligned}\dim H_0(G_n, A) &= \dim(\textit{the space of } G_n\textit{-coinvariants of } A)\\ &= \lambda \cdot p^{(d-c)n} + O(p^{(d-c-1)n})\end{aligned}$$

for some non-zero rational number λ.

In order to connect this result to our main theorem, we note the following:

Lemma 2.4. *There is an exact sequence:*

$$H_2(G_n, \mathbb{F}_p) \to H_0(G_n, \widetilde{H}_1(\mathbb{F}_p)) \to H_1(M_n, \mathbb{F}_p) \to H_1(G_n, \mathbb{F}_p) \to 0.$$

Proof. This follows from a version of the Hochschild–Serre spectral sequence for completed homology (where we have taken into account the fact that $\widetilde{H}_0 = \mathbb{F}_p$, by Example 2.1). □

Note that $\dim H_1(G_n, \mathbb{F}_p) = \delta(G_n) \le d = \dim(G)$ for sufficiently large n, and that $H_2(G_n, \mathbb{F}_p)$ is bounded as a function of n (see [6]). The main theorem is thus automatically true for $d \le 2$. Moreover, when $d \ge 3$, it suffices to prove either that $\widetilde{H}_1(\mathbb{F}_p)$ has codimension ≤ 1, or else that the boundary of M is a union of spheres, that $d = 3$, and that $\widetilde{H}_1(\mathbb{F}_p) = 0$. (This equivalence uses the fact that if $\widetilde{H}_1(\mathbb{F}_p) = 0$, then $H_1(M_n, \mathbb{F}_p) = H_1(G_n, \mathbb{F}_p)$ has dimension $\delta(G_n)$.)

3. The case of closed M

We shall assume in this section that $\partial M = \emptyset$. This case contains all the essential ideas of this paper and is unencumbered by the technical modifications required when M has boundary.

Lemma 3.1. *There is a spectral sequence*

$$\mathrm{Ext}^i(\widetilde{H}_j(\mathbb{F}_p), \Lambda) \Rightarrow \widetilde{H}_{3-i-j}(\mathbb{F}_p).$$

Proof. Apply $\mathrm{Hom}(–, \Lambda)$ to the chain complex $S_\bullet$ described above. □

Remark 3.2. Note that Lemma 3.1 (and its proof) is essentially Poincaré duality "over Λ", the only difference being that Λ (unlike a field) is not cohomologically trivial, so that one obtains a spectral sequence rather than a simple isomorphism between the Λ-dual of $\widetilde{H}_i(\mathbb{F}_p)$ and $\widetilde{H}_{3-i}(\mathbb{F}_p)$.

Let us now study the spectral sequence. Note that $\mathrm{Ext}^i(\mathbb{F}_p, \Lambda) = 0$ unless $i = d$, in which case it equals $\mathbb{F}_p$. Thus the E_2-page of the spectral sequence has the following form:

$$\begin{array}{cccc}
\mathrm{Hom}(\widetilde{H}_2(\mathbb{F}_p), \Lambda) & \mathrm{Ext}^1(\widetilde{H}_2(\mathbb{F}_p), \Lambda) & \dots & \\
\mathrm{Hom}(\widetilde{H}_1(\mathbb{F}_p), \Lambda) & \mathrm{Ext}^1(\widetilde{H}_1(\mathbb{F}_p), \Lambda) & \dots & \\
0 & 0 & \dots & \mathbb{F}_p.
\end{array}$$

The $(0, 0)$ term is stable, and hence, we recover the fact that $\widetilde{H}_3(\mathbb{F}_p) = 0$ when $d \geq 1$ (which we already observed in Example 2.2 above).

As noted at the end of the preceding section, in order to prove our main theorem, we may assume that $d \geq 3$ and that $\widetilde{H}_1(\mathbb{F}_p)$ has codimension at least 2. We henceforth make these assumptions. From the definition of codimension, we find that $\mathrm{Hom}(\widetilde{H}_1(\mathbb{F}_p), \Lambda) = \mathrm{Ext}^1(\widetilde{H}_1(\mathbb{F}_p), \Lambda) = 0$. In particular (since also $d > 1$) we deduce from the spectral sequence that $\widetilde{H}_2(\mathbb{F}_p) = 0$. Then also $\mathrm{Hom}(\widetilde{H}_2(\mathbb{F}_p), \Lambda) = 0$, which, together with the fact that $\mathrm{Ext}^1(\widetilde{H}_1(\mathbb{F}_p), \Lambda) = 0$ and that $d > 2$, allows us to deduce from the spectral sequence that $\widetilde{H}_1(\mathbb{F}_p) = 0$. Thus, the E_2-page of the spectral sequence in fact contains only one non-zero term, namely the $(d, 0)$ term $\mathbb{F}_p$. We therefore conclude that $\widetilde{H}_i(\mathbb{F}_p) = 0$ unless $i = 3 - d$, and that $\widetilde{H}_{3-d}(\mathbb{F}_p) = \mathbb{F}_p$. Since we know that $\widetilde{H}_0(\mathbb{F}_p) = \mathbb{F}_p$, we deduce that $d = 3$. This completes the proof of Theorem 1.1 in the closed case.

4. The case when M has boundary

If M has boundary, then it is still of interest to consider the completed homology groups $\widetilde{H}_i(\mathbb{F}_p)$. However, we also need to consider homology relative to the boundary, and the corresponding completed homology groups:

$$\widetilde{H}_i^{\mathrm{BM}}(\mathbb{F}_p) = \varprojlim H_i(M_n, \partial M_n; \mathbb{F}_p).$$

(Here the superscript "BM" stands for Borel–Moore homology; the reason for using this notation is that the relative homology $H_i(M_n, \partial M_n; \mathbb{F}_p)$ coincides with the Borel–Moore homology of the complement of the boundary $H_i^{\mathrm{BM}}(M_n \setminus \partial M_n, \mathbb{F}_p)$.) Similarly, if we let ∂M denote the boundary of M, then we may define

$$\widetilde{H}_i(\partial, \mathbb{F}_p) = \varprojlim H_i(\partial M_n, \mathbb{F}_p).$$

Lemma 4.1. *There are spectral sequences*

(1) $\mathrm{Ext}^i(\widetilde{H}_j^{\mathrm{BM}}(\mathbb{F}_p), \Lambda) \Rightarrow \widetilde{H}_{3-i-j}(\mathbb{F}_p)$,
(2) $\mathrm{Ext}^i(\widetilde{H}_j(\mathbb{F}_p), \Lambda) \Rightarrow \widetilde{H}^{\mathrm{BM}}_{3-i-j}(\mathbb{F}_p)$,
(3) $\mathrm{Ext}^i(\widetilde{H}_j(\partial, \mathbb{F}_p), \Lambda) \Rightarrow \widetilde{H}_{2-i-j}(\partial, \mathbb{F}_p)$.

Moreover, there is a long exact sequence:

$$\cdots \to \widetilde{H}_j(\partial, \mathbb{F}_p) \to \widetilde{H}_j(\mathbb{F}_p) \to \widetilde{H}_j^{\mathrm{BM}}(\mathbb{F}_p) \to \widetilde{H}_{j-1}(\partial, \mathbb{F}_p) \to \cdots.$$

Proof. For the various spectral sequences we proceed as in the closed case. The long exact sequence is obtained as the projective limit of the usual long exact sequences for the pairs $(M_n, \partial M_n)$. □

Suppose that $n' \geq n$. Although $\partial M_{n'}$ is a finite cover of ∂M_n, typically neither is connected. We will take a moment to discuss the structure of their component sets in terms of the map ϕ, the group G, and so on.

If Σ is a component of ∂M_n, then composing the natural map $\pi_1(\Sigma) \to \pi_1(M_n) = \Gamma_n$ with the map $\phi|_{\Gamma_n} : \Gamma_n \to G_n$ yields a map $\pi_1(\Sigma) \to G_n$. We let H denote the closure of the image of this map. If $n' \geq n$, and if we let $\Sigma_{n'}$ denote the preimage of Σ in $\partial M_{n'}$, then there is a bijection

$$\pi_0(\Sigma_{n'}) \cong G_n / G_{n'} H.$$

We say that the component Σ of ∂M_n splits completely up the tower if each component of $\Sigma_{n'}$ maps homeomorphically to Σ, for each $n' \geq n$, or equivalently, if $\Sigma_{n'}$ has $[G_n : G_{n'}]$ connected components, for each $n' \geq n$, or again equivalently (as one deduces from the preceding description of $\pi_0(\Sigma_{n'})$), if $H = \{1\}$.

Lemma 4.2. *The following are equivalent:*

(1) *For some value of n, the boundary ∂M_n contains a component which splits completely up the tower;*
(2) *$\widetilde{H}_0(\partial, \mathbb{F}_p)$ has codimension 0 as a Λ-module.*

Furthermore, either these equivalent conditions hold, or else $\widetilde{H}_2(\partial, \mathbb{F}_p) = 0$.

Proof. Let $\partial M = \Sigma^1 \coprod \cdots \coprod \Sigma^m$ be the decomposition of ∂M into a disjoint union of connected components. For each n, let Σ_n^i denote the preimage of Σ^i under the covering map $M_n \to M$. If we define $A^i := \varprojlim H^0(\Sigma_n^i, \mathbb{F}_p)$, then $\widetilde{H}_0(\partial, \mathbb{F}_p) = A^1 \oplus \cdots \oplus A^m$, and so $\widetilde{H}_0(\partial, \mathbb{F}_p)$ has codimension 0 if and only if at least one A^i does.

Now if we write H^i to denote the closure of the image of the composite

$$\pi_1(\Sigma^i) \to \pi_1(M) = \Gamma \xrightarrow{\phi} G,$$

then we see from the above discussion that $\pi_0(\Sigma_n^i) = G/G_n H^i$, and thus that

$$A^i = \varprojlim \mathbb{F}_p[G/G_n H^i] = \mathbb{F}_p[[G/H^i]]$$

(with the Λ-module structure being induced by the left action of G on G/H^i). From this description of A^i, one easily verifies that the codimension of A^i is equal to the dimension of H^i. Thus A^i has codimension 0 if and only if H^i has dimension 0, i.e. is finite. This in turn is equivalent to having $H^i \cap G_n = \{1\}$ for sufficiently large n, which is in turn equivalent to one (or equivalently, every) component of Σ_n^i being completely split up the tower. This establishes the equivalence of (1) and (2).

The final statement follows immediately from a consideration of the $(0,0)$-term of the third spectral sequence of Lemma 4.1. We may also prove it more directly: since each ∂M_n is a disjoint union of closed surfaces, by arguing just as in the 3-manifold case considered in Example 2.2, we find that $\widetilde{H}_2(\partial, \mathbb{F}_p) = 0$ unless, for some sufficiently large value of n, there is a component of ∂M_n which splits completely up the tower. $\square$

We now turn to proving Theorem 1.1. Since the 2-sphere is simply connected, any 2-sphere in the boundary of M splits completely up the tower of M_n. Thus we may cap off all these 2-spheres with 3-balls; note that this does not change $H_1(M_n, \mathbb{F}_p)$. Hence we may and do assume that every component of ∂M, and hence every component of ∂M_n for each value of n, has positive genus. As noted in the preceding section, the theorem is automatic if $d \leq 2$, and so we may assume that $d \geq 3$. We have already proved the theorem in the closed case, and thus we may also assume that M has a non-empty boundary (every component of which has positive genus). The theorem will follow if we prove that $\widetilde{H}_1(\mathbb{F}_p)$ has codimension at most one. Thus we assume that $\widetilde{H}_1(\mathbb{F}_p)$ has codimension at least 2, and argue by contradiction.

Since M, and so each M_n, is not closed, we certainly have that $\widetilde{H}_3(\mathbb{F}_p) = 0$, and an argument like that given in the closed case in Example 2.2 shows that also $\widetilde{H}_3^{\mathrm{BM}}(\mathbb{F}_p) = 0$. (Alternatively, this follows from the second spectral sequence of Lemma 4.1.) Furthermore, $\widetilde{H}_0^{\mathrm{BM}}(\mathbb{F}_p) = 0$ (since each M_n is not closed) and $\widetilde{H}_0(\mathbb{F}_p) = \mathbb{F}_p$.

We now collect various consequences of our assumptions:

(1) $\widetilde{H}_1^{\mathrm{BM}}(\mathbb{F}_p) = 0$, as follows from the second spectral sequence of Lemma 4.1, the fact that $d > 1$, and the fact that $\widetilde{H}_1(\mathbb{F}_p)$ has positive codimension.

(2) $\widetilde{H}_2(\partial, \mathbb{F}_p) \cong \widetilde{H}_2(\mathbb{F}_p)$, as follows from the long exact sequence of Lemma 4.1 and the vanishing of $\widetilde{H}_3^{\mathrm{BM}}(\mathbb{F}_p)$ (noted above) and $\widetilde{H}_2^{\mathrm{BM}}(\mathbb{F}_p)$ (noted in the preceding point).

(3) $\widetilde{H}_1(\partial, \mathbb{F}_p)$ has positive codimension, as follows from the fact that it embeds into $\widetilde{H}_1(\mathbb{F}_p)$ (since $\widetilde{H}_2^{\mathrm{BM}}(\mathbb{F}_p)$ vanishes, by (1)), which has positive codimension by assumption.

We now deduce that ∂M_n has no completely split boundary component, for any value of n. Indeed, suppose that M_n did have such a component for some n. By assumption, the genus g of this component would be positive, and so we would find that $\widetilde{H}_1(\partial, \mathbb{F}_p)$ has codimension 0, contradicting point (3) above. Lemma 4.2 then implies that $\widetilde{H}_2(\partial, \mathbb{F}_p) = 0$, and so point (2) above implies that $\widetilde{H}_2(\mathbb{F}_p) = 0$.

A consideration of the second spectral sequence of Lemma 4.1, taking into account our assumptions that $d \geq 3$ and that $\widetilde{H}_1(\mathbb{F}_p)$ has codimension ≥ 2, then shows that $\widetilde{H}_1^{\mathrm{BM}} = 0$. We have already observed that $\widetilde{H}_i^{\mathrm{BM}} = 0$ for $i = 0, 2, 3$, and so we conclude that the first spectral sequence of Lemma 4.1 is identically zero. Consequently, $\widetilde{H}_i(\mathbb{F}_p)$ must vanish for every value of i. This contradicts the fact that $\widetilde{H}_0(\mathbb{F}_p) = \mathbb{F}_p$. Thus in fact the codimension of $\widetilde{H}_1(\mathbb{F}_p)$ is at most one, and the theorem is proved.

5. Examples

Example 5.1. Fix a p-adic Lie group G, let F be a free group of rank ≥ 2, and consider a map $F \to G$ with Zariski dense image. If F_n denotes the preimage of G_n, then F_n will be free of rank $[G : G_n](\mathrm{rank}(F) - 1) + 1$. In particular, the $\mathbb{F}_p$-homology of F_n will grow linearly in the degree.

Now suppose that M is a closed 3-manifold such that $\Gamma = \pi_1(M)$ admits a surjective map $\Gamma \to F$. If we define ϕ to be the composite $\phi\colon \Gamma \to F \to G$, then we are in case (1) of Theorem 1.1. This construction applies more generally: we may take F to be any group admitting a faithful representation into a p-adic Lie group G such that the $\mathbb{F}_p$-homology of any finite index subgroup F_n has rank at least $c \cdot [F : F_n]$ for some constant c independent of n. For example, F could be the fundamental group of a compact surface Σ_g of genus $g \geq 2$. It seems a difficult problem to characterize such groups F. If M is hyperbolic, one might wonder whether this is the only way in which case (1) can occur.

Example 5.2. The prime 3 splits in $\mathbb{Z}[\sqrt{-2}]$, and so there is a surjection $\mathbb{Z}[\sqrt{-2}] \to \mathbb{F}_3 \times \mathbb{F}_3$, inducing a surjection $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}] \to \mathrm{SL}_2(\mathbb{F}_3) \times \mathrm{SL}_2(\mathbb{F}_3)$. If Γ denotes the kernel of this map, then Γ admits a map to $\mathrm{SL}_2(\mathbb{Z}_3^2) \cong \mathrm{SL}_2(\mathbb{Z}_3)^2$ with Zariski dense pro-3 image (induced by the embedding $\mathbb{Z}[\sqrt{-2}] \hookrightarrow \mathbb{Z}_3^2$ given by forming the 3-adic completion of $\mathbb{Z}[\sqrt{-2}]$). The dimension of the target is $d = 6 > 3$. The computations of [12], §4.1, imply that, with respect to the corresponding pro-3 cover, case (2) of the main theorem occurs.

Example 5.3. Suppose that ∂M is the union of N tori. We obtain a map $\phi\colon \Gamma := \pi_1(M) \to \mathbb{Z}^N \hookrightarrow \mathbb{Z}_p^N =: G$. Let $\widetilde{M}$ be the cover of M with covering group $\mathbb{Z}^N$ corresponding to the kernel of ϕ. The $\mathbb{Z}[\mathbb{Z}^N]$-module $H^1(\widetilde{M}, \mathbb{Z})$ is the Alexander invariant of M. There is a natural map $\mathbb{Z}[\mathbb{Z}^N] \to \mathbb{F}_p[[\mathbb{Z}_p^N]]$, and the completed homology $\widetilde{H}_1(\mathbb{F}_p)$ is obtained from the Alexander invariant by base-change along this map:

$$\widetilde{H}_1(\mathbb{F}_p) := \mathbb{F}_p[[\mathbb{Z}_p^N] \otimes_{\mathbb{Z}[\mathbb{Z}^N]} H^1(\widetilde{M}, \mathbb{Z}).$$

Theorem 1.1 shows that $\widetilde{H}_1(\mathbb{F}_p)$, and hence the Alexander invariant, can be trivial only if $N \leq 2$.

If M is a knot complement (in which case $N = 1$), then in fact $\widetilde{H}_1(\mathbb{F}_p) = 0$, as we now show.

Lemma 5.4. *If M is a knot complement in S^3, then $\widetilde{H}_1(\mathbb{F}_p) = 0$, and consequently $H_1(M_n, \mathbb{F}_p) = \mathbb{F}_p$ for all n.*

Proof. We give two proofs, one in the spirit of this note, and the other (which was explained to us by Barry Mazur) using the classical theory of the Alexander polynomial.

First proof. We have $G = \mathbb{Z}_p$, and $G_n = p^n\mathbb{Z}_p$ for each n. Then $\Lambda = \mathbb{F}_p[[T]]$, and the exact sequence of Lemma 2.4 simplifies to the following short exact sequence:

$$0 \to \widetilde{H}_1(\mathbb{F}_p)/T^{p^n}\widetilde{H}_1(\mathbb{F}_p) \to H_1(M_n, \mathbb{F}_p) \to \mathbb{F}_p \to 0.$$

Taking $n = 1$, and noting that $H_1(M, \mathbb{F}_p) = \mathbb{F}_p$ (since M is a knot complement), we find that $\widetilde{H}_1(\mathbb{F}_p)/T\widetilde{H}_1(\mathbb{F}_p) = 0$. Since $\widetilde{H}_1(\mathbb{F}_p)$ is finitely generated over $\mathbb{F}_p[[T]]$, Nakayama's lemma implies that $\widetilde{H}_1(\mathbb{F}_p) = 0$.

Second proof. Write $M = S^3 \setminus \overline{K}$, where $\overline{K}$ is a tubular neighbourhood of the knot K. For any integer $e \geq 1$, let X_e denote the e-fold cover of S^3 ramified along K. If $\Delta(t)$ denotes the Alexander polynomial of M (note that the variables t and T are related via $t = T + 1$), then $\#H_1(X_e, \mathbb{Z}) = \prod_{\zeta^e=1} \Delta(\zeta)$ if the right hand side is non-zero. (If the right hand side vanishes, then $H_1(X_e, \mathbb{Z})$ has positive rank).

Recall that $\Delta(1) = 1$. Thus, taking $e = p^n$, we find that $\#H_1(X_{p^n}, \mathbb{Z}) \equiv 1 \bmod \zeta - 1$, where ζ is a primitive p^nth root of 1. Since $\zeta - 1$ is a non-unit algebraic integer of norm p, while $\#H_1(X_e, \mathbb{Z})$ is an integer, we find that in fact $\#H_1(X_{p^n}, \mathbb{Z}) \equiv 1 \bmod p$. In particular, $H_1(X_{p^n}, \mathbb{Z})$ is finite and p-torsion free, and so by the universal coefficient theorem, $H_1(X_{p^n}, \mathbb{F}_p) = 0$. Now M_n is equal to the complement in X_{p^n} of a tubular neighbourhood of the preimage of $\overline{K}$, and thus $H_1(M_n, \mathbb{F}_p) = \mathbb{F}_p$, generated by the class of a meridian in the preimage of $\overline{K}$.

If $n' > n$, then the map of meridians induced by the map $M_{n'} \to M_n$ has degree $p^{n'-n}$, and hence the map $H_1(M_{n'}, \mathbb{F}_p) \to H_1(M_n, \mathbb{F}_p)$ is zero. Thus we also conclude that $\widetilde{H}_1(\mathbb{F}_p) = 0$. $\square$

Note that the argument in the second proof of the preceding lemma breaks down if e is divisible by more than one prime, since if ζ is a primitive eth root of unity for such a value of e, then $\zeta - 1$ is a unit in the ring of algebraic integers. And indeed, for such e, the cover X_e can have non-zero 1st Betti number, or torsion in H^1 of order that is not coprime to e. (For example, if K is the trefoil knot and $e = 6$ then $H_1(X_6, \mathbb{Z})$ is infinite [13], p. 150.)

If M is the complement of the Hopf link, then $N = 2$, and it is known that the Alexander invariant, and hence $\widetilde{H}_1(\mathbb{F}_p)$, vanishes [13], p. 190. This provides an example in which case (3) of Theorem 1.1 occurs.

Example 5.5. In [3] and [2], a closed arithmetic manifold M is considered which has the property that the 3-adic completion G of its fundamental group Γ is analytic. This implies, for the associated map $\phi\colon \Gamma \to G$, that $\widetilde{H}_1(\mathbb{F}_p) = 0$, and hence that we are in case (4) of Theorem 1.1 (with $p = 3$). Another example is given (with $p = 5$) by a finite cover of the Weeks manifold (see [3], p. 321.

Example 5.6. If M is hyperbolic, then Γ can be realized as a torsion free discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$ and $M \simeq \Gamma\backslash\mathbb{H}^3$. If M has finite volume, then, by Mostow rigidity, we may assume (after possibly conjugating Γ) that $\Gamma \subset \mathrm{SL}_2(E)$ for some (minimal) number field E. Since Γ is finitely generated, there exists a finite set of primes S in E such that $\Gamma \subset \mathrm{SL}_2(\mathcal{O}_{E,S})$, where $\mathcal{O}_{E,S}$ denotes the ring of S-integers in E. From this description of Γ, it follows that Γ is residually finite, and, for all but finitely many primes $\mathfrak{p} \in \mathcal{O}_E$, admits an injective map $\phi_{\mathfrak{p}}\colon \Gamma \to \mathrm{SL}_2(\widehat{\mathcal{O}}_{E,\mathfrak{p}})$ (where $\widehat{\mathcal{O}}_{E,\mathfrak{p}}$ denotes the $\mathfrak{p}$-adic completion of the ring of integers of E). This construction provides a natural source of homomorphisms ϕ of the type considered in this paper.

Note that these maps are of quite a different nature to those considered in Example 5.1 (when the latter exist), since the latter cannot be injective. (A finite volume hyperbolic 3-manifold group cannot be free.)

Example 5.7. Let M be any finite volume hyperbolic 3-manifold. Let ϕ be the direct sum of $\phi_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ above p. The Zariski closure of the image is a group G of dimension at least 6. It follows that M admits a sequence of covers with large mod-p homology growth. Moreover, we see that any hyperbolic 3-manifold M admits a finite cover M' such that the fundamental group Γ' of M' admits a map to a pro-p analytic group G of dimension > 3. If the pro-p completion of Γ' were analytic and isomorphic to G, then $\widetilde{H}_1(\mathbb{F}_p)$ would vanish, contradicting Theorem 1.1. In particular, we see that if Γ is arithmetic, it cannot satisfy the congruence subgroup property. This was first proved for arithmetic lattices in $\mathrm{SL}_2(\mathbb{C})$ by Lubotzky [11].

Remark 5.8. The assumption that M be orientable has been made primarily for simplicity of exposition, and it should not be difficult to extend the proof of Theorem 1.1 to the non-orientable case. Indeed, any manifold is orientable mod 2, while if p is odd, then we may pass to the orientation double covers $\widetilde{M}_n$ of the M_n, each of which is equipped with an orientation reversing involution σ such that $M_n = \widetilde{M}_n/\sigma$, and work with the σ-fixed part of the $\mathbb{F}_p$-homology of $\widetilde{M}_n$ (which is canonically isomorphic to the $\mathbb{F}_p$-homology of M_n). The reader will easily verify that the arguments of the paper remain valid after restricting to σ-invariants. (The key point is that $H^0(\widetilde{M}_n, \mathbb{F}_p)$ is fixed by σ.)

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