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Groups, Geometry, and Dynamics

Subspace arrangements and property T

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To Fritz Grunewald on the occasion of his 60th birthday

Abstract. We reformulate and extend the geometric method for proving the Kazhdan property T developed by Dymara and Januszkiewicz and used by Ershov and Jaikin. The main result says that a group G generated by finite subgroups G_i has property T if the group generated by each pair of subgroups has property T and sufficiently large Kazhdan constant. Essentially, the same result was proven by Dymara and Januszkiewicz; however our bound for "sufficiently large" is significantly better.

As an application of this result, we give exact bounds for the Kazhdan constants and the spectral gaps of the random walks on any finite Coxeter group with respect to the standard generating set, which generalizes a result of Bacher and de la Harpe.

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1. Introduction

One of the aims of this paper is to explain the author's interpretation of the method for proving property T developed by Dymara and Januszkiewicz in [11]. This method reduces proving property T of a group G to "local representation theory" and geometry of configurations of subspaces of a Hilbert space. Here by "local representation theory" we mean studying the representations of (relatively) small subgroups in the group G. The second part of this method can be reduced to an optimization problem in some finite dimensional space, however in almost all cases the dimension is too big and this problem can not be approached directly. Instead, methods from linear algebra and graph theory are used (see [13], [14]). One unfortunate side effect is that the simple geometric idea behind this approach gets "hidden" in the computations.

The main new result in this paper is a solution of the resulting optimization problem in one (relatively easy) special case. In some sense, our solution is optimal, which

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allows us to obtain exact bounds for the Kazhdan constants and spectral gaps in several situations (in the case of any finite Coxeter group with respect to the standard generating set or the group $\mathrm{SO}(n)$ for some specific generating set). The majority of the results were known previously (see [2], [8]), however we were able to obtain them almost without using any representation theory. In particular, we do neither use the classification of the irreducible representations of symmetric groups nor any character estimates.

Let us recall the definition of the Kazhdan property T: a unit vector v in a unitary representation of a group G is called ε-almost invariant under a generating set S if $\|g(v) - v\| \leq \varepsilon$ for any $g \in S$. One way to construct almost invariant vectors is to take small perturbations of invariant vectors. A group G has Kazhdan property T if this is essentially the only way to construct almost invariant vectors. More precisely:

Definition 1.1. The Kazhdan constant, denoted by $\kappa(G, S)$, of a group G with respect to a generating set S is the largest ε such that the existence of an ε-almost invariant vector in a unitary representation implies the existence of an invariant vector. A finitely generated discrete group G is said to have the Kazhdan property T if for some (equivalently any) finite generating set S the Kazhdan constant is positive.

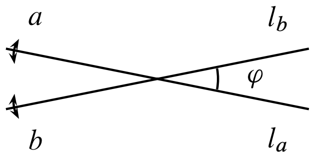
This definition can be extended to locally compact groups by replacing finite generating set with a compact generating set.

It follows almost immediately from the above definition that any finite group has property T. However, computing the Kazhdan constants even for finite groups is very difficult and there are only a few cases where exact values are known [2]. It is well known [4], [20] that many infinite groups also have property T; for example any lattice in a high rank Lie group has property T, typical examples are the groups $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{SL}_n(\mathbb{F}_p[t])$ for $n \geq 3$. Usually this is proved using the representation theory of the ambient Lie group [20], but this approach does not produce any bounds for the Kazhdan constants of this groups. In the last ten years several algebraic methods for proving property T have been developed [13], [18], [19], [23], [24]. One main advantage of these methods is that they provide explicit bounds for the Kazhdan constants of these groups, another is that these methods are applicable in a more general setting.

One of the "smallest" groups which does not have property T is the infinite dihedral group

$$D_\infty \simeq \langle a, b \mid a^2 = b^2 = 1 \rangle.$$

The failure of property T can be easily seen using 2-dimensional representations of D_∞. Let l_a and l_b be two different lines in the Euclidean plane $\mathbb{R}^2$. Such two lines define a representation of D_∞ on $\mathbb{R}^2$, where the generators act as reflection along these lines. Thus, the lines l_a and l_b are the fixed subspaces of the subgroups of D_∞ generated by a and b.



A quick computation shows that this representation does not contain any invariant vectors but contains an ε-almost invariant vector, where $\varepsilon = 2\sin\varphi/2$ (here φ denotes the angle between the lines l_a and l_b). Since the angle φ can be arbitrarily small, we see that for any $\varepsilon > 0$ there exists a representation of D_∞ with ε-almost invariant vectors, but without invariant vectors. In other words, the Kazhdan constant $\kappa(D_\infty, \{a, b\}) = 0$ and the group D_∞ does not have property T.

This example suggests that a group G, generated by two (finite) subgroups G_1 and G_2, has property T if and only if there exists a universal bound for the angle between the fixed spaces $\mathcal{H}^{G_1}$ and $\mathcal{H}^{G_2}$ for any (irreducible) representation $\mathcal{H}$ of G. For example, if we replace D_∞ with D_n by adding the relation $(ab)^n = 1$, then the reflections along l_a and l_b define representation of D_n if and only if $\phi = k\pi/n$, $k \in \mathbb{Z}$, which prevents the angle between $l_a = \mathcal{H}^{G_1}$ and $l_b = \mathcal{H}^{G_2}$ from being arbitrarily small. Of course, this example is not useful since the group D_n is finite, therefore it has property T.

Observation 2.1 allows us to generalize this situation to a group generated by several subgroups, which can be used to show that some groups have property T by solving a "geometric optimization" problem. Before explaining this reduction and stating the main result in this paper, we need to look at the angle between two subspaces from another viewpoint.

We say that the angle[1] between two closed subspaces V_1 and V_2 is larger than φ if, for any vectors $v_i \in V_i$ such that each v_i is perpendicular to the intersection $V_1 \cap V_2$, the angle between v_1 and v_2 is larger than φ. We need the condition $v_i \perp V_1 \cap V_2$ because we want to measure the angle between subspaces that have nontrivial intersection. The motivating example for this definition is the geometric angle between two planes in a 3-dimensional Euclidean space.

An equivalent way of saying this is that for any vector $v \in V$ there is a bound for the distance $d_0(v)$ from v to the intersection $V_1 \cap V_2$ in terms of the distances $d_i(v)$ from v to V_i:

$$d_0(v)^2 \leq \frac{1}{1-\cos\varphi}(d_1(v)^2 + d_2(v)^2).$$

Similar bounds can be used to define angle between many subspaces. We say that the angle $\sphericalangle(V_1, V_2, \ldots, V_n)$ between the subspaces $V_1, \ldots, V_n$ is larger than φ if for

[1] In the theory of Hilbert spaces this angle is known as Friedrichs angle [7].

any vector v the square of distance from v to the intersection $\bigcap V_i$ is bounded by a constant times the sum of the squares of the distances from v to each subspace, i.e.,

$$d(v, \bigcap V_i)^2 \le C_n(\varphi) \sum d(v, V_i)^2,$$

where $C_n(\varphi)$ is an explicitly defined function. (See Section 3 for a precise definition of the angle between several subspaces.) Our main result is a sufficient condition when the angle between many subspaces is positive:

Theorem 1.2. *Let V_i be n closed subspaces in a Hilbert space $\mathcal{H}$. Suppose that for any pair of indices $1 \le i, j \le n$ we have $\cos \sphericalangle(V_i, V_j) \le \varepsilon_{ij}$ and the symmetric matrix*

$$A = \begin{pmatrix} 1 & -\varepsilon_{12} & -\varepsilon_{13} & \dots & -\varepsilon_{1n} \\ -\varepsilon_{21} & 1 & -\varepsilon_{23} & \dots & -\varepsilon_{2n} \\ -\varepsilon_{31} & -\varepsilon_{32} & 1 & \dots & -\varepsilon_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\varepsilon_{n1} & -\varepsilon_{n2} & -\varepsilon_{n3} & \dots & 1 \end{pmatrix}$$

is positive definite. Then the angle $\sphericalangle(V_1, V_2, \dots, V_n) \ge \varphi > 0$, where the constant φ depends only on the matrix A.

Moreover, if the matrix A is not positive definite then there exist a Hilbert space $\mathcal{H}$ and closed subspaces V_i such that $\cos \sphericalangle(V_i, V_j) \le \varepsilon_{ij}$, but the angle $\sphericalangle(V_1, V_2, \dots, V_n)$ is equal to 0.

Weaker forms of Theorem 1.2 were previously known: Dymara and Januszkiewicz [11] proved an analogous statement if $\varepsilon_{ij} \le 12^{-n}$. This result was improved by Ershov and Jaikin [13] to $\varepsilon_{ij} \le (n-1)^{-1}$. Moreover, Ershov and Jaikin [13], Theorem 5.9, also proved an analog of Theorem 1.2 in the case $n = 3$.

The applications of Theorem 1.2 are based on Observation 2.1. A group G generated by (finite) subgroups G_i has property T if and only if for any unitary representation of G in $\mathcal{H}$ there is a bound for the angle between the subspaces $\mathcal{H}^{G_i}$, which does not depend on the representation $\mathcal{H}$. This allows one to prove that G has property T using only information from the representation theory of the groups generated by G_i and G_j. Moreover, a quantitative version of this result (Theorem 5.1) can be used to obtain good bounds for the Kazhdan constant and the spectral gap of the Laplacian; see Theorems 6.1, 6.12 and 6.14. It is remarkable that in some cases the resulting bounds are sharp.

Theorem 1.3. *Let G be a finite Coxeter group with a generating set S. The spectral gap and the Kazhdan constant $\kappa(G, S)$ of G can be computed by considering only the defining representation and are listed in Table 1. In particular, the spectral gap of the Laplacian is equal to $\frac{4}{n}\sin^2(\pi/2h)$, where $n = |S|$ and h denote the Coxeter number of the group G.*

This generalizes results by Bacher and de la Harpe [2] and Bagno [3] and is one of the few results that provide exact values for the Kazhdan constants of non-abelian finite groups.

Another application of this method is a simplified[2] proof of the following:

Theorem 1.4. *The group* $\mathrm{SL}_n(\mathbb{F}_p[t_1, \ldots, t_k])$ *has property T if* $p \geq 5$ *and* $n \geq 3$.

The condition $n \geq 3$ is necessary because the group $\mathrm{SL}_2(\mathbb{F}_p[t])$ does not have property T. On the other side the condition $p \geq 5$ is redundant – it is even possible to replace $\mathbb{F}_p$ with $\mathbb{Z}$. However, removing the condition $p \geq 5$ (and replacing $\mathbb{F}_p$ with $\mathbb{Z}$) requires significant additional work, see [13] and [14]. Theorem 1.4 also can be generalized to show that many Steinberg and Kac–Moody groups have Kazhdan property T.

The proof of Theorem 1.2 can also be used to obtain good bounds for the spectral gaps for some random walks on $\mathrm{SL}_n(\mathbb{F}_p)$, $\mathrm{SO}(n)$; see Theorems 6.12 and 6.14, which in turn can be used to estimate the relaxation and the mixing times of these random walks. Most of these results are only a slight improvement of previous results [8], [9], [18], however the previous proofs involve completely different methods and use "more complicated" representation theory (at least according to the author).

Notation. All the representations in this paper are assumed by unitary. Throughout the paper $\mathcal{H}$ will denote an arbitrary Hilbert space. As usual $\langle \cdot, \cdot \rangle$ will denote the scalar product in $\mathcal{H}$ and $\| \cdot \|$ will be the norm. We we use $\sphericalangle(v, w)$ to denote the angle between two nonzero vectors in $\mathcal{H}$. For a closed subspace V, by $V^{\perp}$ we will denote the orthogonal complement of V in $\mathcal{H}$ and by $P_V : \mathcal{H} \to \mathcal{H}$ the orthogonal projection on $\mathcal{H} \to V$. The notation $d_V(v)$ will be used for the distance between a vector $v \in \mathcal{H}$ and the subspace V, i.e., $d_V(v) = \|v - P_v(v)\|$. We almost never use that $\mathcal{H}$ is a vector space over the complex number, thus we often consider $\mathcal{H}$ only as a Euclidean space. This explains why most of the examples in the paper use finite dimensional Euclidean spaces (over $\mathbb{R}$); of course one can "lift" all these examples to Hilbert spaces by tensoring with $\mathbb{C}$.

Structure. In Section 2, we start with an observation, which connects property T and geometry and use it to outline an approach to prove property T for some groups. The notion of the angle between a collection of subspaces is defined Section 3, which also contains many properties of this notion. The following Section 4 contains (relatively easy) technical results about angles between three subspaces and their intersections. These results are used in Section 5 to prove Theorem 1.2. The final Section 6 describes several applications of Theorem 1.2.

[2] A similar proof in the case $p > (n-1)^2$ can be found in [13]. With a small modification one can extend that proof to the general case.

2. Strategy for proving property T

A key part of this geometric approach to property T is the following observation, which allows us to relate property T to geometry.

Observation 2.1. *Let G be a group and let G_i be a collection of n subgroups in G such that $G = \langle G_1, \dots, G_n \rangle$. Then the Kazhdan constant $\kappa(G, \bigcup G_i)$ is strictly positive if and only if there exists $\alpha > 0$ such that $\sphericalangle(\mathcal{H}^{G_1}, \mathcal{H}^{G_2}, \dots, \mathcal{H}^{G_n}) > \alpha$ for any unitary representation $\mathcal{H}$ of G. In particular, if all G_i are finite subgroups and there exists lower bound for the angle, then G has Kazhdan property T.*

Proof. Suppose that $\sphericalangle(\mathcal{H}^{G_1}, \mathcal{H}^{G_2}, \dots, \mathcal{H}^{G_n}) > \alpha$ for any representation $\mathcal{H}$ of G. Let $\mathcal{H}$ be arbitrary unitary representation of G and let v be a unit vector in $\mathcal{H}$ which is ε-almost invariant with respect to $\bigcup G_i$, for some sufficiently small ε. Since each G_i is a subgroup, the almost invariance under G_i implies that the distance from v to the subspace $\mathcal{H}^{G_i}$ is less than ε. The bound for the angle between the subspaces $\mathcal{H}^{G_i}$ implies that

$$d_{\cap \mathcal{H}^{G_i}}(v)^2 \leq C \sum d_{\mathcal{H}^{G_i}}(v)^2 \leq Cn\varepsilon^2,$$

for some constant C, which depends only on α but not on the representation $\mathcal{H}$. If ε is smaller than $(Cn)^{-1/2}$ then the distance between v and the space of G-invariant vectors $\mathcal{H}^G = \bigcap \mathcal{H}^{G_i}$ is less than 1, therefore there exist nonzero vectors in $\mathcal{H}^G$. This shows that if a representation $\mathcal{H}$ has an $\frac{1}{2\sqrt{Cn}}$-almost invariant vector then $\mathcal{H}$ has an invariant vector, which is equivalent to $\kappa(G, \bigcup G_i) \geq (Cn)^{-1/2}/2 > 0$.

The other direction is similar. Suppose that there is no nontrivial lower bound for the angle $\sphericalangle(\mathcal{H}^{G_1}, \mathcal{H}^{G_2}, \dots, \mathcal{H}^{G_n})$, then for any C there exists a representation $\mathcal{H}$ and a vector v such that

$$d_{\cap \mathcal{H}^{G_i}}(v)^2 \geq C \sum d_{\mathcal{H}^{G_i}}(v)^2 > 0.$$

However, the image $\bar{v} = v / d_{\cap \mathcal{H}^{G_i}}(v) + \mathcal{H}^G$ of the vector $v / d_{\cap \mathcal{H}^{G_i}}(v)$ in $\mathcal{H}/\mathcal{H}^G$ is a unit vector which is a distance less than $C^{-1/2}$ to each of the subspaces $\mathcal{H}^{G_i}/\mathcal{H}^G$. Therefore $\bar{v}$ is $2C^{-1/2}$-almost invariant with respect to $\bigcup G_i$. However, if C is large enough this contradicts with the condition $\kappa(G, \bigcup G_i) > 0$. □

The following outline[3] shows one possible way to apply the above observation and use it to prove that some groups G generated by subgroups G_i have property T, more precisely that the Kazhdan constant $\kappa(G, \bigcup G_i)$ is positive.

Briefly the idea is first to extract enough information (steps 1 and 2) from the "local representation theory" of the group G and translate this information to bounds

[3]This idea goes back to [11] and may be even further to [5], [6], [15]. However, these papers refer to "unnecessary geometric objects", at least according to the author, which makes these ideas difficult to "extract".

on the angles between some of subspaces $\mathcal{H}^{G_i}$. Then, one applies "geometric" arguments (step 3) to show that these conditions imply a bound for the angle between all subspaces $\mathcal{H}^{G_i}$. Although the last step is "geometric" in most cases the proof has an algebraic flavor and heavily uses linear algebra.

The first step in the approach is to study "local representation theory" of the group G, i.e., one can consider the subgroups $G_J = \langle \bigcup_{j \in J} G_j \rangle \subset G$ for some subsets $J \subset \{1, \dots, n\}$. If these groups have property T and there exist good bounds for the Kazhdan constants $\kappa(G_J, \bigcup_{j \in J} G_j)$, then one can apply Observation 2.1 and translate these into bounds for the angles $\sphericalangle\{\mathcal{H}^{G_j} \mid j \in J\}$.

The second step (which is optional, but essential for some applications [13], [14]) is to study the inclusions between the subgroup G_J and translate them into conditions about the intersections of the subspaces $\mathcal{H}^{G_j}$. For example, the inclusion $G_3 \subset \langle G_1, G_2 \rangle$ leads to the condition $\mathcal{H}^{G_1} \cap \mathcal{H}^{G_2} \subset \mathcal{H}^{G_3}$.

The third and final step is to consider all possible configurations of subspaces V_i in some Hilbert space $\mathcal{H}$, which satisfy all the conditions found in the first two steps. If one can prove that $\sphericalangle(V_1, V_2, \dots, V_n) \geq \alpha$ for any "allowed" configuration, then we will get that $\sphericalangle(\mathcal{H}^{G_1}, \mathcal{H}^{G_2}, \dots, \mathcal{H}^{G_n}) \geq \alpha$ for any representation $\mathcal{H}$, which by Observation 2.1 implies that the group G has a variant of property T. In most cases this geometric problem is best attacked using tools from linear algebra (and some times graph theory).

Although the third step refers to subspaces in an arbitrary Hilbert space, it is possible to reduce it to a question about subspaces in a finite dimensional Euclidean space: the angle $\sphericalangle(V_1, V_2, \dots, V_n)$ is determined using the distances between a vector v and the subspaces V_i (of course one needs to take a supremum over all vectors v). However, since we work with only one vector at a time, without loss of generality we can assume that the Hilbert space $\mathcal{H}$ is spanned by the projections v_J of v onto the subspaces $\bigcap_{j \in J} V_j$ for all subsets $J \subset \{1, \dots, n\}$, i.e., we can assume that $\dim \mathcal{H} \leq 2^n$.

Therefore, it is sufficient to consider all possible configurations of subspaces V_i in a 2^n-dimensional Euclidean space satisfying the conditions found in the first two steps. Thus, the last step can be reduced to an optimization problem on some finite dimensional space. Unfortunately, even for a small n, it is very difficult to formulate this optimization problem and solve it directly; for example, proving Corollary 4.5 using this idea will involve considering configurations of three 3-dimensional subspaces in a 6-dimensional Euclidean space. The resulting optimization problem will involve optimizing function defined on a subset of 7×7 semi-positive definite symmetric matrices, satisfying certain conditions. In short, the author does not expect this reduction to be used in practice (unless it can be implemented on a computer).

As we have already mentioned this approach for proving property T is not new and there are several examples in the literature, where a similar program has been carried out:

Dymara and Januszkiewicz [11] essentially proved that $\sphericalangle(V_1, V_2, \ldots, V_n) > \alpha > 0$ if $\cos \sphericalangle(V_i, V_j) < \varepsilon$ for any $1 \le i, j \le n$ and a sufficiently small ε, i.e., if the subspaces V_i are pairwise almost perpendicular. They combined this result with bounds coming from the representation theory of rank 2 groups over finite fields to prove that 2-spherical Kac–Moody groups have property T if the defining field is finite and sufficiently large.

Ershov and Jaikin [13] proved a spectral criterion for property T for groups having a decomposition as graph of groups. This criterion can be translated into the language outlined above by considering the fixed subspaces of all vertex and edge groups. The bounds for the co-distances at each vertex are equivalent to bounds for the angles between the edge spaces. Also, the graph of groups decomposition imposes restrictions between the intersections of these subspaces.

Ershov and Jaikin also applied this spectral criterion to improve the result of Dymara and Januszkiewicz mentioned above. They proved that if $\cos \sphericalangle(V_i, V_j) < \frac{1}{n-1}$ for any $1 \le i, j \le n$ then $\sphericalangle(V_1, V_2, \ldots, V_n) > \alpha > 0$ and obtained a precise result in the case $n = 3$, which is used to prove a variant of Theorem 1.4. One of the main results in this paper, Theorem 1.2, improves that result and provides some geometric interpretation.

The main result in [13], Theorem 5.5, gives bounds for the Kazhdan constant for groups "graded by root systems of type A_2" with respect to the union of their root subgroups. Its proof again "follows" the general outline described above, but this is not easily seen since the conditions found in the first two steps are complicated and can not be easily translated in a geometric language. Instead the proof is written in an "algebraic" language and heavily uses linear algebra and graph theory. This result is generalized in [14] for groups graded by arbitrary root systems.

3. Angle between subspaces

In this section we define the angle between two (and several) closed subspaces of a Hilbert space. We start with a geometric definition, which is motivated by the standard notion of angles between lines and planes in a 3-dimensional Euclidean space. Then, we find an equivalent definition using the spectrum of certain operators, which later will be used to define "angle" between several subspaces.[4]

3.1. Geometric definition. We start with the usual definition of an angle between a vector and a closed subspace:

Definition 3.1. The angle $\sphericalangle(v, V)$ between a closed subspace V and a nonzero vector $v \in \mathcal{H}$ is defined to be the angle between v and its projection onto V (or $\pi/2$ if the

[4] It is not clear what is the precise geometric meaning of the angle between several subspaces. Our definition is closely related to the notion of a co-distance used in [13], see Remark 3.22.

projection is zero). Equivalently one can use

$$\|P_V(v)\| = \|v\| \cos \sphericalangle(v, V).$$

It is clear that $\sphericalangle(v, V) = 0$ if and only if $v \in V$. Notice that if we fix the subspace V the function $v \to \sphericalangle(v, V)$ is a continuous function defined on $\mathcal{H} \setminus \{0\}$.

This definition can be extended to angles between two subspaces. There are several natural ways, known as Friedrichs and Dixmier angles [7], to do that, which are equivalent if the two subspaces have trivial intersection. Our approach is to "ignore" the intersection by factoring it out, or equivalently by considering only the orthogonal complement to the intersection. The definition used in this paper differs from the similar one used in [11], [13].

Definition 3.2. Let V_1 and V_2 be two closed subspaces in a Hilbert space $\mathcal{H}$. If neither of the spaces V_i is contained in the other one, then the (Friedrichs) angle between V_1 and V_2 (denoted by $\sphericalangle(V_1, V_2)$) is defined to be the infimum of the angles between nonzero vectors v_1 and v_2, where $v_i \in V_i$ and $v_i \perp (V_1 \cap V_2)$, i.e.,

$$\cos \sphericalangle(V_1, V_2) = \sup\{|\langle v_1, v_2\rangle| \mid \|v_i\| = 1, v_i \in V_i, v_i \perp (V_1 \cap V_2)\}.$$

Alternatively, one can define the angle as infimum of the angles between vector and a subspace:

Lemma 3.3. *The angle between V_1 and V_2 is equal to the infimum of the angles between V_2 and non-zero (or unit) vector in V_1, which are perpendicular to the intersection, i.e.,*

$$\sphericalangle(V_1, V_2) = \inf\{\sphericalangle(v, V_2) \mid v \in V_1, v \perp (V_1 \cap V_2)\}.$$

Proof. This follows from the observation that if $v \perp (V_1 \cap V_2)$ and $v \in V_1$ then $P_{V_2}(v) \perp (V_1 \cap V_2)$. $\square$

Remark 3.4. We need the condition that neither of V_i is a subset of the other one, because if $V_2 \subset V_1$ there are no vectors in V_2 which are perpendicular to the intersection of V_1 and V_2. However, using Lemma 3.3 one can see that the angle "should be equal" to $\pi/2$ in the case when $V_2 \subset V_1$ and $V_2 \neq V_1$.

Corollary 3.5. *Let v_1 be a nonzero vector in V which is perpendicular to the intersection of $V_1 \cap V_2$. Then*

a) $\|P_{V_2}(v_1)\| \leq \|v_1\| \cos \sphericalangle(V_1, V_2)$;

b) $\|P_{V_2^{\perp}}(v_1)\| \geq \|v_1\| \sin \sphericalangle(V_1, V_2)$;
c) $\|P_{V_2}(v_1)\| \leq \|P_{V_2^{\perp}}(v_1)\| \cot \sphericalangle(V_1, V_2)$.

The next result is well known consequence of the compactness of unit sphere in a finite dimensional Hilbert space.

Lemma 3.6. *If one of the subspaces V_1 and V_2 is finite dimensional then there exist vectors $v_1, v_2 \perp (V_1 \cap V_2)$, $v_i \in V_i$, such that $\sphericalangle(V_1, V_2) = \sphericalangle(v_1, v_2) > 0$.*

Proof. Assume that V_1 is finite dimensional. By Lemma 3.3 the angle between V_1 and V_2 is equal to the infimum of the function $\theta\colon v \to \sphericalangle(v, V_2)$ defined on the unit sphere in $V_1 \cap (V_1 \cap V_2)^{\perp}$. The function θ is a continuous function with compact domain therefore the infimum is achieved. □

Corollary 3.7. *If at least one of the subspaces V_i is finite dimensional, then the angle $\sphericalangle(V_1, V_2)$ is positive. It is not difficult to construct examples of two infinite dimensional closed subspaces V_i with trivial intersection such that $\sphericalangle(V_1, V_2) = 0$. It is easy to see that if $\sphericalangle(V_1, V_2) > 0$ then $V_1 + V_2$ is a closed subspace.*

Remark 3.8. If the dimensions of V_1, V_2 and $\mathcal{H}$ are finite and fixed, then the angle between the subspaces V_1 and V_2 can be considered as a function defined on the subset of the product of two Grassmannians. It is important to notice that this function is NOT continuous. The dimension of the intersection divides the domain into cells and each cell is open in it closure. The restriction of the angle to each cell is a continuous function, which tends to zero as one approaches the boundary of each cell.

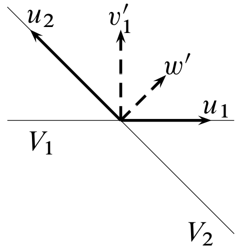
The following lemma plays in central role in the rest of the paper; roughly speaking it allows to exchange all subspace with their orthogonal complements. As a consequence, we can replace intersections of subspaces with their sums.

Lemma 3.9. *The angle between the orthogonal complements $V_1^{\perp}$ and $V_2^{\perp}$ is equal to the angle between the subspaces V_1 and V_2.*

Proof. Assume that $\sphericalangle(V_1, V_2) > 0$. Let v_1' is a non-zero vector in $V_1^{\perp}$ which is perpendicular to $V_1^{\perp} \cap V_2^{\perp}$. Since $\left(V_1^{\perp} \cap V_2^{\perp}\right)^{\perp} = \overline{V_1 + V_2} = V_1 + V_2$ we have $v_1' \in V_1 + V_2$, i.e., one can write $v_1' = u_1 + u_2$, where $u_i \in V_i$. Without loss of generality, we can assume that $u_2 \perp V_1 \cap V_2$. Let w' be the unique vector of the form $v_1' + \lambda u_2$ which is perpendicular to u_2.

Claim 3.10. We have the inequality

$$\|w'\| \leq \|v_1'\| \cos \sphericalangle(V_1, V_2).$$



Proof. The angle between v_1' and u_2 is not more than

$$\pi/2 - \sphericalangle(u_2, V_1) \geq \pi/2 - \sphericalangle(V_1, V_2).$$

Therefore, the angle between w' and v_1' is at least $\sphericalangle(V_1, V_2)$. □

Notice that by the construction of u_2 and w' we have $P_{V_2^\perp}(v_1') = P_{V_2^\perp}(w')$. Therefore,

$$\|P_{V_2^\perp}(v_1')\| \leq \|w'\| \leq \|v_1'\| \cos \sphericalangle(V_1, V_2),$$

i.e., the angle between v' and the subspaces $V_2^\perp$ is at least $\sphericalangle(V_1, V_2)$. This shows that $\sphericalangle(V_1^\perp, V_2^\perp) \geq \sphericalangle(V_1, V_2)$. The opposite inequality follows from the symmetry, thus we have that $\sphericalangle(V_1^\perp, V_2^\perp) = \sphericalangle(V_1, V_2)$. □

Remark 3.11. This lemma (and its proof) is essentially the same as [13], Lemma 2.4. The statement is somehow cleaner because we do not need to check whether the intersections of $V_1 \cap V_2$ and $V_1^\perp \cap V_2^\perp$ are trivial or not. Exactly the same result can be found in [7].

3.2. Spectral definition. Before generalizing the notion of angle to several subspaces, we need to "replace" the geometric definition with an "algebraic" one.

Consider the addition operator

$$\mathrm{sum}\colon V_1 \oplus V_2 \to \mathcal{H}, \quad \mathrm{sum}(v_1, v_2) = v_1 + v_2,$$

and its "square" $\Sigma = \mathrm{sum}^* \circ \mathrm{sum}$, where $\mathrm{sum}^*\colon \mathcal{H} \to V_1 \oplus V_2$ denotes the transpose of the operator sum. It is easy to see that

$$\mathrm{sum}^*(v) = (P_{V_1}(v), P_{V_2}(v)) \quad \text{and} \quad \Sigma(v_1, v_2) = (P_{V_1}(v_1 + v_2), P_{V_2}(v_1 + v_2)).$$

Here $V_1 \oplus V_2$ denotes the external direct sum of V_1 and V_2, which is naturally a Hilbert space.

By definition, Σ is a positive self adjoint operator. The triangle inequality implies that the norm of Σ is bounded above by 2, and the point 2 appears in the discrete spectrum of Σ if and only if the subspaces V_1 and V_2 have nontrivial intersection.

Lemma 3.12. *The spectrum of Σ is invariant under reflection at 1, i.e., $\lambda \in \mathrm{Spec}(\Sigma)$ if and only if $2 - \lambda \in \mathrm{Spec}(\Sigma)$.*

Proof. A direct computation shows that if (v_1, v_2) is an eigenvector of Σ with eigenvalue λ then $(v_1, -v_2)$ is also an eigenvector, but with eigenvalue $2-\lambda$. This shows that the discrete part of the spectrum of Σ is invariant under the reflection. Using "approximate eigenvectors" one can obtain a similar result for the continuous part of the spectrum. □

Lemma 3.13. *The spectrum of* Σ *is contained in*

$$[1-\cos\sphericalangle(V_1,V_2), 1+\cos\sphericalangle(V_1,V_2)] \cup \{0,2\}.$$

Moreover the points $1 \pm \cos\sphericalangle(V_1,V_2)$ *are in the spectrum.*

Proof. The eigenspace W_2 corresponding to the eigenvalue 2 consists of all vectors (v,v) for $v \in V_1 \cap V_2$, similarly the eigenspace W_0 corresponding to 0 consists of all vectors $(v,-v)$ for $v \in V_1 \cap V_2$. Let $w = (v_1, v_2) \in V_1 \oplus V_2$ be a vector perpendicular to $W_0 \oplus W_2$, i.e., $v_1, v_2 \perp V_1 \cap V_2$. Then

$$\begin{aligned}\langle \Sigma(w), w\rangle &= \|v_1+v_2\|^2 = \|v_1\|^2 + \|v_2\|^2 + 2\|v_1\|\cdot\|v_2\|\cos(\phi)\\ &\le (1+|\cos(\phi)|)(\|v_1\|^2+\|v_2\|^2) - |\cos(\phi)|(\|v_1\|-\|v_2\|)^2\\ &\le (1+|\cos(\phi)|)\|w\|^2,\end{aligned}$$

where ϕ denotes the angle between the vectors v_1 and v_2. Thus, on $(W_0+W_2)^\perp$ the operator Σ is bounded by $1+\cos\sphericalangle(V_1,V_2)$, i.e., the spectrum of Σ is contained in $[0, 1+\cos\sphericalangle(V_1,V_2))] \cup \{2\}$. By Lemma 3.12 the interval $(0, 1-\cos\sphericalangle(V_1,V_2))$ is also not part of the spectrum.

Let $v_{1,i}$ and $v_{2,i}$ be sequences of unit vectors, in V_1 and V_2 respectively, perpendicular to $V_1 \cap V_2$ such that $\sphericalangle(v_{1,i}, v_{2,i}) \to \sphericalangle(V_1,V_2)$. By the above computation we see that

$$\langle \Sigma(w_i), w_i\rangle \to (1+\cos\sphericalangle(V_1,V_2))\|w_i\|^2,$$

where $w_i = (v_{1,i}, v_{2,i})$. Thus, the point $1+\cos\sphericalangle(V_1,V_2)$ is in $\mathrm{Spec}(\Sigma)$. □

Remark 3.14. The above two lemmas, together with the observation that for any two closed subspaces V_1 and V_2 and any $v \in \mathcal{H}$ one has the equality

$$\|\operatorname{sum}^*_{V_1,V_2}(v)\|^2 + \|\operatorname{sum}^*_{V_1^\perp,V_2^\perp}(v)\|^2 = 2\|v\|^2,$$

give an alternative proof of Lemma 3.9.

Lemma 3.13 allows us to define the angle between V_1 and V_2 using the spectral gap of the operator Σ:

Definition 3.15. The angle $\sphericalangle(V_1,V_2) \in [0,\pi]$ between two subspaces V_1 and V_2 is defined by

$$1+\cos\sphericalangle(V_1,V_2) = \sup\{\mathrm{Spec}(\Sigma) \setminus \{2\}\}.$$

Remark 3.16. This definition does not require that neither of the subspaces is contained in the other one and allows us to define the angle between two subspaces in these degenerate cases:

a) if $V_1 \subset V_2$ but $V_1 \neq V_2$ then the spectrum of Σ consists 0, 1 and 2, therefore $\sphericalangle(V_1, V_2) = \pi/2$. Notice that this agrees with Lemma 3.3;
b) if $V_1 = V_2$ then the spectrum of Σ consists 0 and 2, therefore $\sphericalangle(V_1, V_2) = \pi$.

Remark 3.17. The following example shows why the angle is not a continuous function on the product of the Grassmaniann s: Let us consider the angle between a plane and a line in a 3-dimensional space. If the line and the plane are in general position then the angle is equal to the "geometrically defined one", i.e., the angle between the line and its projection onto the plane. However, if the line lies in the plane the angle is equal to $\pi/2$.

In terms of the spectrum of Σ we have: if the line is not inside the plane then the spectrum of Σ is $\{1 - \cos\phi, 1, 1 + \cos\phi\}$, when ϕ goes to 0 the spectrum becomes $\{0, 1, 2\}$ and the spectral gap near 2 suddenly increases from $1 - \cos\phi$ to 1.

Remark 3.18. Using the spectral definition of the angle between two subspaces it is very easy to see why the angle $\sphericalangle(V_1, V_2)$ is always positive if both V_1 and V_2 are finite dimensional. In this case the domain of Σ is a finite dimensional vector space, thus, $\mathrm{Spec}(\Sigma)$ is a finite subset of $[0, 2]$, therefore it does not contain the interval $(2 - \varepsilon, 2)$ for some $\varepsilon > 0$.

Remark 3.19. Another way to define the angle between V_1 and V_2 is to use the spectrum of the operator $\Sigma' = P^* \circ P$ where $P : V_1 \to V_2$ is the orthogonal projection from V_1 to V_2. In the non-degenerate cases we have

$$\cos \sphericalangle(V_1, V_2) = \sup\{\mathrm{Spec}(\Sigma') \setminus \{1\}\}.$$

The only reason we chose to use the operator Σ instead of Σ' is to preserve the symmetry between the two subspaces.

One minor difference is that if one uses the operator Σ' to define the angle, the "natural" extension of angle to the case $V_1 = V_2$ will give a different answer $\sphericalangle(V_1, V_2) = \pi/2$. In this paper, we will not deal with this degenerate case, so it does not matter how one defines the angle if the two subspaces are equal.

3.3. Angle between several subspaces. An analog of Definition 3.15 can be used to define an "angle" between a collection of several subspaces. It is not clear what the exact geometric meaning of the angle defined below is; the definition is equivalent to the one described in the Introduction, see Remark 3.25. The same notion is studied in [1], where the authors define several ways to measure the "angle" between several subspaces. Our definition of angle is the same as the *Friedrichs number* used in [1].

Definition 3.20. Let $V_1, V_2, \ldots, V_n$ be closed subspaces of a Hilbert space $\mathcal{H}$. Let

$$\mathrm{sum}\colon V_1 \oplus V_2 \oplus \cdots \oplus V_n \to \mathcal{H}, \quad \mathrm{sum}(v_1, v_2, \ldots, v_n) = v_1 + v_2 + \cdots + v_n,$$

denote the addition operator. The "angle" $\sphericalangle(V_1, V_2, \ldots, V_n)$ between the subspaces $\{V_i\}$ is defined by

$$1 + (n-1)\cos\sphericalangle(V_1, V_2, \ldots, V_n) = \sup\{\mathrm{Spec}(\Sigma) \setminus \{n\}\},$$

where $\Sigma = \mathrm{sum}^* \circ \mathrm{sum}$. The reason of this unusual normalization is to make the angle lying between 0 and $\pi/2$, unless all V_i are equal to each other.

Remark 3.21. The triangle inequality shows that $\mathrm{Spec}(\Sigma) \subset [0, n]$. The point n is in the discrete part of $\mathrm{Spec}(\Sigma)$ if and only if the intersection $\bigcap V_i$ is not trivial. Similarly, 0 is in the spectrum if and only if there exist vectors $v_i \in V_i$ which are linearly dependant. It can be shown that unless all V_i are the same, $\mathrm{Spec}(\Sigma)$ contains points in the interval $[1, n)$, i.e., $\sphericalangle(V_1, V_2, \ldots, V_n) \leq \pi/2$.

Remark 3.22. Our definition of angle is closely related to the notion of co-distance used in [13]. If the intersection $\bigcap V_i$ is trivial, one has

$$\cos\sphericalangle(V_1, V_2, \ldots, V_n) = \frac{n\rho(V_1, \ldots, V_n) - 1}{n-1},$$

where $\rho(V_1, \ldots, V_n)$ denotes the co-distance between V_i as defined in [13].

Example 3.23. If the spaces V_i are pairwise orthogonal and have pairwise trivial intersections, for example if the V_i are the coordinate lines in n-dimensional Euclidean space, then $\sphericalangle(V_1, V_2, \ldots, V_n) = \pi/2$.

If the orthogonal complements of the V_i are pairwise orthogonal and have pairwise trivial intersections, for example if the V_i are the coordinate hyperplanes in n-dimensional Euclidean space, then $\cos\sphericalangle(V_1, V_2, \ldots, V_n) = 1 - \frac{1}{n-1}$.

This example shows that the analog of Lemma 3.9 does not hold for $n > 2$, i.e., it is not true in general that $\sphericalangle(V_1, V_2, \ldots, V_n) = \sphericalangle(V_1^\perp, V_2^\perp, \ldots, V_n^\perp)$.

3.4. Distance estimates. In this section we will assume that V_1 and V_2 are two closed subspaces in $\mathcal{H}$ such that $\sphericalangle(V_1, V_2) > 0$. As mentioned before, this condition implies that the subspace $V_1 + V_2$ is closed.

For a vector $w \in \mathcal{H}$ with w_0, w_1, w_2 and w_{12} we will denote the projections of w onto the subspaces $V_1 \cap V_2$, V_1, V_2 and $V_1 + V_2$, respectively. Similarly, with $d_0(w)$, $d_1(w)$, $d_2(w)$, $d_{12}(w)$ we will denote the distances of w to these four subspaces.

Lemma 3.24. *The distances $d_i(w)$ satisfy the inequality*

$$d_0(w)^2 \leq \frac{1}{1 - \cos\sphericalangle(V_1, V_2)}(d_1(w)^2 + d_2(w)^2).$$

Moreover, if there exists a constant $K > 0$ *such that* $d_0(w)^2 \leq K(d_1(w)^2 + d_2(w)^2)$ *for any vector* $w \in \mathcal{H}$*, then*

$$\cos \sphericalangle(V_1, V_2) \leq 1 - K^{-1}.$$

Proof. The vector $w' = w - w_0$ is perpendicular to the intersection $V_1 \cap V_2$ which is the eigenspace of $\mathrm{sum} \circ \mathrm{sum}^*$ corresponding to the eigenvalue 2. Since the spectra of the operators $\mathrm{sum} \circ \mathrm{sum}^*$ and $\mathrm{sum}^* \circ \mathrm{sum} = \Sigma$ are almost the same (they might differ only at 0) and w' is perpendicular to the eigenspace corresponding to 2, we have

$$\langle w', \mathrm{sum} \circ \mathrm{sum}^*(w') \rangle \leq (1 + \cos \sphericalangle(V_1, V_2)) \|w'\|^2.$$

The left-hand side is equal to

$$\begin{aligned}
\| \mathrm{sum}^*(w') \|^2 &= \|(P_{V_1}(w'), P_{V_2}(w'))\| \| P_{V_1}(w')\|^2 + \|P_{V_2}(w')\|^2 \\
&= 2\|w'\|^2 - (\|w'\|^2 - \|P_{V_1}(w')\|^2) - (\|w'\|^2 - \|P_{V_2}(w')\|^2) \\
&= 2d_0(w)^2 - d_1(w)^2 - d_2(w)^2.
\end{aligned}$$

Therefore, $(1 - \cos \sphericalangle(V_1, V_2)) d_0(w)^2 \leq d_1(w)^2 + d_2(w)^2$.

The second part follows from the observation that for any $\varepsilon > 0$ there exists a vector $w' \perp (V_1 \cap V_2)$ such that $\|\mathrm{sum}^*(w)\|^2 > (1 + \cos \sphericalangle(V_1, V_2) - \varepsilon)\|w\|^2$ and by the above computation for such vector one has

$$(1 - \cos \sphericalangle(V_1, V_2) - \varepsilon) d_0(w)^2 \geq d_1(w)^2 + d_2(w)^2. \qquad \square$$

Remark 3.25. The previous proof can be generalized to the case of n subspaces. Let $d_i(w)$ denote the distance between w and V_i, and $d_0(w)$ the distance between w and $\bigcap V_i$. Then we have

$$d_0(w)^2 \leq \frac{1}{(n-1)(1 - \cos \sphericalangle(V_1, V_2, \ldots, V_n))} \sum d_i(w)^2.$$

Similarly, any bound of the form $d_0(w)^2 \leq \frac{1}{(n-1)\varepsilon} \sum d_i(w)^2$ valid for all vectors $w \in \mathcal{H}$ implies that $\cos \sphericalangle(V_1, V_2, \ldots, V_n) \leq 1 - \varepsilon$.

This explains why the definition of the angle give in the introduction is equivalent to the Definition 3.20, see [1] for a detailed proof.

One can obtain a slightly more precise estimate than in Lemma 3.24 using the following lemma:

Lemma 3.26. *The distance* $\|w_1 - w_0\|$ *between the projection of* w *onto* V_1 *and the intersection* $V_1 \cap V_2$ *is bounded by*

$$\|w_1 - w_0\| \leq \frac{\cos \sphericalangle(V_1, V_2) d_1(w) + d_2(w)}{\sin \sphericalangle(V_1, V_2)}.$$

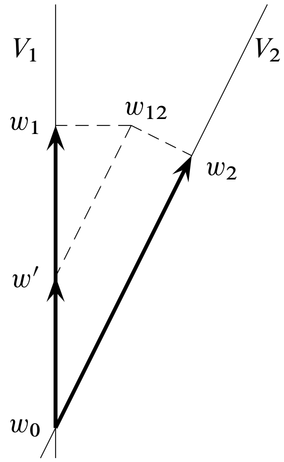
Proof. Let w' be the vector in V_1 such that $w_{12} - w'$ is in V_2 and is perpendicular to $V_1 \cap V_2$. Then $w_1 - w' = P_{V_1}(w_{12} - w')$ and $w_{12} - w_1 = P_{V_1^\perp}(w_{12} - w')$, thus by Corollary 3.5 we have

$$\|w_1 - w'\| \leq \cot \sphericalangle(V_1, V_2)\|w_{12} - w_1\|.$$

Similarly $w_{12} - w_2 = P_{V_2}(w' - w_0)$, i.e.,

$$\|w' - w_0\| \leq \frac{1}{\sin \sphericalangle(V_1, V_2)}\|w_{12} - w_2\|.$$

Therefore,



$$\begin{aligned}
\|w_1 - w_0\| &\leq \|w_1 - w'\| + \|w' - w_0\| \\
&\leq \cot \sphericalangle(V_1, V_2)\|w_{12} - w_1\| + \frac{1}{\sin \sphericalangle(V_1, V_2)}\|w_{12} - w_2\| \\
&\leq \cot \sphericalangle(V_1, V_2) d_1(w) + \frac{1}{\sin \sphericalangle(V_1, V_2)} d_2(w).
\end{aligned}$$
□

The following improvement of Lemma 3.24 is a special case of Theorem 5.1:

Lemma 3.27. *Let* $\varepsilon = \cos \sphericalangle(V_1, V_2) < 1$. *Then*

a) $d_0(w)^2 \leq \begin{pmatrix} d_1(w) & d_2(w) \end{pmatrix} \begin{pmatrix} 1 & -\varepsilon \\ -\varepsilon & 1 \end{pmatrix}^{-1} \begin{pmatrix} d_1(w) \\ d_2(w) \end{pmatrix}$;

b) $\|w_{12}\|^2 \leq \begin{pmatrix} \|w_1\| & \|w_2\| \end{pmatrix} \begin{pmatrix} 1 & -\varepsilon \\ -\varepsilon & 1 \end{pmatrix}^{-1} \begin{pmatrix} \|w_1\| \\ \|w_2\| \end{pmatrix}$.

Proof. a) We have $d_0(w)^2 = \|w_1 - w_0\|^2 + d_1(w)^2$. Lemma 3.26 gives us a bound for $\|w_1 - w_0\|^2$,

$$\begin{aligned}
d_0(w)^2 &\leq \left(\frac{\cos \sphericalangle(V_1, V_2) d_1(w) + d_2(w)}{\sin \sphericalangle(V_1, V_2)}\right)^2 + d_1(w)^2 \\
&= \frac{1}{\sin^2 \sphericalangle(V_1, V_2)} \left(d_1(w)^2 + 2 d_1(w) d_2(w) \cos \sphericalangle(V_1, V_2) + d_2(w)^2\right),
\end{aligned}$$

which is equal to the bound in the statement of the lemma.

b) Follows from part a) applied to the subspaces $V_1^\perp$ and $V_2^\perp$. □

Remark 3.28. Part b) of the previous lemma implies that the subspace $V_1 + V_2$ is closed if $\sphericalangle(V_1, V_2) > 0$. The analog of this statement is not true for more than two subspaces: there exist closed subspaces V_i such that $\sphericalangle(V_1, V_2, \ldots, V_n) > 0$, but $V_1 +$

$V_2+\cdots+V_n$ is not closed. However, Remark 3.25 implies that if $\sphericalangle(V_1, V_2, \ldots, V_n) > 0$, then $V_1^{\perp} + V_2^{\perp} + \cdots + V_n^{\perp}$ is closed and is equal to $(\bigcap V_i)^{\perp}$.

Remark 3.29. The bounds in Lemma 3.26 and Corollary 3.27 do not make sense when $V_1 = V_2$ and $\sphericalangle(V_1, V_2) = \pi$, because the matrix $\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)$ is not invertible. However, these bounds are still valid if one resolves the undefined fraction by taking the limit $\sphericalangle(V_1, V_2) \to \pi$. The resulting bounds

$$\|w_1 - w_0\| = 0, \quad d_0(w)^2 = d_1(w)^2 = d_2(w)^2, \quad \|w_{12}\|^2 = \|w_1\|^2 = \|w_2\|^2$$

hold by trivial geometric arguments.

4. Three subspaces

In this section we study the configuration of three subspaces in a Hilbert spaces $\mathcal{H}$. Corollary 4.5 gives a bound for the angle between the three subspaces in terms of the angles between each pair. This result is a special case of Theorem 1.2.

4.1. Bounds for the angles. Let V_i be three closed subspaces in $\mathcal{H}$ such that $0 < \alpha_{ij} \leq \sphericalangle(V_i, V_j)$ and $\varepsilon_{ij} = \cos\alpha_{ij}$. The next several lemmas give bounds for the angles between intersections of the subspaces V_i. Similar results with weaker bounds can be found in [11], [13]. Informally, these lemmas show that if W_i are three planes in $\mathbb{R}^3$ such that $\alpha_{ij} = \sphericalangle(W_i, W_j)$, then the angle between two intersections of the V_i's is bounded by the angle between the corresponding intersection of the W_i's.

Lemma 4.1. *The angles* $\sphericalangle(V_1 + V_2, V_3)$ *and* $\sphericalangle(V_1 \cap V_2, V_3)$ *satisfy the inequalities*

a) $\cos^2 \sphericalangle(V_1 + V_2, V_3) \leq \frac{\varepsilon_{13}^2+\varepsilon_{23}^2+2\varepsilon_{12}\varepsilon_{23}\varepsilon_{13}}{1-\varepsilon_{12}^2}$;

b) $\cos^2 \sphericalangle(V_1 \cap V_2, V_3) \leq \dfrac{\varepsilon_{13}^2 + \varepsilon_{23}^2 + 2\varepsilon_{12}\varepsilon_{23}\varepsilon_{13}}{1 - \varepsilon_{12}^2}$.

Proof. a) Let $v_3 \in V_3$ be a vector perpendicular to the intersection $(V_1 + V_2) \cap V_3$. By Lemma 3.27 we can bound the length of the projection $P_{V_1+V_2}(v_3)$ of v_3 onto $V_1 + V_2$ using the length of the projections $P_{V_1}(v_3)$ and $P_{V_2}(v_3)$. However, v_3 is perpendicular to both $V_1 \cap V_3$ and $V_2 \cap V_3$, therefore we have

$$\|P_{V_1}(v_3)\| \leq \varepsilon_{13}\|v_3\| \quad \text{and} \quad \|P_{V_2}(v_3)\| \leq \varepsilon_{23}\|v_3\|.$$

Thus,

$$\begin{aligned}\|P_{V_1+V_2}(v_3)\|^2 &\leq \|v_3\|^2 \begin{pmatrix}\varepsilon_{13} & \varepsilon_{23}\end{pmatrix} \begin{pmatrix}1 & -\varepsilon_{12}\\ -\varepsilon_{12} & 1\end{pmatrix}^{-1} \begin{pmatrix}\varepsilon_{13}\\ \varepsilon_{23}\end{pmatrix}\\ &= \frac{\varepsilon_{13}^2 + \varepsilon_{23}^2 + 2\varepsilon_{12}\varepsilon_{23}\varepsilon_{13}}{1 - \varepsilon_{12}^2}\|v_3\|^2.\end{aligned}$$

By definition, any bound of $\|P_{V_1+V_2}(v_3)\|/\|v_3\|$, which is independent of the vector v_3, is also a bound for $\cos\sphericalangle(V_1+V_2, V_3)$.

Part b) is an application of part a) to the subspaces $V_i^{\perp}$ and of Lemma 3.9 several times. □

Lemma 4.2. *The angles* $\sphericalangle(V_1+V_3, V_2+V_3)$ *and* $\sphericalangle(V_1\cap V_3, V_2\cap V_3)$ *satisfy the inequalities*

a) $\cos\sphericalangle(V_1+V_3, V_2+V_3) \le \frac{\varepsilon_{12}+\varepsilon_{13}\varepsilon_{23}}{\sqrt{1-\varepsilon_{13}^2}\sqrt{1-\varepsilon_{23}^2}}$;

b) $\cos\sphericalangle(V_1\cap V_3, V_2\cap V_3) \le \frac{\varepsilon_{12}+\varepsilon_{13}\varepsilon_{23}}{\sqrt{1-\varepsilon_{13}^2}\sqrt{1-\varepsilon_{23}^2}}$.

Proof. a) Let w_1 and w_2 be vectors in V_1+V_3 and V_2+V_3 that are perpendicular to the intersection $(V_1+V_3)\cap(V_2+V_3)\supset V_3+(V_1\cap V_2)$.

Observe that there is a canonical isomorphism between $V_i\cap(V_i\cap V_3+V_1\cap V_2)^{\perp}$ and $(V_i+V_3)\cap(V_3+V_1\cap V_2)^{\perp}$ given by $v\to v-P_{V_3+V_1\cap V_2}(v)$. Using the inverse of this isomorphism we can find vectors $v_i\in V_i\cap(V_i\cap V_3+V_1\cap V_2)^{\perp}$ such that $w_i=v_i-P_{V_3+V_1\cap V_2}(v_i)$. Then we have

$$\|w_i\|^2=\|v_i\|^2-\|P_{V_3+V_1\cap V_2}(v_i)\|^2\ge\left(1-\varepsilon_{i3}^2\right)\|v_i^2\|$$

because $\sphericalangle(V_i, V_3)\le\sphericalangle(V_i, V_3+V_1\cup V_2)$. Therefore

$$\begin{aligned}\langle w_1,w_2\rangle &= \langle v_1-P_{V_3+V_1\cap V_2}(v_1), v_2-P_{V_3+V_1\cap V_2}(v_2)\rangle\\ &= \langle v_1,v_2\rangle-\langle P_{V_3+V_1\cap V_2}(v_1), P_{V_3+V_1\cap V_2}(v_2)\rangle\\ &\le \varepsilon_{12}\|v_1\|\,\|v_2\|+\|P_{V_3+V_1\cap V_2}(v_1)\|\,\|P_{V_3+V_1\cap V_2}(v_2)\|\\ &\le (\varepsilon_{12}+\varepsilon_{13}\varepsilon_{23})\,\|v_1\|\,\|v_2\|.\end{aligned}$$

Therefore,

$$\frac{\langle w_1,w_2\rangle}{\|w_1\|\,\|w_2\|}\le\frac{\varepsilon_{12}+\varepsilon_{13}\varepsilon_{23}}{\sqrt{1-\varepsilon_{13}^2}\sqrt{1-\varepsilon_{23}^2}}.$$

Again, part b) can be obtained by applying part a) to the subspaces $V_i^{\perp}$ and using Lemma 3.9. □

4.2. Relations with spherical geometry. All the bounds obtained in the previous section are "sharp" and have an easy geometric interpretation. Let us start with the observation that these bounds are nontrivial only if ε_{ij} satisfy the inequality

$$\varepsilon_{12}^2+\varepsilon_{23}^2+\varepsilon_{13}^2+2\varepsilon_{12}\varepsilon_{23}\varepsilon_{13}<1.$$

This condition is equivalent to the positive definiteness of the matrix

$$\begin{pmatrix}1&-\varepsilon_{12}&-\varepsilon_{13}\\-\varepsilon_{12}&1&-\varepsilon_{23}\\-\varepsilon_{13}&-\varepsilon_{23}&1\end{pmatrix},$$

which in turn is equivalent to $\alpha_{12} + \alpha_{23} + \alpha_{13} > \pi$.

Notice that this is equivalent to the existence of three unit vectors $w_i \in \mathbb{R}^3$ such that $\langle w_i, w_j \rangle = -\varepsilon_{ij}$ (such configuration of vectors is unique up to isometry of $\mathbb{R}^3$). These vectors define three lines $W_i = \mathbb{R}w_i$ and three planes $W_i' = W_i^\perp$.

It is a well-known fact from spherical geometry that the cosine of the angle between the line W_3 and the plane $W_1 + W_2$ is given by the formula in part a) of Lemma 4.1. Similarly the cosine of the angle between the planes $W_1 + W_3$ and $W_2 + W_3$ is equal to the expression in part a) of Lemma 4.2. The analogous formulas in parts b) of these lemmas correspond to the angles between the intersections constructed starting from the planes W_i'.

One way to prove the second fact is the use of the Gram–Schmidt process. Let $w_1' = w_1 + \lambda_1 w_3$ and $w_2' = w_2 + \lambda_2 w_3$ be the projections of w_1 and w_2 onto the plane perpendicular to w_3, where

$$\lambda_1 = -\frac{\langle w_1, w_3 \rangle}{\langle w_3, w_3 \rangle} = \varepsilon_{13}, \quad \lambda_2 = -\frac{\langle w_2, w_3 \rangle}{\langle w_3, w_3 \rangle} = \varepsilon_{23}.$$

These vectors are in the planes $W_1 + W_3$ and $W_2 + W_3$ and by construction are perpendicular to their intersection $(W_1 + W_3) \cap (W_2 + W_3) = W_3$. Therefore, the angle between these vectors is equal to the angle between the two planes. A direct computation shows that

$$\cos \sphericalangle(w_1', w_2') = \frac{\langle w_1', w_2' \rangle}{\|w_1'\| \, \|w_2'\|} = -\frac{\varepsilon_{12} + \varepsilon_{13}\varepsilon_{23}}{\sqrt{1 - \varepsilon_{13}^2}\sqrt{1 - \varepsilon_{23}^2}}.$$

If $\alpha_{12} + \alpha_{23} + \alpha_{13} \leq \pi$ then the bounds in Lemmas 4.1 and 4.2 are trivial. The reason for that is that it is possible to construct subspaces V_i in $\mathbb{R}^3$ such that $\sphericalangle(V_i, V_j) \leq \alpha_{ij}$, where the angles $\sphericalangle(V_1, V_2 \cap V_3)$ and $\sphericalangle(V_1 \cap V_3, V_2 \cap V_3)$ are arbitrarily small.

4.3. Distance estimates. Let V_i be three subspaces in a Hilbert space $\mathcal{H}$. For a vector $v \in \mathcal{H}$ let $d_i(v)$ denote the distance between v and the subspace V_i. Also let $d_0(v)$ denote the distance between v and the intersection $\bigcap V_i$.

Lemma 4.3. *If* $\varepsilon_{12}^2 + \varepsilon_{23}^2 + \varepsilon_{13}^2 + 2\varepsilon_{12}\varepsilon_{23}\varepsilon_{13} < 1$, *then*

$$d_0(v)^2 \leq \begin{pmatrix} d_1(v) & d_2(v) & d_3(v) \end{pmatrix} \begin{pmatrix} 1 & -\varepsilon_{12} & -\varepsilon_{13} \\ -\varepsilon_{12} & 1 & -\varepsilon_{23} \\ -\varepsilon_{13} & -\varepsilon_{23} & 1 \end{pmatrix}^{-1} \begin{pmatrix} d_1(v) \\ d_2(v) \\ d_3(v) \end{pmatrix}.$$

Proof. Let $v_3 = P_{V_3}(v)$ denote the projection of v on to V_3. By construction we have $d_0(v)^2 = d_3(v)^2 + d_0(v_3)^2$, thus our goal is to bound the distance $d_0(v_3)$.

Let $W_1 = V_1 \cap V_3$ and $W_3 = V_2 \cap V_3$. Observe that $W_1 \cap W_2 = \cap V_i$, hence by Lemma 3.27 we can bound $d_0(v_3)$ using the distances between v_3 and the subspaces

W_1 and W_2, and the angle $\sphericalangle(W_1, W_2)$. Lemmas 3.26 and 4.2 provide us with bounds for all these. Substituting everything we obtain

$$
\begin{aligned}
d_0(v)^2 &= d_0(v_3)^2 + d_3(v)^2 \\
&\leq \begin{pmatrix} d_{W_1}(v_3) & d_{W_2}(v_3) \end{pmatrix} \begin{pmatrix} 1 & -\varepsilon \\ -\varepsilon & 1 \end{pmatrix}^{-1} \begin{pmatrix} d_{W_1}(v_3) \\ d_{W_2}(v_3) \end{pmatrix} + d_3(v)^2,
\end{aligned}
$$

where $\varepsilon = \cos \sphericalangle(W_1, W_2)$,

$$
d_{W_i}(v_3) \leq \frac{\cos \sphericalangle(V_1, V_3) d_3(v) + d_i(v)}{\sin \sphericalangle(V_1, V_3)}
$$

and

$$
\cos \sphericalangle(W_1, W_2) \leq \frac{\cos \sphericalangle(V_1, V_2) + \cos \sphericalangle(V_1, V_3) \cos \sphericalangle(V_1, V_3)}{\sin \sphericalangle(V_1, V_3) \sin \sphericalangle(V_1, V_3)}.
$$

A long and boring computation shows that the resulting expression is exactly equal to the formula in the statement of the lemma. The proof of Theorem 5.1 follows the same idea and shows how to avoid doing this long computation. □

Remark 4.4. The bound in the above lemma is sharp. Let W_i' be the three planes in $\mathbb{R}^3$ such that the angles between them are equal to α_{ij}. It is easy to see that for any three positive numbers d_i there exists a vector $v \in \mathbb{R}^3$ such that $d_{W_i'}(v) = d_i$ and $\|v\|^2$ is given by the formula above.

An immediate application of the above bound is the following corollary, which is a special case of Theorem 1.2.

Corollary 4.5. *If* $\varepsilon_{12}^2 + \varepsilon_{23}^2 + \varepsilon_{13}^2 + 2\varepsilon_{12}\varepsilon_{23}\varepsilon_{13} < 1$, *then*

$$
1 - \cos \sphericalangle(V_1, V_2, V_3) \geq \frac{\lambda}{2},
$$

where λ is the smallest eigenvalue of the (positive definite) matrix

$$
A_\varepsilon = \begin{pmatrix} 1 & -\varepsilon_{12} & -\varepsilon_{13} \\ -\varepsilon_{12} & 1 & -\varepsilon_{23} \\ -\varepsilon_{13} & -\varepsilon_{23} & 1 \end{pmatrix}.
$$

In particular there is a (nontrivial) lower bound for $\sphericalangle(V_1, V_2, V_3)$ which depends only on ε_{ij}.

5. Main result

5.1. Proof of Theorem 1.2. We start with a quantitative variant of Theorem 1.2 that will be used in Section 6 to obtain bounds for the Kazhdan constants and spectral gaps.

Theorem 5.1. *Let V_i be n closed subspaces of a Hilbert space $\mathcal{H}$. Suppose that the symmetric $(n \times n)$-matrix*

$$A = \begin{pmatrix} 1 & -\varepsilon_{12} & -\varepsilon_{13} & \dots & -\varepsilon_{1n} \\ -\varepsilon_{21} & 1 & -\varepsilon_{23} & \dots & -\varepsilon_{2n} \\ -\varepsilon_{31} & -\varepsilon_{32} & 1 & \dots & -\varepsilon_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\varepsilon_{n1} & -\varepsilon_{n2} & -\varepsilon_{n3} & \dots & 1 \end{pmatrix},$$

where $\varepsilon_{ij} = \cos \sphericalangle(V_i, V_j)$, is positive definite. Then for any $v \in \mathcal{H}$ we have

$$d_0(v)^2 \leq \boldsymbol{d}_v^t A^{-1} \boldsymbol{d}_v,$$

where $d_0(v)$ denotes the distance between v and $\bigcap V_i$, and $\boldsymbol{d}_v$ is the column vector with entries the distances $d_{V_i}(v)$.

Proof. The proof is by induction on n. The base case $n = 2$ is Lemma 3.27. The induction step follows the idea of Lemma 4.3. Let $V_i' = V_i \cap V_n$ and let $v' = P_{V_n}(v)$. Using Lemma 4.2 one can bound the angles between V_i' and apply these bounds to form an $(n-1) \times (n-1)$-matrix A'. Also, the distances between v' and V_i' can be bounded by Lemma 3.26 and these bounds can be combined in a vector $\boldsymbol{d}_v'$. In order to complete the induction step we need to show that 1) the matrix A' is positive definite and 2) the equality

$$\boldsymbol{d}_v^t A^{-1} \boldsymbol{d}_v = \boldsymbol{d}_v'^t A'^{-1} \boldsymbol{d}_v' + d_{V_n}(v)^2 \tag{1}$$

holds. The matrix A can be written as the product

$$A = \begin{pmatrix} \mathrm{Id} & -\boldsymbol{\varepsilon}_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{A} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ -\boldsymbol{\varepsilon}_n^t & 1 \end{pmatrix},$$

where $\tilde{A}$ is an $(n-1) \times (n-1)$-matrix with diagonal entries $1 - \varepsilon_{in}^2$ and the off diagonal entries $-\varepsilon_{ij} - \varepsilon_{in}\varepsilon_{jn}$. Here $\boldsymbol{\varepsilon}_n$ denotes the column vector with entries ε_{in}.

The decomposition of A as a product implies that

$$\boldsymbol{d}_v^t A^{-1} \boldsymbol{d}_v = \boldsymbol{d}_v^t \begin{pmatrix} \mathrm{Id} & 0 \\ \boldsymbol{\varepsilon}_n^t & 1 \end{pmatrix} \begin{pmatrix} \tilde{A} & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{Id} & \boldsymbol{\varepsilon}_n \\ 0 & 1 \end{pmatrix} \boldsymbol{d}_v = \tilde{\boldsymbol{d}}_v^t \tilde{A}^{-1} \tilde{\boldsymbol{d}}_v + d_{V_n}(v)^2,$$

where $\tilde{\boldsymbol{d}}$ is the vector defined by

$$\begin{pmatrix} \tilde{\boldsymbol{d}}_v \\ d_n \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & \boldsymbol{\varepsilon}_n \\ 0 & 1 \end{pmatrix} \boldsymbol{d}_v.$$

Equality (1) now follows from observations that are immediate consequences of the definitions of A' and $\boldsymbol{d}_v'$:

(a) the matrices A' and $\tilde{A}$ are related by $A' = D\tilde{A}D$, where D is a diagonal matrix with entries $1/\sqrt{1-\varepsilon_{in}^2}$,

(b) the vectors $\boldsymbol{d}'_v$ and $\tilde{\boldsymbol{d}}_v$ satisfy $\boldsymbol{d}'_v = D\tilde{\boldsymbol{d}}_v$.

For this implies that

$$\boldsymbol{d}'^t_v A'^{-1}\boldsymbol{d}'_v = \tilde{\boldsymbol{d}}'^t_v DA'^{-1}D\tilde{\boldsymbol{d}}'_v = \tilde{\boldsymbol{d}}^t_v\tilde{A}^{-1}\tilde{\boldsymbol{d}}_v.$$

The first observation also proves that A' is a positive definite matrix since $\tilde{A}$ is. □

Remark 5.2. The geometric interpretation of the above theorem and its proof is the following: let w_i be unit vectors in $\mathbb{R}^n$ such that $\langle w_i, w_j\rangle = -\varepsilon_{ij}$ (such vectors exist since A is positive definite) and let w be a vector such that $\langle w, w_i\rangle = d_{P_{V_i}}(v)$. Then $d_0(v) \le \|w\|$.

The proof is by induction. The induction step uses the Gram–Schmidt process: one projects the vectors w_i onto the hyperplane perpendicular to w_n and then "normalize" the resulting vectors w'_i to have unit length. By the Lemma 4.2 the angle between w'_i and w'_j is a bound for the angle between the intersections $V_i \cap V_n$ and $V_j \cap V_n$. Similarly, by Lemma 3.26, the distance between the projection $P_{V_n}(v)$ to the intersection $V_i \cap V_n$ is bounded by $\frac{\langle w', w'_i\rangle}{\|w'_i\|^2}$, where w' is the projection of w onto the hyperplane perpendicular to w_n. The induction step is completed by

$$d_0(v)^2 = d_0(P_{V_n}(v))^2 + d_{V_n}(v)^2 \le \|w'\|^2 + \langle w, w_n\rangle^2 = \|w\|^2,$$

where the inequality follows by the induction assumption.

Remark 5.3. The inequality in Theorem 5.1 can be rephrased as follows.

The $(n+1)\times(n+1)$-matrix

$$B = \begin{pmatrix} d_0(v)^2 & \boldsymbol{d}^t_v \\ \boldsymbol{d}_v & A \end{pmatrix}$$

is not positive definite.

Proof of Theorem 1.2. If the matrix A is positive definite, then Theorem 5.1 applies. Therefore, we have

$$d_0(v)^2 \le \boldsymbol{d}^t_v A^{-1}\boldsymbol{d}_v \le \frac{1}{\lambda}\sum d_{V_i}(v)^2,$$

where λ is the smallest eigenvalue of the matrix A. By Remark 3.25 this bound implies that

$$\cos\sphericalangle(V_1, V_2, \ldots, V_n) \le 1 - \frac{\lambda}{n-1}, \quad \text{i.e.,} \quad \sphericalangle(V_1, V_2, \ldots, V_n) \ge \alpha,$$

where $\alpha = \cos^{-1}(1 - \frac{\lambda}{n-1})$. This completes the proof since λ and α depend only on ε_{ij}. □

Remark 5.4. In the special case when all $\varepsilon_{ij} = \varepsilon$ are the same we obtain that if $\varepsilon \leq \frac{1}{n-1}$ then $\sphericalangle(V_1, V_2, \ldots, V_n) > \alpha$, where $\cos\alpha = \frac{n-2}{n-1} + \varepsilon$, because the smallest eigenvalue λ of the matrix A is equal to $1 - (n-1)\varepsilon$, which is equivalent to Corollary 5.3 from [13].

Example 5.5. If V_i are pairwise orthogonal subspaces in $\mathcal{H}$ then the matrix A is equal to the identity matrix, and by Theorem 1.2 we have

$$\cos \sphericalangle(V_1, V_2, \ldots, V_n) \leq 1 - \frac{1}{n-1}.$$

In fact, by Example 3.23 equality holds if V_i are the coordinate hyperplanes in n-dimensional Euclidean space.

5.2. Geometric interpretation. Theorem 1.2 can be rephrased as follows: Let Δ be a spherical simplex such the internal angle between any two faces F_i and F_j is equal to α_{ij}. Then the angle between any collection of subspaces V_i such that $\sphericalangle(V_i, V_j) \geq \alpha_{ij}$ is bounded by

$$\sphericalangle(V_1, V_2, \ldots, V_n) \geq \sphericalangle(\widetilde{F}_1, \widetilde{F}_2, \ldots, \widetilde{F}_n) > 0,$$

where $\widetilde{F}_n$ is the affine subspace that contains the face F_i. A slight modification of the proof also gives that a similar inequality holds for the angle between intersections of the V_i's.

Theorem 5.1 has a similar interpretation: Let p be any point in the interior of the simplicial cone defined by Δ. Then for any $v \in \mathcal{H}$ such that $d_{V_i}(v) \leq d_{\widetilde{F}_i}(p)$ we have that

$$d_{\cap V_i}(v) \leq d_{\cap \widetilde{F}_i}(p) = \|p\|.$$

6. Applications

6.1. Kazhdan constants and spectral gap for Coxeter groups. Let G be a finite group generated by a symmetric set S, i.e., $S = S^{-1}$. Let $\pi\colon G \to U(L^2(G))$ denote the regular representation of the group G. The operator

$$\Delta_S = \frac{1}{|S|} \sum_{s \in S} (\mathrm{Id} - \pi(s)) : L^2(G) \to L^2(G)$$

is called Laplacian[5] on G. An equivalent way to define this operator is to take the Laplacian of the Cayley graph of the group G with respect to the generating set S.

[5] The operator Δ can be defined even if the group is not finite, however in this setting there is no direct connection between Δ_S and a graph Laplacian. One can even define Δ_μ when G as a group and μ is a measure on G. In these more general situations there is also a connection between the spectral gap of Δ (if positive) and the relaxation time of some random walk.

This operator is positive definite and has an eigenvalue 0 with multiplicity 1 (the eigenvector is the constant function). The smallest nontrivial eigenvalue λ_S of Δ_S is called the spectral gap of the Laplacian and is closely related to the relaxation time of the random walk on Cayley graph. Thus, a bound for the spectral gap can be used to estimate the mixing time of this random walk.

A Coxeter group G generated by a set $S = \{s_1, \dots, s_n\}$ is defined by numbers $m_{ij} \geq 2$ and has a presentation

$$G \simeq \langle s_i \mid s_i^2 = 1,\ (s_i s_j)^{m_{ij}} = 1 \rangle.$$

It is known that G has a *defining representation* on an n-dimensional vector space V where each generator s_i acts as a reflection with respect to a hyperplane V_i. Moreover, if G is finite there is a G-invariant Euclidean structure on V and the angle between the hyperplanes V_i and V_j is equal to π/m_{ij}.

Theorem 6.1. *Let G be a finite Coxeter group. Then the Kazhdan constant $\kappa(G, S)$ and the spectral gap of the Laplacian can be computed using the defining representation of G.*

Proof. The group G is generated by n subgroups $G_i = \{1, s_i\}$ of order 2. From the presentation of G it is clear that the group generated by G_i and G_j is the dihedral group $D_{m_{ij}}$. Therefore, for any unitary representation of G in $\mathcal{H}$ the angle between $\mathcal{H}^{G_i}$ and $\mathcal{H}^{G_j}$ is bounded below by π/m_{ij}. It is a classical fact [16] that the finiteness of the group G is equivalent to the positive definiteness of the matrix

$$A = \begin{pmatrix} 1 & -\varepsilon_{12} & -\varepsilon_{13} & \dots & -\varepsilon_{1n} \\ -\varepsilon_{21} & 1 & -\varepsilon_{23} & \dots & -\varepsilon_{2n} \\ -\varepsilon_{31} & -\varepsilon_{32} & 1 & \dots & -\varepsilon_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\varepsilon_{n1} & -\varepsilon_{n2} & -\varepsilon_{n3} & \dots & 1 \end{pmatrix},$$

where $\varepsilon_{ij} = \cos \pi/m_{ij}$. This allows us to apply Theorem 5.1.

Let v be an ε-almost invariant unit vector in $\mathcal{H}$. Then, by Theorem 5.1, the distance between v and the space of G-invariant vectors is bounded by

$$d_{\mathcal{H}^G}(v)^2 \leq \left(\frac{\varepsilon}{2}\right)^2 \mathbf{1}^t A^{-1} \mathbf{1},$$

where $\mathbf{1}$ is the column vector consisting of n ones, because each generator s_i moves v by $2d_{\mathcal{H}^{G_i}}(v) \leq \varepsilon$. Thus, if $\varepsilon < 2(\mathbf{1}^t A^{-1} \mathbf{1})^{-1/2}$ there is a nontrivial invariant vector in $\mathcal{H}$. This shows that $\kappa(G, S) \geq \varepsilon_0 = 2(\mathbf{1}^t A^{-1} \mathbf{1})^{-1/2}$. However, it is easy to see that equality holds because the defining representation of G contains a unit vector which is ε_0-almost invariant.

A similar argument can be used to obtain bounds for the spectral gap of the Laplacian: the operator $\mathrm{Id} - \pi(s_i)$ is equal to two times the projection onto $(\mathcal{H}^{G_i})^\perp$. Thus, for a vector v we have $(\mathrm{Id} - \pi(s_i)(v) = 2P_{(\mathcal{H}^{G_i})^\perp}(v)$, i.e.,

$$\langle \Delta_S(v), v\rangle = \frac{1}{|S|}\sum \langle 2P_{(\mathcal{H}^{G_i})^\perp}(v), v\rangle = \frac{2}{|S|}\sum d_{\mathcal{H}^{G_i}}(v)^2.$$

If the vector v has a trivial projection on the space of G-invariant vectors we have

$$\|v\|^2 = d_{\mathcal{H}^G}(v)^2 \le \boldsymbol{d}_v^t A^{-1}\boldsymbol{d}_v \le \lambda^{-1}\|\boldsymbol{d}_v\|^2 = \lambda^{-1}\sum d_{\mathcal{H}^{G_i}}(v)^2,$$

where λ is the smallest eigenvalue of the matrix A. Combining the above inequalities yields

$$\langle \Delta_S(v), v\rangle \ge \frac{2}{n}\lambda\|v\|^2,$$

i.e., the spectral gap of Δ_S is at least $\frac{2\lambda}{n}$. Again, it is easy to see that equality holds since the smallest eigenvalue of Δ_S in the defining representation is equal to $\frac{2\lambda}{n}$. $\square$

Example 6.2. Let the Coxeter group G be of type A_n, i.e., $G \simeq \mathrm{Sym}(n+1)$ and S consists of transpositions $(1,2), (2,3), \dots, (n, n+1)$. In this case the matrix A has the form

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 & \dots & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \dots & 0 \\ 0 & -\frac{1}{2} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

A standard computation shows that this matrix has eigenvalues $\lambda_k = 2\sin^2(\frac{k\pi}{2n+2})$ with eigenvectors $v_k = (\dots \sin\frac{k\pi i}{n+1}\dots)^t$. Thus, the spectral gap of the Laplacian is

$$\frac{2\lambda_1}{n} = \frac{4}{n}\sin^2\frac{\pi}{2n+2} \sim \frac{\pi^2}{n^3},$$

which implies that the relaxation time of the random walk on symmetric group is of order n^3.

The eigenvalues and the eigenvectors of A can be used to compute the value of $\mathbf{1}^t A^{-1}\mathbf{1}$, however it is easier[6] to write down explicitly the matrix A^{-1}, and calculate that $\mathbf{1}^t A^{-1}\mathbf{1} = (n^3 + 3n^2 + 2n)/6$, which implies that the Kazhdan constant of the symmetric group is

$$\kappa(\mathrm{Sym}(n+1), \{(1,2), (2,3), \dots, (n, n+1)) = \sqrt{\frac{24}{n^3 + 3n^2 + 2n}}.$$

[6] It is possible to bypass this computation by constructing in the defining representation a vector that at the same distance forms each of the fixed subspaces and use it to evaluate $\mathbf{1}^t A^{-1}\mathbf{1}$.

Remark 6.3. For any Coxeter group the smallest eigenvalue of the matrix A is equal to $2\sin^2\frac{\pi}{2h}$, where h is the Coxeter number of G. This implies that the spectral gap of Δ_S is equal to $\frac{2}{n}(1-\cos\frac{\pi}{h}) = \frac{4}{n}\sin^2\frac{\pi}{2h}$.

The Kazhdan constant $\kappa(G,S)$ is equal to $2\left(\mathbf{1}^t A^{-1}\mathbf{1}\right)^{-1/2}$. In the simply laced case the number $M = \mathbf{1}^t A^{-1}\mathbf{1}$ is equal to the Dynkin index [12] of the canonical embedding of $\mathfrak{sl}_2$ in the simple Lie algebra corresponding to the Coxeter group G, and by [22] is equal to $nh(h+1)/6$. We do not know of any similar formula in the non-simply laced case. However, it is not difficult to compute the Kazhdan constant in each case $\kappa(G,S) = 2M^{-1/2}$, where $M = \mathbf{1}^t A^{-1}\mathbf{1}$ is given in Table 1.

Table 1. Kazhdan constants for Coxeter groups.

type	rank	Coxeter num.	M	order of M
A_n	n	$n+1$	$n(n+1)(n+2)/6$	$n^3/6$
B_n	n	$2n$	$n(2n^2+3(\sqrt{2}-1)n+4-3\sqrt{2})/3$	$2n^3/3$
D_n	n	$2(n-1)$	$n(n-1)(2n-1)/3$	$2n^3/3$
E_6	6	12	156	
E_7	7	18	399	
E_8	8	30	1240	
F_4	4	12	$56+36\sqrt{2}$	
H_3	3	10	$31+12\sqrt{5}$	
H_4	4	30	$332+144\sqrt{5}$	
$I_2(m)$	2	m	$2(1-\cos\frac{\pi}{m})^{-1}$	$4m^2/\pi^2$

Remark 6.4. Bacher and de la Harpe [2] computed the Kazhdan constant $\kappa(G,S)$ when the Coxeter group G is of type[7] A_n, i.e., $G = \mathrm{Sym}(n+1)$. Their proof uses the representation theory of the symmetric group and character estimates. Bounds for the spectral gap of the Laplacian on the symmetric group were obtained by Diaconis and Shahshahani [10], and Diaconis and Saloff-Coste [8].

Bagno [3] extended the methods from [2] to the case of Coxeter groups of types B_n and D_n, but he was not able to compute the exact value of the Kazhdan constants. For the exceptional Coxeter groups results of this type can be verified by long computation.

Remark 6.5. It seems that Theorem 6.1 can be generalized to classical finite complex reflection groups. In order to do that one first needs to prove that for any unitary representation of any rank 2 complex reflection group the angle between the fixed subspaces of the two generating pseudo-reflection groups is the same as the angle in the

[7] They also considered the type $I_2(m)$, when G is the dihedral group D_m.

defining representation. Since this is clearly the case for the classical rank 2 complex reflection groups, one can easily extend Theorem 6.1 to the groups of type $G_{m,p,n}$.

Remark 6.6. One of the reasons why Theorem 6.1 gives exact bounds for the Kazhdan constants and spectral gap is the existence of the defining representation, where each "generating" subgroup fixes a hyperplane and the angles between these hyperplanes are lower bounds for the angles between the fixed subspaces of these subgroups in any unitary representation of the group G. Theorem 6.14 is another example where a similar configuration of subspaces allows us to compute the exact value of the spectral gap.

6.2. Property T for Steinberg groups. Theorem 5.1 can be used to prove property T for groups G which have generating set S consisting of several (finite) subgroups, $S = \bigcup G_i$. Before applying it, one only needs to understand the representation theory of the subgroups of G generated by any pair of G_i and G_j. One situation where this method works nicely is the following:

Theorem 6.7. *The Steinberg group* $\mathrm{St}_n(\mathbb{F}_p\langle t_1,\dots,t_k\rangle)$ *has property T, provided that* $n \geq 3$ *and* $p \geq 5$.

Proof. For an associative ring R the Steinberg group $\mathrm{St}_n(R)$ is generated by the elements $x_{ij}(r)$, where $1 \leq i \neq j \leq n$ and $r \in R$ subject to the defining relations

$$x_{ij}(r_1)x_{ij}(r_2) = x_{ij}(r_1 + r_2), \quad [x_{ij}(r), x_{jk}(s)] = x_{ik}(rs), \quad [x_{ij}(r), x_{kl}(s)] = 1.$$

The group $G = \mathrm{St}_n(\mathbb{F}_p\langle t_1,\dots,t_k\rangle)$ contains n finite subgroups $G_1,\dots,G_n$: for $i = 1,\dots,n-1$ the subgroup G_i consists of the elements $x_{i,i+1}(a\cdot 1)$ for $a \in \mathbb{F}_p$, and the group G_n consists of $x_{n1}(a_0 1 + a_1 t_1 + \dots + a_k t_k)$ for $a_i \in \mathbb{F}_p$. An easy computation by induction shows that the subgroups G_i generate the group G.

If $i, j < n$ and $|i - j| > 1$ then the subgroups G_i and G_j commute. Thus for any unitary representation of G the fixed subspaces are perpendicular, that is, $\sphericalangle(\mathcal{H}^{G_i}, \mathcal{H}^{G_j}) = \pi/2$. If $j = i + 1 < n$ the G_i and G_j generate a Heisenberg group H_p of order p^3. It can be shown [13], Section 4, that in any representation of H_p the angle between the fixed subspaces of any two non-central cyclic subgroups of order p is larger than $\cos^{-1}(p^{-1/2})$, thus $\cos\sphericalangle(\mathcal{H}^{G_i}, \mathcal{H}^{G_{i+1}}) \leq p^{-1/2}$.

If one of the subgroups is G_n, the argument is almost the same: G_n commutes with $G_2,\dots,G_{n-2}$ and the groups $\langle G_1, G_n\rangle$ and $\langle G_{n-1}, G_n\rangle$ are generalized Heisenberg groups. Thus, we have obtained bounds for all angles $\sphericalangle(\mathcal{H}^{G_i}, \mathcal{H}^{G_j})$ and the matrix A has the form

$$\begin{pmatrix} 1 & -\varepsilon & 0 & \dots & 0 & -\varepsilon \\ -\varepsilon & 1 & -\varepsilon & \dots & 0 & 0 \\ 0 & -\varepsilon & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -\varepsilon \\ -\varepsilon & 0 & 0 & \dots & -\varepsilon & 1 \end{pmatrix}, \tag{2}$$

where $\varepsilon = p^{-1/2}$. For $p \geq 5$ this matrix is positive definite and its smallest eigenvalue is equal to $1 - 2\varepsilon$, the corresponding eigenvector is $(1, 1, \ldots, 1)^t$.

By Theorem 5.1 we have that

$$d_{\mathcal{H}^G}(v)^2 \leq \frac{1}{1 - 2p^{-1/2}} \sum d_{\mathcal{H}^{G_i}}(v)^2.$$

for any $v \in \mathcal{H}$. If the unit vector v is ε-almost invariant under the generating set $\bigcup G_i$ then $d_{\mathcal{H}^{G_i}}(v) \leq \varepsilon/\sqrt{2}$ for each i, because any unit vector in a unitary representation of a group H is moved by more than $\sqrt{2}$ by some element in H. Thus

$$d_{\mathcal{H}^G}(v)^2 \leq \frac{1}{1 - 2p^{-1/2}} \times \frac{n\varepsilon^2}{2}.$$

If $\varepsilon < \sqrt{\frac{2(1-2p^{-1/2})}{n}}$ we have that $d_{\mathcal{H}^G}(v) < 1$, i.e., $\mathcal{H}$ has invariant vectors. This shows that

$$\kappa(G, \cup G_i) \geq \sqrt{\frac{2(1 - 2p^{-1/2})}{n}} \geq \sqrt{\frac{1}{5n}} > 0.$$

In particular G has the Kazhdan property T since $\bigcup G_i$ is finite. □

Remark 6.8. Theorem 6.7 can be generalized to (higher rank) Steinberg groups of other types (over commutative rings) and the proof is essentially the same. The groups G_i for $i = 1, \ldots, n-1$ are part of the root subgroups corresponding to the simple roots, and G_n is a subgroup of the root subgroup corresponding to the largest negative root. In the simply laced case the matrix A is related to the Cartan matrix corresponding to the extended Dynkin diagram. It will be positive definite if $p \geq 5$ and the smallest eigenvalue will be again equal to $1 - 2p^{-1/2}$.

Remark 6.9. One can use the method in the proof of Theorem 6.7 to show that the simply laced Kac–Moody groups over finite fields corresponding to k-regular graphs have property T if $p \geq k^2$. These groups are generated by subgroups G_i, indexed by the vertices of the graph. Each group G_i is isomorphic to $\mathrm{SL}_2(\mathbb{F}_p)$ and the group generated by G_i and G_j is either isomorphic to $\mathrm{SL}_2(\mathbb{F}_p) \times \mathrm{SL}_2(\mathbb{F}_p)$ or $\mathrm{SL}_3(\mathbb{F}_p)$, depending on whether the vertices are connected or not. These conditions lead to bounds for the angles between the fixed spaces for G_i. By Theorem 1.2 and Observation 2.1 the positive definiteness of the matrix A_ε implies that the resulting Kac–Moody group has property T. However, the matrix A_ε is positive definite because its smallest eigenvalue is equal to $1 - kp^{-1/2} > 0$.

Proof of Theorem 1.4. The group $\mathrm{SL}_n(\mathbb{F}_p[t_1, \ldots, t_k])$ is a quotient of the Steinberg group $\mathrm{St}_n(\mathbb{F}_p\langle t_1, \ldots, t_k\rangle)$, which has property T by Theorem 6.7. However, property T is inherited by quotients, so $\mathrm{SL}_n(\mathbb{F}_p[t_1, \ldots, t_k])$ also has the Kazhdan property T. □

Remark 6.10. This is not the first proof that the group $\mathrm{St}_n(\mathbb{F}_p\langle t_1,\dots,t_k\rangle)$ has property T. The case $k = 1$ is very old and goes back to Kazhdan [20]; in this case G has T because it is a lattice in a high rank Lie group over $\mathbb{F}_p((t^{-1}))$. In the case of commuting variables the author and N. Nikolov [19] have shown that the group has property τ, which is a weak form of property T. Also, in the commutative case Y. Shalom [23] proved that G has T if $k \le n-2$. This condition was replaced by Vaserstein [25] with $n \ge 3$. Recently Ershov and Jaikin [13] extended these results by showing that $\mathrm{St}_n(\mathbb{Z}\langle t_1,\dots,t_k\rangle)$ has property T for $n \ge 3$.

Essentially, the same proof as above valid in the case $p > n^2$ can also be found in [13]. As mentioned in the introduction, the aim of this paper is not to prove new results but to explain the author's interpretation of the ideas in [11] and [13].

Remark 6.11. Using results about the relative property T one can also works with group G_i which are not finite. This approach yields that the Steinberg groups $\mathrm{St}_n(R)$ have property T if $n \ge 3$, the ring R is finitely generated and R does not have $\mathbb{F}_q$ as a quotient, for $q \le 4$. It is possible to remove the condition that R does not have small quotients, however this requires significantly more complicated arguments instead of Theorem 1.2; see [13], [14].

6.3. Spectral gaps and mixing times of some random walks. The proofs of the following two theorems are very similar to the proofs of Theorems 6.1 and Theorem 6.7 so that we will only sketch the main steps of the proofs.

Theorem 6.12. *Let G denote the group $\mathrm{SL}_n(\mathbb{F}_p)$ for $n \ge 3$ and $p \ge 5$, and let $G_{i,j}$ be the root subgroup $\mathrm{Id} + \mathbb{F}_p e_{ij}$ inside G. The spectral gap of Δ_S is bounded by*

a) $1/10n$ *if* $S = G_{1,2} \cup G_{2,3} \cup \cdots \cup G_{n-1,n} \cup G_{n,1}$;
b) $1/10n$ *if* $S = \bigcup G_{i,j}$;
c) $1/200n$ *if* $S = \{\mathrm{Id} \pm e_{ij} \mid |i-j| \le 1 (\mathrm{mod}\ n)\}$.

These bounds imply that the random walks of $\mathrm{SL}_n(\mathbb{F}_p)$ with respect to the generating set described above have mixing time bounded by $Cn^3 \log p$.

Proof. a) The proof is essentially the same as the one of Theorem 6.7. Every pair of groups G_i and G_j either commute or generate a Heisenberg group. The resulting matrix is the same as the one in (2) and its smallest eigenvalue is equal to $1-2p^{-1/2}$. Now using that

$$\Big\langle \tfrac{1}{|G_i|} \sum_{g\in G_i} (\mathrm{Id} - \pi(g))v, v \Big\rangle = d(v, \mathcal{H}^{G_i})^2$$

we obtain that the spectral gap of Δ is bounded by

$$\Delta_S > \frac{1}{n}(1-2p^{-1/2}) > \frac{1}{10n}.$$

b) This follows from part a) by decomposing the complete graph as union of n-cycles and observing that the Laplacian is the average of the Laplacians corresponding to the union of the root subgroups in each cycle.

c) First, one uses Selberg's theorem to bound the spectral gap of the Laplacian on $\mathrm{SL}_2(\mathbb{F}_p)$ with respect to the generating set consisting of $\mathrm{Id} \pm e_{12}$ and $\mathrm{Id} \pm e_{12}$. This bound implies that if a vector v in any unitary representation of $\mathrm{SL}_n(\mathbb{F}_p)$ is ε-almost invariant with respect to S then it is 20ε-almost invariant with respect to $\bigcup_{|i-j|=1} G_{ij}$, which combined with part a) completes the proof. □

Remark 6.13. Theorem 6.12 implies that the Kazhdan constant of $\mathrm{SL}_n(\mathbb{F}_p)$ with respect to the generating set $S = \bigcup G_{i,j}$ is bounded below by a function of order $n^{-1/2}$. A bound of this type was found in [18], however this argument gives a slightly better constant. It is easy to construct representations of $\mathrm{SL}_n(\mathbb{F}_p)$ with $n^{-1/2}$-almost invariant vectors which shows that there is an upper bound for the Kazhdan constant of the same order.

Our final example is slightly different because it involves a compact Lie group. Let G denote the group $\mathrm{SO}(n)$ (with its standard action on $\mathbb{R}^n$). This group contains the subgroups $G_{ij} \simeq \mathrm{SO}(2)$ consisting of rotations in the coordinate plane $\mathrm{span}\{e_i, e_j\}$. Kac [17] studied the random walk on $\mathrm{SO}(n)$ with respect to $\bigcup G_{ij}$. Maslen [21] computed the spectral gap in the case $S = \{G_{ij} \mid 1 \le i, j \le n\}$, and Diaconis and Saloff-Coste [9] obtained bounds in the case $S = \left\{G_{i,i+1} \mid 1 \le i < n\right\}$. Since the group G is not finite, one needs to slightly modify the definition of the Laplacian Δ_S. Instead of averaging over the generating set one uses an integral with respect to some measure μ. S is then a union of circles and μ is just the average of the uniform measures on each circle.

Theorem 6.14. *The spectral gap of Δ_S is*

a) *equal to δ if $S = \bigcup_{1 \le i \le n} G_{i,i+1}$;*
b) *bounded below by δ if $S = \bigcup_{1 \le i,j \le n+1} G_{ij}$,*

where $\delta = \frac{2}{n} \sin^2\left(\frac{\pi}{2n+2}\right) \sim \frac{\pi^2}{2n^3}$.

Remark 6.15. Part a) improves the bound found in [9] by a constant factor. On the other hand, the bound in part b) is significantly weaker than the exact value of the gap $\frac{n+3}{2n(n+1)} \sim \frac{1}{2n}$; see [21], Theorem 2.1.

Proof. The proof of part a) is similar to Theorem 6.1 and Example 6.2. The measure $\mu_{G_{ij}}$ on G_{ij} is uniform, thus

$$\int_{\mu_{G_{ij}}} (\mathrm{Id} - \pi(g))(v)\, d\mu = P_{(\mathcal{H}^{G_{ij}})^\perp}(v).$$

Therefore, if we denote $G_i = G_{i,i+1}$, we have

$$\langle \Delta_S(v), v\rangle = \frac{1}{n} \sum \langle P_{(\mathcal{H}^{G_i})^\perp}(v), v\rangle = \frac{1}{n} \sum d_{\mathcal{H}^{G_i}}(v)^2.$$

The group G_i and G_j commute if $|i-j|>1$, so $\cos\sphericalangle(\mathcal{H}^{G_i},\mathcal{H}^{G_j})\le 0$ (the representation $\mathcal{H}$ might not have any invariant vectors under G_i, in which case the angle will be equal to π). If $j=i+1$ the groups G_i and G_j generate a group isomorphic to SO(3). Using the representation theory of SO(3) one can show [21], Lemma 3.2, that $\cos\sphericalangle(\mathcal{H}^{G_i},\mathcal{H}^{G_j})\le 1/2$.

Thus, the matrix A is the same as in Example 6.2, and its smallest eigenvalue is equal to $\lambda = 2\sin^2\left(\frac{\pi}{2n+2}\right)$, which implies that the spectral gap of Δ_S is bounded below by

$$\frac{2}{n}\sin^2\left(\frac{\pi}{2n+2}\right)\sim\frac{\pi^2}{2n^3}.$$

Actually equality holds because the spectral gap of the Laplacian on the representation of $\mathrm{SO}(n+1)$ on the space of harmonic homogeneous polynomials of degree 2 is exactly equal to δ. This representation contains a subspace V of dimension n which contains n hyperplanes $\mathcal{H}^{G_i}$, and the angle between $\mathcal{H}^{G_i}$ and $\mathcal{H}^{G_j}$ is either $\pi/2$ or $\pi/3$.

Part b) follows immediately from part a) by writing the generating set as union of several generating sets for part a). □

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References

[1] C. Badea, S. Grivaux V. Muller, The rate of convergence in the method of alternating projections. Preprint 2010. arXiv:1006.2047

[2] R. Bacher and P. de la Harpe, Exact values of Kazhdan constants for some finite groups. *J. Algebra* **163** (1994), 495–515. Zbl 0842.20013 MR 1262716

[3] E. Bagno, Kazhdan constants of some colored permutation groups. *J. Algebra* **282** (2004), 205–231. Zbl 1062.20043 MR 2095580

[4] B. Bekka, P. de la Harpe, and A. Valette. *Kazhdan's property (T)*, New Math. Monogr. 11, Cambridge University Press, Cambridge 2008. Zbl 1146.22009 MR 2415834

[5] M. Burger, Kazhdan constants for $\mathrm{SL}(3,\mathbb{Z})$. *J. Reine Angew. Math.* **413** (1991), 36–67. Zbl 0704.22009 MR 1089795

[6] D. I. Cartwright, W. Młotkowski, and T. Steger, Property (T) and $\tilde{A}_2$ groups. *Ann. Inst. Fourier (Grenoble)* **44** (1994), 213–248. Zbl 0792.43002 MR 1262886

[7] F. Deutsch, The angle between subspaces of a Hilbert space. In *Approximation theory, wavelets and applications*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 454, Kluwer Acad. Publ., Dordrecht 1995, 107–130. Zbl 0848.46010 MR 1340886

[8] P. Diaconis and L. Saloff-Coste, Comparison techniques for random walk on finite groups. *Ann. Probab.* **21** (1993), 2131–2156. Zbl 0790.60011 MR 1245303

[9] P. Diaconis and L. Saloff-Coste, Bounds for Kac's master equation. *Comm. Math. Phys.* **209** (2000), 729–755. Zbl 0953.60098 MR 1743614

[10] P. Diaconis and M. Shahshahani, Generating a random permutation with random transpositions. *Z. Wahrscheinlichkeitstheorie verw. Geb.* **57** (1981), 159–179. Zbl 0485.60006 MR 626813

[11] J. Dymara and T. Januszkiewicz, Cohomology of buildings and their automorphism groups. *Invent. Math.* **150** (2002), 579–627. Zbl 1140.20308 MR 1946553

[12] E. B. Dynkin, Semisimple subalgebras of semisimple Lie algebras. *Mat. Sb.* (*N.S.*) **30 (72)** (1952), 349–462; English transl. Amer. Math. Soc. Transl. (2) **6** (1957) , 111–243. Zbl 0048.01701 MR 0047629

[13] M. Ershov and A. Jaikin-Zapirain, Property (T) for noncommutative universal lattices. *Invent. Math.* **179** (2010), 303–347. Zbl 05661758 MR 2570119

[14] M. Ershov, A. Jaikin-Zapirain, and M. Kassabov, Property T for groups graded by root systems. In preparation.

[15] H. Garland, p-adic curvature and the cohomology of discrete subgroups of p-adic groups. *Ann. of Math.* (2) **97** (1973), 375–423. Zbl 0262.22010 MR 0320180

[16] J. E. Humphreys, *Reflection groups and Coxeter groups*. Cambridge Stud. Adv. Math. 29, Cambridge University Press, Cambridge 1990. Zbl 0768.20016 MR 1066460

[17] M. Kac, Foundations of kinetic theory. In *Proc. Third Berkeley Sympos. Math. Statist. Probability*, Vol. 3 , University of California Press, Berkeley and Los Angeles 1956, 171–197. Zbl 0072.42802 MR 0084985
http://www.lib.berkeley.edu/math/services/symposium.html

[18] M. Kassabov, Kazhdan constants for $\mathrm{SL}_n(\mathbb{Z})$. *Internat. J. Algebra Comput.* **15** (2005), 971–995. Zbl 1097.22007 MR 2197816

[19] M. Kassabov and N. Nikolov, Universal lattices and property tau. *Invent. Math.* **165** (2006), 209–224. Zbl 1139.19003 MR 2221141

[20] D. A. Každan, Connection of the dual space of a group with the structure of its closed subgroups. *Funktsional. Anal. i Prilozhen.* **1** (1967), 71–74; English transl. *Functional Anal. Appl.* **1** (1967), 63–65. Zbl 0168.27602 MR 0209390

[21] D. K. Maslen, The eigenvalues of Kac's master equation. *Math. Z.* **243** (2003), 291–331. Zbl 1016.82016 MR 1961868

[22] D. I. Panyushev, On the Dynkin index of a principal $\mathfrak{sl}_2$-subalgebra. *Adv. Math.* **221** (2009), 1115–1121. Zbl 1163.17013 MR 2518633

[23] Y. Shalom, Bounded generation and Kazhdan's property (T). *Inst. Hautes Études Sci. Publ. Math.* **90** (1999), 145–168. Zbl 0980.22017 MR 1813225

[24] Y. Shalom, The algebraization of Kazhdan's property (T). In *Internat. Congress Mathematicians*, Vol. II, European Math. Soc. Publ. House, Zürich 2006, 1283–1310. Zbl 1109.22003 MR 2275645

[25] L. Vaserstein, Bounded reduction of invertible matrices over polynomial rings by addition operations. Preprint 2007.

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