Geodesic manifolds with a transitive subset of smooth biLipschitz maps

Enrico Le Donne

Abstract. This paper is connected with the problem of describing path metric spaces that are homeomorphic to manifolds and biLipschitz homogeneous, i.e., whose biLipschitz homeomorphism group acts transitively.

Our main result is the following. Let $X = G/H$ be a homogeneous manifold of a Lie group G and let d be a geodesic distance on X inducing the same topology. Suppose that there exists a subgroup G_S of G that acts transitively on X such that each element $g \in G_S$ induces a locally biLipschitz homeomorphism of the metric space (X, d). Then the metric is locally biLipschitz equivalent to a sub-Riemannian metric. Any such metric is defined by a bracket generating G_S-invariant sub-bundle of the tangent bundle.

The result is a consequence of a more general fact that requires a transitive family of uniformly biLipschitz diffeomorphisms with a control on their differentials. It will be relevant that the group acting transitively on the space is a Lie group and so it is locally compact, since, in general, the whole group of biLipschitz maps, unlikely the isometry group, is not locally compact.

Our method also gives an elementary proof of the following fact. Given a Lipschitz sub-bundle of the tangent bundle of a Finsler manifold, both the class of piecewise differentiable curves tangent to the sub-bundle and the class of Lipschitz curves almost everywhere tangent to the sub-bundle give rise to the same Finsler–Carnot–Carathéodory metric, under the condition that the topologies induced by these distances coincide with the manifold topology.

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1. Introduction

In the last twenty years there has been a surge of interest in the geometry of non-smooth spaces and in their corresponding biLipschitz analysis. This movement arose from the interaction between active areas of mathematics concerning the theory of analysis on metric spaces [Sem96], [HK98], [Che99], [AK00], [LP01], [Hei01], [Laa02], geometric analysis [GS92], [BGP92], [CC97] along with geometric group theory,

rigidity, and quasiconformal homeomorphisms. One focus of this research has been the study of mappings between non-Riemannian metric structures such as Carnot groups and boundaries of hyperbolic groups [Pan89], [BM91], [MM95], [KL97], [Gro99], [BP00], [KB02].

In the present paper, we focus on the rigidity of certain non-smooth metric structures on manifolds, namely, geodesic metrics on manifolds that have a transitive group of biLipschitz homeomorphisms; for short they are called biLipschitz homogeneous geodesic manifolds. Every known example is locally biLipschitz equivalent to a homogeneous space G/H equipped with a Carnot–Carathéodory metric; here G is a connected Lie group and H is a closed subgroup. Any such metric, also called a sub-Riemannian metric, is defined by a bracket generating sub-bundle of the tangent bundle, also known as completely non-holonomic distribution. For surveys of this area, including how the jargon interchanges between sub-Riemannian geometry and Carnot–Carathéodory geometry, see [BBI01], [Mon02] and the papers [Gro96], [Mit85], [Bel96].

All of the 2-dimensional examples, known so far, are locally biLipschitz equivalent to the Euclidean plane. One may therefore ask whether these are in fact the only examples. Our main goal in this paper is to show that under additional assumptions this is indeed the case.

The motivation for this question comes from several sources. First, one can view this as an analogue, in the biLipschitz category, of Hilbert's fifth problem on the characterization of Lie groups, solved in [MZ74], or the conjectural Bing–Borsuk characterization of topological manifolds. Another source of motivation is the work of Berestovskii [Ber88], [Ber89a], [Ber89b], who showed that a finite dimensional geodesic metric space with transitive isometry group is isometric to an example G/H as above, except that in the general case one has to use a Finsler–Carnot–Carathéodory metric as opposed to a Riemann–Carnot–Carathéodory metric. In addition, a coarse version of this question in the two dimensional case has arisen in several situations in Geometric Group Theory, and is directly related with the problem considered. For example in [KK06], with the purpose of analyzing 2-dimensional Poincaré duality groups over commutative rings, Michael Kapovich and Bruce Kleiner studied quasi-homogeneous quasi-planes, i.e., simply-connected metric cell complexes satisfying a coarse Poincaré duality in dimension 2 on which the group of quasi-isometries acts transitively. Quasi-planes also appeared in [KK05].

We consider length metrics since otherwise there are many metrics with transitive isometry group that are not locally biLipschitz to the standard one, e.g., even on the real line $\mathbb{R}$, all distances $d(s,t) := \sqrt[d]{|s-t|}$, for $d > 1$, are translation invariant. In fact, biLipschitz homogeneous curves have been studied deeply in [Bis01], [GH99].

We show that, locally, the examples mentioned above are the only examples if the biLipschitz maps giving homogeneity come from a Lie group acting by diffeomorphisms. This assumption is equivalent to the space being homeomorphic to a homogeneous space $X = G/H$, with G containing a transitive subgroup of biLipschitz homeomorphisms. To be precise, we prove the following:

Theorem 1.1. *Let G be a Lie group and H be a closed subgroup. Let $X = G/H$ be the corresponding homogeneous manifold equipped with a geodesic distance d inducing its natural topology. Suppose that there exists a subgroup G_S of G that acts transitively on X, and that acts by maps that are locally biLipschitz with respect to d. Then there exists a completely non-holonomic G_S-invariant distribution such that any Carnot–Carathéodory metric coming from it gives a metric that is locally biLipschitz equivalent to d.*

Notice that we do not assume uniform bounds on the biLipschitz constants. Indeed, if one assumes that X is a geodesic metric space with a transitive group G of L-biLipschitz homeomorphisms, then by taking the supremum over the G-orbit of the distance function, and then the associated path metric, one gets an L-biLipschitz equivalent metric with respect to which G acts by isometries. One can then apply Berestovskii's result in [Ber89b] mentioned above. However, without the extra hypothesis about uniformity of biLipschitz constants, the argument breaks down altogether. Our work can be considered as the first step toward a coarse version of Berestovskii's result. In fact, the main steps of our proof and Berestovskii's strategy share some common features, although his method is more algebraic.

Our assumption is connected with the fact that, in general, the full group of biLipschitz maps is not a locally compact group. On the other hand, as a consequence of the Ascoli–Arzelà Theorem, the isometry group of the space X is a locally compact topological group. Berestovskii uses this fact in the case that the isometry group acts transitively to apply the celebrated Gleason–Montgomery–Zippin Theorem and subsequent work [MZ74], Chapter III. The result is a reduction to the case when the action of the isometry group is topologically conjugate to a transitive smooth action of a Lie group on a smooth manifold, in fact on a homogeneous space $X = G/H$. Thus, in the case of biLipschitz homogeneous geodesic manifolds, we shall assume that we already have a similar structure, in the sense that the biLipschitz maps giving homogeneity are coming from a Lie group G acting on a quotient G/H.

Since the problem is in fact local, another formulation is as follows. Suppose that we have a geodesic metric on a neighborhood of the origin in $\mathbb{R}^n$, and a collection of biLipschitz maps that sends the origin to any point in this neighborhood. Furthermore, suppose that these biLipschitz maps are, in fact, elements of a smooth "local" action. Then we can conclude that in a neighborhood of the origin this metric is biLipschitz equivalent to a Carnot–Carathéodory metric. See the next section for more general statements.

Here is a concrete application of Theorem 1.1.

Example (Affine maps giving homogeneity). Suppose that we have a geodesic distance on the plane such that for any two points there exists an affine map that is locally biLipschitz with respect to the geodesic distance and sends the first point to the second. Then we can conclude that the distance is locally biLipschitz equivalent to the Euclidean one.

Indeed, one can apply Theorem 1.1 where G is the group of affine diffeomorphisms and $H = \mathrm{GL}_2(\mathbb{R})$. The group G_S is the intersection between G and the group of maps that are locally biLipschitz with respect to d. The metric is in fact Riemannian since there is no proper sub-Riemannian structures in dimension 2.

Several references [Ber88], [Gro96], [Mon02] state that in the definition of Carnot–Carathéodory metrics, one gets the same metric when considering either piecewise continuously differentiable, or Lipschitz (or absolutely continuous curves), as horizontal curves. However, we could not find any proof of this fact in the literature. An element of the proof of Theorem 1.1, detailed in step 5 below, can be used to give a simple proof of this fact in the case that the two topologies are assumed to coincide. See the Appendix for details.

Theorem 1.2. *Let X be a Finsler manifold, equipped with a locally Lipschitz sub-bundle (of the tangent bundle). Then both the class of piecewise $C^{1,1}$-curves tangent to the sub-bundle and the class of Lipschitz curves almost everywhere tangent to the sub-bundle give rise to the same Finsler–Carnot–Carathéodory metric, under the condition that the topologies induced by these distances coincide with the manifold topology.*

1.1. An outline of the proof of Theorem 1.1. *Step* 1. Argue that we can assume that the group G is embedded and closed in the homeomorphism group of the space $X = G/H$. Thus, every time we will apply the Ascoli–Arzelà theorem, the limits of C^0-converging subsequences will still be elements of the group and the convergence will be, in fact, C^∞.

Step 2. Apply a Baire category argument to get a locally transitive set of elements of the group that are C^k close to id_X and are uniformly biLipschitz with respect to both the metric d and any Riemannian metric on X that we fixed.

Step 3. Prove that, locally, the distance d is greater than a multiple of some smooth Riemannian distance. Therefore, the geodesics for d are Lipschitz maps for the smooth distance; thus, they are differentiable almost everywhere.

Step 4. Define a sub-bundle of the tangent bundle related to the set of velocities of the geodesics. Use it to define a Carnot–Carathéodory metric d_{CC}. It will be easy to argue that $d \geq \mathrm{Constant} \cdot d_{\mathrm{CC}}$ locally.

Step 5. Prove that $d \leq \mathrm{Constant} \cdot d_{\mathrm{CC}}$ locally.

1.2. Organization of the paper. In the next section, we discuss generalizations and variations of Theorem 1.1., Indeed, steps 3, 4, and 5 above constitute a more general fact, specified in Theorem 2.1, that implies Theorem 1.1. In Section 3, we show that locally the distance d is greater than a multiple of any smooth distance. An immediate consequence is the absolute continuity of geodesics: any curve that is rectifiable with respect to the geodesic distance d is differentiable almost everywhere. In Section 4, we define a subset of the tangent bundle, and prove that it is, in fact, a sub-bundle;

this allows us to define an associated Carnot–Carathéodory metric. In Section 5, we prove the biLipschitz equivalence between the geodesic metric d, and the newly defined Carnot–Carathéodory metric. In Section 6, we complete steps 1 and 2 above to show how Theorem 1.1 follows from Theorem 2.1. In Appendix 6.3, we repeat the argument to prove Theorem 1.2.

2. The general criterion and some consequences

Theorem 1.1 is a consequence of a more general fact. Given a geodesic metric on a domain in $\mathbb{R}^n$, whenever there is a transitive family of uniformly biLipschitz diffeomorphisms with a control on their differentials, the metric is locally biLipschitz equivalent to a Carnot–Carathéodory metric. Namely, this conclusion can be reached when there is a family $\mathcal{F}$ of C^2-diffeomorphisms and a base point p_0 such that the orbit $\mathcal{F}(p_0)$ is a neighborhood of p_0, the elements of $\mathcal{F}$ are uniformly biLipschitz for both the distance d and a fixed Riemannian metric, and the family of the differentials has a uniform modulus of continuity.

We will denote by $\|A\|_{\mathrm{Lin}(\mathbb{R}^n)}$ the norm as linear operator of a matrix $A \in \mathrm{Lin}(\mathbb{R}^n)$.

Theorem 2.1. *Let $N \subset \mathbb{R}^n$ be a compact neighborhood of $0 \in \mathbb{R}^n$, equipped with a geodesic distance d that induces the standard topology. Suppose that there exists a family $\mathcal{F} \subset C^2(N, \mathbb{R}^n)$ of local C^2-diffeomorphisms satisfying:*

***Homogeneity**: for any $p \in N$, there is $f \in \mathcal{F}$, so that $f(0) = p$.*

***Uniform biLipschitz**: there exists $k \in \mathbb{R}$ such that each $f \in \mathcal{F}$ is a k-biLipschitz map on $N \cap f^{-1}(N)$, with respect to both the Euclidean metric and the distance d.*

***Uniform modulus of continuity of derivatives**: the family $\{df\}_{f \in \mathcal{F}}$ has a uniform modulus of continuity, i.e., there exists an increasing function $\eta \colon [0, \infty) \to [0, \infty)$, with $\eta(0) = 0$, so that, for any $f \in \mathcal{F}$ and $x, y \in N$,*

$$\|(df)_x - (df)_y\|_{\mathrm{Lin}(\mathbb{R}^n)} \leq \eta(|x - y|). \tag{1}$$

Then there exists an $\mathcal{F}$-invariant, C^1-sub-bundle $\Delta \subset TN$ such that if d_Δ is any sub-Riemannian metric coming from Δ, then the geodesic metric d is locally biLipschitz equivalent to d_Δ.

The conclusion of the above theorem is that the regularity of the bundle is C^1. However, we will prove that the distribution of Theorem 1.1 is smooth in Proposition 6.8.

The following variation shows how Theorem 2.1 can be used:

Theorem 2.2. *Let d be a geodesic metric on a neighborhood N of the origin in $\mathbb{R}^n$. Suppose that there exists a C^2-map*

$$F : \Lambda \times U \to N, \quad (\lambda, p) \mapsto F_\lambda(p),$$

where $U \subset N \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}^N$ are neighborhoods of the origin, with the property that, for some neighborhood of the origin $\Lambda_0 \subset \Lambda$, the set

$$\{F_\lambda(0) : F_\lambda \text{ is biLipschitz with respect to } d, \lambda \in \Lambda_0\}$$

is a neighborhood of the origin. Then in a (possible smaller) neighborhood of the origin the metric d is biLipschitz equivalent to a Carnot–Carathéodory metric.

Proof. Clearly, we may assume $F_0 = \mathrm{id}_U$, and U and Λ_0 compact sets. We want to show that the hypotheses of Theorem 2.1 apply. Indeed, first, since the map F is C^2, the condition about the family of differentials having a uniform modulus of continuity is immediately verified. Next, call m-biLipschitz those maps that are m-biLipschitz with respect to both the Euclidean distance and the distance d, and consider the sets

$$A_m := \{p \in U : F_\lambda(0) = p, \, F_\lambda \, m\text{-biLipschitz}, \, \lambda \in \Lambda_0\}.$$

Each A_m is closed. Indeed, take $F_{\lambda_j}(0) \in A_m \to p$, with F_{λ_j} an m-biLipschitz map, then by Ascoli–Arzelà Theorem, some subsequence converges: $F_{\lambda_j} \to F$ uniformly and F is still an m-biLipschitz map. Since here we have assumed Λ_0 compact, $\lambda_j \to \lambda_\infty \in \Lambda_0$ up to subsequence, so $F = F_{\lambda_\infty}$, so $p \in A_m$, i.e., A_m is closed. Thus, by the Baire Category Theorem, for some $m \in \mathbb{N}$, the set $\{F_\lambda(0) : F_\lambda \, m\text{-biLipschitz}\}$ is a neighborhood of a point in U that we can assume to be the origin. Theorem 2.1 can be applied to conclude. □

We now give a generalization of Theorem 1.1 that points out the importance of the assumption that the group acting transitively on the space is a Lie group and so is locally compact. Notice that the whole group of biLipschitz maps, unlikely the isometry group, is not in general locally compact. According to Montgomery–Zippin's work [MZ74], if a locally compact group acts continuously, effectively, and transitively on a manifold, then it is a Lie group. Here and in what follows, manifolds are supposed to be connected, however, Lie groups can have infinitely many components. Thus, Theorem 1.1 yields the following generalization:

Theorem 2.3. *Let (M, d) be a manifold endowed with a geodesic metric (inducing the same topology). Let G be a locally compact group with a countable base. Let $G \times M \to M$ be a continuous, effective action of G on M. Suppose that there exists a subgroup of G acting transitively on M by biLipschitz maps (with respect to the metric d). Then (M, d) is locally biLipschitz equivalent to a homogeneous space equipped with a Carnot–Carathéodory metric.*

Proof. Following, [MZ74], any locally compact group G has the property of having an open subgroup $G' < G$ that is the inverse limit of Lie groups; with the language of Montgomery–Zippin's book, G' has property A.

First, we claim that, for any $q \in M$, the orbit $G' \cdot q$ of q under G' is open. This is because the projection $G \to G/H$, where H is the stabilizer of the action, is open and the orbit action $G/H \to M$ is a homeomorphism by a standard argument [Hel01], Theorem 3.2, p. 121.

Now we show that the G'-action is still transitive. Indeed, fix a point $p \in M$, suppose by contradiction that $G' \cdot p \neq M$. Hence,

$$M = (G' \cdot p) \bigsqcup (\bigcup_{q \notin G' \cdot p} G' \cdot q)$$

is a disjoint union of two non-empty open sets of M. This contradicts the fact that M is connected.

Thus G' satisfies the hypotheses of Montgomery–Zippin's Theorem [MZ74], p. 243, so G' is a Lie group. Therefore G' does not have small subgroups, thus neither does G. By the Gleason–Yamabe Theorem, cf. [MZ74], Chapter III, G is a Lie group. Theorem 1.1 can be applied to conclude.   □

3. The given metric is greater than a Riemannian metric

We now start the proof of Theorem 2.1. In particular this section is devoted to showing that curves that are rectifiable with respect to the geodesic distance d, are differentiable almost everywhere. To prove this we will show (cf. Proposition 3.3) that the given metric d is locally greater than a Riemannian metric.

We begin with a couple of lemmas. The first asserts that, if we just want C^1-maps, we may assume continuity of the differentials when the evaluation at a base point goes to the base point.

Lemma 3.1. *Under the assumptions of Theorem 2.1, there exists a family $\{f_p\}_{p \in N} \subset C^1(N, \mathbb{R}^n)$ with $f_p(0) = p$, that is uniformly biLipschitz, satisfies the condition that the family $\{df_p\}$ has a uniform modulus of continuity, and satisfies*

Continuity of $(df_p)_0$ ***at*** 0: *the map*

$$N \to \mathrm{Lin}(\mathbb{R}^n), \quad p \mapsto (df_p)_0,$$

is continuous at $p = 0$. In other words,

$$\|(df_0)_0 - (df_p)_0\|_{\mathrm{Lin}(\mathbb{R}^n)} \to 0, \quad as\ p \to 0. \tag{2}$$

Proof. This is another application of the Baire Category Theorem. Set $A_0 := N$. For each $j \in \mathbb{N}$, let $\{V_m^{(j)}\}_{m \in \mathbb{N}}$ be a countable cover of $\mathrm{Lin}(\mathbb{R}^n)$ by balls of diameter

$< j^{-1}$, with respect to some fixed metric inducing the standard topology. Inductively define sets $A_j \subseteq \mathbb{R}^n$ as follows. Let $\mathcal{F}$ be the family from Theorem 2.1 and let

$$A_{j,m} = \{p \in A_j : \text{there exists } f \in \mathcal{F} \text{ such that } f(0) = p, \ (df)_0 \in V_m^{(j)}\}.$$

Then $A_j = \bigcup_m A_{j,m}$. By the Baire Category Theorem, there is $m_j \in \mathbb{N}$ so that there exists a ball A_{j+1} of radius $< j^{-1}$ in the closure of A_{j,m_j}. Let q be in the intersection of the closures of the A_j's. By post-composing with a map that sends q to 0 we may assume that $q = 0$.

For any $p \neq 0$ in N, we have $p \in A_j \setminus A_{j+1}$, for some j. Then we define f_p as follows. Since $p \in A_j$, there exists sequences $\{p_k\} \subseteq A_{j-1}, \{f_k\} \subseteq \mathcal{F}$ such that $p_k \to p$, $f_k(0) = p_k$, and $(df_k)_0 \in V_{m_{j-1}}^{(j-1)}$. Note that the functions f_k are uniformly Lipschitz and converge at 0, and that the sequence $\{(df_k)_0\}$ is equicontinuous and uniformly bounded at 0. Thus, by the Ascoli–Arzelà Theorem, a subsequence of $\{f_k\}$ converges in the C^1-topology. Let f_p be one such limit. We can apply this same argument to any sequence $\{f_{p_j}\}$ with $p_j \to 0$ and define f_0 as one of the limits of some sequence $\{f_{p_{j_i}}\}$ converging in the C^1-topology. All of the constructed maps f_p's are C^1 and still are uniformly biLipschitz and their derivatives have uniform modulus of continuity. Moreover, since $\bigcap_j V_{m_j}^{(j)}$ is a single point,

$$(df_p)_0 \to (df_0)_0, \quad \text{as } p \to 0. \qquad \square$$

The next lemma gives uniform control on the deviation of elements in the family $\mathcal{F}$ from their linear approximations.

Lemma 3.2 (Uniform distortion control)**.** *If a family $\mathcal{F}$ has the property that the family $\{df\}_{f \in \mathcal{F}}$ is equicontinuous, so (1) holds, then there exists an increasing function $\omega(t)$ such that, for any element $f \in \mathcal{F}$ and $y \in N$,*

$$|(df)_0(y) + f(0) - f(y)| \leq \omega(|y|), \tag{3}$$

and $\frac{\omega(t)}{t} \to 0$ as $t \to 0$.

Proof. By assumption, there exists an increasing function $\eta \colon [0, \infty) \to [0, \infty)$, with $\eta(0) = 0$, so that

$$\|(df)_x - (df)_y\|_{\mathrm{Lin}(\mathbb{R}^n)} \leq \eta(|x - y|)$$

for any $f \in \mathcal{F}$ and $x, y \in N$. Let $\omega(t) := \eta(t)t$. Thus we just need to show (3) for any $f \in \mathcal{F}$ and $y \in N$. Consider the function $t \in \mathbb{R} \mapsto f(ty)$. By the fundamental theorem of calculus and the chain rule,

$$f(y) - f(0) = \int_0^1 \frac{d}{dt} f(ty) \, dt = \int_0^1 (df)_{ty} \cdot y \, dt.$$

So,

$$|(df)_0(y) + f(0) - f(y)| \leq \int_0^1 |(df)_0 \cdot y - (df)_{ty} \cdot y| \, dt$$

$$\leq \int_0^1 \|(df)_0 - (df)_{ty}\| \, |y| \, dt$$

$$\leq \int_0^1 \eta(|ty|)|y| \, dt$$

$$\leq \int_0^1 \eta(|y|)|y| \, dt = \omega(|y|). \qquad \square$$

This next proposition is the core of the paper. We get the first important relation between the geodesic distance d and the Euclidean metric.

Proposition 3.3. *Let (N, d) satisfy the assumptions of Theorem 2.1. Then some multiple of the distance d is greater than the Euclidean one, i.e., there exists $C > 0$ such that*

$$\|p - q\| \leq C \cdot d(p, q) \tag{4}$$

for any $p, q \in N$.

Proof. It suffices to prove (4) for p, q in a neighborhood V_0 of the origin. Indeed, since the family $\mathcal{F}$ acts transitively by k-biLipschitz maps with respect to both metrics, it would follow that any point $p \in N$ has a neighborhood V where $\|q_1 - q_2\| \leq Ck^2 d(q_1, q_2)$ for $q_1, q_2 \in V$. Let $f \in \mathcal{F}$ so that $f(0) = p$. Then, for any $q_1, q_2 \in V := f(V_0)$, we have

$$\|q_1 - q_2\| \leq k\|f^{-1}(q_1) - f^{-1}(q_2)\| \leq Ckd(f^{-1}(q_1), f^{-1}(q_2)) \leq Ck^2 d(q_1, q_2).$$

Now, since both distances are geodesic, we conclude that $\|p - q\| \leq Ck^2 d(p, q)$ for all points $p, q \in N$.

Suppose that (4) is not true. So there is a sequence of pairs of points p_n, q_n where the ratio of the metrics is smaller and smaller,

$$\frac{d(p_n, q_n)}{\|p_n - q_n\|} < \frac{1}{n},$$

for all $n \in \mathbb{N}$. We can assume that $q_n = 0$ since we can move it to 0 using the transitivity of k-biLipschitz maps. Indeed, let f_n be a k-biLipschitz diffeomorphism such that $f_n(q_n) = 0$, for each $n \in \mathbb{N}$. Now,

$$\frac{1}{n} > \frac{d(p_n, q_n)}{\|p_n - q_n\|} \geq \frac{\frac{1}{k}d(f_n(p_n), f_n(q_n))}{k\|f_n(p_n) - f_n(q_n)\|} = \frac{1}{k^2} \frac{d(f_n(p_n), 0)}{\|f_n(p_n)\|}.$$

After replacing p_n with $f_n(p_n)$ and possibly changing indices, we get a sequence of points $p_n \in N$ with the property

$$\frac{d(p_n, 0)}{\|p_n\|} < \frac{1}{n}. \tag{5}$$

Note that $p_n \to 0$ as $n \to \infty$. Indeed, p_n is in the bounded set N, so $d(p_n, 0) \leq \frac{1}{n}\|p_n\| \leq \frac{1}{n}\operatorname{Diam}(N) \to 0$. Moreover, d gives the usual topology, so $p_n \to 0$.

By Lemma 3.1, there is a neighborhood U of 0 such that

$$\|(df_p)_0 - (df_0)_0\|_{\operatorname{Lin}(\mathbb{R}^n)} \leq \frac{1}{4} \quad \text{for all } p \in U.$$

Thus, since $f_0 = \operatorname{id}$, we have, for any vector $v \in \mathbb{R}^2$,

$$\|(df_p)_0(v) - v\| \leq \|(df_p)_0 - \operatorname{Id}\|_{\operatorname{Lin}(\mathbb{R}^n)}\|v\| \leq \frac{1}{4}\|v\| \quad \text{for all } p \in U. \tag{6}$$

All of the balls in the argument shall have the origin as center. Consider a Euclidean ball of radius R contained in the neighborhood U. Since the topologies induced by the two distances are the same, we can find a d-ball of radius $r > 0$ inside the Euclidean one. We want to use the fact that the points p_n go to zero, and are different from 0, to construct a 'pseudo' geodesic as a chain of a controlled number of points, whose final point $H(p_n)$ will be outside the d ball of radius r.

Let ω be the function from Lemma 3.2 that controls the extent to which maps in $\mathcal{F}$ fail to be linear at 0. Now, fix $n \in \mathbb{N}$ so large that $\omega(\|p_n\|) < \frac{1}{4}\|p_n\|$. Since p_n is now fixed, we will drop the subscript $p = p_n$. Therefore by estimates (6) and (3), we have

$$\|p + f(0) - f(p)\| \leq \frac{1}{2}\|p\| \quad \text{for all } f \in \mathcal{F}. \tag{7}$$

Starting with $f_0 = \operatorname{id}$, by recurrence, for $j \in \mathbb{N}$, choose $f_j \in \mathcal{F}$ such that $f_j(0) = f_{j-1}(p)$. We claim that there exists $j \leq \frac{2R}{\|p\|}$ for which $f_j(p)$ is not in the Euclidean ball of radius R. In fact, each $f_j(p)$ is quantitatively farther from 0 than $f_{j-1}(p)$. Indeed, we consider the projection $\pi_p(f_j(p))$ of $f_j(p)$ on the direction of p and prove that

$$\pi_p(f_j(p)) := \frac{\langle f_j(p)|p\rangle}{\|p\|} \geq \frac{j}{2}\|p\|. \tag{8}$$

We prove (8) by induction using the estimates (7) and the fact that $\langle u|v\rangle \geq \langle w|v\rangle - \|u - w\| \cdot \|v\|$ for all $u, v, w \in \mathbb{R}^2$. For $j = 1$,

$$\langle f_j(p)|p\rangle \geq \langle p + p|p\rangle - \|f_1(p) - p - p\| \cdot \|p\|$$

$$\geq 2\|p\|^2 - \frac{1}{2}\|p\|^2 \geq \frac{1}{2}\|p\|^2.$$

Assume that (8) is proved for j. Then

$$
\begin{aligned}
\langle f_{j+1}(p)|p\rangle &\geq \langle f_j(p) + p|p\rangle - \|f_{j+1}(p) - f_j(p) - p\| \cdot \|p\| \\
&\geq \langle f_j(p)|p\rangle + \langle p|p\rangle - \|f_{j+1}(p) - f_{j+1}(0) - p\| \cdot \|p\| \\
&\geq \frac{j}{2}\|p\|^2 + \|p\|^2 - \frac{1}{2}\|p\|^2 \geq \frac{j+1}{2}\|p\|^2.
\end{aligned}
$$

Therefore, for $j^* := \frac{2R}{\|p\|}$, we have

$$
\frac{\langle f_{j^*}(p)|p\rangle}{\|p\|} \geq \frac{2R}{2\|p\|}\|p\| = R.
$$

So $f_{j^*}(p)$ is not in the Euclidean ball of radius R. Let $H(p)$ be the first point in the sequence $\{f_j(p)\}_{j=0}^{j^*}$ that leaves the ball. Such ball contains the ball of radius r with respect to the distance d. Therefore,

$$
\begin{aligned}
r \leq d(H(p_n), 0) &\leq \sum_{j=0}^{j^*} d(f_j(0), f_j(p)) \\
&\leq \sum_{j=0}^{j^*} k\,d(0, p) \leq k(j^* + 1)d(0, p) \leq k\frac{2R}{\|p\|}d(0, p) + k\,d(0, p).
\end{aligned}
$$

Consider now that $p = p_n$ satisfies (5), so we have

$$
0 < r \leq k\frac{2R}{\|p_n\|}d(p_n, 0) + k\,d(0, p_n) \leq \frac{2kR}{n} + k\,d(0, p_n).
$$

But then we have a contradiction since the value r, on the left, is constant and greater than 0, while the right-hand side goes to 0 as $n \to \infty$. $\square$

Remark 3.4. From Proposition 3.3 we know that some dilation of the distance d is greater than the Euclidean one. So, rescaling the metric d if necessary, we may assume that $\|\cdot\| \leq d$ in N. From this we can conclude that any d-geodesic γ is a 1-Lipschitz map with respect to the Euclidean distance, since

$$
|t_1 - t_2| = d(\gamma(t_1), \gamma(t_2)) \geq \|\gamma(t_1) - \gamma(t_2)\|.
$$

More generally, if γ is a d-rectifiable curve parametrized by (finite) speed, say smaller than s, then

$$
s|t_1 - t_2| \geq L_d(\gamma(t_1, t_2)) \geq d(\gamma(t_1), \gamma(t_2)) \geq \|\gamma(t_1) - \gamma(t_2)\|.
$$

In other words, γ is an s-Lipschitz map with respect to the Euclidean distance, so it is Lipschitz in each coordinate. At this point we are allowed to use a classical fact in Lipschitz analysis, i.e., Rademacher's Theorem: on $\mathbb{R}$, any Lipschitz function is differentiable almost everywhere. Hence any d-rectifiable curve is differentiable almost everywhere, in particular, it is rectifiable with respect to the Euclidean distance.

4. The construction of the Carnot–Carathéodory metric

To prove Theorem 1.1 we need to find a sub-bundle Δ of the tangent bundle. As a result of Proposition 3.3 (cf. Remark 3.4), we know that any d-rectifiable curve is differentiable almost everywhere, thus it makes sense to look at the set of velocities of d-rectifiable curves. For any $p \in N$, we can now define a subset of $T_p N$ as

$$\Delta_p := \{\dot{\gamma}(0) : \gamma(0) = p, \, \gamma \, d\text{-rectifiable with finite speed and differentiable at } 0\}, \tag{9}$$

and we let $\Delta = \bigcup_p \Delta_p \subseteq TN$.

   In the next lemma we prove some properties of Δ such as the fact that it is a sub-bundle, together with some control estimates, needed later, on some family of curves Γ representative of Δ.

Lemma 4.1 (Control on curves Γ representing Δ). *At any point $p \in N$, the set Δ_p is a vector space whose dimension is independent of p. The set $\Delta = \bigcup_p \Delta_p \subset TN$ is invariant under $\mathcal{F}$. Moreover, there exists a special class of curves Γ and a constant $S > 0$ with the following property: for any $p \in N$ and any $v \in \Delta_p$, there exists $\gamma \in \Gamma$ such that $\gamma(0) = p$, $\dot{\gamma}(0) = v$, and for any $t \in \mathbb{R}$,*

$$\text{length}_d(\gamma[0,t]) \leq S\|v\|t. \tag{10}$$

   Moreover, there is also an increasing function $\omega_\Gamma \colon \mathbb{R} \to \mathbb{R}$ such that

$$\|\gamma(t) - (\gamma(0) + \dot{\gamma}(0)t)\| \leq \omega_\Gamma(t\|\dot{\gamma}(0)\|) \tag{11}$$

for any $\gamma \in \Gamma$, and $\frac{\omega_\Gamma(t)}{t} \to 0$ as $t \to 0$.

Proof. Take $w_1, \ldots, w_m \in \Delta_0$ a maximal set of linearly independent vectors coming from paths $\gamma_1, \ldots, \gamma_m$. We may assume that the γ_j's are parametrized by finite d-speed s_j, and $\|w_j\| = \|\dot{\gamma}_j(0)\| = 1$ and $\text{length}_d(\gamma_j[0,t]) \leq s_j t$ for all $j = 1, \ldots, m$ and all t where the curves are defined. Note that for each γ_j there is an increasing function ω_j so that

$$\|\gamma_j(t) - (\gamma_j(0) + \dot{\gamma}_j(0)t)\| \leq \omega_j(t).$$

Let $S := \max_j s_j$ and $\omega_\Gamma(t) := \max_j\{\omega_j(t)\}$. Then each γ_j satisfies (10) and (11).

   To prove that Δ_0 is a vector space we will show that, for any $v \in \text{Span}(\Delta_0)$, a limit of "zig-zag" curves, constructed using the γ_j's, is still a rectifiable curve and such limit has tangent at zero equal to v.

   For simplicity of exposition, assume $v = \frac{1}{2}(w_1 + w_2)$. The zig-zag curves are defined recursively by

$$\sigma_\varepsilon(t) := \begin{cases} \gamma_1(t), & 0 \leq t < \varepsilon, \\ f_{\sigma_\varepsilon(n\varepsilon)}(\gamma_i(t - n\varepsilon)), & n\varepsilon \leq t < (n+1)\varepsilon, \end{cases} \tag{12}$$

where $i = n \pmod 2$. Each curve $\sigma_\varepsilon(t)$ is d-rectifiable with uniformly bounded speed, i.e., (10) holds. By the Ascoli–Arzelà Theorem, the curves σ_ε converge, up to subsequence. Moreover the limit, denoted by $\sigma(t) = \sigma^{(v)}(t)$, is a d-rectifiable curve parameterized with same finite speed. In other words, (10) holds for $\sigma^{(v)}$ too.
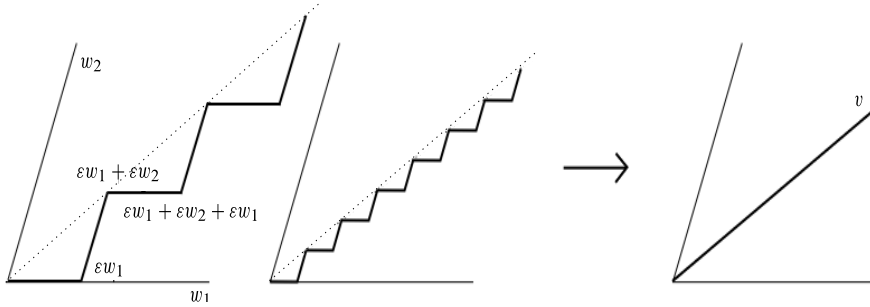

Figure 1. The zigzag curves converge to a curve whose tangent at the origin is parallel to the sum of the two vectors w_1 and w_2.

For a better understanding of the curves $\sigma_\varepsilon(t)$ and their limit, consider first the easier case when the γ_j's were lines, $\gamma_j(t) = w_j t$, and the f_p's were translations, see Figure 1 above. In this case, the zig-zag curve is

$$\hat{\sigma}_\varepsilon(t) := \varepsilon w_1 + \varepsilon w_2 + \varepsilon w_1 + \cdots + \bar{t} w_i,$$

where $t = n\varepsilon + \bar{t}$ and $i = n \pmod 2$. In the sum there are $n + 1$ terms. Thus,

$$\hat{\sigma}_\varepsilon(t) = \frac{n}{2}\varepsilon w_1 + \frac{n}{2}\varepsilon w_2 + o(\varepsilon) = \frac{t}{2}w_1 + \frac{t}{2}w_2 + o(\varepsilon) = tv + o(\varepsilon)$$

is ε-close to the line tv. Therefore, the curve $\hat{\sigma}(t) := \lim_{\varepsilon \to 0} \hat{\sigma}_\varepsilon(t)$ is such that

$$\frac{d\hat{\sigma}}{dt}(0) = v.$$

Define now another auxiliary curve. Let $f_m := f_{\sigma_\varepsilon(m\varepsilon)}$ and M_m be the matrix $(df_m)_0$. Set

$$\hat{\hat{\sigma}}_\varepsilon(t) := \varepsilon w_1 + \varepsilon M_1 w_2 + \varepsilon M_2 w_1 + \cdots + \bar{t} M_i w_i.$$

By (2), there is a neighborhood U of 0 where $\|(df)_0 - \mathrm{Id}\| < \delta$ for all $f \in \mathcal{F}$ with $f(0) \in U$, and so $\|(df)_0\| < 1 + \delta$ for all $f \in \mathcal{F}$. Let $t^* > 0$ such that

$$\sigma_\varepsilon(t), \hat{\sigma}_\varepsilon(t), \hat{\hat{\sigma}}_\varepsilon(t) \in U \quad \text{for all } t \in [0, t^*].$$

Then we have

$$\|\hat{\sigma}_\varepsilon(t) - \hat{\hat{\sigma}}_\varepsilon(t)\| \le \sum_{i=1}^{n} \|\mathrm{Id} - M_i\| \varepsilon \|w_i\| \le n\varepsilon\delta \le \frac{t}{\varepsilon}\varepsilon\delta = t\delta.$$

Also,

$$\begin{aligned}
\|\sigma_\varepsilon(t) - \hat{\hat{\sigma}}_\varepsilon(t)\| &= \big\| f_n(\gamma_i(\bar{t})) - \varepsilon w_1 - \varepsilon M_1 w_2 - \varepsilon M_2 w_1 - \cdots - \bar{t} M_i w_i \big\| \\
&\leq \| f_n(\gamma_i(\bar{t})) - (df_n)_0 \gamma_i(\bar{t}) - f_n(0) \| \\
&\quad + \| (df_n)_0 \gamma_i(\bar{t}) - \bar{t}(df_n)_0 \dot{\gamma}_i(0) \| \\
&\quad + \| f_{n-1}(\gamma_{i-1}(\bar{t})) - \varepsilon w_1 - \varepsilon M_1 w_- \cdots - \bar{t} M_{i-1} w_{i-1} \| \\
&:= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

Consider the last equation as the sum of three terms: I, II, and III. Note that the first term can be bounded, cf. (3), by

$$\mathrm{I} \leq \omega(\|\gamma_i(\bar{t})\|) \leq \omega(\varepsilon).$$

The second term is bounded by

$$\mathrm{II} \leq \|(df_n)_0\| \|\gamma_i(\bar{t}) - \bar{t}\dot{\gamma}_i(0)\| \leq (1+\delta)\omega_\gamma(\varepsilon).$$

Finally, the third term is similar to the initial term: the right-hand side in the calculation above, except that we have one term less. Therefore, we can iterate the procedure and get

$$\|\sigma_\varepsilon(t) - \hat{\hat{\sigma}}_\varepsilon(t)\| \leq n(\omega(\varepsilon) + (1+\delta)\omega_\gamma(\varepsilon)) \leq (t+1)\Big(\frac{\omega(\varepsilon)}{\varepsilon} + (1+\delta)\frac{\omega_\gamma(\varepsilon)}{\varepsilon}\Big).$$

We are now ready to calculate the derivative at 0 of $\sigma(t)$. Indeed, for all $t \in U$,

$$\begin{aligned}
\Big\| \frac{\sigma(t)}{t} - v \Big\| &= \lim_{\varepsilon \to 0} \Big\| \frac{\sigma_\varepsilon(t)}{t} - v \Big\| \\
&\leq \lim_{\varepsilon \to 0} \Big(\Big\| \frac{\hat{\sigma}_\varepsilon(t)}{t} - v \Big\| + \Big\| \frac{\hat{\sigma}_\varepsilon(t) - \hat{\hat{\sigma}}_\varepsilon(t)}{t} \Big\| + \Big\| \frac{\hat{\hat{\sigma}}_\varepsilon(t) - \sigma_\varepsilon(t)}{t} \Big\| \Big) \\
&\leq 0 + \frac{t\delta}{t} + \lim_{\varepsilon \to 0} \frac{t+1}{t}\Big(\frac{\omega(\varepsilon)}{\varepsilon} + (1+\delta)\frac{\omega_\gamma(\varepsilon)}{\varepsilon} \Big) \\
&= \delta.
\end{aligned}$$

Since δ is arbitrarily small, we have $\frac{d\sigma}{dt}(0) = v$ for $t \to 0$.

Therefore $v \in \Delta_0$, i.e., Δ_0 is a vector space.

From the transitive action of biLipschitz maps we have that Δ is invariant under $\mathcal{F}$, so the dimension of Δ_p is constant.

The class Γ is defined to be the curves $\{ f_p \circ \sigma^{(v)} \}$ for p in a neighborhood of the origin and $v \in \Delta_0$. Such curves satisfy inequality (11) for a suitable ω_Γ, since it is true for the γ_j's that are in a finite number, so also for the zig-zag limits and finally for all curves in Γ, using that (3) implies that the f's have a controlled distortion. For a similar reason, the curves in Γ have uniform bound on the speed, i.e., (10) holds.

$\square$

From the previous lemma we have that, for each $p \in N$, the set Δ_p is a k-dimensional plane. So the map $p \mapsto (p, \Delta_p)$ is a (not necessarily continuous) section of the Grassmannian bundle. In the next subsection we will prove that this map is C^1 and so Δ is a C^1-sub-bundle.

We have noticed, more than once, that the d-geodesics in our setting are Lipschitz curves with respect to the Euclidean metric, therefore they are absolutely continuous functions, i.e., they are differentiable almost everywhere and each curve is the integral of its derivative that is a priori just an L^1 function. On the other hand, each absolutely continuous curve can be reparametrized to be Lipschitz with respect to the Euclidean metric.

Definition 4.2. Fixing a distribution Δ, a curve is called *horizontal* if it is absolutely continuous and it derivative lies in the distribution Δ wherever it exists.

We can now consider another distance on N. In the literature, this distance has many different names: *Carnot–Carathéodory metric*, *sub-Riemannian metric*, *geometric control metric*, *nonholonomic mechanical metric*.

$$d_{\mathrm{CC}}(p, q) = \inf\{\mathrm{length}_{\|\cdot\|}(\sigma) : \sigma \text{ horizontal from } p \text{ to } q\}, \qquad (13)$$

where $\mathrm{length}_{\|\cdot\|}$ denotes the length with respect to the Euclidean metric.

4.1. Continuity of the sub-bundle. To prove that the sub-bundle Δ is in fact a C^1-sub-bundle, we will use a result that give a characterization of C^1-sub-manifold as ambiently C^1-homogeneous compacta.

A set $A \subset \mathbb{R}^n$ is said to be *ambiently C^1-homogeneous* if for every pair of points $x, y \in A$, there exist neighborhoods O_x and O_y in $\mathbb{R}^n$ and a C^1-diffeomorphism

$$h : (O_x, O_x \cap A, x) \to (O_y, O_y \cap A, y).$$

Theorem 4.3 ([RSŠ96]). *Let $A \subset \mathbb{R}^n$ be compact. Then A is ambiently C^1-homogeneous if and only if A is a C^1 submanifold of $\mathbb{R}^n$.*

The original proof of this result in [RSŠ96] requires the Rademacher Theorem. Shchepin and Repovš [SR00] simplify the proof by eliminating the need to invoke Rademacher.

Clearly, since the statement of Theorem 4.3 is local, the assumption of compactness can be replaced by local compactness.

Proposition 4.4. *Let $\Delta \subset TN$ be a (not necessarily continuous) distribution. Suppose that $\mathcal{F} = \{f_p\}_{p \in N}$ is a transitive family of local C^1-diffeomorphisms of N, $f_p(0) = p$, that leaves invariant Δ, i.e., $(df)(\Delta) = \Delta$ for any $f \in \mathcal{F}$. If the map $p \mapsto (df_p)_0$ is continuous at $p = 0$, then Δ is a continuous sub-bundle. If moreover there is another transitive family of C^2-diffeomorphisms that leaves invariant Δ, then Δ is a C^1-sub-bundle of the tangent bundle.*

Proof. The continuity of Δ at the origin is consequence of the continuity of $p \mapsto (df_p)_0$. Then continuity everywhere follows from the invariance under the transitive family of C^1-diffeomorphisms.

Let us denote by $h(p) := \Delta_p$ the k-dimensional plane as an element in the k-dimensional Grassmannian $\mathrm{Gr}_k(T_pN)$. We want to prove that h is a C^1-function. Consider the map

$$\sigma \colon p \mapsto (p, h(p)),$$

i.e., the induced section of the Grassmannian bundle $\mathrm{Gr}_k(TN)$.

First notice that, since we proved that Δ is continuous, h is continuous. Being σ the graph of the continuous function h, the image $M := \sigma(N)$ is closed in $\mathrm{Gr}_k(TN)$. Now, if f is C^2, then its differential df induces a C^1-map f_* of $\mathrm{Gr}_k(TN)$. Moreover if a family of C^2-diffeomorphisms acts transitively on N preserving Δ, the induced maps on the Grassmannian bundle act transitively on M preserving M. Thus, we use Theorem 4.3: since M is a closed, and so locally compact, subset of the manifold $\mathrm{Gr}_k(TN)$ that is ambiently C^1-homogeneous, it follows that M is a C^1-sub-manifold of $\mathrm{Gr}_k(TN)$.

Let $V \subseteq N$ be a neighborhood that trivializes the bundle

$$\mathrm{Gr}_k(TV) = V \times \mathbb{R}^m.$$

Call $\pi \colon V \times \mathbb{R}^m \to V$ the projection on the first space. Let $M' := M \cap (V \times \mathbb{R}^m)$. Since M is a C^1-manifold, it follows that

$$\rho := \pi_*|_{TM'} \colon TM' \to TV$$

is a bundle map. Now in case that there exists a point $x \in M'$ such that

$$\rho_x(T_xM') = TN, \tag{14}$$

then, by the Implicit Function Theorem, in a neighborhood U of x, there is a C^1-function $\tilde{h} \colon U \to \mathbb{R}^m$ such that, locally,

$$\{(p, \tilde{h}(p)) : p \in U\} = M = \{(p, h(p)) : p \in U\}.$$

Thus $h = \tilde{h}$ and so h is C^1 on the neighborhood U. Using again that M is C^1-homogeneous, we have that each point in M has a neighborhood on which h is C^1.

If (14) is not true for any $x \in M$, we will arrive at a contradiction. Indeed, in this case

$$\dim(\ker \rho_x) > 0 \quad \text{for all } x \in M.$$

One can then find an open set $A \subset M$ such that $\dim(\ker \rho_a) = c$ for all $a \in A$, for some constant $c > 0$. From general theory of sub-bundles [Ati67], we have that $\ker \rho \subset TA \subset TM$ is a sub-bundle of $TA \subset TM$ over $A \subset M$. From this we have that locally there is a non-trivial section $X \colon A \to \ker \rho$. In other words, X is a vector field on A such that $X_a \in \ker(\pi_*|_{T_aA})$ for all $a \in A$. This means that X_a is of the

form $(0, X_a^{(2)}) \in TM \times T\mathbb{R}^m$. If now γ is an integral curve for X, then we may assume that γ is not constant since X is non-trivial. However,

$$\gamma(t) = \gamma(0) + \int_0^t X_{\gamma(t)} dt = \gamma(0) + \left(0, \int_0^t X_{\gamma(t)}^{(2)} dt\right).$$

But this contradicts the fact that M is a graph. $\qquad\square$

5. Proof of biLipschitz equivalence

In the previous section we used the fact that, by Proposition 3.3, the d-rectifiable curves are differentiable almost everywhere to construct a distribution Δ coming from the derivatives of such curves. Now, with the next result, we conclude the proof of Theorem 2.1.

Theorem 5.1. *Let (N, d) satisfy the assumptions of Theorem 2.1. Let Δ be the distribution defined above in (9). Let d_{CC} be any Carnot–Carathéodory metric associated to Δ. Then we have, locally,*

$$\frac{1}{L} d_{CC}(p, q) \leq d(p, q) \leq L d_{CC}(p, q)$$

for some $L > 0$.

Proof. We need to prove two inequalities.

5.1. The first inequality. This is straightforward. Given $p \in N$, let γ_p be a d-geodesic from 0 to p. Since by Proposition 3.3 the d distance is greater than the Euclidean distance, γ_p is Lipschitz, thus it is differentiable almost everywhere. We may parametrize γ_p by arc length with respect to d, so $\gamma_p : [0, T] \to N$, where $T := d(0, p)$. We claim that $\|\dot\gamma_p\|(t) \leq 1$, for almost every t. Indeed, for any point t of differentiability,

$$\|\dot\gamma_p(t)\| = \lim_{h \to 0} \frac{\|\gamma_p(t+h) - \gamma_p(t)\|}{h} \leq \lim_{h \to 0} \frac{d(\gamma_p(t+h), \gamma_p(t))}{h} = \frac{h}{h} = 1.$$

Therefore,

$$d_{CC}(p, 0) \leq \text{length}_{\|\cdot\|}(\gamma_p) = \int_0^T \|\dot\gamma_p(t)\| \, dt \leq \int_0^T 1 \, dt = T = d(0, p).$$

5.2. The second inequality. Given a point $p \in N$, we want to construct a d-rectifiable curve σ that starts at 0 and ends arbitrarily close to p, whose d-length is close to the CC-distance of p from 0. This will be enough since the metric d gives

the standard topology. To construct such a curve, we will use the curves of the family Γ defined in Lemma 4.1. For any $v \in \Delta$ there is a pre-chosen curve $\gamma_v \in \Gamma$ such that $\dot{\gamma}_v(0) = v$, and these curves have a common bound for the speed (10) and for the distance from the linear approximation (11).

Take any $\eta \colon [0, T] \to \mathbb{R}^n$ that is a Lipschitz curve, almost everywhere tangent to the distribution Δ, with $\eta(0) = 0$, $\eta(T) = p$, i.e., one of the candidate curves in the calculation of the CC-distance between 0 and p. We can suppose that η is parametrized by arc length, i.e., $\|\dot{\eta}\| = 1$, so $T = \mathrm{length}_{\|\cdot\|}(\eta)$. Our goal is to show that T is greater than a fixed constant times $d(0, p)$.

Recall that Δ is C^1. Thus, in a neighborhood of 0, that we will still call N, we can find a C^1-framing $\{X_1, \ldots, X_k\}$ of Δ, i.e., each X_j is a C^1-vector field and

$$\Delta_p = \mathrm{span}\{(X_1)_p, \ldots, (X_k)_p\} \quad \text{for all } p \in N.$$

We may assume that $\{(X_1)_p, \ldots, (X_k)_p\}$ is an orthonormal basis of Δ_p. Since N is compact, for some $C > 0$, each vector field X_j is C-Lipschitz, i.e.,

$$\|(X_j)_{q_1} - (X_j)_{q_2}\| \leq C \, \|q_1 - q_2\| \quad \text{for all } q_1, q_2 \in N.$$

A consequence is that if $v \in \Delta_{q_1}$ is such that $\|v\| \leq 1$ then there is $w \in \Delta_{q_2}$ with $\|w\| \leq 1$ such that $\|v - w\| \leq C \, \|q_1 - q_2\|$. Indeed, since $\{(X_j)_{q_1}\}$ is an orthonormal basis, there are numbers a_j such that $v = \sum_j a_j (X_j)_{q_1}$ with $\sum_j |a_j| \leq 1$. Thus $w := \sum_j a_j (X_j)_{q_2}$ satisfies

$$|v - w| \leq \sum_j |a_j| \, \|(X_j)_{q_1} - (X_j)_{q_2}\| \leq C \|q_1 - q_2\|.$$

What we conclude is that each $v \in \Delta_{q_1}$ with $\|v\| \leq 1$ has distance less than $C \|q_1 - q_2\|$ from the unit ball in Δ_{q_2}. Denoting by $U(\Delta_q)$ the unit ball in Δ_q, we write

$$\mathrm{dist}(U(\Delta_{q_2}), U(\Delta_{q_2})) \leq C \|q_1 - q_2\|. \tag{15}$$

5.2.1. The construction of σ. Take $\varepsilon > 0$. Construct piece-by-piece a curve σ in the following way. Start at $0 = \eta(0)$. After a suitable choice of a vector $v_0 \in \Delta_0$, we will take the curve $\gamma_{v_0}(t) \in \Gamma$, where Γ is the fixed family of curves from Lemma 4.1, and then we will define the first piece of $\sigma(t)$ as, for $0 \leq t \leq \varepsilon$,

$$\sigma(t) := \gamma_{v_0}(t). \tag{16}$$

By (15), since $\|\dot{\eta}(t)\| \leq 1$ for a.e. t, we have that, for a.e. $t \leq \varepsilon$,

$$\mathrm{dist}(U(\Delta_0), \dot{\eta}(t)) \leq C \|\eta(t)\| \leq C |t| \leq C \varepsilon,$$

since η is parametrized by arc length. Since the unit ball $U(\Delta_0)$ is convex, we have

$$\mathrm{dist}(U(\Delta_0), \tfrac{1}{\varepsilon} \textstyle\int_0^\varepsilon \dot{\eta}(t)dt) \leq C\varepsilon.$$

Therefore there exists a $v_0 \in \Delta_0$ with $\|v_0\| \leq 1$ such that

$$\left\| v_0 - \frac{1}{\varepsilon} \int_0^\varepsilon \dot{\eta}(t)dt \right\| \leq C\varepsilon. \tag{17}$$

So for $0 \leq t \leq \varepsilon$, the curve σ has been defined by (16).

For the inductive construction of σ suppose that for any $t \leq n\varepsilon$, the value $\sigma(t)$ has been defined. We shall define σ, for $n\varepsilon < t \leq (n+1)\varepsilon$, as

$$\sigma(t) := \gamma_{v_n}(t - n\varepsilon),$$

for a suitable choice of $v_n \in \Delta_{\sigma(n\varepsilon)}$ and its related $\gamma_{v_n} \in \Gamma$.

First note that $\lim_{t \to n\varepsilon+} \sigma(t) = \gamma_{v_n}(n\varepsilon - n\varepsilon) = \gamma_{v_n}(0) = \sigma(n\varepsilon)$. Therefore the new piece agrees with the previous one, i.e., the path is continuous. Moreover, $\sigma((n+1)\varepsilon) = \gamma_{v_n}(\varepsilon)$. Then calculate the (right)-derivative at $n\varepsilon$:

$$\frac{d}{dt}\sigma(t)\Big|_{t=n\varepsilon+} = \dot{\gamma}_{v_n}(n\varepsilon - n\varepsilon) = \dot{\gamma}_{v_n}(0) = v_n.$$

Again using (15), we have that there exists $w_n \in \Delta_{\eta(n\varepsilon)}$ with $\|w_n\| \leq 1$ such that

$$\left\| w_n - \frac{1}{\varepsilon} \int_{n\varepsilon}^{(n+1)\varepsilon} \dot{\eta}(t)dt \right\| \leq C\varepsilon.$$

Also, since

$$\mathrm{dist}(U(\Delta_{\sigma(n\varepsilon)}), w_n) \leq C \|\sigma(n\varepsilon) - \eta(n\varepsilon)\|,$$

there exists a vector $v_n \in \Delta_{\sigma(n\varepsilon)}$ with $\|v_n\| \leq 1$ such that

$$\|w_n - v_n\| \leq C\|\sigma(n\varepsilon) - \eta(n\varepsilon)\|.$$

So

$$\left\| v_n - \frac{1}{\varepsilon} \int_{n\varepsilon}^{(n+1)\varepsilon} \dot{\eta}(t)dt \right\| \leq C \|\sigma(n\varepsilon) - \eta(n\varepsilon)\| + C\varepsilon. \tag{18}$$

Let us now estimate $\|\eta(T) - \sigma(T)\|$. We will show that we have a system of the following type:

$$\begin{cases} \|\eta(\varepsilon) - \sigma(\varepsilon)\| \leq o(\varepsilon), \\ \|\eta(n\varepsilon) - \sigma(n\varepsilon)\| \leq (1 + C\varepsilon)\|\eta((n+1)\varepsilon) - \sigma((n+1)\varepsilon)\| + o(\varepsilon) \end{cases} \tag{19}$$

for all $n \in \mathbb{N}$, where $\frac{o(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$. Observe that a sequence of the form

$$\begin{cases} a_1 = \alpha, \\ a_n = \beta a_{n-1} + \alpha \end{cases} \tag{20}$$

has solution $a_n = \alpha(\beta^{n-1} + \cdots + 1) = \alpha\frac{1-\beta^n}{1-\beta}$. So, from (19),

$$\|\eta(T) - \sigma(T)\| = \left\|\eta\left(\frac{T}{\varepsilon}\varepsilon\right) - \sigma\left(\frac{T}{\varepsilon}\varepsilon\right)\right\|$$

$$\leq o(\varepsilon)\frac{1 - (1 - \varepsilon C)^{\frac{T}{\varepsilon}}}{1 - (1 - \varepsilon C)}$$

$$\simeq \frac{o(\varepsilon)}{\varepsilon}(1 - e^{CT}) \longrightarrow 0, \qquad \text{as } \varepsilon \to 0.$$

One big triangular inequality. Now let us do the calculation showing (19). The case $n = 1$ is shown by considering the following four curves and comparing them at time $t = \varepsilon$:

(1) $\eta(t)$,
(2) $t\frac{1}{\varepsilon}\int_0^\varepsilon \dot{\eta}(s)ds$,
(3) tv_0,
(4) $\sigma(t) = \gamma_{v_0}(t)$.

Step by step,

1 *and* 2: At time ε, the curves are at the same point, by the Fundamental Theorem of Calculus.

2 *and* 3: By (17), we have

$$\left\|\varepsilon v_0 - \varepsilon\frac{1}{\varepsilon}\int_0^\varepsilon \dot{\eta}(t)dt\right\| \leq C\varepsilon^2.$$

3 *and* 4: Since $\gamma \in \Gamma$, by (11) we have $\|\varepsilon v_0 - \gamma_{v_0}(\varepsilon)\| < \omega_\Gamma(\|v_0\|\varepsilon) < \omega_\Gamma(\varepsilon)$.

Thus putting everything altogether with the triangle inequality we obtain

$$\|\eta(\varepsilon) - \sigma(\varepsilon)\| \leq C\varepsilon^2 + \omega_\Gamma(\varepsilon).$$

For $n > 1$, more estimates are needed. We compare the following five curves at time $t = (n + 1)\varepsilon$:

(1) $\eta(t)$,
(2) $(t - n\varepsilon)\frac{1}{\varepsilon}\int_{n\varepsilon}^{(n+1)\varepsilon} \dot{\eta}(t)dt + \eta(n\varepsilon)$,
(3) $(t - n\varepsilon)\frac{1}{\varepsilon}\int_{n\varepsilon}^{(n+1)\varepsilon} \dot{\eta}(t)dt + \sigma(n\varepsilon)$,
(4) $(t - n\varepsilon)v_n + \sigma(n\varepsilon)$,
(5) $\sigma(t) = \gamma_{v_n}(t - n\varepsilon)$.

Step by step,

1 *and* 2: At time $(n + 1)\varepsilon$, as before, the curves are at the same point:

$$\eta((n + 1)\varepsilon) = \eta(n\varepsilon) + \varepsilon\frac{1}{\varepsilon}\int_{n\varepsilon}^{(n+1)\varepsilon} \dot{\eta}(t)dt.$$

2 *and* 3: One is just a translations of the other by $\|\eta(n\varepsilon) - \sigma(n\varepsilon)\|$.

3 *and* 4: As before, by (18),

$$\left\| \varepsilon v_n - \varepsilon \frac{1}{\varepsilon} \int_{n\varepsilon}^{(n+1)\varepsilon} \dot{\eta}(t) dt \right\| \leq C\varepsilon \|\sigma(n\varepsilon) - \eta(n\varepsilon)\| + C\varepsilon^2.$$

4 *and* 5: From (11), the distance between the fourth and fifth curve is

$$\|\varepsilon v_n + \sigma(n\varepsilon) - \gamma_{v_n}(\varepsilon)\| = \|\varepsilon \dot{\gamma}_{v_n}(0) + \gamma_{v_n}(0) - \gamma_{v_n}(\varepsilon)\| \leq \omega_\Gamma(\|v_n\|\varepsilon) \leq \omega_\Gamma(\varepsilon).$$

Thus, putting everything together with the triangle inequality:

$$\begin{aligned}
\|\eta((n+1)\varepsilon) - \sigma((n+1)\varepsilon)\| &\leq C\varepsilon^2 + \|\eta(n\varepsilon) - \sigma(n\varepsilon)\| \\
&\quad + \varepsilon C \|\sigma(n\varepsilon) - \eta(n\varepsilon)\| + \omega_\Gamma(\varepsilon) \\
&\leq (1 + \varepsilon C)\|\eta(n\varepsilon) - \sigma(n\varepsilon)\| + C\varepsilon^2 + \omega_\Gamma(\varepsilon).
\end{aligned}$$

Thus, with the terminology of the system (20), $\beta = 1 + \varepsilon C$ and $\alpha = C\varepsilon^2 + \omega_\Gamma(\varepsilon) = o(\varepsilon)$. Then, as we observed after (20), $\|\eta(T) - \sigma(T)\| \to 0$, as $\varepsilon \to 0$. This shows that we can choose ε to have $\sigma(T)$ as close to $\eta(T)$ as we want.

Now we calculate $d(0, \sigma(T))$:

$$\begin{aligned}
d(0, \sigma(T)) &\leq \sum_{n < T/\varepsilon} d(\sigma(n\varepsilon), \sigma((n+1)\varepsilon)) \\
&= \sum d(\gamma_{v_n}(0), \gamma_{v_n}(\varepsilon)) \\
&\leq \sum_1^{T/\varepsilon} S\|v_n\|\varepsilon \\
&\leq \sum_1^{T/\varepsilon} S\varepsilon = S\varepsilon\frac{T}{\varepsilon} = ST = S \text{ length}_{\|\cdot\|}(\eta),
\end{aligned}$$

where we used, in this order, the triangle inequality, the definition of σ, i.e., the fact that $\gamma_{v_n}(t) = \sigma(n\varepsilon + t)$, that γ_{v_n} is d-rectifiable parametrized by (uniformly) bounded speed, i.e., (10) holds, and, finally, the bound for $\|v_n\|$.   □

6. The case of biLipschitz maps coming from a Lie group action

We now describe how Theorem 1.1 can be proved using Theorem 2.1. What we need to show is that the properties of the transitive action can be improved, i.e., steps 1 and 2 of the outlined argument in the introduction can be done. Let G, H, and d be as in Theorem 1.1.

6.1. Getting a closed and embedded subgroup of Homeo(G/H). Any element of G induces a diffeomorphism of G/H. Without loss of generality, we can assume that G acts effectively, so that it may be viewed as a subgroup of $\mathrm{Diff}(G/H)$: the space of all C^∞-diffeomorphisms of G/H equipped with the C^∞-topology given by uniform convergence on compact sets of the functions together with all of their derivatives. So G has two different natural topologies: the first one as a subset of $\mathrm{Diff}(G/H)$ and the second one (weaker) as a subset of $\mathrm{Homeo}(G/H)$: the space of all homeomorphisms of G/H equipped with the C^0-topology, i.e., uniform convergence on compact sets. The first topology is more helpful since it gives control on the derivatives, however, the second one is easier to control by category arguments.

The following proposition tells us that we may assume that the inclusion $\iota\colon G \hookrightarrow \mathrm{Homeo}(G/H)$ is an embedding and that $\iota(G)$ is closed. In other words, for any sequence of elements of G, viewed as a sequence of maps on G/H, that converges uniformly on compact sets, the limit map is still an element of $\iota(G)$, and the convergence is, in fact, as elements of G, and so the sequence converges as maps in $\mathrm{Diff}(G/H)$.

In general a Lie group G acting on G/H can fail to have the above property. Here is an example. Let G be the group of isometries of $\mathbb{R}^4$ generated by the translations and a non-closed 1-parameter subgroup of $O(4)$. So G is a connected Lie group of dimension 5, acting on $\mathbb{R}^4$. Thus, to have a group that is closed in $\mathrm{Homeo}(R^4)$, one has to extend the group to a bigger group, in this case the closure of G in $\mathrm{Isom}(G)$. What is not trivial in general is that such larger group can be chosen to be still a Lie group.

Proposition 6.1. *Let G be a Lie group and H be a closed subgroup. Then there exists a Lie group $\widehat{G}$ that extends the action of G on G/H and is embedded in $\mathrm{Homeo}(G/H)$ as a closed set. (Moreover, $G/H = \widehat{G}/\widehat{H}$, for some closed subgroup $\widehat{H}$.)*

The rest of this section is devoted to the proof of Proposition 6.1. Let X be the homogeneous space G/H. After taking the quotient of G by the kernel of the action, we can suppose G acts effectively on X. Then we can replace G by its universal cover, so it is a simply connected Lie group acting on X effectively in a neighborhood of the identity $e \in G$.

Let V denote the subspace of vector fields on X that corresponds to the Lie algebra of G. In other words, for each $\xi \in L(G) := T_e G$, the one-parameter subgroup of G,

$$t \mapsto \exp(t\xi) \in G,$$

acts on X by translation. So, for any $x \in X$ and $t \in \mathbb{R}$, we can consider the flow on X,

$$\Phi_\xi(t, x) := \exp(t\xi) \cdot x.$$

Differentiating we obtain a vector field on X that gives the above flow: for $x \in X$,

$$\xi(x) := d(\exp(t\xi) \cdot x)_{t=0} \in T_x X. \tag{21}$$

Abusing terminology the vector field is still called ξ since we can identify V and $L(G)$. Indeed, V is isomorphic to $L(G)$ as vector spaces (and even the bracket operation, up to sign, is preserved, as shown in [Hel01]). In particular, we point out that there is also a one-to-one correspondence of the above flows with elements in V (or $L(G)$). Indeed,

$$\text{if } \Phi_\xi(t, \cdot) = \Phi_{\xi'}(t, \cdot) \in \text{Homeo}(X) \text{ for all } t \in \mathbb{R} \text{ then } \xi = \xi' \in V, \qquad (22)$$

because of the local effectivity of the action: for t small enough, $\exp(t\xi) \cdot x = \exp(t\xi') \cdot x \in G/H$ implies $\exp(t\xi) = \exp(t\xi') \in G$ and then $t\xi = t\xi' \in L(G)$ since exp is a local diffeomorphism at the origin in $L(G)$.

The vectors in the Lie algebra of H correspond to those vector fields in V that vanish at the origin $[e] \in X$:

$$
\begin{aligned}
\xi \in L(H) &\iff \exp(t\xi) \in H \quad \text{for all } t \in \mathbb{R} \\
&\iff \exp(t\xi)[e] = [e] \quad \text{for all } t \in \mathbb{R} \\
&\iff \Phi_\xi(t, [e]) = [e] \quad \text{for all } t \in \mathbb{R} \\
&\iff \xi([e]) = 0.
\end{aligned}
\qquad (23)
$$

Note that if $g \in G$, then the translation $\tau_g \colon X \longrightarrow X$, induced by the left translation, $xH \mapsto gxH$, preserves the vector fields in V; this is just another manifestation of the adjoint representation[1] of G: we have the formula $g \exp(\xi) g^{-1} = \exp(\text{ad}_g \xi)$, see [Kna02], p. 53, so $\text{ad}_g \xi \in L(G)$ is the push-forward vector field. However, we shall be interested in the fact that τ_g preserves the flows of vector fields in V; indeed, we get $g \exp(t\xi) = \exp(t \, \text{ad}_g \xi) g$ and so

$$\tau_g(\Phi_\xi(t, x))) = \Phi_{\text{ad}_g \xi}(t, \tau_g(x)). \qquad (24)$$

The new group $\hat{G}$ extending the action of G will come from the set of homeomorphisms of X, that, as the elements of G in (24), preserve the flows of vector fields in V.

If $\{g_k\} \subseteq G$ is a sequence that converges as maps in $\text{Homeo}(X)$ uniformly on compact sets to a homeomorphism $f \colon X \longrightarrow X$, then we claim that f also *preserves* V, in the sense that for any $\xi \in V$ the flow $\Phi_\xi(t, x)$ is conjugated by f to the flow $\Phi_{\xi'}(t, x)$ for some $\xi' \in V$, i.e., for any $t \in \mathbb{R}$, the diagram

$$
\begin{array}{ccc}
\Phi_\xi(t, \cdot) \colon X & \longrightarrow & X \\
f \downarrow & & \downarrow f \\
\Phi_{\xi'}(t, \cdot) \colon X & \longrightarrow & X
\end{array}
$$

---

[1]The map $h \mapsto g^{-1}hg$ is differentiable and fixes the origin. Its differential at the origin is a homomorphism of the Lie algebra called ad_g. The map

$$G \to \text{GL}(T_e G), g \mapsto \text{ad}_g,$$

is a representation of G inside the algebra homomorphisms of the Lie algebra.

commutes. Since, because of (24), any $g \in G$ preserves V, the claim is a consequence of the more general lemma:

Lemma 6.2. *The space of homeomorphisms preserving V is C^0-closed.*

Proof. Let $\{g_k\}_{k \in N}$ be a sequence of homeomorphisms (not necessarily coming from the G action) preserving V, i.e., for any $\xi \in V$ and any k there exists a $\xi_k \in V$ such that the diagram

$$
\begin{array}{ccc}
\Phi_\xi(t, \cdot) \colon X & \longrightarrow & X \\
g_k \downarrow & & \downarrow g_k \\
\Phi_{\xi_k}(t, \cdot) \colon X & \longrightarrow & X
\end{array}
$$

commutes. If $g_k \to f$ in $\mathrm{Homeo}(X)$, then the above diagram (for fixed ξ and t), converges uniformly on compact sets to

$$
\begin{array}{ccc}
\Phi_\xi(t, \cdot) \colon X & \longrightarrow & X \\
f \downarrow & & \downarrow f \\
\Phi_\infty(t, \cdot) \colon X & \longrightarrow & X.
\end{array}
$$

Recall that $L(G)$ is finite dimensional, so, after passing to a subsequence, either ξ_k converges in direction, i.e., the sequence of unit vectors $\frac{\xi_k}{\|\xi_k\|}$ converges to some v_1 or is zero for all k. In the second case, there is nothing else to prove since, for any k, g_k is the identity, and so is the limit. In the first case, we can complete v_1 to a basis $v_1, \ldots, v_n$ of $L(G)$. Defining $T_k := \|\xi_k\|$, write $\xi_k = T_k \sum_{j=1}^n a_{kj} v_j$. Thus, for $j = 2, \ldots n$, $a_{kj} \to 0$ and $a_{k1} \to 1$ as $k \to \infty$.

Note that $\Phi_\xi(0, \cdot) = \mathrm{Id}$, and so $\Phi_\infty(0, \cdot) = \mathrm{Id}$. Pick $p \in X$. Since Φ_∞ is continuous in t, for any $\varepsilon > 0$, there exists δ such that

$$\mathrm{Diam}(\Phi_\infty([0, \delta], p)) < \varepsilon.$$

We denoted by Diam the diameter of a set with respect to some fixed metric on X inducing the same topology. Similarly, $\mathrm{Nbhd}_\varepsilon(A)$ denotes the ε-neighborhood with respect to the auxiliary distance. Now, by uniform convergence, for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$\Phi_{\xi_k}([0, \delta], p) \subset \mathrm{Nbhd}_\varepsilon(\Phi_\infty([0, \delta], p))$$

for any $k > K$. Therefore,

$$\Phi_{\xi_k}([0, \delta], p) \subset \mathrm{Nbhd}_{2\varepsilon}(p). \tag{25}$$

We may assume that p is not a fixed point of the vector field v_1. Take a time $t > 0$ such that $q := \Phi_{v_1}(t, p) \neq p$. Suppose that we choose $\varepsilon < |p - q|/3$. Assume by

contradiction that $T_k \to \infty$. Take k big enough such that $\frac{t}{T_k} < \delta$. For $s_k = \frac{t}{T_k}$ very small, we have

$$
\begin{aligned}
\Phi_{\xi_k}(s_k, p) &= \Phi_{T_k \sum_{j=1}^n a_{kj} v_j}(s_k, p) \\
&= \Phi_{\sum_{j=1}^n a_{kj} v_j}(T_k s_k, p) \\
&= \Phi_{\sum_{j=1}^n a_{kj} v_j}(t, p) \to \Phi_{v_1}(t, p) = q.
\end{aligned}
$$

But this contradicts (25), which says that, for all k, the points $\Phi_{\xi_k}(s_k, p)$ lie in a neighborhood of p and so, by our choice of ε, outside the ball $B(q, \frac{\varepsilon}{3})$, and therefore they cannot converge to q.

From the contradiction we deduce that the sequence ξ_k is bounded. Thus, after passing to a subsequence, it converges to some ξ' and Φ_∞ has to be the flow of ξ' (by uniqueness of limit). In particular ξ' is uniquely determined by Φ_∞ by (22). We proved that every subsequence has a convergent sub-subsequence, and the limit is independent of the choice of the subsequence; therefore ξ_k actually converges to a fixed $\xi' \in V$, giving the conclusion of the lemma. $\qquad\square$

By (22), the vector field ξ' of the lemma is uniquely determined by Φ_∞ and so by f and ξ. Therefore we have a well-defined function $f_*: V \to V$ such that

$$
f(\Phi_\xi(t, x))) = \Phi_{f_*\xi}(t, f(x)).
$$

Note that this induced map on the space V is functorial, i.e., $f_* \circ g_* = (f \circ g)_*$ for any such maps f and g. If g is an element in G, then $g_* = \mathrm{ad}_g$, so g_* is a Lie algebra homomorphisms of V. Now, suppose that $g_k \in \mathrm{Homeo}(X)$ have the similar property that the maps $\xi \mapsto (g_k)_*\xi$ are Lie algebra homomorphisms of V. Then if $g_k \to f$ in $\mathrm{Homeo}(X)$, the map $\xi \mapsto f_*\xi$ is also a Lie algebra homomorphism of V, because $f_*\xi = \lim_{k \to \infty} (g_k)_*\xi$. In other words, fixing a base for $L(G)$, the maps $(g_k)_*$ are square matrices converging pointwise to a square matrix f_*.

Moreover, if the origin $[e] \in X$ is preserved by f, then f_* preserves $L(H)$, i.e.,

$$
\text{if } f([e]) = [e] \text{ then } f_*(L(H)) = L(H); \tag{26}
$$

the reason is just the characterization (23): $\xi \in L(H)$ if and only if $\xi([e]) = 0$ if and only if, for every $t \in \mathbb{R}$, $[e] = f([e]) = f(\Phi_\xi(t, [e])) = \Phi_{f_*\xi}(t, f([e])) = \Phi_{f_*\xi}(t, [e])$ if and only if $f_*\xi([e]) = 0$ if and only if $f_*\xi \in L(H)$.

We can consider the group Homeo_V of homeomorphisms that preserve V, in the sense of the lemma above and induce a Lie algebra homomorphism on V.

Definition 6.3 (Homeo_V). The set Homeo_V is the group of homeomorphisms $f \in \mathrm{Homeo}(X)$ such that there exists a Lie algebra homomorphism $f_*: L(G) \to L(G)$ with the property $f(\Phi_\xi(t, x))) = \Phi_{f_*\xi}(t, f(x))$, or, explicitly, for all $t \in \mathbb{R}, \xi \in V$, and $x \in X$,

$$
f(\exp(t\xi)x) = \exp(t f_*\xi) f(x). \tag{27}
$$

Lemma 6.2 just says that the closure of G in $\mathrm{Homeo}(X)$ is contained in Homeo_V, and, more generally, Homeo_V is closed in $\mathrm{Homeo}(X)$.

Lemma 6.4. *The group Homeo_V is generated by left translations by elements of G and automorphisms of G that fix H. In particular, any element $f \in \mathrm{Homeo}_V$ can be written uniquely as the composition of a translation and such an automorphism. In fact, if $[g] = f([e])$, with $g \in G$, then*

$$f = \tau_g \circ \hat{\Lambda}_{\mathrm{ad}_g^{-1} \circ f_*}, \tag{28}$$

where τ_g is the translation by g and $\hat{\Lambda}_{\mathrm{ad}_g^{-1} \circ f_*}$ is the map induced on the quotient by the (unique) group automorphism of G with differential $\mathrm{ad}_g^{-1} \circ f_*$.

Proof. We first argue that if a map $f \in \mathrm{Homeo}_V$ fixes $[e]$ and $f_* = \mathrm{Id}_V$ then in fact $f = \mathrm{id}_X$. Indeed, we claim that the set of fixed points $F := \{gH \in G/H : f(gH) = gH\}$ is non-empty, closed and open, and so it is all of G/H, i.e., the function f is the identity. Indeed, F is non-empty since the class of the identity is in it by assumption and it is closed since it is defined by a closed relation. The fact that F is open is a consequence of exp being locally invertible. Indeed, take any g' that is close enough to $g \in F$ so that it can be written as $g' = \exp(\xi)g$ for some $\xi \in L(G)$. Then

$$f(g'H) = f(\exp(\xi)gH) = \exp(f_*\xi)f(gH) = \exp(\xi)gH = g'H.$$

It is a classical fact, [Kna02], p. 49, that since G is simply connected, for any (Lie algebra) homomorphism ψ of $L(G)$, there exists a unique smooth (group) homomorphism Λ_ψ of G such that $(d\Lambda_\psi)_e = \psi$. Moreover, in our setting, when Λ_ψ is H-invariant (so it passes to the quotient $X = G/H$), then $\Lambda_\psi \in \mathrm{Homeo}_V$ and we point out that $(\Lambda_\psi)_* = \psi$. Indeed, since Λ_ψ is a homomorphism,

$$\Lambda_\psi(\exp(t\xi)x) = \Lambda_\psi(\exp(t\xi))\Lambda_\psi(x) = \exp(t\psi\xi)f(x).$$

Suppose now that $f : X \to X$ is any map belonging to Homeo_V. Take $g \in G$ such that $[g] = f([e])$ and pre-compose f by the translation τ_g^{-1}, so that $(\tau_g^{-1} \circ f)([e]) = [e]$. Take the automorphism $\Lambda : G \to G$ whose induced automorphism $V \to V$ is the inverse of $(\tau_g^{-1} \circ f)_*$. Explicitly, we take $\Lambda = \Lambda_{((\tau_g^{-1})_* \circ f_*)^{-1}} = \Lambda_{f_*^{-1} \circ \mathrm{ad}_g}$. Moreover, $\tau_g^{-1} \circ f$ fixes the origin and so $(\tau_g^{-1} \circ f)_*$ fixes $L(H)$, and so Λ fixes H. Therefore, passing to the quotient G/H, we have a homeomorphism $\hat{\Lambda} : X \to X$. Then $\hat{\Lambda} \circ \tau_g^{-1} \circ f : X \to X$ fixes $[e] \in X$ and maps each left invariant vector field to itself, i.e., $(\hat{\Lambda} \circ \tau_g^{-1} \circ f)_* = \mathrm{Id}$. Hence, from what we showed at the beginning of the proof, $\hat{\Lambda} \circ \tau_g^{-1} \circ f$ is the identity, i.e., $f = \tau_g \circ \hat{\Lambda}^{-1}$. The uniqueness comes from the fact that the intersection between translations and automorphisms is trivial. $\square$

Lemma 6.5. *The group* Homeo_V *is a Lie group of diffeomorphisms and the inclusion* $\mathrm{Homeo}_V \hookrightarrow \mathrm{Homeo}(X)$ *is an embedding with closed image.*

Proof. The previous lemma says that every element of Homeo_V is a diffeomorphism. In fact, the lemma is claiming more: observe that the group of left translations and the group of automorphisms of G fixing H are both Lie groups; the first one is equivalent to G itself, and the second one is a closed subgroup of $\mathrm{Aut}(G)$ and $\mathrm{Aut}(G)$ is a Lie group since automorphisms of a Lie group come from automorphisms of the Lie algebra, i.e., linear transformations of a finite dimensional vector space. Let K_1 be the group of left translations and K_2 be the group of automorphisms of G fixing H. The previous lemma says that as sets $\mathrm{Homeo}_V = K_1 K_2$. Note that K_1 is normal in Homeo_V. Indeed, for any $\tau_g \in K_1$ left translation by $g \in G$ and any $\phi \in K_2$, we have

$$
\begin{aligned}
(\phi^{-1} \circ \tau_g \circ \phi)(xH) &= (\phi^{-1} \circ \tau_g)(\phi(x)\phi(H)) \\
&= \phi^{-1}(g\phi(x)H) \\
&= \phi^{-1}(g)(\phi^{-1} \circ \phi)(x)\phi^{-1}(H) \\
&= \phi^{-1}(g)xH = \tau_{\phi^{-1}(g)}(xH).
\end{aligned}
$$

So $\phi^{-1} \circ \tau_g \circ \phi \in K_1$. Thus, we have $\mathrm{Homeo}_V = K_1 \rtimes K_2$ is a semi-direct product of Lie groups, so it is a Lie group.

Now the fact that the inclusion has closed image is just Lemma 6.2. However, we must show that it is an embedding. This comes from the fact that V is finite dimensional and every sequence of matrices converges C^∞ as soon as it converges point-wise. Indeed, take $f_k \in \mathrm{Homeo}_V$; we need to show that, under the hypothesis that $\{f_k\}_k$ converges in $\mathrm{Homeo}(X)$, it converges in Homeo_V. (Recall that $\mathrm{Homeo}(X)$ has the C^0 topology, but Homeo_V has the C^∞ one.) Since $f_k \in \mathrm{Homeo}_V$, there are associated maps $(f_k)_* \in \mathrm{GL}(V)$. The proof of Lemma 6.2 shows that for any $\xi \in V$ the sequence $(f_k)_*\xi$ converges to $f_*\xi$, where $f \in \mathrm{Homeo}_V$ is the C^0-limit of f_k. Since $\{(f_k)_*\}_k$ are linear endomorphisms of the finite dimensional vector space V that converge pointwise, the convergence is in fact in $C^\infty(V)$.

So, since by assumption we have $f_k \xrightarrow{C^0} f$, in particular we have convergence at the point $[e] = H$, i.e., $g_k H := f_k(H) \xrightarrow{G/H} f(H) =: gH$, for some $g_k \in G$. This means that there exist $h_k \in H$ such that $g_k h_k \xrightarrow{G} g$. Call $g' := g_k h_k$, so $[g'_k] = [g_k] = f_k([e])$, thus we can use the formula (28) and have

$$
f_k = \tau_{g'_k} \circ \Lambda_{\mathrm{ad}_{g'_k}^{-1} \circ f_*}.
$$

Now, since $g'_k \xrightarrow{G} g$, we have $\tau_{g'_k} \xrightarrow{C^\infty} \tau_g$ and $\mathrm{ad}_{g'_k}^{-1} \xrightarrow{C^\infty} \mathrm{ad}_g^{-1}$. From this last formula and from the fact that $(f_k)_* \xrightarrow{C^\infty} f_*$, we know that, defining $\hat{f}_k := \tau_{g'_k}^{-1} \circ$

$f_k = \Lambda_{\mathrm{ad}_{g_k'}^{-1} \circ f_*}$, we have that $\hat{f}_k \xrightarrow{C^\infty} \tau_g^{-1} \circ f$, and so that $\{f_k\}_k$ is converging in the C^∞-topology. $\qquad\square$

End of the proof of Proposition 6.1. Consider the space V of vector fields given by (21). Let $\widehat{G} := \mathrm{Homeo}_V$ be the group as in Definition 6.3, of the homeomorphisms of the homogeneous space preserving V in the sense of (27) and inducing homomorphisms on the algebra. Then from Lemma 6.5, we know that $\widehat{G}$ extends the action of G, consists of C^∞-diffeomorphisms, is a Lie group and is a closed, embedded subgroup of $\mathrm{Homeo}(X)$. $\qquad\square$

Remark 6.6. By Proposition 6.1, without loss of generality we may assume that G has the following property: as soon as a sequence of elements $g_n \in G$ converges as maps of X in the C^0-topology, the limit is a map coming from G, the convergence is also in the topology of G itself, and, moreover, since the action is smooth, the sequence also converges in the C^∞-topology.

6.2. Getting uniformly biLipschitz maps close to the identity. Remember that G acts by diffeomorphisms on G/H. Suppose now that, for any two points in a neighborhood of the identity, there is an element of G, sending the first point to the second one, that is biLipschitz with respect to the geodesic metric d. The following lemma tells us that, by staying enough close to the origin $[e]$, we may assume that these elements of G give maps that are uniformly biLipschitz and are C^∞-close to the identity as much as we want. Such argument is based on the Baire Category Theorem and has been used several times in the theory of homogenous compacta, e.g. in [MNP98], Theorem 3.1, or [Hoh85], Theorem 6.1.

Lemma 6.7. *Let d be a metric on a homogeneous space $X = G/H$ inducing the usual topology. Suppose that G is closed and embedded in* $\mathrm{Homeo}(G/H)$. *Let U be a compact neighborhood of the origin $[e] \in G/H$ where the elements of G that are biLipschitz with respect to the metric d, act transitively. Then there exists a constant k such that, for any $\varepsilon > 0$, there exists a smaller neighborhood U_ε of the origin where the elements of G that are k-biLipschitz with respect to both a fixed Riemannian metric on X and the metric d and are ε close to the identity with respect to a fixed C^∞-distance act transitively.*

Proof. The group G can be seen as a subset of $\mathrm{Diff}(G/H)$. Since the action is smooth, the topology of G is the same as that of any of those induced by any $C^\infty(G/H)$-distance.[2] Fix one such distance and fix $\varepsilon > 0$. Consider a cover of G by a countable

---

[2]We recall that f_n converges to f in $C^\infty(X)$ if, for any $p \in X$, for any charts ϕ and ψ at p and $f(p)$ respectively, for any compact $K \subset \mathbb{R}^n$ inside the domain of ϕ, and any multi-index $\alpha \in \mathbb{N}^n$, we have that the associated seminorm goes to zero, i.e.,

$$\sup_K D^\alpha(\phi^{-1} \circ f_n \circ \phi - \phi^{-1} \circ f \circ \phi) \longrightarrow 0.$$

number of closed balls D_n for $n \in \mathbb{N}$. We will say later how to choose the cover depending on ε. So $G = \bigcup_{n \in \mathbb{N}} D_n$ with D_n closed. Since any element of G is smooth, it is in particular locally biLipschitz with respect to any Riemannian metric on X. Fix one such Riemannian metric. We will use the term m-biLipschitz to describe to maps that are m-biLipschitz for both the Riemannian metric and the metric d, and the set of such maps will be denoted as biLip^m. We denote by 0 the origin $[e] \in U$. Now consider the orbit set of 0 in U under m-biLipschitz maps in D_n, i.e.,

$$A_{n,m} := (D_n \cap \text{biLip}^m)0$$
$$:= \{p \in U \mid f(0) = p \text{ for some } m\text{-biLipschitz map } f \in D_n\}.$$

By transitivity, we have $U = \bigcup_{n,m \in \mathbb{N}} A_{n,m}$. We claim that the $A_{n,m}$ are closed. Indeed, take $p_j \in A_{n,m}$ converging to $p \in U$. Choose the $f_j \in D_n$ such that $f_j(0) = p_j$. The f_j's are m-biLipschitz and $f_j(0) = p_j$ converges. The Ascoli–Arzelà argument implies that, after passing to a subsequence, the f_j converge to some f, uniformly on compact sets and the limit function is m-biLipschitz (for both metrics). Since G is assumed to be closed in $\text{Homeo}(G/H)$, it follows that $f \in G$. From Remark 6.6, the convergence is in the C^∞-topology, so, since D_n is C^∞-closed, the limit function belongs to it and its value at 0 is p. Therefore, $f(0) = p$ for an $f \in D_n \cap \text{biLip}^m$. In other words, $p \in A_{n,m}$, so $A_{n,m}$ is closed.

The Baire Category Theorem implies that one of the $A_{n,m}$ has non-empty interior. So there exists a compact $V \subset U$, which is a neighborhood of some point q, such that $V \subset A_{\bar{n},\bar{m}}$ for some $\bar{n}, \bar{m} \in \mathbb{N}$. Consider $f_q \in D_{\bar{n}} \cap \text{biLip}^{\bar{m}}$ such that $f_q(0) = q$.

We claim that we can take $U_\varepsilon := f_q^{-1}(V)$ as our new neighborhood and $k := \bar{m}^4$ as our required constant. Indeed, for any points $p_1, p_2 \in f_q^{-1}(V)$, for $i = 1, 2$, $f_q(p_i) \in V \subset A_{\bar{n},\bar{m}}$, so there exists $f_i \in D_{\bar{n}} \cap \text{biLip}^{\bar{m}}$ such that $f_i(0) = f_q(p_i)$. Thus

$$p_2 = (f_q^{-1} \circ f_2 \circ f_1^{-1} \circ f_q)(p_1)$$

and $f_q^{-1} \circ f_2 \circ f_1^{-1} \circ f_q \in G$ is k-biLipschitz. Moreover, $f_q^{-1} \circ f_2 \circ f_1^{-1} \circ f_q$ is ε close to the identity in $\text{Diff}(G/H)$, if we had previously made a good choice of the cover $\{D_n\}$, considering that the function $h \circ f \circ g^{-1} \circ h^{-1}$ is continuous in $f, g, h \in G$.

Now we explain how to choose the cover, given ε. Consider the map

$$C^\infty(X) \times C^\infty(X) \to C^\infty(X), \quad (g, h) \mapsto g \circ h \circ g^{-1}.$$

It is continuous and sends $C^\infty(X) \times \{\text{id}\}$ to the identity function. Given a fixed ε, there exists a neighborhood V_1 of the identity in $C^\infty(X)$ such that $C^\infty(X) \times V_1$ goes

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In general, if $(\rho_n)_{n \in N}$ is a sequence of seminorms defining a (locally convex) topological vector space E, then

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\rho_n(x - y)}{1 + \rho_n(x - y)}$$

is a metric defining the same topology.

into the ε-neighborhood of the identity function (the convergence is in $C^\infty(X)$ with respect to the C^∞-metric that we fixed). Now consider the map

$$C^\infty(X) \times C^\infty(X) \to C^\infty(X), \quad (g, h) \mapsto g \circ h^{-1}.$$

It is continuous and sends the diagonal Δ to the identity function. Given the neighborhood V_1 of before, there exists a neighborhood V_2 of Δ that is sent by the map into V_1.

So if we had chosen the cover so that $D_n \times D_n \subset V_2$ for any $n \in \mathbb{N}$, then $f \circ g^{-1} \in V_1$ for any $f, g \in D_n$. Thus for any $h \in D_n$, we have that $h \circ f \circ g^{-1} \circ h^{-1}$ lies in an ε-neighborhood of the identity. This was what was left to prove. $\qquad\square$

6.3. The end of the proof of Theorem 1.1 using Theorem 2.1. We can focus on a neighborhood N of a point in the manifold that, after fixing a coordinate chart, is a neighborhood of 0 in $\mathbb{R}^n$. We can also transfer the geodesic metric imposing that the chart is an isometry. From now on we will identify the neighborhood in the manifold and that one in $\mathbb{R}^n$.

We need to construct now a family of maps. The idea is to use Lemma 6.7 to select, for each p in the neighborhood, a biLipschitz diffeomorphism f_p whose differential $d(f_p)$ differs from the identity by an error that depends only on $d(0, p)$, and that tends to zero, as $d(0, p)$ tends to zero.

Consider the neighborhoods $U_{1/n}$ given by Lemma 6.7, we can suppose that $N = U_1$. Note that $\bigcap_n U_{1/n} = \{0\}$. For $p = 0$, choose $f_0 = \mathrm{Id}$; for $p \in U_{1/n} \setminus U_{1/(n+1)}$, Lemma 6.7 gives the existence of a k-biLipschitz map f_p so that $f_p(0) = p$ and which is $1/n$ close to the identity.

We need to show that the hypotheses of Theorem 2.1 apply. Since uniformly biLipschitz homogeneity is clear, we have left to show that condition (1) holds. It is satisfied, since in N the second derivatives of the f's are equibounded, say by C, so

$$\left| \frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(y) \right| \leq \int_0^{|x-y|} \frac{d}{dt} \frac{\partial f_i}{\partial x_j} \left(x + t \frac{x-y}{|x-y|} \right) dt$$

$$\leq \int_0^{|x-y|} \sum_{k=1}^n \frac{\partial^2 f_i}{\partial x_k \partial x_j} \left(x + t \frac{x-y}{|x-y|} \right) \frac{(x-y)_k}{|x-y|} dt$$

$$\leq nC |x - y|.$$

Hence all of the hypotheses of Theorem 2.1 apply. Thus, there exists a sub-bundle $\Delta \subset TN$, defined explicitly in (9), such that if d_Δ is any sub-Riemannian metric coming from Δ, then the geodesic metric d is locally biLipschitz equivalent to d_Δ. The fact that Δ is invariant under the action of a transitive subset of a Lie group, will imply, by next proposition, that Δ is not just C^1, but in fact smooth. $\quad\square$

Proposition 6.8. *Assume that S is a set of elements of a Lie group G that is transitive on the space $X := G/H$. Let Δ be a distribution on X preserved by the action of S. Then Δ must be real analytic.*

Proof. Since $S \subset G$ preserves the distribution Δ, any product of its elements does. Call G_S the group generated by S. Consider $G_1 := \bar{G}_S$ the closure of G_S in G. The set G_1 is a Lie group and it preserves the distribution Δ too.

Fix a point $p \in X$. Look at the orbit map of p under G_1,

$$\Phi \colon G_1 \to X, \quad g \mapsto g(p).$$

This is a smooth map, so we can take the derivative at the identity, i.e., the differential from the tangent space at the identity $L := T_e G_1$, the Lie algebra, and the tangent at p, $T_p X$,

$$d\Phi_e \colon L \to T_p X.$$

Since this map is surjective, we can find a subspace W of the Lie algebra of G_1 such that the orbit map restricted to this subspace is an isomorphism of vector spaces,

$$d\Phi_e|_W \colon W \to T_p X.$$

Moreover, $\exp(W)$ is, locally, an analytic sub-variety. The orbit map Φ restricted to $\exp(W)$ gives an analytic map

$$\Phi|_{\exp(W)} \colon \exp(W) \to X.$$

By the Implicit Function Theorem, this is locally an analytic isomorphism.

From the hypothesis we know that, for any $g \in G_1$, $\Delta_{g(p)} = g\Delta_p$. Now, $g = \exp(v)$, for some $v \in W$, so we have the formula

$$\Delta_{(\exp(v))(p)} = \exp(v)\Delta_p.$$

For any other point $q \in X$, let Φ_{loc}^{-1} be a local inverse of Φ in a neighborhood of q, so $q = \Phi(\exp(v)) = (\exp(v))(p)$ and $\exp(v) = \Phi_{\text{loc}}^{-1}(q)$. Then

$$\Delta_q = \Delta_{(\exp(v))(p)} = \exp(v)\Delta_p = \Phi_{\text{loc}}^{-1}(q)\Delta_p.$$

This implies that Δ is smooth because Δ_q depends analytically on q. $\qquad\square$

Appendix: Equality of piecewise $C^{1,1}$ and Lipschitz Carnot–Carathéodory metrics

One can define the Carnot–Carathéodory distance using as horizontal curves either Lipschitz curves tangent almost everywhere or piecewise $C^{1,1}$-curves, i.e., C^1-curves with Lipschitz derivative, tangent to the distribution. We prove now that piecewise $C^{1,1}$-horizontal curves or Lipschitz horizontal curves yield to the same Carnot–Carathéodory distances, when they both induce the manifold topology.

Theorem A.1. *Let X be a Finsler manifold and let Δ be a locally Lipschitz sub-bundle of the tangent bundle. Let $d_{CC}^{C^{1,1}}$ and d_{CC}^{Lip} be the Finsler–Carnot–Carathéodory metrics where the horizontal curves are chosen to be the class of curves that are, respectively, piecewise $C^{1,1}$ and Lipschitz, tangent to the distribution almost everywhere. Suppose that the two topologies induced by the two distances are the same as the topology of the manifold. Then the distances are the same, i.e., $d_{CC}^{C^{1,1}} = d_{CC}^{\text{Lip}}$.*

Proof. The fact that $d_{CC}^{C^{1,1}} \geq d_{CC}^{\text{Lip}}$ is obvious since all piecewise $C^{1,1}$-curves are Lipschitz. A priori, the infimum over all of the Lipschitz curve can be strictly smaller.

Since both metrics are path metrics, it suffices to prove the statement locally. So we may suppose that we are in $\mathbb{R}^n$ with a fixed norm $\|\cdot\|$. In general, the space could be covered by small balls in which charts give a $(1 + \varepsilon)$-biLipschitz approximations. The ε goes to zero as the diameters of the balls go to zero.

Take $\eta\colon [0, T] \to \mathbb{R}^n$ to be a Lipschitz curve, almost everywhere tangent to the distribution Δ, with $\eta(0) = 0$, $\eta(T) = p$, i.e., one of the candidate to calculate the Lipschitz CC-distance between 0 and p. We can suppose that η is parametrized by arclength, i.e., $\|\dot{\eta}\| = 1$ a.e., so $T = \text{length}(\eta)$.

We will construct a sequence of piecewise $C^{1,1}$-curves whose length is smaller than or equal to the length of η, going from 0 to a sequence of points that converges to p. Since the topologies are the same, this will give the conclusion.

Take $\varepsilon > 0$. Construct piece-by-piece a curve σ in a way similar to that of Section 5.2. Start at $0 = \eta(0)$. After a suitable choice of a vector $v_0 \in \Delta_0$, we will take a curve $\gamma_{v_0}(t)$, as the next Lemma A.2 says, and then we will define the first piece of $\sigma(t)$ as

$$\sigma(t) := \gamma_{v_0}(t)$$

for $0 \leq t \leq \varepsilon$. Each curve σ that we will construct will have length less than $2T$. On the Euclidean ball of radius $2T$, the distribution Δ is C-Lipschitz for some $C > 0$. By the same discussion that in Section 5.2 led us to (15), we have that there exists $v_0 \in \Delta_0$ with $\|v_0\| \leq 1$ such that

$$\left\| v_0 - \frac{1}{\varepsilon} \int_0^\varepsilon \dot{\eta}(t)dt \right\| \leq C\varepsilon. \tag{A.1}$$

For the inductive construction of σ suppose that $\sigma(t)$ has been defined for any $t \leq n\varepsilon$. We shall define σ, for $n\varepsilon < t \leq (n + 1)\varepsilon$, as

$$\sigma(t) := \gamma_{v_n}(t - n\varepsilon),$$

for a suitable choice of $v_n \in \Delta_{\sigma(n\varepsilon)}$ and its related $\gamma_{v_n}(t)$ given by Lemma A.2.

Take $w_n \in \Delta_{\eta(n\varepsilon)}$ with $\|w_n\| \leq 1$ such that

$$\left\| w_n - \frac{1}{\varepsilon} \int_{n\varepsilon}^{(n+1)\varepsilon} \dot{\eta}(t)dt \right\| \leq C\varepsilon.$$

Since Δ is C-Lipschitz, there exists a vector $v_n \in \Delta_{\sigma(n\varepsilon)}$ with $\|v_n\| \leq 1$ such that

$$\|w_n - v_n\| \leq C \|\sigma(n\varepsilon) - \eta(n\varepsilon)\|.$$

So

$$\left\| v_n - \frac{1}{\varepsilon} \int_{n\varepsilon}^{(n+1)\varepsilon} \dot{\eta}(t)dt \right\| \leq C \|\sigma(n\varepsilon) - \eta(n\varepsilon)\| + C\varepsilon. \tag{A.2}$$

To estimate $\|\eta(T) - \sigma(T)\|$, one uses system (19). The strategy of the proof for bounding the terms in system (19) is the same as that one in Section 5.2, except that instead of using (11), one uses Lemma A.2. Also, estimates (17) and (18) are replaced by (A.1) and (A.2).

The conclusion is that, as we observed after (20), $\|\eta(T) - \sigma(T)\| \to 0$ as $\varepsilon \to 0$. This shows that we can choose ε to have $\sigma(T)$ as close as we want to $\eta(T)$.

Now we calculate the length of $\sigma(0, T)$:

$$\begin{aligned}
\text{length}(\sigma[0, T]) &\leq \sum_{n < T/\varepsilon} \text{length}(\sigma[n\varepsilon], \sigma((n+1)\varepsilon)) \\
&= \sum \text{length}(\gamma_{v_n}[0, \varepsilon]) \\
&\leq \sum_1^{T/\varepsilon} \|v_n\| \, \varepsilon \\
&\leq \sum_1^{T/\varepsilon} \varepsilon = \varepsilon \frac{T}{\varepsilon} = T = \text{length}(\eta[0, T]),
\end{aligned}$$

where we used, in this order, the triangle inequality, the definition of σ, and that γ_{v_n} is parametrized by (uniformly) bounded speed, as Lemma A.2 says. $\square$

Lemma A.2. *Given $v \in \Delta$ there exists a $C^{1,1}$-curve γ_v starting at v tangent to the distribution Δ, parametrized by speed smaller than $\|v\|$ and such that*

$$\|\gamma_v(t) - [\gamma_v(0) + \dot{\gamma}_v(0)t]\| \leq Ct^2.$$

Proof. Extend v to a vector field $X \subset \Delta$ using the orthogonal projection:

$$X_p := \pi_{\Delta_p}(v).$$

The sub-bundle Δ is C-Lipschitz, thus X has the properties of being C-Lipschitz, $v \in X$ and $\|X_p\| \leq \|v\|$. Let γ_v be the integral curve of X starting at v, i.e., $\dot{\gamma}_v(0) = v$ and $\dot{\gamma}_v(t) = X_{\gamma_v(t)}$. Then

$$\text{length}(\gamma_v[0, t]) = \int_0^t \dot{\gamma}_v(s)ds \leq \int_0^t \|\dot{\gamma}_v(s)\|ds = \int_0^t \|X_{\gamma_v(t)}\|ds \leq \int_0^t \|v\|ds = t\|v\|.$$

In other words, γ_v is of speed smaller than $\|v\|$. The rest of the conclusion of the lemma is clear: since X is C-Lipschitz, it is differentiable a.e. with derivative bounded by C. Thus γ_v is a C^2-curve a.e. with second derivative bounded by C. $\square$

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E. Le Donne, ETH Zurich, Departement Mathematik, Rämistrasse 101, 8092 Zurich, Switzerland

E-mail: enrico.ledonne@math.ethz.ch