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Geometric cycles, Albert algebras and related cohomology classes for arithmetic groups

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Dedicated to Fritz on the occasion of his 60th birthday†

Abstract. We discuss the construction of totally geodesic cycles in locally symmetric spaces attached to arithmetic subgroups in algebraic groups G of type F_4 which originate with reductive subgroups of the group G. In many cases, it can be shown that these cycles, to be called geometric cycles, yield non-vanishing (co)homology classes. Since the cohomology of an arithmetic group is related to the automorphic spectrum of the group, this geometric construction of non-vanishing classes leads to results concerning the existence of specific automorphic forms.

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Introduction

Let G denote a connected semi-simple algebraic group defined over $\mathbb{Q}$, and let X_G be the corresponding symmetric space of maximal compact subgroups of the group of real points of G. A torsion-free arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$ acts properly and freely on X, and the quotient space X/Γ is a complete Riemannian manifold of finite volume. A given reductive $\mathbb{Q}$-subgroup H of G gives rise to a natural map

$$j_{H|\Gamma}\colon X_H/\Gamma_H \longrightarrow X/\Gamma,$$

where $\Gamma_H = \Gamma \cap H(\mathbb{Q})$. Basic results (see Sect. 5.1) guarantee that this map (by passing to a finite covering if necessary) is a proper, injective, closed embedding, and so that each connected component of the image is an orientable, totally geodesic submanifold of X/Γ, to be called a geometric cycle in X/Γ.

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†Since this paper was written my friend Fritz Grunewald has passed away. I dedicate this paper to him with admiration.

Several techniques may be used to prove that such a geometric cycle represents a nontrivial class in homology or homology with closed supports, as necessary. One method is by showing that the cycle intersects a second geometric cycle, of complementary dimension, in a single point with multiplicity ± 1. Unfortunately, geometric cycles of complementary dimension usually intersect in a more complicated set, possibly of dimension greater than zero. To handle this situation, the theory of "excess intersections" as developed in [29], Sects. 3 and 4, has turned out to be useful.

In this paper we begin an investigation of this geometric approach to constructing (co-) homology classes in $H^*(X/\Gamma, \mathbb{R})$ in the case of arithmetic subgroups in algebraic groups of type F_4. These groups arise as the groups of automorphisms of exceptional simple Jordan algebras of dimension 27, also called Albert algebras.

The locally symmetric spaces we will consider originate in the following way: Let k be a totally real algebraic number field of degree $d = [k : \mathbb{Q}]$. Given an Albert algebra A over k we denote by G' the algebraic $\mathbb{Q}$-group $\mathrm{Res}_{k/\mathbb{Q}}\, G$ obtained from the group G of k-algebra automorphisms of A by restriction of scalars. Then the group of real points of G' is of the form of a direct product (ranging over the archimedean places of k) $G'(\mathbb{R}) \xrightarrow{\sim} \prod_{v \in V_\infty} G_v$ where, depending on the underlying Albert algebra A, the real Lie group G_v is isomorphic to one of the simple groups $F_{4(4)}$, $F_{4(-20)}$ or $F_{4(-52)}$ respectively. The latter group is compact, whereas the former groups are of real rank 4 and 1, respectively. These groups are simply connected and have trivial center.

In this exposition, for the sake of simplicity, we mainly focus on the following case: Suppose that $d = [k : \mathbb{Q}] \geq 2$, and suppose that there is an integer $1 \leq r < d$ so that G_{v_j} is isomorphic to $F_{4(-20)}$ for the places $v_j \in V_\infty$, $j = 1, \ldots, r$, and $G_{v_j} \cong F_{4(-52)}$ is a compact group for all other places $v_j \in V_\infty$, $r < j \leq d$. Let Γ be a torsion-free arithmetic subgroup of $G(k)$. As a consequence of the compactness criterion [4], Thm. 12.3, since there is at least one place $v \in V_\infty$ such that G_v is a compact group, the arithmetic quotient X/Γ is compact. The dimension of this compact complete Riemannian manifold is $16r$. In this situation we prove the following (see Section 5)

Theorem. *Let A be an Albert algebra over a totally real algebraic number field $k \neq \mathbb{Q}$ of degree d, and let G be the group of k-algebra automorphisms of A. Suppose that there is an integer $1 \leq r < d$ so that G_{v_j} is isomorphic to $F_{4(-20)}$ for the places $v_j \in V_\infty$, $j = 1, \ldots, r$, and $G_{v_j} \cong F_{4(-52)}$ is a compact group for all other places $v_j \in V_\infty$, $r < j \leq d$. Let $\Gamma \subset G(k)$ be an arithmetic subgroup. Then the cohomology $H^*(X/\Gamma, \mathbb{R})$ contains (up to a subgroup $\Gamma' \subset \Gamma$ of finite index) a non-trivial cohomology class of degree $8r$. By duality, this class, to be denoted $[c_{H/\Gamma_H}]$, is detected by the fundamental class of a totally geodesic submanifold X_H/Γ_H in X/Γ with H a reductive subgroup of G defined over k. We note that $H_\infty \cong \prod \mathrm{Spin}(8,1)$ (up to compact factors). This geometric cycle arises as a fixed point component under a suitable rational involution σ_u on the group G, u an idempotent in A. This class $[c_{H/\Gamma_H}]$ cannot be obtained as the restriction of a continuous class from the underlying Lie group $G_\infty \cong G'(\mathbb{R})$.*

Interpreting this result in the frame work of the theory of automorphic forms this geometric construction of a non-bounding cycle in $H^{8r}(X/\Gamma, \mathbb{R})$ which is not a G_∞-invariant class implies the existence of certain automorphic representations occurring with non-zero multiplicities in $L^2(G_\infty/\Gamma)$.

We outline the content of the paper: In Section 1 we give a brief account of the necessary background in the theory of Albert algebras and their automorphism groups. Then, in Section 2, given an arithmetic subgroup Γ of a connected reductive group G defined over an algebraic number field k we discuss various realizations of the cohomology groups attached to an arithmetic quotient X/Γ. In particular, we describe how the de Rham cohomology groups are related to the relative Lie algebra cohomology groups. It is because of this isomorphism that representation theoretic methods can be used to describe the cohomology of X/Γ. In Section 3 we will be concerned with the real Lie group $F_{4(-20)}$ and its associated symmetric space, also called the Cayley hyperbolic plane. We determine the cohomology of the compact dual X_u of the latter one. It can be viewed as the space of G_∞-invariant forms on X. Based on a brief review of the constructive approach to the classification [37] of irreducible unitary representations of a real semi-simple Lie group with non-zero cohomology we enumerate these representations (up to infinitesimal equivalence) in the case $F_{4(-20)}$ and give their cohomology in Section 4. In Section 5, we give an overview of the general construction of geometric cycles in arithmetic quotients. Then we discuss the case of quotients attached to arithmetic subgroups of algebraic groups of type F_4 as indicated above. We conclude with some remarks on the existence of other types of geometric cycles in the case as hand but also for other exceptional groups.

Notation

(1) Let F be an arbitrary finite extension of the field $\mathbb{Q}$ of degree $d = [F : \mathbb{Q}]$, and denote by $\mathcal{O}_F$ its ring of integers. The set of places will be denoted by V, while V_∞ (resp. V_f) will refer to the set of archimedean (resp. non-archimedean) places of F. The completion of F at a place $v \in V$ is denoted by F_v, and its ring of integers by $\mathcal{O}_v (v \in V_f)$.

Let S be the set of distinct embeddings $\sigma_i : F \to \mathbb{C}$, $1 \leq i \leq d$. Among these embeddings some factor through $F \to \mathbb{R}$. Let $\sigma_1, \ldots, \sigma_s$ denote the real embeddings $F \to \mathbb{R}$. Given one of the remaining embeddings $\sigma : F \to \mathbb{C}$, $\sigma(k) \not\subset \mathbb{R}$, to be called imaginary, there is the conjugate one $\bar{\sigma} : F \to \mathbb{C}$, defined by $x \mapsto \overline{\sigma(x)}$, where $\bar{z}$ denotes the usual complex conjugation of the complex number z. Then the number of imaginary embeddings is an even number, which we denote by $2t$. We number the $d = s + 2t$ embeddings $\sigma_i : F \to \mathbb{C}$, $i = 1, \ldots, d$, in such a way that, as above, σ_i is real for $1 \leq i \leq s$, and $\bar{\sigma}_{s+i} = \sigma_{s+i+t}$ for $1 \leq i \leq t$.

The set V_∞ of archimedean places of F is naturally identified with the set of embeddings $\{\sigma_i\}_{1 \leq i \leq s+t}$ that is, we take all real embeddings and one representative for each pair of conjugate imaginary embeddings.

(2) The algebraic groups we consider will be linear groups, i.e., such a group G defined over a field k is affine viewed as an algebraic variety. It comes with an embedding $\rho\colon G \to \mathrm{GL}_n$ (defined over k) of G into some general linear group.

Let k be an algebraic number field. If G is an algebraic group defined over k, then a subgroup Γ of the group $G(k)$ of k-valued points of G is arithmetic or arithmetically defined if, given an embedding $\rho\colon G \to \mathrm{GL}_n$ over k, the group $\rho(\Gamma)$ is commensurable with $\rho(G) \cap \mathrm{GL}_n(\mathcal{O}_k) =: G_{\mathcal{O}_k}$, that is, the intersection $\rho(\Gamma) \cap G_{\mathcal{O}_k}$ has finite index both in $\rho(\Gamma)$ and $G_{\mathcal{O}_k}$. This notion is independent of the choice of a faithful representation $\rho\colon G \to \mathrm{GL}_n$.

1. Albert algebras and their automorphism group

Algebraic groups of type F_4 arise as groups of automorphisms of exceptional simple Jordan algebras of dimension 27. The structure of these algebras, also called Albert algebras, and the corresponding automorphism groups were systematically studied by Albert and Jacobson [2], [18] and Springer [32], [33].

1.1. Composition algebras. Let C be a composition algebra over a field k, that is, C is a finite dimensional k-algebra $(C, +, \cdot)$ with identity element e and endowed with a non-degenerate quadratic form N which is multiplicative (or permits composition, as one says), $N(xy) = N(x)N(y)$ for all $x, y \in C$. The quadratic form is often referred to as the norm of C, it is already uniquely determined by the structure of $(C, +, \cdot)$ as a k-algebra. Let $b_C(x, y) = N(x+y) - N(x) - N(y)$, $x, y \in C$, be the associated bilinear form. By definition, a composition subalgebra D of C is a non-singular linear subspace D of $(C, +)$ which contains e and is closed under multiplication. A composition algebra C is endowed with a conjugation defined by the assignment $x \mapsto \bar{x} := b_C(x, e)e - x$, $x \in C$. A composition subalgebra of C is necessarily closed under conjugation.

As a consequence of the structure theory for composition algebras ([19], 33.17, or [33], 1.6.2) there exist only composition algebras of dimension 1, 2, 4 or 8. Composition algebras of dimension 1 or 2 are commutative and associative, those of dimension 4 are the quaternion algebras over k (these are associative but not commutative) and those of dimension 8 are neither associative nor commutative. The latter ones, to be called Cayley algebras (or octonion algebras) over k can be constructed by a doubling process from a quaternion composition subalgebra. This construction is obtained by gluing together two copies of a quaternion algebra (Q, N_Q) in the following way. Let $\lambda \in k^*$ be an arbitrary non-zero element. Then $C = Q \oplus Q$, endowed with the multiplication

$$(x, y) \cdot (x', y') := (xx' + \lambda \overline{y'} y, y'x + y\overline{x'}), \tag{1.1}$$

for x, y, x', and $y' \in Q$ and the quadratic form

$$N(x, y) := N_Q(x) - \lambda N_Q(y), \quad x, y \in Q, \tag{1.2}$$

is a composition algebra of dimension 8, to be denoted $\mathrm{CD}(Q, \lambda)$. It contains Q as a composition subalgebra.

We note that a Cayley algebra (or, more generally, a composition algebra) is determined by its norm form, that is, two Cayley algebras C and C' are isomorphic if the corresponding quadratic spaces (C, N_C) and $(C', N_{C'})$ are isometric. Thus, the classification of Cayley algebras over the field k is reduced to the determination of equivalence classes of quadratic forms of the given type. Let C be a Cayley algebra over k with norm form N. If N is isotropic, i.e., there exists $x \in C$, $x \neq 0$, with $N(x) = 0$, then C contains zero divisors. In this case the norm form has maximal Witt index 4. However, in such a case C is uniquely determined up to k-isomorphism. A representative of this unique isomorphism class is the Cayley algebra $\mathrm{CD}(M_2(k), -1)$, to be called the split Cayley algebra over k. If the norm form N is anisotropic, i.e., $N(x) \neq 0$ for all $x \in C$, $x \neq 0$, or, equivalently, the Witt index of N is 0, then each x has an inverse element with respect to multiplication. In such a case, C is called an octonion division algebra.

Suppose that C is a Cayley algebra over the field $k = \mathbb{C}$. Since a non-degenerate form over $\mathbb{C}$ is isotropic, C is $\mathbb{C}$-isomorphic to the split Cayley algebra over $\mathbb{C}$. If $k = \mathbb{R}$, the norm form of a Cayley algebra over $\mathbb{R}$ is isotropic or positive definite. Consequently, there are exactly two possibilities up to $\mathbb{R}$-isomorphism, the split case, and the algebra $\mathbb{O} = \mathrm{CD}(\mathbb{H}, -1)$ of Graves–Cayley octonions respectively.

If k is a perfect field of cohomological dimension ≤ 2, e.g., a totally imaginary number field or a p-adic field, then any Cayley algebra C over k is isomorphic to the split Cayley algebra over k because the norm form of C is isotropic in this case [33], 1.10.

1.2. Albert algebras. Let k be a field with $\mathrm{char}(k) \neq 2, 3$, and let (C, N) be a Cayley algebra over k. For $A = (x_{i,j}) \in M_3(C)$ let $\overline{A} = (\bar{x}_{i,j})$ where $x \mapsto \bar{x}$ denotes conjugation in C. Given a diagonal matrix $\delta = \mathrm{diag}(d_1, d_2, d_3) \in \mathrm{GL}_3(k)$, the assignment $A \mapsto \delta^{-1}\overline{A}^t\delta$ defines an involution on $M_3(C)$, to be denoted Δ. We may replace δ by $\mathrm{diag}(td_1, td_2, td_3)$ for any $t \in k$, $t \neq 0$, without changing this involution. Let

$$\mathcal{H}(C, \Delta) = \{A \in M_3(C) \mid A = A^{\Delta}\} \tag{1.3}$$

be the set of all Δ-Hermitian matrices in $M_3(C)$. Endowed with the usual matrix addition, scalar multiplication and with the product

$$A \cdot B = \tfrac{1}{2}(AB + BA), \tag{1.4}$$

where AB is the usual matrix product in $M_3(C)$, $\mathcal{H}(C, \Delta)$ is a commutative, non-associative algebra of dimension 27 over k. It is a central simple algebra with the identity matrix as unit element. We call an algebra of the form $\mathcal{H}(C, \Delta)$ or a twisted form of $\mathcal{H}(C, \Delta)$ an Albert algebra. This algebra is naturally equipped with a non-degenerate quadratic form q, the quadratic trace, to be defined by

$$q(A) = \tfrac{1}{2}\,\mathrm{trace}(A^2).$$

It is worth noting that the multiplication $(A, B) \mapsto A \cdot B$ satisfies the identity $((A \cdot A) \cdot B) \cdot A = (A \cdot A) \cdot (B \cdot A)$ for all $A, B \in \mathcal{H}(C, \Delta)$. In fact, the algebra $\mathcal{H}(C, \Delta)$ and twisted forms of $\mathcal{H}(C, \Delta)$ are central simple exceptional Jordan algebras. In turn, as proved by Albert, any central simple exceptional Jordan algebra is a twisted form of $\mathcal{H}(C, \Delta)$ for some Cayley algebra C over k. By [2], Thm. 3, two algebras $\mathcal{H}(C, \Delta)$ and $\mathcal{H}(C', \Delta')$ are isomorphic only if C and C' are isomorphic. Thus, one usually calls C the coordinate algebra of $\mathcal{H}(C, \Delta)$. This result is supplemented by a set of necessary and sufficient conditions on Δ and Δ' under which there exists an isomorphism between $\mathcal{H}(C, \Delta)$ and $\mathcal{H}(C, \Delta')$. As a consequence, two algebras $\mathcal{H}(C, \Delta)$ and $\mathcal{H}(C', \Delta')$ with isomorphic coordinate algebras $C \cong C'$ are isomorphic if the quadratic forms q on $\mathcal{H}(C, \Delta)$ and q' on $\mathcal{H}(C', \Delta')$ are equivalent over k [33], 5.8.1.

This result provides a constructive approach to enumerate (up to isomorphism) all Albert algebras over specific fields. For interest of us are the cases $k = \mathbb{R}$, a local field, or an algebraic number field.

If $k = \mathbb{R}$, there are (up to isomorphism) exactly two Cayley algebras over $\mathbb{R}$, the split Cayley algebra $\mathrm{CD}(M_2(\mathbb{R}), -1) =: C_s$ and the algebra $\mathbb{O}$ of Graves–Cayley octonions. In the former case, there is only one isomorphism class of Albert algebras whose coordinate algebra is C_s. A representative of this class is $\mathcal{H}(C_s, \Delta_s)$ where Δ_s is the involution given by $\delta = \mathrm{diag}(1, -1, 1)$. In the latter case, there are two isomorphism classes with $\mathbb{O}$ as coordinate algebra. These classes can be represented by $\mathcal{H}(C_a, \Delta_0)$ and $\mathcal{H}(C_a, \Delta_1)$ respectively, where $\delta_0 = \mathrm{diag}(1, 1, 1)$ and $\delta_1 = \mathrm{diag}(1, -1, 1)$ for any non-split Cayley algebra C_a over $\mathbb{R}$.

Let k be an algebraic number field of degree $n = s + 2t$ where s resp. t denotes the number of real resp. complex places $v \in V$ of k. Let $A = \mathcal{H}(C, \Delta)$ be an Albert algebra defined over k. Given a place $v \in V$ there is the local analogue

$$A_v = A \otimes_k k_v$$

of A given as the tensor product over k of A with the local field k_v. If v is a non-archimedean place there is only one isomorphism class, the one determined by the split Cayley algebra over k_v. The same assertion is true if v is a complex place. If v is a real place, there are three isomorphism classes. As a result, there are 3^s different isomorphism classes of Albert algebras over k. Given two Albert algebras $A = \mathcal{H}(C, \Delta)$ and $A' = \mathcal{H}(C, \Delta')$ with the same coordinate algebra there are conditions on the matrices Δ and Δ' under which the algebras A and A' are isomorphic [2], Thm. 5.

1.3. The automorphism group of an Albert algebra. Let A be an Albert algebra defined over some field k with $\mathrm{char}(k) \neq 2, 3$. Then the group $\mathrm{Aut}_k(A)$ of k-algebra automorphisms of A is the group of k-rational points of an algebraic group G defined over k. This group is a connected simple algebraic group of type F_4.

Suppose that k is an algebraic number field. Given an Albert algebra A over k, the corresponding k-group of automorphisms is denoted by G. Let $G' = \mathrm{Res}_{k/\mathbb{Q}}(G)$

be the algebraic $\mathbb{Q}$-group obtained from G by restriction of scalars. Then $G'(\mathbb{R})$ is isomorphic to the product of real Lie groups $G_v = G^{\sigma_v}(k_v)$ of type F_4, $v \in V_\infty$. According to the classification of Albert algebras over $\mathbb{R}$ [2], Sect. 13, the following three possibilities can occur for G_v if $v \in V_\infty$, v real:

- If $A_v = \mathcal{H}(C_s, \Delta_s)$ (where Δ_s is the involution given by $\delta = \mathrm{diag}(1, -1, 1)$) is the split Albert algebra over $\mathbb{R}$, the corresponding Lie group G_v is a simple Lie group of real rank 4, to be denoted $F_{4(4)}$, following the notation of Cartan.
- If $A_v = \mathcal{H}(C_a, \Delta_1)$ where Δ_1 is the involution given by $\delta = \mathrm{diag}(1, -1, 1)$ for any non-split Cayley algebra C_a over $\mathbb{R}$ the Lie group G_v is of real rank 1, to be denoted $F_{4(-20)}$. Recall that C_a is isomorphic to the algebra $\mathbb{O}$ of Graves–Cayley octonions.
- If $A_v = \mathcal{H}(C_a, \Delta_0)$ where Δ_0 is the involution given by $\delta = \mathrm{diag}(1, 1, 1)$ for any non-split Cayley algebra C_a over $\mathbb{R}$ the Lie group G_v is of real rank 0, to be denoted $F_{4(-52)}$. This group is compact.

The real Lie group of type $F_{4(4)}$ gives rise to an irreducible symmetric space of dimension 28 whereas the symmetric space corresponding to $F_{4(-20)}$ is of dimension 16. In fact, it is the Cayley hyperbolic plane [25], Sect. 19.

If $v \in V_\infty, v$ complex, there is only the split Cayley algebra over $\mathbb{C}$ in this case. Thus, Albert algebras over $\mathbb{C}$ form one isomorphism class, and G_v is uniquely determined.

2. Arithmetic quotients

In this section we discuss various realizations of the cohomology groups attached to an arithmetic subgroup Γ of a connected reductive group G defined over an algebraic number field k and the corresponding quotient X/Γ. In particular, we describe how the de Rham cohomology groups are related to the relative Lie algebra cohomology groups. It is because of this isomorphism that representation theoretic methods can be used to describe the cohomology of X/Γ.

2.1. Generalities. Let G be a connected reductive algebraic group defined over an algebraic number field k. We choose an embedding $\rho\colon G \to \mathrm{GL}_N$ and write $G_{\mathcal{O}_k} = G(k) \cap \mathrm{GL}_N(\mathcal{O}_k)$ for the group of integral points with respect to ρ.

For every archimedean place $v \in V_\infty$ corresponding to the embedding $\sigma_v : k \to \bar{k}$ there are given a local field $k_v = \mathbb{R}$ or $\mathbb{C}$ and a real Lie group $G_v = G^{\sigma_v}(k_v)$. The group

$$G_\infty = \prod_{v \in V_\infty} G_v,$$

viewed as the topological product of the groups G_v, $v \in V_\infty$, is isomorphic to the group of real points $G'(\mathbb{R})$ of the algebraic $\mathbb{Q}$-group $G' = \mathrm{Res}_{k/\mathbb{Q}}(G)$ obtained from G by restriction of scalars. In G_∞, we identify $G(k)$ resp. $G_{\mathcal{O}_k}$ with the set of elements $(g^{\sigma_v})_{v \in V_\infty}$ with $g \in G(k)$ resp. $g \in G_{\mathcal{O}_k}$. If Γ is an arithmetic subgroup of G then Γ is a discrete subgroup in G_∞.

Each of the groups G_v has finitely many connected components. The factor G_v has maximal compact subgroups, and any two of these are conjugate by an inner automorphism. Thus, if K_v is one of them, the homogeneous space $K_v \backslash G_v = X_v$ may be viewed as the space of maximal compact subgroups of G_v. Since X_v is diffeomorphic to $\mathbb{R}^{\mathrm{d}(G_v)}$, where $\mathrm{d}(G_v) = \dim G_v - \dim K_v$, the space X_v is contractible. Notice that if G is semi-simple, the space X_v is the symmetric space associated to G_v. We let

$$X = \prod_{v \in V_\infty} X_v$$

(or we write X_G emphasizing the underlying k-group G) resp. $\mathrm{d}(G) = \sum_{v \in V_\infty} \mathrm{d}(G_v)$.

A torsion-free arithmetic subgroup Γ of G acts properly discontinuously and freely on X and the quotient X/Γ is a smooth manifold of dimension $\mathrm{d}(G)$. There is the result of Borel and Harish-Chandra [4], Thm. 12.3, to the effect that X/Γ has finite volume if and only if G has no non-trivial rational character, and it is compact if and only if , in addition, every rational unipotent element belongs to the radical of G. If X/Γ is of finite volume but not compact, the adjunction of corners [5] provides a compact manifold $\overline{X}/\Gamma$ so that the inclusion $X/\Gamma \to \overline{X}/\Gamma$ is a homotopy equivalence.

We now turn to various realizations of the cohomology of these arithmetic quotients. On one hand, since X is a contractible space, X/Γ is an Eilenberg–MacLane space $K(\Gamma, 1)$. Its cohomology (or homology) is isomorphic to the Eilenberg–MacLane cohomology of Γ. More precisely, if E is a Γ-module, we denote the corresponding local system on X/Γ by $\widetilde{E}$. Then there are canonical isomorphisms

$$H_q(\Gamma, E) = H_q(X/\Gamma, \widetilde{E}) \quad \text{resp.} \quad H^q(\Gamma, E) = H^q(X/\Gamma, \widetilde{E}),$$

for any degree q. As a consequence, the cohomological dimension $\mathrm{cd}(\Gamma)$ of Γ is at most $\mathrm{d}(G)$. If X/Γ is compact, we have $\mathrm{cd}(\Gamma) = \mathrm{d}(G)$, otherwise $\mathrm{cd}(\Gamma) < \mathrm{d}(G)$. In fact, by [5], $\mathrm{cd}(\Gamma) = \mathrm{d}(G) - \mathrm{rk}_k(G)$ in the latter case.

On the other hand, let (ν, E) be a finite dimensional irreducible representation of the real Lie group G_∞ on a real or complex vector space E.

The group G_∞ operates on X and on the complex $(\Omega^*(X, E), d)$ of smooth E-valued forms on X. Given a torsion-free arithmetic subgroup Γ of G_∞, the cohomology $H^*(X/\Gamma, \widetilde{E})$ of the manifold X/Γ with coefficients in the local system defined by (ν, E) is canonically isomorphic to the de Rham cohomology $H^*(\Omega(X, E)^\Gamma)$.

2.2. An interpretation in Lie algebra cohomology. The de Rham cohomology groups $H^*(\Omega(X, E)^\Gamma)$ are related in a natural way to relative Lie algebra cohomology groups. It is this transition by which some questions on the cohomology of arithmetic groups are turned into questions about cohomological properties of unitary representations of the underlying Lie group G_∞. However, since this reinterpretation in terms of relative Lie algebra cohomology only relies on Lie theoretic data we work in this context.

Let G be a real Lie group with finitely many connected components, $\mathfrak{g}$ its Lie algebra, let K be a compact subgroup of G, $\mathfrak{k}$ its Lie algebra. We denote the natural projection map $G \to K\backslash G$ by π. Let (ν, E) be a finite dimensional irreducible representation of G on a real or complex vector space E. We want to study the complex $\Omega^*(K\backslash G, E)$ of smooth E-valued differential forms in terms of representation theory. There is a natural identification of the tangent space at the point $e \in G$ with $\mathfrak{g}$. This gives rise to an identification of complexes

$$\Omega^*(G, E) \xrightarrow{\sim} \mathrm{Hom}(\Lambda^*\mathfrak{g}, C^\infty(G) \otimes E). \tag{2.1}$$

The pullback map $\pi^*\colon \Omega^*(K\backslash G, E) \to \Omega^*(G, E)$ of the C^∞-map π is compatible with differentials thus an inclusion of complexes. We endow $C^\infty(G) \otimes E$ with the G-module structure given as the tensor product of the left regular representation l of G on $C^\infty(G)$ and of (ν, E). Then a q-form ω in $\mathrm{Hom}(\Lambda^*\mathfrak{g}, C^\infty(G) \otimes E)$ is in the image of π^* if ω is annihilated by the interior products i_Y, $Y \in \mathfrak{k}$, and ω lies in $\mathrm{Hom}_K(\Lambda^*\mathfrak{g}, C^\infty(G) \otimes E)$ where K acts on $\Lambda^*\mathfrak{g}$ by the adjoint action.

Then the space $C^\infty(G)_K$ of all C^∞-vectors f for which $l(K)f$ spans a finite dimensional subspace of $C^\infty(G)$ is preserved by the action of $\mathfrak{g}$ (obtained by differentiation of l) and compatible with the action of K. Moreover $C^\infty(G)_K$ is locally finite as a K-module. Thus, $C^\infty(G)_K$ is a $(\mathfrak{g}, K)$-module. Then there is an isomorphism of graded complexes of $\Omega^*(K\backslash G, E)$ onto $C^*(\mathfrak{g}, K, C^\infty(G)_K \otimes E)$

Given any discrete torsion-free subgroup Γ of G the space of functions invariant by Γ acting on the right is a $(\mathfrak{g}, K)$-submodule of $C^\infty(G)_K$. One obtains an isomorphism of $\Omega^*(K\backslash G, E)^\Gamma$ onto $C^*(\mathfrak{g}, K, C^\infty(G/\Gamma)_K \otimes E)$. Thus, there is a canonical isomorphism

$$H^*(K\backslash G/\Gamma, \widetilde{E}) = H^*(\Omega(K\backslash G, E)^\Gamma) \xrightarrow{\sim} H^*(\mathfrak{g}, K, C^\infty(G/\Gamma)_K \otimes E). \tag{2.2}$$

We refer, for example, to [6], Chap. VII, or [36] for a more thorough treatment.

2.3. A result of Matsushima. Suppose G is a real reductive Lie group with finitely many connected components, $K \subset G$ is a maximal compact subgroup and $\Gamma \subset G$ is a torsion-free discrete subgroup so that the quotient G/Γ is compact. In that case the left regular representation of G on the space $L^2(G/\Gamma)$ of square integrable functions (modulo the center) on G/Γ decomposes as a direct Hilbert sum of irreducible unitary representations (π, H_π) of G with finite multiplicities

$$L^2(G/\Gamma) = \widehat{\bigoplus_{\pi \in \widehat{G}}} m(\pi, \Gamma) H_\pi. \tag{2.3}$$

Here $\widehat{G}$ denotes the unitary dual of G, and the multiplicity $m(\pi, \Gamma)$ with which (π, H_π) occurs in $L^2(G/\Gamma)$ is a non-negative integer for each π. Given such an irreducible unitary Hilbert space representation (π, H_π) the space $H_{\pi,K}$ of all C^∞-vectors $v \in H_\pi$ such that $\pi(K)v$ spans a finite dimensional subspace of H_π carries a natural $(\mathfrak{g}, K)$-module structure. Since (π, H_π) is irreducible, $H_{\pi,K}$ is irreducible as a $(\mathfrak{g}, K)$-module. The space $H_{\pi,K}$ of K-finite vectors is dense in the space H_π^∞ of C^∞-vectors for H_π, and there is an inclusion

$$\widehat{\bigoplus_{\pi\in\widehat{G}}}\, m(\pi,\Gamma)\ H_{\pi,K} \longrightarrow C^\infty(G/\Gamma)_K. \tag{2.4}$$

Let (ν, E) be a finite dimensional irreducible real of complex representation of G. Then this inclusion induces an isomorphism

$$\begin{aligned} &H^*(K\backslash G/\Gamma, \widetilde{E}) \\ &\quad\cong H^*(\mathfrak{g}, K, C^\infty(G/\Gamma)_K \otimes E) \xrightarrow{\sim} \bigoplus_{\pi\in\widehat{G}} m(\pi,\Gamma) H^*(\mathfrak{g}, K, H_{\pi,K} \otimes E), \end{aligned}$$

where the right hand side is a finite direct algebraic sum. This result is due to Matsushima [23]. The representations which can possibly contribute to the sum on the right hand side are usually called representations with non-vanishing (Lie algebra) cohomology. By [6], I, 4.2, the cohomology $H^*(\mathfrak{g}, K, H_{\pi,K} \otimes E)$ can only be non-zero if the center of the enveloping algebra of $\mathfrak{g}$ acts on $H_\pi \otimes E$ as in the trivial representation. As a consequence, given (ν, E) there are (up to infinitesimal equivalence) only finitely many irreducible representations (π, H_π) of G with $H^*(\mathfrak{g}, K, H_{\pi,K} \otimes E) \neq 0$. Only those (π, H_π) might occur whose infinitesimal character χ_π coincides with the one of the contragredient representation of (ν, E).

If one drops the assumption that the quotient G/Γ is compact these results are not true any more. However, by replacing the coefficient module $C^\infty(G/\Gamma)$ by appropriate spaces of functions which satisfy certain growth conditions, they hold in a modified form.

3. The real Lie group $F_{4(-20)}$

3.1. The Cayley hyperbolic plane. Within the classification of Riemannian globally symmetric spaces, symmetric spaces of negative curvature and rank one are real, complex and quaternionic hyperbolic spaces and the Cayley hyperbolic plane. In the latter case the corresponding Riemannian symmetric pair (G, K) of non-compact type (in the sense of [15], Chap. VI) is the real simple Lie group, as introduced in 1.3,

$$G = \mathcal{H}(C_a, \Delta_1),$$

where Δ_1 is the involution given by $\delta = \mathrm{diag}(1, -1, 1)$ for any non-split Cayley algebra C_a over $\mathbb{R}$. This Lie group G is of real rank 1, to be denoted $F_{4(-20)}$. Recall that C_a is isomorphic to the algebra $\mathbb{O}$ of Graves–Cayley octonions. We fix a maximal compact subgroup $K \cong \mathrm{Spin}(9)$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding

Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. There is an associated involutive automorphism $\theta\colon \mathfrak{g} \to \mathfrak{g}$, called the Cartan involution, which acts as identity Id on the subalgebra $\mathfrak{k}$ and as $-(\mathrm{Id})$ on $\mathfrak{p}$. Let G_u be a maximal compact subgroup of the complexification $G_{\mathbb{C}}$ of G which contains K. We may suppose that the Lie algebra $\mathfrak{g}_u$ of G_u admits the Cartan decomposition

$$\mathfrak{g}_u = \mathfrak{k} \oplus \mathfrak{p}_u \quad \text{with } \mathfrak{p}_u = i\,\mathfrak{p}.$$

Note that $\mathfrak{g}_u$ is a compact real form of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. If B denotes the Killing form of $\mathfrak{g}_{\mathbb{C}}$, its restrictions to $\mathfrak{g} \times \mathfrak{g}$ and $\mathfrak{g}_u \times \mathfrak{g}_u$ are the Killing forms of $\mathfrak{g}$ and $\mathfrak{g}_u$ respectively. The homogeneous space $X_u := K\backslash G_u$ endowed with the unique Riemannian structure induced by the restriction of $-B$ to $i\,\mathfrak{p} \times i\,\mathfrak{p}$ is a compact symmetric Riemannian space, the compact dual symmetric space of $X = K\backslash G$. Note that there is a natural identification of the tangent space $T_{Ke}(X_u)$ at the origin with the Lie algebra $\mathfrak{p}_u = i\,\mathfrak{p}$ where $i\,\mathfrak{p}$ is viewed as a real subspace of $\mathfrak{p}_{\mathbb{C}}$.

3.2. Invariant forms. The compact dual X_u is the quotient of a compact connected group modulo a closed connected subgroup. Thus the cohomology $H^*(X_u, \mathbb{R})$ admits a natural interpretation as the space $I^*_{G_u}$ of G_u-invariant $\mathbb{R}$-valued C^∞-forms on X_u, that is, there is an isomorphism $H^*(X_u, \mathbb{R}) \xrightarrow{\sim} I^*_{G_u}$.

By evaluating a form at the origin we obtain an isomorphism

$$I_{G_u} \xrightarrow{\sim} \mathrm{Hom}_K(\Lambda(\mathfrak{p}_u), \mathbb{R}) = (\Lambda(\mathfrak{g}_u/\mathfrak{k})^*)^K$$

of the space I_{G_u} of G_u-invariant $\mathbb{R}$-valued C^∞-forms on X_u onto the complex $C(\mathfrak{g}_u, \mathfrak{k}, \mathbb{R}) = \mathrm{Hom}_K(\Lambda(\mathfrak{p}_u), \mathbb{R}) = (\Lambda(\mathfrak{g}_u/\mathfrak{k})^*)^K$. However, there is a natural $\mathbb{R}$-linear isomorphism

$$(\Lambda\,\mathfrak{p}^*)^K \longrightarrow (\Lambda\,\mathfrak{p}_u^*)^K.$$

In view of the identification of the former space with the space I^*_G of G-invariant $\mathbb{R}$-valued smooth forms on X, we obtain an isomorphism

$$H^*(X_u, \mathbb{R}) \xrightarrow{\sim} I^*_{G_u} \xrightarrow{\sim} I^*_G. \tag{3.1}$$

Given a torsion-free discrete subgroup $\Gamma \subset G$, a G-invariant $\mathbb{R}$-valued form ω on X naturally descends to a form on the quotient X/Γ, thus there is a map

$$H^*(X_u, \mathbb{R}) \xrightarrow{\sim} I^*_G \longrightarrow \Omega^*(X, \mathbb{R})^\Gamma. \tag{3.2}$$

The space I^*_G consists of closed (even harmonic) forms, thus we get a homomorphism

$$\beta^*_\Gamma\colon H^*(X_u, \mathbb{R}) \longrightarrow H^*(\Omega(X, \mathbb{R})^\Gamma) \xrightarrow{\sim} H^*(X/\Gamma, \mathbb{R}). \tag{3.3}$$

Now we suppose that the discrete group $\Gamma \subset G$ gives rise to a compact quotient X/Γ. In this case, by Hodge theory, a harmonic form $\omega \neq 0$ cannot be a coboundary, therefore the homomorphism β_Γ is injective.

We are led to determine the cohomology of the compact dual symmetric space X_u. This can be easily derived from a result of Borel–Chevalley [3].

Proposition. *The real cohomology $H^*(X_u, \mathbb{R})$ of the compact dual of the Cayley hyperbolic plane is given as*

$$H^*(X_u, \mathbb{R}) = \begin{cases} \mathbb{C} & \text{if } j = 0,\ 8,\ 16, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By [3], Sect. 2, the Poincaré polynomial $P(X_u, t) = \sum_{i \geq 0} \dim H^i(X_u, \mathbb{R})$ has the form $1 + t^8 + t^{16}$. This implies that the cohomology groups vanish in odd degrees. As X_u is a compact manifold of dimension 16, $H^i(X_u, \mathbb{R}) = \mathbb{C}$ for $i = 0$, 16. Since the Euler characteristic $\chi(X_u) = \frac{|W_{G_u}|}{|W_K|} = 3$ all other cohomology groups have to vanish except in degree $i = 8$. $\square$

4. Unitary representations with non-vanishing cohomology

In this section, we briefly discuss the constructive approach to the classification [37] of irreducible unitary representations of a connected real semi-simple Lie group with non-vanishing relative Lie algebra cohomology. This general result allows us to enumerate (up to infinitesimal equivalence) the irreducible unitary $(\mathfrak{g}, K)$-modules with non-vanishing Lie algebra cohomology in the case $G = F_{4(-20)}$ in an explicit way. They are parametrized by θ_K-stable parabolic subalgebras $\mathfrak{q}$ of $\mathfrak{g}$. Given such an irreducible unitary $(\mathfrak{g}, K)$-module $A_\mathfrak{q}$ we also determine $H^*(\mathfrak{g}, K, A_\mathfrak{q})$.

4.1. The classification up to infinitesimal equivalence. Let G be a connected real semi-simple Lie group with finite center, $K \subset G$ a maximal compact subgroup. Let θ_K be the Cartan involution corresponding to $K \subset G$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. By definition a θ_K-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ is a parabolic subalgebra of $\mathfrak{g}_\mathbb{C}$ such that $\theta_K \mathfrak{q} = \mathfrak{q}$, and $\bar{\mathfrak{q}} \cap \mathfrak{q} = \mathfrak{l}_\mathbb{C}$ is a Levi subalgebra of $\mathfrak{q}$ where $\bar{\mathfrak{q}}$ refers to the image of $\mathfrak{q}$ under complex conjugation with respect to the real form $\mathfrak{g}$ of $\mathfrak{g}_\mathbb{C}$. Write $\mathfrak{u}$ for the nilradical of $\mathfrak{q}$. Then $\mathfrak{l}_\mathbb{C}$ is the complexification of a real subalgebra $\mathfrak{l}$ of $\mathfrak{g}$. The normalizer of $\mathfrak{q}$ in G is connected since G is, and it coincides with the connected Lie subgroup L of G with Lie algebra $\mathfrak{l}$. The Levi subgroup L has the same rank as G, is preserved by the Cartan involution θ_K, and the restriction of θ_K to L is a Cartan involution. Moreover, the group L contains a maximal torus $T \subset K$. We will indicate below a construction of all possible θ_K-stable parabolic subalgebras $\mathfrak{q}$ in $\mathfrak{g}$ up to conjugation by K. There are only finitely many K-conjugacy classes of θ_K-stable parabolic subalgebras $\mathfrak{q}$ in $\mathfrak{g}$.

Via cohomological induction, a θ_K-stable parabolic subalgebra $\mathfrak{q}$ gives rise to an irreducible unitary representation $\mathcal{R}_\mathfrak{q}^S(\mathbb{C}) = A_\mathfrak{q}$ of G with $S = \dim \mathfrak{u} \cap \mathfrak{k}$ [37], Thm. 2.5. It is uniquely determined up to infinitesimal equivalence by the K-conjugacy class of $\mathfrak{q}$. We denote the Harish-Chandra module of $A_\mathfrak{q}$ by the same letter or by $A_{\mathfrak{q},K}$. One has

$$H^j(\mathfrak{g}, K, A_{\mathfrak{q},K}) = \mathrm{Hom}_{L\cap K}(\wedge^{j-R}(\mathfrak{l}\cap\mathfrak{p}), \mathbb{C}) \tag{4.1}$$

where $R = R(\mathfrak{q}) := \dim(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})$. In particular, note that the Lie algebra cohomology with respect to the representation $A_{\mathfrak{q}}$ vanishes in degrees below $\dim(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})$ and above $\dim(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}}) + \dim(\mathfrak{l}\cap\mathfrak{p}_{\mathbb{C}})$.

If $\mathfrak{l} \subset \mathfrak{k}_{\mathbb{C}}$, the representation $A_{\mathfrak{q}}$ belongs to the discrete series of G. Then $L = L\cap K$, and the cohomology $H^j(\mathfrak{g}, K, A_{\mathfrak{q},K})$ vanishes in all degrees $j \neq R(\mathfrak{q}) = \dim(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})$, that is, $j \neq \frac{1}{2}\dim\,(G/K)$. One has $H^{R(\mathfrak{q})}(\mathfrak{g}, K, A_{\mathfrak{q},K}) = \mathbb{C}$ (see e.g. [36], Sect. 3). If the θ_K-stable parabolic subalgebra $\mathfrak{q}$ is $\mathfrak{g}_{\mathbb{C}}$ then L coincides with G; we take $A_{\mathfrak{q}} = \mathbb{C}$.

Suppose that (π, H_π) is an irreducible unitary representation (π, H_π) of G with

$$H^*(\mathfrak{g}, K, H_{\pi,K}) \neq 0.$$

Then there is a θ_K-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ so that $\pi \cong A_{\mathfrak{q}}$. This construction [37], Thm. 4.1, is essentially a consequence of results of Parthasarathy [26]. One finds a proof of the unitarity of the representation $A_{\mathfrak{q}}$ in [35].

Following [37] and [36], Sect. 4, we outline a construction of all θ_K-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ up to conjugation by K. Fix a maximal torus T in K. The centralizer H of T in G is a Cartan subgroup. According to the Cartan decomposition of $\mathfrak{g}$ we may write $H = TA$ with $A = H\cap(\exp\mathfrak{p})$. We denote the Lie algebra of T by $\mathfrak{t}_c$.

Let $\Phi(\mathfrak{k}, \mathfrak{t}_c) =: \Phi_c$ be the system of roots for $\mathfrak{t}_c$ in $\mathfrak{k}_{\mathbb{C}}$, and fix a system $\Phi_c^+ := \Phi^+(\mathfrak{k}, \mathfrak{t}_c) \subset \Phi(\mathfrak{k}, \mathfrak{t}_c)$ of positive roots. Similarly, we write Φ_n for the set of non-zero weights of $\mathfrak{t}_c$ on $\mathfrak{p}_{\mathbb{C}}$.

Fix an element $x \in i(\mathfrak{t}_c)_{\mathbb{R}}$ that is dominant for K, that is, $\gamma(x) \geq 0$ for all $\gamma \in \Phi^+(\mathfrak{k}, \mathfrak{t}_c)$. Then the θ_K-stable parabolic subalgebra associated to x is defined by

$$\mathfrak{q}_x = \mathfrak{h}_{\mathbb{C}} \oplus \sum_{\gamma\in\Phi, \gamma(x)\geq 0} \mathfrak{g}_{\mathbb{C},\gamma}$$

with $\Phi := \Phi_n \cup \Phi_c$. The corresponding Levi subalgebra $\overline{\mathfrak{q}_x} \cap \mathfrak{q}_x = (\mathfrak{l}_x)_{\mathbb{C}}$ is

$$(\mathfrak{l}_x)_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \sum_{\gamma\in\Phi, \gamma(x)=0} \mathfrak{g}_{\mathbb{C},\gamma}.$$

4.2. The case of the real rank one Lie group $F_{4(-20)}$. The real rank one form $G = \mathscr{H}(C_a, \Delta_1)$ of type F_4, also denoted by $F_{4(-20)}$, is a connected, simply connected Lie group. Write θ_K for the Cartan involution fixing the maximal compact subgroup $K \cong \mathrm{Spin}(9)$. Let $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. One has $\mathfrak{k} \cong \mathfrak{so}(9)$. The symmetric space X corresponding to the pair (G, K) is of dimension 16.

Since $\mathrm{rk}_{\mathbb{R}}(G) = \mathrm{rk}_{\mathbb{R}}(K)$ the group G has discrete series representations. We denote by $G_{d,\mathbb{C}}$ the set of equivalence classes of irreducible discrete series representations of $G = F_{4(-20)}$ whose infinitesimal character coincides with the one of the trivial representation. By the enumeration of discrete series representations, this set contains exactly $|W_G/W_K|$ elements where W_G and W_K denote the Weyl group of G and K respectively. Therefore we have (up to infinitesimal equivalence) three elements in $G_{d,\mathbb{C}}$, to be denoted by $A_{\mathfrak{q}_i}$, $i = 1, 2, 3$. In these cases the Lie algebra $\mathfrak{q}_i$ is a θ_K-stable Borel subalgebra. The trivial representation $\mathbb{C}$ of G is described by $A_{\mathfrak{q}_0}$ with $\mathfrak{q}_0 = \mathfrak{g}_{\mathbb{C}}$. Up to conjugation by K, the other θ-stable parabolic subalgebras $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}$ so that L contains T can be represented by the elements in the set $\mathcal{Q} = \{\mathfrak{q}_4, \ldots, \mathfrak{q}_6, \mathfrak{q}_7^{\pm}\}$ where, given $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}$, the corresponding Levi subalgebra $\mathfrak{l}_{\mathbb{C}}$ is characterized by

$$[\mathfrak{l}_4, \mathfrak{l}_4] = \mathfrak{sp}(2,1), \quad [\mathfrak{l}_5, \mathfrak{l}_5] = \mathfrak{so}(6,1), \quad [\mathfrak{l}_6, \mathfrak{l}_6] = \mathfrak{so}(4,1)$$

and, finally,

$$[\mathfrak{l}_7^{\pm}, \mathfrak{l}_7^{\pm}] = \mathfrak{sl}(2, \mathbb{R}).$$

Note that there are two θ-stable parabolic subalgebras in this set which both give rise to the same Levi subalgebra $\mathfrak{l}_7$. In addition, these representations $A_{\mathfrak{q}}$ with $\mathfrak{q} \in \mathcal{Q}$ are all non-tempered. In [8], one finds a parametrization of all irreducible admissible representations (not necessarily unitary) of G with non-zero relative Lie algebra cohomology. Then the list above is supplemented by six non-unitary representations (up to infinitesimal equivalence). However, we obtain the following result:

Proposition. *If (π, H_π) is an irreducible unitary representation of G with non-zero cohomology $H^*(\mathfrak{g}, K, H_{\pi,K}) \neq 0$ then the Harish-Chandra module of (π, H_π) is (up to infinitesimal equivalence) one of the representations*

$$A_{\mathfrak{q}_i}, \ i = 0, \ldots, 6, \quad or \quad A_{\mathfrak{q}_7^{\pm}}.$$

We have the following non-vanishing result, where $d = 16$ denotes the dimension of the symmetric space attached to the pair $(F_{4(-20)}, \mathrm{Spin}(9))$:

$$\text{If } i = 4, \ldots, 7, \text{ then } H^j(\mathfrak{g}, K, A_{\mathfrak{q}_i} \otimes \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } j = i,\ d-i, \\ 0 & \text{otherwise}; \end{cases}$$

$$\text{If } i = 1, \ldots, 3, \text{ then } H^j(\mathfrak{g}, K, A_{\mathfrak{q}_i} \otimes \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } j = \tfrac{1}{2}d, \\ 0 & \text{otherwise}; \end{cases}$$

$$\text{If } i = 0, \text{ then } H^j(\mathfrak{g}, K, A_{\mathfrak{q}_0} \otimes \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } j = 0,\ \tfrac{1}{2}d, \text{ or } d, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. The actual computation of the relative Lie algebra cohomology in the first case relies on identifying the Langlands parameter of the irreducible unitary representations $A_{\mathfrak{q}}$, $\mathfrak{q} \in \mathcal{Q}$, and analyzing an attached short exact sequence of $(\mathfrak{g}, K)$-modules.

For Lie groups of real rank one this procedure is outlined in [6], VI. For the group in question the final result of this tedious computation is also given in [8].

Another approach is based on a direct calculation, using the general formula $H^j(\mathfrak{g}, K, A_{\mathfrak{q},K}) = \mathrm{Hom}_{L\cap K}(\bigwedge^{j-R}(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C})$ where $R = \dim(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}})$. The second assertion concerns discrete series representations; these were discussed in 4.1. The third assertion follows from 3.2 since $I_G \xrightarrow{\sim} \mathrm{Hom}_K(\Lambda(\mathfrak{p}), \mathbb{R}) \xrightarrow{\sim} H^*(X_u, \mathbb{R})$. □

5. Geometric cycles

First, in this section, we give an overview of the general construction of geometric cycles in arithmetic quotients X/Γ as outlined in [31], Sects. 6 and 9. Second, in the specific case of interest for us, we use one of the techniques developed in [29] to show that certain geometric cycles exist and represent a non-zero homology class for the underlying manifold X/Γ. This relies on the approach via "excess intersections".

5.1. The construction of geometric cycles. Let G denote a connected semi-simple algebraic group defined over an algebraic number field k, $\Gamma \subset G(k)$ an arithmetic subgroup.

Let H be a reductive k-subgroup of G, let K_H be a maximal compact subgroup of the real Lie group H_∞, and let $X_H = K_H\backslash H_\infty$ be the space associated to H_∞, as in 2.1. If $x_0 \in X$ is fixed under the natural action of $K_H \subset G_\infty$ on X, then the assignment $h \mapsto x_0\, h$ defines a closed embedding

$$X_H = K_H\backslash H_\infty \longrightarrow X,$$

that is, the orbit map identifies X_H with a totally geodesic submanifold of X. Thus, we also have a natural map

$$j_{H|\Gamma} \colon X_H/\Gamma_H \longrightarrow X/\Gamma,$$

where $\Gamma_H = \Gamma \cap H(k)$. We consider the composite of the inclusion $i\colon H_\infty/\Gamma_H \to G_\infty/\Gamma$ and the projection $\pi\colon G_\infty/\Gamma \to X/\Gamma$. The latter map is proper since it is the projection of a locally trivial fibration with compact fibers. The same is true for the surjective map $\pi_H\colon H_\infty/\Gamma_H \to X_H/\Gamma_H$. The composite $j_{H|\Gamma} \circ \pi_H$ coincides with the map $\pi \circ i$. It is known [31], Sect. 6, that the map $j_{H|\Gamma}$ is proper.

Now we are interested in situations in which for a given subgroup H and a torsion-free arithmetic subgroup Γ of G, the corresponding map $j_{H|\Gamma}$ is an injective immersion. Thus, by being proper, $j_{H|\Gamma}$ is an embedding, and the image $j_H(X_H/\Gamma_H)$ of X_H/Γ_H is a submanifold in X/Γ. This submanifold is totally geodesic, to be called a *geometric cycle* in X/Γ. The following theorem, stated in [31], Sect. 6, Thm. D, with an outline of its proof, is a combination of a result by Raghunathan [9], Sect. 2, and a result in [29].

Theorem. *Let G be a connected semi-simple algebraic k-group, let $H \subset G$ be a connected reductive k-subgroup, and let Γ be an arithmetic subgroup of $G(k)$. Then there exists a subgroup Γ' of finite index in Γ such that if Γ is replaced by Γ' the map*

$$j_{H|\Gamma'} \colon X_H/\Gamma'_H \longrightarrow X/\Gamma'$$

is a proper, injective, closed embedding, and so that each connected component of the image is an orientable, totally geodesic submanifold of X/Γ'.

For example, such geometric cycles naturally arise as fixed point components of an automorphism μ of finite order on X/Γ which is induced by a rational automorphism of G. It is known (see e.g. [31], 6.4) that the connected components of the fixed point set $\mathrm{Fix}(\langle\mu\rangle, X/\Gamma)$ are totally geodesic closed submanifolds in X/Γ of the form $F(\gamma) = X(\gamma)/\Gamma(\gamma)$ where γ ranges over a set of representatives for the classes in the non-abelian cohomology set $H^1(\langle\mu\rangle, \Gamma)$. Such a connected component is of the form $X(\gamma)/\Gamma(\gamma)$ where $X(\gamma)$ is the set of fixed points of the action of μ on X twisted by the cocycle γ. The component originates with the group $G(\gamma)$ of elements fixed by the γ-twisted μ-action on G. Occasionally one also writes $X_{G(\gamma)}$ for $X(\gamma)$. As first noted in [27] resp. [28] in specific cases, the map $j_{G(\gamma)|\Gamma}$ is injective in such a case.

In general, we are interested in cases where a geometric cycle Y is orientable and its fundamental class is not homologous to zero in X/Γ, in singular homology or homology with closed supports, as necessary. As stated in the Theorem, there exists a subgroup of finite index in Γ such that the corresponding cycles are orientable. Thus we may suppose that the geometric cycles we consider have this property.

One way to go about the second question is to construct an orientable submanifold Y' of complementary dimension such that the intersection product (if defined) of its fundamental class with that of Y is non-zero. In doing so, if X/Γ is non-compact, we have to assume that at least one of the cycles Y, Y' is compact, while the other need not be.

A simple method to prove that such a geometric cycle represents a nontrivial homology class is by showing that the cycle intersects a second geometric cycle, of complementary dimension, in a single point with multiplicity ± 1. Unfortunately, geometric cycles of complementary dimension usually intersect in a more complicated set, possibly of dimension greater than zero. The theory of "excess intersections" as developed in [29], Sects. 3 and 4, is helpful in such a situation. In particular, it provides a formula for the intersection number of a pair of two such geometric cycles Y and Y' which intersect perfectly. By definition, Y and Y' intersect perfectly if the connected components of the intersection are immersed submanifolds in X/Γ and for each of the components F of $Y \cap Y'$ the tangent bundle TF of F coincides with the intersection of the restriction of the tangent bundles of Y and Y' to F, that is, $TF = TY_{|F} \cap TY'_{|F}$. If the intersection is compact the intersection number of two such cycles can be expressed as the sum of the Euler numbers of the excess bundles corresponding to the connected components of the intersection [29], Prop. 3.3. A detailed analysis of the intersection number might then enable us to show that the underlying geometric cycles are indeed non-bounding cycles. In order to find a non-zero intersection product, if at all possible, it is often necessary to replace the arithmetic group Γ by a suitable subgroup of finite index.

5.2. The case of arithmetic subgroups in algebraic groups of type F_4. Let k be a totally real algebraic number field of degree $d = [k : \mathbb{Q}]$. Given an Albert algebra A over k we denote by G' the algebraic $\mathbb{Q}$-group $\mathrm{Res}_{k/\mathbb{Q}}(G)$ obtained from the group G of k-algebra automorphisms of A by restriction of scalars. There is an isomorphism $G'(\mathbb{R}) \to \prod_{v \in V_\infty} G_v$ for the group of real points of G' where $G_v = G^{\sigma_v}(k_v)$. Given a place $v \in V$, by the enumeration in 1.3, the real Lie group G_v is isomorphic to one of the simple groups $F_{4(4)}$, $F_{4(-20)}$ or $F_{4(-52)}$ respectively. These groups are simply connected and have trivial center. In this exposition, for the sake of simplicity, we assume that we are in one of the two cases:

(I) The real Lie group G_{v_1} is isomorphic to $F_{4(-20)}$ for the place v_1 corresponding to $\sigma_1 = \mathrm{Id}$; for all other places $v \in V_\infty$, $v \neq v_1$, the group G_v is isomorphic to the compact group $F_{4(-52)}$.

(II) The real Lie group G_{v_1} is isomorphic to $F_{4(4)}$ for the place v_1 corresponding to $\sigma_1 = \mathrm{Id}$; for all other places $v \in V_\infty$, $v \neq v_1$, the group G_v is isomorphic to the compact group $F_{4(-52)}$.

Then $G'(\mathbb{R})$ is the product of the real non-compact group G_{v_1} of $\mathbb{R}$-rank 1 or 4 and a finite (possibly empty) product of compact Lie groups. The associated symmetric space $X = K\backslash G'(\mathbb{R})$, $K \subset G'(\mathbb{R})$ a maximal compact subgroup, is of dimension 16 or 28. If $\Gamma \subset G(k)$ is a torsion-free arithmetic subgroup the quotient X/Γ is a locally symmetric space of finite volume.

From the Lie theoretic point of view the arithmetic subgroup $\Gamma \subset G(k)$ gives rise to a lattice in the Lie group $G'(\mathbb{R})$. However, by Margulis [22] in the case $G'(\mathbb{R}) = F_{4(4)}$ and by [12] in the case $G'(\mathbb{R}) = F_{4(-20)}$ (up to compact factors), each lattice in these groups arises in this way.

These lattices fall naturally into two classes, according to whether X/Γ is compact or not. We give two examples:

Example I. Let k be a totally real algebraic number field of degree $d = [k : \mathbb{Q}] \geq 2$. Given an Albert algebra A over k let G be the group of k-algebra automorphisms of A, $G' = \mathrm{Res}_{k/\mathbb{Q}}(G)$. Then there is an isomorphism $G'(\mathbb{R}) \to \prod_{v \in V_\infty} G_v$ where $G_v = G^{\sigma_v}(k_v)$. Suppose that there is an integer $1 \leq r < d$ so that G_{v_j} is isomorphic to $F_{4(-20)}$ for the places v_j corresponding to σ_j, $j = 1, \ldots, r$, and $G_{v_j} \cong F_{4(-52)}$ is a compact group for all other places $v_j \in V_\infty$, $r < j \leq d$. Let Γ be a torsion-free arithmetic subgroup of $G(k)$. As a consequence of the compactness criterion [4], Thm. 12.3, since there is at least one place $v \in V_\infty$ such that G_v is a compact group, the arithmetic quotient X/Γ is compact. The dimension of this compact manifold is $16r$.

Example II. Let $k = \mathbb{Q}$ be the field of rational numbers, and let be A be an Albert algebra over $\mathbb{Q}$ which is split at the only archimedean place, that is, $A \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathcal{H}(C_s, \Delta_s)$, using the notation in Sect. 1. The group G of $\mathbb{Q}$-algebra automorphisms of A is a split simple $\mathbb{Q}$-group of $\mathbb{Q}$-rank 4; its group $G(\mathbb{R})$ of real points is isomorphic to $F_{4(4)}$. Let Γ be a torsion-free arithmetic subgroup of $G(\mathbb{Q})$. The locally symmetric space X/Γ is non-compact but of finite volume. This space may be viewed as the interior of a compact manifold $\overline{X}/\Gamma$ with boundary [5], Thm. 9.3; the inclusion is a homotopy equivalence. In fact, the compactification $\overline{X}/\Gamma$ is a manifold with corners, and the boundary is glued together out of faces $e'(P)$, one for each Γ-conjugacy class of proper parabolic $\mathbb{Q}$-subgroups of G.

5.3. An explicit construction of a non-bounding geometric cycle. Let k be a totally real algebraic number field of degree $d = [k : \mathbb{Q}] \geq 2$. Let A be an Albert algebra A over k, let G be the group of k-algebra automorphisms of A, $G' = \mathrm{Res}_{k/\mathbb{Q}}(G)$ the algebraic $\mathbb{Q}$-group obtained from G by restriction of scalars. Suppose we are in case I of 5.2, that is, $G'(\mathbb{R}) \rightarrow \prod_{v \in V_\infty} G_v$ with $G_{v_1} = F_{4(-20)}$ for the place v_1 corresponding to $\sigma_1 = \mathrm{Id}$, and $G_{v_j} \cong F_{4(-52)}$ is a compact group for all other places $v_j \in V_\infty$, $1 < j \leq d$. Given a torsion-free arithmetic subgroup of $G(k)$ the arithmetic quotient X/Γ is compact manifold of dimension 16.

Theorem. *Given an arithmetically defined subgroup Γ in the algebraic $\mathbb{Q}$-group* $\mathrm{Res}_{k/\mathbb{Q}}(G)$ *obtained from the group G of k-algebra automorphisms of the Albert algebra A (subject to the conditions as given above) by restriction of scalars, its cohomology $H^*(X/\Gamma, \mathbb{R})$ contains (up to a subgroup $\Gamma' \subset \Gamma$ of finite index) a non-trivial cohomology class of degree* 8. *By duality, this class, to be denoted $[c_{H/\Gamma_H}]$, is detected by the fundamental class of a totally geodesic submanifold X_H/Γ_H in X/Γ with H a reductive subgroup of G defined over k. We note that $H_\infty \cong$* Spin(8, 1) *(up to compact factors). This geometric cycle arises as a fixed point component under a suitable rational involution on the group G. This class $[c_{H/\Gamma_H}]$ is not in the image of the injective homomorphism*

$$\beta_\Gamma^* : H^*(X_u, \mathbb{R}) \longrightarrow H^*(\Omega(X, \mathbb{R})^\Gamma) \xrightarrow{\sim} H^*(X/\Gamma, \mathbb{R}),$$

that is, it cannot be represented by a G_∞-invariant $\mathbb{R}$-valued form on X.

By Matsushima's theorem and the results in Section 4, this geometric construction of a non-bounding cycle in $H^8(X/\Gamma, \mathbb{R})$ which is not a $G(\mathbb{R})$-invariant class shows that the sum of the multiplicities with which the irreducible unitary representations $A_{\mathfrak{q}_i}, i = 1, \ldots, 3$, i.e., the discrete series representations in 4.2, occur in $L^2(G_\infty/\Gamma)$ is non-zero.

Proof. The Albert algebra A defined over k is also a J-algebra over k in the sense of [33], 5.1, that is, A is a finite dimensional commutative, not necessarily associative, k-algebra with identity element e, equipped with a non-degenerate quadratic form q such that the following identities hold:

$q(x^2) = q(x)^2$ for all $x \in A$ with $\langle x, e\rangle = 0$;
$\langle xy, z\rangle = \langle x, yz\rangle$ for all $x, y, z \in A$;
$q(e) = \frac{3}{2}$.

Here $\langle\ ,\ \rangle$ denotes the bilinear form defined by $\langle x, y\rangle = q(x + y) - q(x) - q(y)$, $x, y \in A$. We call this bilinear form associated to q the inner product on A.

If k' is an extension field of k then the algebra $A \otimes_k k'$, endowed with the natural extensions of the product structure and the quadratic form q is an J-algebra defined over k'.

Since the field of definition of A is an algebraic number field, the J-algebra A is reduced as proved by Albert [1]. By definition, the property of being reduced includes the existence of idempotent elements $v \neq 0, e$. If $v, v \neq 0, e$, is such an idempotent in A then $\det(v) = 0$ and $q(v) = \frac{1}{2}$ or $q(v) = 1$. One has that $e - v$ is an idempotent in A with $v(e - v) = 0$, $\langle v, e - v\rangle = 0$ and $q(e - v) = \frac{3}{2} - q(v)$. Thus, if A contains an idempotent $\neq 0, e$ it also contains an idempotent u with $q(u) = \frac{1}{2}$. Such an idempotent is called a primitive idempotent.

We fix a primitive idempotent u in A. The restriction of the quadratic form q to the subspace $k\ e \oplus k\ u$ is non-degenerate, hence the restriction $q_{|E}$ of q on its complement

$$E := (ke \oplus ku)^{\perp}$$

is also non-degenerate. Given $x \in E$ we have $\langle ux, e\rangle = \langle x, ue\rangle = 0$ and $\langle ux, u\rangle = \langle x, u^2\rangle = \langle x, u\rangle = 0$. Therefore the assignment $x \mapsto ux$, $x \in E$, defines a linear map

$$t_u\colon E \longrightarrow E.$$

By [33], 5.3.1, $\langle t_u(x), y\rangle = \langle x, t_u(y)\rangle$ for all $x, y \in E$ and $t_u t_u = \frac{1}{2} t_u$. It follows that t_u has possible eigenvalues 0 and $\frac{1}{2}$, and E decomposes as an orthogonal sum

$$E = E_0 \oplus E_1$$

of the corresponding eigenspaces. The restrictions of q to E_0 and E_1 respectively are non-degenerate quadratic forms. One usually calls E_0 and E_1 the zero space and the half space respectively of the idempotent u.

We denote by $\mathrm{Aut}(A)_u$ the group of automorphisms of A that fix the given primitive idempotent $u \in A$. If $\phi \in \mathrm{Aut}(A)_u$, i.e., $\phi(u) = u$, then ϕ leaves invariant the eigenspaces E_i, $i = 0, 1$. Since an automorphism ϕ of the Albert algebra leaves invariant the quadratic form q, ϕ induces orthogonal linear transformations $\phi_i\colon E_i \to E_i$, $i = 0, 1$.

As proved in [18], Thm. 11 (see also [33], 7.1.3), the assignment

$$\phi \mapsto \phi_0 := \phi_{|E_0}$$

defines a homomorphism

$$\chi_u\colon\ \mathrm{Aut}(A)_u \longrightarrow O'(E_0, q_{|E_0})$$

of $\mathrm{Aut}(A)_u$ onto the reduced orthogonal group of the quadratic space $(E_0, q_{|E_0})$. The kernel of χ_u contains exactly two elements.

The group $\mathrm{Aut}(A)_u$, the stabilizer of the primitive idempotent $u \in A$, is an algebraic k-group, to be denoted G_u. It is isomorphic to the spin group $\mathrm{Spin}(E_0, q_{|E_0})$, also called the reduced Clifford group of $(E_0, q_{|E_0})$. This implies that G_u is a connected quasi simple algebraic group of type B_4. Its center $Z(G_u)$ consists of two elements. We denote the non-trivial element in $Z(G_u)$ by z_u. Conjugation with z_u defines an involution σ_u on $G = \mathrm{Aut}(A)$. The group $G(\sigma_u)$ of fixed points under σ_u coincides with G_u. By our assumption on A, that is, $G_{v_1} \cong F_{4(-20)}$, the quadratic form $q_{|E_0}$ has signature (8, 1) over $k_{v_1} \cong \mathbb{R}$. This implies that $G(\sigma_u)(\mathbb{R}) \cong \mathrm{Spin}(8, 1)$.

Choose a maximal subgroup K of $G(\mathbb{R})$, stable under the involution σ_u. Then σ_u induces an involution on the symmetric space $X = K \setminus G(\mathbb{R})$. We denote by $X(\sigma_u) := \mathrm{Fix}(\langle\sigma_u\rangle, X)$ the set of fixed points of σ_u. It is a subsymmetric space of dimension 8 in X. Choose a "rational point" x in $X(\sigma_u)$, that is, the corresponding Cartan involution θ_x is an automorphism of G defined over k. We observe that θ_x fixes the maximal subgroup K_x corresponding to the choice of x. Then the composition $\tau_u := \sigma_u \circ \theta_x$ is an involution on G, defined over k, which commutes with σ_u. Thus $\tau_u(X(\sigma_u)) \subset X(\sigma_u)$. It follows that the fixed point set $X(\tau_u)$ intersects with $X(\sigma_u)$ in the points fixed by σ_u, τ_u and θ_x. This implies

$$X(\tau_u) \cap X(\sigma_u) = \{x\}.$$

Let $\Theta = \langle\sigma_u, \tau_u\rangle$ denote the abelian group generated by σ_u and τ_u. Let Γ be a Θ-stable torsion-free arithmetic subgroup of $G(\mathbb{Q})$. Then there is a natural action of Θ on the locally symmetric space X/Γ as well as one of σ_u and τ_u respectively. By [29] or [31], 6.4, for $\mu = \sigma_u, \tau_u$ the fixed point set $\mathrm{Fix}(\langle\mu\rangle, X/\Gamma)$ is a disjoint union of connected components $F(\gamma) = X(\gamma)/\Gamma(\gamma)$ where $\gamma \in H^1(\langle\mu\rangle, \Gamma)$ ranges over a set of representatives for the classes in the non-abelian cohomology of $\langle\mu\rangle$ in Γ. By using the natural map $H^1(\langle\mu\rangle, \Gamma) \to H^1(\langle\mu\rangle, G(\mathbb{R}))$ we can view a representing cocycle γ of $\langle\mu\rangle$ in Γ as a representative for a class in $H^1(\langle\mu\rangle, G(\mathbb{R}))$. If two cocycles γ and γ' which are not equivalent give rise to the same class in $H^1(\langle\mu\rangle, G(\mathbb{R}))$ then $X(\gamma)$ is a translate of the totally geodesic Riemannian submanifold $X(\gamma')$ under an element in $G(\mathbb{R})$. In particular, $X(\gamma)$ and $X(\gamma')$, and hence $F(\gamma)$ and $F(\gamma')$, have the same dimension.

However, the connected component corresponding to the base point $1_{\langle\mu\rangle}$ in the non-abelian Galois cohomology set $H^1(\langle\mu\rangle, \Gamma)$ coincides with the geometric cycle

$$C(\mu, \Gamma) := X(1_{\langle\mu\rangle})/\Gamma(1_{\langle\mu\rangle}), \quad \mu = \sigma_u, \tau_u.$$

given as the image of the map

$$j_{G(\mu)|\Gamma} \colon X_{G(\mu)}/\Gamma_{G(\mu)} \longrightarrow X/\Gamma, \quad \mu = \sigma_u, \tau_u.$$

We note that this map is injective in such a case. Each of these cycles has dimension 8. The cycles $Y := C(\langle\sigma_u\rangle, \Gamma)$ and $Y' := C(\langle\tau_u\rangle, \Gamma)$ arise as fixed point components of two involutions which commute with one another thus, by [29], Lemma 1.4, they intersect perfectly.

Since $G(\sigma_u)(\mathbb{R})$ and $G(\tau_u)(\mathbb{R})$ are connected groups we may assume (by passing to a subgroup of finite index if necessary) that the connected components are orientable (see [29]). Thus, since $X(\tau_u)$ and $X(\sigma_u)$ are of complementary dimension and intersect in exactly one point, the assumptions of [29], Thm. 4.11, are satisfied. This general result implies that there is a Θ-stable arithmetic subgroup Γ' of G, $\Gamma' \subset \Gamma$ of finite index, so that the intersection number of the two cycles in question is given by

$$[C(\langle\sigma_u\rangle, \Gamma')]\,[C(\langle\tau_u\rangle, \Gamma')] = \sum_{\gamma \in \ker \operatorname{res}_{(\sigma_u, \tau_u)}} \chi(F(\gamma)) \neq 0.$$

Moreover, all connected components $F(\gamma)$ have the same dimension modulo 4. Note that the sum ranges over a set of representatives in the kernel of the natural map

$$\operatorname{res}_{(\sigma_u,\tau_u)} \colon H^1(\Theta, \Gamma) \longrightarrow H^1(\langle\sigma_u\rangle, \Gamma) \times H^1(\langle\tau_u\rangle, \Gamma).$$

Consequently the geometric cycles $[C(\langle\sigma_u\rangle, \Gamma')]$ and $[C(\langle\tau_u\rangle, \Gamma')]$ give rise via duality to non-vanishing classes of degree 8 in the cohomology of the arithmetic quotient X/Γ'. By passing to a subgroup of finite index if necessary, it follows from a general result of Millson–Raghunathan [24], Thm. 2.1, that we can achieve that the classes so obtained are not in the image of the map $\beta^*_{\Gamma'}$, that is, they cannot be represented by a G_∞-invariant $\mathbb{R}$-valued form on X. This proves our claim. □

5.4. Remarks. (1) As we have seen, the stabilizer G_u in G of a primitive idempotent $u \in A$, being realized as the group of fixed points of the rational involution σ_u, gave rise to this geometric construction of a non-bounding cycle. In G one finds other natural subgroups H which can be used to an analogous construction. One of these is of type $C_1 \times C_3$ and occurs as the fixed point set of an involution whereas another one, of type $A_2 \times A_2$, can be interpreted as the group $G(\rho)$ of fixed points under an automorphism ρ of G of order 3. These will be discussed at another occasion. In particular, it is of interest to analyze how this geometric construction is related to the work of Burger–Li–Sarnak [7] on Ramanujan duals and automorphic spectrum.

(2) In case (II), as listed in 5.2, given an arithmetic subgroup $\Gamma \subset G(k)$, $[k : \mathbb{Q}] \geq 2$, such that X/Γ is compact, one can construct a geometric cycle in an analogous way. It originates with the stabilizer $G_u = G(\sigma_u) \subset \operatorname{Aut}(A)$ attached to a primitive idempotent $u \in A$. Since the underlying Albert algebra splits at $v_1 \in V_\infty$ the quadratic form $q_{|E_0}$ on E_0 has Witt index 4, thus $G(\sigma_u)(\mathbb{R}) \cong \operatorname{Spin}(5, 4)$. The corresponding cycle $C(\sigma_u, \Gamma)$ in X/Γ (with $\dim X/\Gamma = 28$) is of dimension 20. It is expected that this cycle is non-bounding as well.

(3) The theory of constructing non-bounding cycles in cases where the arithmetic quotient is non-compact (but of finite volume) needs even more attention. The question to which extent and in which way geometric cycles contribute non-trivially to the

cohomology groups $H^*(X/\Gamma, \mathbb{R})$ is open in any generality. This circle of problems is also related to the description of the "cohomology at infinity" via the theory of Eisenstein series as initiated by Harder in [13], [14] and pursued in, for example, [30], [10], or [11].

5.5. Other exceptional groups. Along the same lines one can deal with special cycles in compact arithmetic quotients attached to exceptional groups of other types. As an application of the main result in [29], Sect. 4, we find among these k-forms of the exceptional Lie groups (up to compact factors) $E_{6(-14)}$, $E_{6(-26)}$, $E_{7(-28)}$. This relies on the following general result in Lie theory [29], 4.7, which allows us to resolve orientability questions: Suppose that σ is an automorphism of finite order (defined over $\mathbb{R}$) of a connected simply connected semi-simple algebraic $\mathbb{R}$-group. If the group $G(\mathbb{R})$ of real points of G is simply connected as a topological group, then the group $G(\sigma)(\mathbb{R})$ of fixed points is connected. It is worth noting that all split exceptional groups as well as $E_{6(2)}$, $E_{7(-5)}$, $E_{8(-24)}$ do not fall in the class addressed in this assertion.

In the case of compact arithmetic quotients X/Γ, let us say of dimension 8 for the sake of simplicity, attached to groups of type G_2, Waldner discusses in his thesis work [38] (see also [31], Sect. 10) the geometric construction of non-bounding cycles. His results include non-vanishing results for the cohomology $H^i(X/\Gamma, \mathbb{R})$ in degrees $i = 3, 4, 5$. The classes originate with cycles of different nature, some are components of fixed point sets of suitable involutions, some are not.

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