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Commensurability classes of discrete arithmetic hyperbolic groups

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Abstract. In this paper we show that the commensurability classes of discrete arithmetic subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ of the simplest type, i.e., those coming from quadratic forms of signature $(n,1)$ over totally real number fields, can be parametrised by isomorphisms classes of quaternion algebras. This is applied to some low dimensional examples to show the commensurability of certain Coxeter groups.

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1. Introduction

All hyperbolic n-manifolds and orbifolds arise as quotient spaces $\mathbb{H}^n/\Gamma$ where $\mathbb{H}^n$ is n-dimensional hyperbolic space and Γ is a discrete subgroup of the group of isometries $\mathrm{Isom}(\mathbb{H}^n)$. When $n = 2$ or 3, $\mathbb{H}^n$ can be modelled by upper-half space in $\mathbb{R}^2$ or $\mathbb{R}^3$ so that $\mathrm{Isom}^+(\mathbb{H}^n)$, the orientation-preserving subgroup, is identified with $\mathrm{PSL}(2,\mathbb{R})$ or $\mathrm{PSL}(2,\mathbb{C})$ respectively. Thus, in these cases, discrete subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ give rise in a natural way to subalgebras of $M_2(\mathbb{R})$ or $M_2(\mathbb{C})$ which turn out to be quaternion algebras (e.g. [19], §3.2). When the groups are arithmetic, the quaternion algebras are defined over number fields and the commensurability classes of these discrete arithmetic subgroups correspond precisely to the isomorphism classes of the quaternion algebras (see [25], [17], [27]). We show that this type of correspondence persists for all even n in that the commensurability classes of discrete arithmetic subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ are parametrised by the isomorphism classes of certain quaternion algebras over totally real number fields (see Theorem 7.2). As the isomorphism classes of quaternion algebras are determined by their ramifications sets, which are finite sets of places of the defining field, we obtain, in particular, the following result (Corollary 7.3)

Theorem 1.1. *When n is even, the commensurability classes of discrete arithmetic subgroups of* $\mathrm{Isom}(\mathbb{H}^n)$ *are parametrised for each totally real field $k \subset \mathbb{R}$, by the* *sets* $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_r\}$ *of prime ideals in the ring of integers* R_k *where*

$$r \equiv \begin{cases} 0 \pmod 2 & \text{if } n \equiv 0 \pmod 8, \\ [k:\mathbb{Q}] - 1 \pmod 2 & \text{if } n \equiv 2 \pmod 8, \\ [k:\mathbb{Q}] \pmod 2 & \text{if } n \equiv 4 \pmod 8, \\ 1 \pmod 2 & \text{if } n \equiv 6 \pmod 8. \end{cases}$$

When n is even, these discrete arithmetic subgroups are all of the simplest type in the language of Vinberg [30], Chap. 6, as they all arise from suitably defined quadratic spaces of dimension $n+1$ over totally real number fields. When n is odd, we obtain parametrising sets similarly defined via quaternion algebras for the commensurability classes of those discrete arithmetic subgroups of the simplest type. The parametrising sets in the cases where n is odd are a little less neat than in the even cases (Theorem 7.4) and this is already manifest in the case $n = 3$ (see [18], [9]). For all odd n, there is another construction of discrete arithmetic subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ [30], Chap. 6. These groups together with all those of simplest type cover all discrete arithmetic subgroups for all odd n apart from 3 and 7. The problem of determining parametrising sets for the groups obtained by this second construction is apparently open.

The parametrising sets are obtained via the Clifford algebras of the related quadratic spaces and the even/odd dichotomy referred to above is partly a reflection of the structural differences between Clifford algebras which depend on the parity of the dimension of the quadratic space. The well-known invariants of Witt and Hasse for quadratic spaces are elements of the Brauer group of the defining totally real number field and, as such, can be represented in the Brauer group by quaternion algebras. It is by using these invariants that we obtain our parametrising sets.

Extensive investigations into discrete subgroups of finite covolume in $\mathrm{Isom}(\mathbb{H}^n)$ have been made (see e.g. [30] and references therein); in particular, in small dimensions using Coxeter groups generated by reflections in the faces of Coxeter polytopes in $\mathbb{H}^n$. If such groups are arithmetic, they are necessarily of the simplest type and necessary and sufficient conditions for such groups to be arithmetic have been given by Vinberg [28]. These results can be readily adapted to describe the quadratic spaces involved. For those that do turn out to be arithmetic, the parameters defining the commensurability classes can be easily calculated. We pursue this in §9 and illustrate the application with examples from dimensions 4 and 5. In the cases where the polytopes are simplices, the commensurability classes of all the Coxeter groups have been described in [11].

2. Background

Let the vector space $\mathbb{R}^{n+1}$ be endowed with a quadratic form Q of signature $(n, 1)$, so that, with respect to a suitable basis,

$$Q(\boldsymbol{x}) = x_1^2 + x_2^2 + \cdots + x_n^2 - x_{n+1}^2.$$

Consider the cone

$$C = \{\boldsymbol{x} \in \mathbb{R}^{n+1} \mid Q(\boldsymbol{x}) < 0\}$$

which has two components. The Lobachevski model of hyperbolic n-space can then be identified with the projective space of one component, $C^+/\mathbb{R}^+$, with metric induced from $ds^2 = dx_1^2 + dx_2^2 + \cdots + dx_n^2 - dx_{n+1}^2$. Let $\mathrm{O}(n,1)$ denote the group of isometries of the quadratic space $(\mathbb{R}^{n+1}, Q)$ so that

$$\mathrm{O}(n,1) = \{T \in \mathrm{GL}(n+1,\mathbb{R}) \mid Q(T(\boldsymbol{x})) = Q(\boldsymbol{x}) \text{ for all } \boldsymbol{x} \in \mathbb{R}^{n+1}\}.$$

Thus if $\mathrm{O}^+(n,1)$ denotes the subgroup preserving the components of the cone then $\mathrm{Isom}(\mathbb{H}^n) = \mathrm{O}^+(n,1)$.

Discrete arithmetic subgroups of $\mathrm{O}^+(n,1)$ can be constructed as follows: Let k be a real subfield of $\mathbb{R}$ which is a totally real number field. Let (V, q) be an $(n+1)$-dimensional quadratic space over k such that (V, q) has signature $(n, 1)$. Furthermore, for any embedding $\sigma \colon k \to \mathbb{R}$, $\sigma \neq \mathrm{Id}$, the induced quadratic space $({}^\sigma V, {}^\sigma q)$ is required to have signature $(n+1, 0)$, i.e., be positive definite. Under these circumstances, there exists $X \in \mathrm{GL}(n+1,\mathbb{R})$ such that $q(X(\boldsymbol{x})) = Q(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^{n+1} \cong V \otimes \mathbb{R}$. Thus if we define

$$\mathrm{O}(V,q) = \{T \in \mathrm{GL}(n+1,\mathbb{R}) \mid q(T(\boldsymbol{x})) = q(\boldsymbol{x}) \text{ for all } \boldsymbol{x} \in V \otimes \mathbb{R}\}$$

then X induces an isomorphism $X^* \colon \mathrm{O}^+(V,q) \to \mathrm{O}^+(n,1)$ by $X^*(T) = X^{-1}TX$. Now let $\mathrm{O}(V,q;k)$ denote the k-points of the subgroup of the algebraic group $\mathrm{O}(V,q)$ and let L be a complete R_k-lattice in V, where R_k is the ring of integers in k. Define

$$\mathrm{O}(L) = \{T \in \mathrm{O}^+(V,q;k) \mid T(L) = L\}.$$

Then $\mathrm{O}(L)$ is a discrete subgroup of finite covolume in $\mathrm{O}^+(V,q)$ [3]. Furthermore, for any two complete lattices L_1, L_2 in V, the groups $\mathrm{O}(L_1)$, $\mathrm{O}(L_2)$ are commensurable. Thus $X^*(\mathrm{O}(L))$ is a discrete subgroup of finite covolume in $\mathrm{O}^+(n,1)$, and any subgroup of $\mathrm{O}^+(n,1)$ commensurable with some such $X^*(\mathrm{O}(L))$ is a discrete arithmetic subgroup of $\mathrm{O}^+(n,1)$. The groups so obtained will be referred to subsequently as *arithmetic groups of the simplest type* (see [30]).

Under the general definition of discrete arithmetic subgroups of a Lie group (e.g. [3], [8]), all discrete arithmetic subgroups of $\mathrm{O}^+(n,1)$ are arithmetic groups of the simplest type in the cases where n is even [15]. In the above discussion, X is not uniquely defined, but any other will differ by an element of $\mathrm{O}^+(n,1)$. We thus always define commensurable to mean commensurable in the wide sense, so that two subgroups Γ_1, Γ_2 of $\mathrm{O}^+(n,1)$ are commensurable if there exists $A \in \mathrm{O}^+(n,1)$ such that $\Gamma_1 \cap A\Gamma_2A^{-1}$ has finite index in both Γ_1 and in $A\Gamma_2A^{-1}$.

3. Commensurability and similarity

Let us suppose that Γ_1, Γ_2 are arithmetic groups of the simplest type in $O^+(n,1)$ and that Γ_1 and Γ_2 are commensurable. Thus there exist totally real number fields k_1, k_2 with $k_1, k_2 \subset \mathbb{R}$ and quadratic spaces (V_1, q_1), (V_2, q_2) over k_1, k_2 respectively, of dimensions $n+1$ such that (V_1, q_1), (V_2, q_2) have signature $(n,1)$ and for any non-identity Galois embeddings σ, σ' of k_1 or k_2, $({}^{\sigma}V_1, {}^{\sigma}q_1)$, $({}^{\sigma'}V_2, {}^{\sigma'}q_2)$ are positive definite. In the remainder of the paper, the totally real fields will always be regarded as being embedded in $\mathbb{R}$ and so come equipped with an embedding into $\mathbb{R}$. Furthermore, the statement that the quadratic space "*has the appropriate signatures*" will be used to cover the conditions just stated on the signature of the quadratic space and its images under the real embeddings. Then for L_i a complete lattice in V_i and $X_i \in \mathrm{GL}(n+1, \mathbb{R})$ such that $q_i X_i = Q$, Γ_i is commensurable with $X_i^*(\mathrm{O}(L_i))$ in $O^+(n,1)$, $i = 1, 2$.

In these circumstances, $k_1 = k_2 := k$ and there exists a totally positive $\lambda \in k^*$ such that (V_1, q_1) is isometric to $(V_2, \lambda q_2)$ over k (see [8]).

Thus the (wide) commensurability classes of these discrete arithmetic groups of the simplest type in $O^+(n,1)$ are in one-to-one correspondence with the similarity classes of quadratic spaces (V, q) over totally real subfields $k \subset \mathbb{R}$ such that (V, q) has appropriate signatures, and two such spaces are similar, if there exists a totally positive $\lambda \in k^*$ such that (V_1, q_1) and $(V_2, \lambda q_2)$ are isometric over k.

Thus, in order to parametrise these commensurability classes we need to parametrise the similarity classes of quadratic spaces, which we will denote by $Q(k, n)$, for each totally real field k. There are well-established invariants for isometry classes of quadratic spaces – the Witt invariant and the Hasse invariant. These are elements of the Brauer group $\mathrm{Br}(k)$ of k and, as such, can be represented by quaternion algebras since k is a number field (see e.g. [13]). Thus our parametrising sets will turn out to be, in essence, isomorphism classes of certain quaternion algebras. That quaternion algebras parametrise commensurability classes of arithmetic subgroups of $O^+(n,1)$ is known for $n = 2$ [25]. They are also used in the parametrisation of arithmetic groups of the simplest type in the case $n = 3$ (e.g. [18]) and the commensurability classes of all discrete arithmetic subgroups of $O^+(3,1)$ are parametrised by quaternion algebras (e.g. [17], [19]). The neat classification of quaternion algebras over number fields via their ramification sets (see Theorem 4.1 below) makes them particularly useful as parametrising sets.

4. Quaternion algebras

We briefly recall some information on quaternion algebras and prove a result concerning them which will be required subsequently. For more details, see [19], [27]. A quaternion algebra over k is a four-dimensional central simple algebra. It has a basis of the form $1, i, j, ij$ where $i^2 = c$, $j^2 = d$, $ij = -ji$, $c, d \in k^*$. This is referred to as a standard basis and the quaternion algebra B can be represented by a Hilbert symbol $\left(\frac{c,d}{k}\right)$. A quaternion algebra over a local field K, $(\neq \mathbb{C})$, is either isomorphic to $M_2(K)$ or to a unique division algebra. If k_ν denotes the completion of k at the place corresponding to the valuation ν, then B is said to be *ramified* at ν if $B_\nu := B \otimes_k k_\nu$ is a division algebra. Also B is said to *split* at ν if $B_\nu \cong M_2(k_\nu)$.

Theorem 4.1 (Classification Theorem). *Let B be a quaternion algebra over a number field k. Then:*

- *B is ramified at a finite set of places, called the ramification set, and the cardinality of the set is even.*
- *Two quaternion algebras over k are isomorphic if and only if their ramification sets are equal.*
- *Let S be any finite set of places of k of even cardinality, excluding the complex places. Then there exists a quaternion algebra over k whose ramification set is precisely S.*

Denote the ramification set of B by $\mathrm{Ram}(B)$, the subset of Archimedean ramified places by $\mathrm{Ram}_\infty(B)$ and the subset of non-Archimedean or $\mathcal{P}$-adic ramified places by $\mathrm{Ram}_f(B)$.

If B_1, B_2 are quaternion algebras over a number field k, then there exists a quaternion algebra D over k such that

$$B_1 \otimes_k B_2 \cong D \otimes_k M_2(k). \tag{1}$$

Thus in the Brauer group $\mathrm{Br}(k)$, $B_1 \cdot B_2 = D$. For quaternion algebras B_1, B_2, D as at (1),

$$\mathrm{Ram}(D) = (\mathrm{Ram}(B_1) \cup \mathrm{Ram}(B_2)) \setminus (\mathrm{Ram}(B_1) \cap \mathrm{Ram}(B_2)). \tag{2}$$

Later, in establishing the parametrisation of commensurability classes of arithmetic groups of the simplest type, we require the following result. The necessary information on quaternion algebras can be found in [19] (cf. [19], Theorem 9.5.5).

Lemma 4.2. *Let k be a totally real number field and let $\delta \in k^*$ be such that $L = k(\sqrt{\delta})$ is a quadratic extension field of k. Let B_1, B_2 be quaternion algebras over k such that* $\mathrm{Ram}_\infty(B_1) = \mathrm{Ram}_\infty(B_2)$ *and*

$$B_1 \otimes_k L \cong B_2 \otimes_k L.$$

Then there exists a totally positive $\lambda \in k^*$ such that

$$B_1 \otimes_k B_2 \cong \left(\frac{\lambda, \delta}{k}\right) \otimes_k M_2(k).$$

Proof. Let $A = B_i \otimes_k L$ so that A is a quaternion algebra over L. Let $\mathcal{P}$ be an ideal of L and let $p = \mathcal{P} \cap R_k$. Then

$$A \otimes_L L_{\mathcal{P}} \cong (B_i \otimes_k L) \otimes_L L_{\mathcal{P}} \cong (B_i \otimes_k k_p) \otimes_{k_p} L_{\mathcal{P}}.$$

Clearly, if B_i splits at p, then A splits at $\mathcal{P}$. If p is ramified or inert in $L|k$, then $[L_{\mathcal{P}} : k_p] = 2$. If B_i is ramified at p, the division algebra $B_i \otimes_k k_p$ over the local field k_p splits under any quadratic extension. Thus A cannot be ramified at any prime $\mathcal{P}$ such that p is inert or ramified in $L|k$. If p decomposes in $L|k$, then $k_p \to L_{\mathcal{P}}$ is an isomorphism. Thus, in these cases, A is ramified at $\mathcal{P}$ if and only if B_i is ramified at p, if and only if A is ramified at $\mathcal{P}'$ where $pR_L = \mathcal{P}\mathcal{P}'$. Thus $\text{Ram}_f(A) = \{\mathcal{P}_1, \mathcal{P}_1', \ldots, \mathcal{P}_r, \mathcal{P}_r'\}$ where $p_1, p_2, \ldots, p_r$ are prime ideals of k which decompose in $L|k$ as $p_i R_L = \mathcal{P}_i \mathcal{P}_i'$. Furthermore, $\mathcal{T}_0 = \{p_1, p_2, \ldots, p_r\} \subset \text{Ram}_f(B_i)$, $i = 1, 2$.

Let $\text{Ram}_f(B_1) = \mathcal{T}_0 \cup \mathcal{T}_1$, $\text{Ram}_f(B_2) = \mathcal{T}_0 \cup \mathcal{T}_2$ where $\mathcal{T}_1$, $\mathcal{T}_2$ are subsets of those primes of k which are inert or ramified in the extension $L|k$. Now, if D is as defined at (1), then by (2),

$$\text{Ram}_f(D) = (\mathcal{T}_1 \setminus \mathcal{T}_2) \cup (\mathcal{T}_2 \setminus \mathcal{T}_1)$$

and $\text{Ram}_\infty(D) = \emptyset$.

Since $L \otimes_k k_\nu$ is a field for all $\nu \in \text{Ram}(D)$, L embeds in D. Let $u \in D$ be such that $u^2 = \delta$. Then there exists $v \in D$ such that $vuv^{-1} = -u$ by the Skolem–Noether theorem. Then $v^2 = \lambda \in k^*$ and $1, u, v, uv$ is a standard basis of D. So $D \cong \left(\frac{\lambda,\delta}{k}\right)$. Now v can be replaced by $v' = v(a + bu)$, $a, b \in k$, and then $v'^2 = v^2 N_{L|k}(a+bu) = \lambda(a^2 - \delta b^2)$. Suppose that λ is positive at real embeddings ν_i, $i \in S_1$, and negative at ν_i, $i \in S_2$. Since $\text{Ram}_\infty(D) = \emptyset$, $\sigma_i(\delta)$ must be positive for $i \in S_2$. So, for $i \in S_1$ pick $a_i, b_i \in k_{\nu_i}$ such that $a_i^2 - \sigma_i(\delta) b_i^2 > 0$ and for $i \in S_2$, pick $a_i, b_i \in k_{\nu_i}$ such that $a_i^2 - \sigma_i(\delta) b_i^2 < 0$. By the Approximation Theorem, there exist $a, b \in k$ arbitrarily close to a_i, b_i respectively for all $i \in S_1 \cup S_2$. Replacing v by $v' = v(a + bu)$ gives $v'^2 = \lambda'$ where λ' is totally positive and $D \cong \left(\frac{\lambda',\delta}{k}\right)$. □

5. Clifford algebras

Our parametrisation of the sets $Q(k, n)$ is established by using Clifford algebras. These are described in detail in [13]. We make extensive use of this theory and adopt some notation from [13], parts of which are recalled below.

If (V, q) is a quadratic space over a field k, then the Clifford algebra, $C(V)$, is an associative algebra over k with 1 which contains V and which is universal for the property that its multiplication is compatible with the form q in that $\boldsymbol{x}^2 = q(\boldsymbol{x}) \cdot 1$ for all $\boldsymbol{x} \in V$. If V has dimension r, then $C(V)$ has dimension 2^r and, for a regular quadratic space, it is a central simple $\mathbb{Z}_2$-graded algebra with even and odd parts denoted by $C_0(V)$ and $C_1(V)$ respectively. Note that $C_0(V)$ is a k-subalgebra. If

$\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_r\}$ is an orthogonal basis of V, then the element $z = \boldsymbol{x}_1\boldsymbol{x}_2 \ldots \boldsymbol{x}_r$ is such that $z^2 = (-1)^{r(r-1)/2} d(V) = \delta \in k^*$ where $d(V)$ is the determinant of (V, q). Then δ, regarded as an element of k^*/k^{*2} is termed the *signed determinant*. The structure of $C(V)$ depends on the parity of r. When r is odd, z lies in the center of $C(V)$ and $C_0(V)$ is a central simple algebra over k. The subalgebra $k + kz$ is $\mathbb{Z}_2$-graded by defining $\partial(k) = 0$, $\partial(kz) = 1$. Carrying this grading, the subalgebra is denoted by $k\langle\sqrt{\delta}\rangle$. Then, as a graded algebra $C(V) \cong (C_0(V)) \hat{\otimes} k\langle\sqrt{\delta}\rangle$ where $\hat{\otimes}$ denotes the graded tensor product and the notation $(C_0(V))$ indicates that the algebra $C_0(V)$ is graded so that it is concentrated at zero. When r is even, z lies in the center of $C_0(V)$ and $C(V)$ is a central simple algebra over k. If B is a quaternion algebra over k with Hilbert symbol $\left(\frac{c,d}{k}\right)$ and standard basis $1, i, j, ij$, we use the notation $\left\langle\frac{c,d}{k}\right\rangle$ to indicate that it has been $\mathbb{Z}_2$-graded by $\partial(1) = \partial(ij) = 0$ and $\partial(i) = \partial(j) = 1$.

The structure of these graded Clifford algebras is compatible with taking orthogonal sums in that

$$C(V \perp V') \cong C(V) \hat{\otimes} C(V').$$

We will make use of this and related orthogonal decomposition theorems (see [13], Chap. 5). The Witt invariant, $c(V)$, of a regular quadratic space (V, q) is the element of the Brauer group $\mathrm{Br}(k)$ of k defined by

$$c(V) = \begin{cases} [C_0(V)] & \text{if } \dim(V) \text{ is odd,} \\ [C(V)] & \text{if } \dim(V) \text{ is even.} \end{cases}$$

The Hasse invariant, $s(V)$, also in $\mathrm{Br}(k)$, is obtained from a diagonalisation $\{a_1, a_2, \ldots, a_r\}$ of (V, q) as the element in $\mathrm{Br}(k)$ given by the product of the quaternion algebras with Hilbert symbols $\left(\frac{a_i, a_j}{k}\right)$ over $i < j$. The Hasse invariant does not depend on the choice of diagonalisation and is related to the Witt invariant in $\mathrm{Br}(k)$ as follows:

$$\begin{aligned}
\dim(V) &\equiv 1, 2 \pmod 8\colon & c(V) &= s(V), \\
\dim(V) &\equiv 3, 4 \pmod 8\colon & c(V) &= s(V) \cdot \left(\frac{-1, -d(V)}{k}\right), \\
\dim(V) &\equiv 5, 6 \pmod 8\colon & c(V) &= s(V) \cdot \left(\frac{-1, -1}{k}\right), \\
\dim(V) &\equiv 7, 8 \pmod 8\colon & c(V) &= s(V) \cdot \left(\frac{-1, d(V)}{k}\right).
\end{aligned} \tag{3}$$

We further note that if k is a number field, then $c(V), s(V)$ can always be represented by an element $[B]$ where B is a quaternion algebra. Thus it is not surprising that our parametrising sets, defined in the following sections, are sets of isomorphism classes of quaternion algebras.

6. Invariants

Recall that $Q(k,n)$ is the set of similarity classes of quadratic spaces (V,q) over the totally real field k, of dimension $n+1$ such that (V,q) has the appropriate signatures and $(V,q),(V',q')$ are similar if there exists a totally positive $\lambda \in k^*$ such that $(V,\lambda q),(V',q')$ are isometric over k. We continue to use results and notation on Clifford algebras from [13].

Let $I(k,n)$ denote the set of isomorphism classes of k-algebras of dimension 2^n. Define

$$\Theta\colon\, Q(k,n) \to I(k,n) \quad \text{by } \Theta([(V,q)]) = C_0(V)$$

where $[(V,q)]$ denotes the similarity class of (V,q). This is well defined since $C_0(V)$ is unchanged in its isomorphism class under scaling. We will show that Θ is a perfect invariant by establishing that Θ is injective. The proof depends on the parity of n.

***n* even.** In this case, $\dim(V) = n+1$ is odd so that $C_0(V)$ is a central simple algebra over k.

Theorem 6.1. *When n is even, Θ is an injection.*

Proof. Let (V,q), (V',q') be representatives of elements of $Q(k,n)$. Thus they both have appropriate signatures and so both have the same signature at all real places. This will, of course, remain true under scaling by a totally positive element of k^* and we can scale by such a scalar to assume that $d(V) = d(V')$. If $\Theta(C_0(V)) \cong \Theta(C_0(V'))$, then $c(V) = c(V')$. Thus in the Brauer group $\mathrm{Br}(k_{\mathcal{P}})$ for any prime ideal $\mathcal{P}$, $s(V) = s(V')$ since the dimensions and the determinants are also equal. Thus (V,q), (V',q') are isometric over $k_{\mathcal{P}}$ for each $\mathcal{P}$. Then, by the Hasse–Minkowski theorem, the quadratic spaces $(V,q),(V',q')$ are isometric over k. □

***n* odd.** In this case, $\dim(V) = n+1$ is even and $C(V)$ is a central simple algebra over k.

Let $V = V_0 \perp V_1$ where $V_1 = \langle \boldsymbol{x}_1 \rangle$ is one-dimensional with $q(\boldsymbol{x}_1)$ totally positive. First scale (V,q) so that $q(\boldsymbol{x}_1) = 1$. Then $d(V) = d(V_0)$ and $\delta = -\delta_0$ for the signed determinants of V and V_0. Now $C(V) \cong C(V_0) \mathbin{\hat{\otimes}} C(V_1)$ and $C(V_1) \cong k\langle\sqrt{1}\rangle$ as $\mathbb{Z}_2$-graded algebras. Furthermore, $C(V_0) \cong (C_0(V_0)) \mathbin{\hat{\otimes}} k\langle\sqrt{\delta_0}\rangle$. Thus

$$C(V) \cong (C_0(V_0)) \mathbin{\hat{\otimes}} \left\langle \frac{-\delta, 1}{k} \right\rangle. \tag{4}$$

Now $C_0(V_0)$ is a central simple algebra over k, so there exists a quaternion algebra B over k such that $C_0(V_0) \cong M_r(B)$ where $r = (n-3)/2$. Since, in (4), $(C_0(V_0))$ is concentrated at zero, the graded tensor product is just the tensor product. Also, as $\left(\frac{-\delta,1}{k}\right) \cong M_2(k)$, it follows that $c(V) = [B]$ in $\mathrm{Br}(k)$.

Theorem 6.2. *When n is odd, Θ is injective.*

Proof. Let (V, q), (V', q') represent elements of $Q(k, n)$ where n is odd and suppose that $\Theta(C_0(V)) \cong \Theta(C_0(V'))$. Thus their centers are isomorphic and $\delta = \delta'$ as elements of k^*/k^{*2}. Now $C_0(V) \cong C_0(V_0) \otimes C(-\delta_0 V_1) \cong M_r(B) \otimes k(\sqrt{\delta})$. If $\delta \notin k^{*2}$, then $C_0(V)$ is a central simple algebra over $k(\sqrt{\delta})$ which will be represented in the Brauer group of $k(\sqrt{\delta})$ by $B \otimes k(\sqrt{\delta})$. Thus $B \otimes k(\sqrt{\delta}) \cong B' \otimes k(\sqrt{\delta})$. At all real places, the invariants $s(V_0), s(V_0')$ are trivial. Note that $[B] = c(V_0)$. Since $\delta = \delta'$, $d(V_0) = d(V_0')$ so that, from (3), $\mathrm{Ram}_\infty(B) = \mathrm{Ram}_\infty(B')$. By Lemma 4.2, there exists a totally positive λ such that $B = B'\left(\frac{\lambda, \delta}{k}\right)$ in $\mathrm{Br}(k)$. Now scale V' by λ so that $c(\lambda V') = c(V')\left(\frac{\lambda, \delta}{k}\right) = c(V)$ in $\mathrm{Br}(k)$. Arguing as in Theorem 6.1, it follows that (V, q) and $(V', \lambda q')$ are isometric. If $\delta \in k^{*2}$, then $C_0(V) \cong M_r(B) \times M_r(B)$. But an isomorphism $M_r(B) \times M_r(B) \to M_r(B') \times M_r(B')$ induces an isomorphism $M_r(B) \to M_r(B')$ by Schur's lemma. Since $C(V) \cong M_{r+1}(B), C(V') \cong M_{r+1}(B')$, it follows that $c(V) = c(V')$ and the argument is as before. □

7. Parametrising sets

We have shown that Θ maps injectively into the isomorphism classes of k-algebras. In this section, we obtain a precise description of the image of Θ in terms of quaternion algebras, thus establishing the parametrising sets for the commensurability classes of discrete arithmetic subgroups of $\mathrm{O}^+(n, 1)$ of the simplest type.

Let (V, q) be a quadratic space over k with appropriate signatures and recall the definitions of the Witt invariant and Hasse invariant. Choose a diagonalisation $\{a_1, a_2, \ldots, a_{n+1}\}$ of (V, q) in such a way that $a_1 < 0$, $a_j > 0$ for all $j > 1$ and $\sigma(a_i) > 0$ for all i and embeddings $\sigma \neq \mathrm{Id}$. Then the quaternion algebra $\left(\frac{a_i, a_j}{k}\right)$ is unramified at all real places. Thus the Hasse invariant, $s(V)$, is represented by a quaternion algebra which has no real ramification by (2). But then, from (3), the quaternion algebra B representing the Witt invariant of (V, q) satisfies

$$\mathrm{Ram}_\infty(B) = \begin{cases} \emptyset & n \equiv 0, 1 \pmod 8, \\ \Omega_\infty \setminus \{\mathrm{Id}\} & n \equiv 2, 3 \pmod 8, \\ \Omega_\infty & n \equiv 4, 5 \pmod 8, \\ \{\mathrm{Id}\} & n \equiv 6, 7 \pmod 8, \end{cases} \tag{5}$$

where Ω_∞ denotes the set of all real places of k.

Theorem 7.1. *Let n be even, k be a totally real number field and let B be a quaternion algebra over k such that B satisfies (5). Then $M_{2^{(n-2)/2}}(B) \in \Theta(Q(k, n))$.*

Proof. For $n = 2, 4$ we construct from B, a central simple $\mathbb{Z}_2$-graded algebra C such that its odd part C_1 contains a subspace V of dimension $n + 1$ on which multiplication defines a quadratic form with the appropriate signatures and such that $c(V) = B$.

The remaining cases are then obtained from $V \perp V_r$ where we use the notation V_r to denote a space of dimension r with diagonalisation $\{1, 1, \ldots, 1\}$.

We begin with $n = 2$ (where the result is already known). Thus let B have real ramification at all real places except the identity and let $B = \left(\frac{c,d}{k}\right)$ with standard basis $1, i, j, ij$. Choose $\delta \in k^*$ such that $\delta > 0$ and $\sigma(\delta) < 0$ for all $\sigma \neq \mathrm{Id}$. Then $k\langle\sqrt{\delta}\rangle$ is a $\mathbb{Z}_2$-graded algebra. Let $C_0 = B$ and define $C = (C_0) \hat{\otimes} k\langle\sqrt{\delta}\rangle$ so that C is a central simple $\mathbb{Z}_2$-graded algebra whose even part is isomorphic to B.

Let B^o denote the 3-dimensional subspace of pure quaternions in B and set

$$V = \{b_0 \otimes x\sqrt{\delta} \mid b_0 \in B^o,\ x \in k\}.$$

Thus V is a 3-dimensional subspace of C_1 and for $\alpha = b_0 \otimes x\sqrt{\delta} \in V$, $\alpha^2 = x^2 b_0^2 \delta \in k^*$. Thus (V, q') with $q'(\alpha) = \alpha^2$ is a quadratic space over k with orthogonal basis $i \otimes \sqrt{\delta}$, $j \otimes \sqrt{\delta}$, $ij \otimes \sqrt{\delta}$. It thus has the diagonalisation $\{\delta c,\ \delta d,\ -\delta cd\}$ and so has the appropriate signatures. By uniqueness, $C \cong C(V)$ and $C_0(V) \cong B$ so that $B \in \Theta(Q(k, 2))$.

Now consider the case $n = 4$ and take a quaternion algebra B over k which is ramified at all real places. Choose $\delta \in k^*$ such that $\delta < 0$ and $\sigma(\delta) > 0$ for all $\sigma \neq \mathrm{Id}$. Let $k\langle\sqrt{\delta}\rangle$ denote the field $k(\sqrt{\delta})$ with $\mathbb{Z}_2$-grading as defined earlier. Let C_0 be the central simple algebra $M_2(B)$ and define $C = (C_0) \hat{\otimes} k\langle\sqrt{\delta}\rangle$. In this case, take

$$V = \left\{ \begin{pmatrix} a & b \\ \bar{b}/\delta & -a \end{pmatrix} \otimes x\sqrt{\delta} \mid a, x \in k,\ b \in B \text{ with conjugate } \bar{b} \right\}.$$

Then V is 5-dimensional subspace and for $\alpha = \begin{pmatrix} a & b \\ \bar{b}/\delta & -a \end{pmatrix} \otimes x\sqrt{\delta} \in V$ we have $\alpha^2 = x^2(\delta a^2 + b\bar{b}) \in k^*$. Thus (V, q') with $q'(\alpha) = \alpha^2$ is a quadratic space with orthogonal basis $\{X \otimes \sqrt{\delta}\}$, where

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1/\delta & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i/\delta & 0 \end{pmatrix}, \begin{pmatrix} 0 & j \\ -j/\delta & 0 \end{pmatrix}, \begin{pmatrix} 0 & ij \\ -ij/\delta & 0 \end{pmatrix}.$$

This yields the diagonalisation $\{\delta, 1, -c, -d, cd\}$ so that (V, q') has the appropriate signatures. Again by uniqueness, $C \cong C(V)$ and $C_0(V) \cong M_2(B)$ so that $M_2(B) \in \Theta(Q(k, 4))$.

For $n = 6$, let B be a quaternion algebra over k ramified at the identity real place. Let B' be the quaternion algebra over k such that, in the Brauer group $\mathrm{Br}(k)$, $B = B' \cdot \left(\frac{-1,-1}{k}\right)$. Thus B' is ramified at all real places except the identity. Thus from the case $n = 2$, we can construct a 3-dimensional quadratic space V' having the appropriate signatures and such that $C_0(V') \cong B'$. Now let $V = V' \perp V_4$. Then V has the appropriate signatures and $C_0(V) \cong C_0(V') \otimes C(-\delta V_4)$. The signed determinant of $-\delta V_2$ is -1 so that

$$C(-\delta V_4) \cong C(-\delta V_2) \otimes C(\delta V_2) \cong \left(\frac{-\delta, -\delta}{k}\right) \otimes \left(\frac{\delta, \delta}{k}\right) \cong \left(\frac{-1, -1}{k}\right) \otimes M_2(k).$$

Thus $M_4(B) \in \Theta(Q(k, 6))$.

For $n = 8$, a similar argument using the construction for the case $n = 4$ yields the result.

Finally, let $n = n_0 + 8m$ where $n_0 = 2, 4, 6, 8$, $m \geq 1$, and let B be a quaternion algebra over k as described at (5). Let V' be the quadratic space of dimension $n_0 + 1$ and with the appropriate signatures such that $c(V') = B$ in $\mathrm{Br}(k)$. Let V be obtained by taking the orthogonal sum of V' with m copies of V_8. Then V has the appropriate signatures. Furthermore $C(-\delta V_8)$ is isomorphic to the tensor product of four copies of $C(-\delta V_1 \perp \delta V_1) \cong M_2(k)$. Thus $M_{2^{(n-2)/2}}(B) \in \Theta(Q(k, n))$. □

Thus assembling these results we have:

Theorem 7.2. *When n is even, the commensurability classes of discrete arithmetic subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ are in one-to-one correspondence with the isomorphism classes of quaternion algebras over totally real number fields k which satisfy the conditions on their real ramification given at (5).*

Since the isomorphism class of a quaternion algebra over k is determined by its ramification set (Theorem 4.1) and the real ramification is prescribed by the dimension n, we can describe the parametrising sets as follows:

Corollary 7.3. *When n is even, the commensurability classes of discrete arithmetic subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ are parametrised for each totally real field $k \subset \mathbb{R}$, by the sets $\{\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_r\}$ of prime ideals of R_k where*

$$r \equiv \begin{cases} 0 \pmod 2 & \text{if } n \equiv 0 \pmod 8, \\ [k:\mathbb{Q}] - 1 \pmod 2 & \text{if } n \equiv 2 \pmod 8, \\ [k:\mathbb{Q}] \pmod 2 & \text{if } n \equiv 4 \pmod 8, \\ 1 \pmod 2 & \text{if } n \equiv 6 \pmod 8. \end{cases}$$

Now consider the cases where n is odd. Then $\Theta([(V, q)]) = C_0(V)$ which has center $k + kz$ with $z^2 = \delta \in k^*$ where

$$(-1)^{n(n+1)/2}\delta < 0 \quad \text{and} \quad (-1)^{n(n+1)/2}\sigma(\delta) > 0 \quad \text{for all } \sigma \neq \mathrm{Id}. \tag{6}$$

When $\delta \notin k^{*2}$, then $C_0(V)$ is a matrix algebra over $B \otimes k(\sqrt{\delta})$ for some quaternion algebra B which represents the class of $C_0(V_0)$ in $\mathrm{Br}(k)$ under the decomposition $V = V_0 \perp V_1$ as described in §6. Thus (V_0, q) has dimension n and appropriate signatures. In particular $\mathrm{Ram}_\infty(B)$ satisfies (5). Conversely, let B satisfy (5) and let δ satisfy (6) such that $\delta \notin k^{*2}$. The construction in Theorem 7.1 gives a quadratic space V_0 of dimension n, signed determinant $-\delta$, having appropriate signatures and $C_0(V_0) \cong M_r(B)$. Let $V = V_0 \perp V_1$. Then V has signed determinant δ, the appropriate signatures and $C_0(V) \cong M_r(B \otimes k(\sqrt{\delta}))$. If $\delta \in k^{*2}$, then $k = \mathbb{Q}$ and $n \equiv 1 \pmod 4$. In that case, $C_0(V)$ is a direct sum of two matrix algebras of the same dimension over B, where again B is a quaternion algebra with real ramification as at

(5). Conversely, for $n \equiv 1 \pmod 4$, let B be a quaternion algebra over $\mathbb{Q}$ satisfying (5). As in Theorem 7.1, construct a quadratic space V_0 of dimension n over $\mathbb{Q}$, signed determinant -1 and $C_0(V) \cong M_r(B)$. Then let $V = V_0 \perp V_1$, which has signed determinant 1 and $C_0(V) \cong M_r(B) \times M_r(B)$. This then proves the following result:

Theorem 7.4. *When n is odd, the commensurability classes of discrete arithmetic subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ of the simplest type are in one-to-one correspondence with the isomorphism classes over $k(\sqrt{\delta})$ of quaternion algebras of the form $B \otimes k(\sqrt{\delta})$ where B is a quaternion algebra over k satisfying (5) and $k(\sqrt{\delta})$ is a quadratic extension of k where δ satisfies (6) together with, in the cases where $n \equiv 1 \pmod 4$, the isomorphism classes of quaternion algebras over $\mathbb{Q}$ which satisfy (5).*

The number of real places of $k(\sqrt{\delta})$ at which $B \otimes k(\sqrt{\delta})$ is ramified will always be even and the finite places at which it is ramified are of the form $\{\mathcal{P}_1, \mathcal{P}_1', \ldots, \mathcal{P}_s, \mathcal{P}_s'\}$ where $\mathcal{P}_i \cap R_k = \mathcal{P}_i' \cap R_k = p_i$ by the proof of Lemma 4.2 and $\{p_1, \ldots, p_s\} \subset \mathrm{Ram}_f(B)$. Thus we can describe the parametrising sets as follows:

Corollary 7.5. *When n is odd, the commensurability classes of discrete arithmetic subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ of the simplest type are parametrised by the triples $\{k, \delta, \{p_1, \ldots, p_s\}\}$ where $k \subset \mathbb{R}$ is a totally real number field, $\delta \in k$ satisfies (6) and $k(\sqrt{\delta})$ is a quadratic extension of k and $\{p_1, \ldots, p_s\}$ are prime ideals in k which split in $k(\sqrt{\delta})$, together with, in the cases where $n \equiv 1 \pmod 4$, the pairs $\{\mathbb{Q}, \{p_1, \ldots, p_r\}\}$ where $p_1, \ldots, p_r$ are rational primes such that*

$$r \equiv \begin{cases} 0 \pmod 2 & \text{if } n \equiv 1 \pmod 8, \\ 1 \pmod 2 & \text{if } n \equiv 5 \pmod 8. \end{cases}$$

8. Non-cocompact groups

The discrete non-cocompact arithmetic subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ are all of the simplest type [15]. A non-cocompact arithmetic group of the simplest type occurs if and only if the defining spaces are isotropic (see e.g. [8]). Since (V, q) is required to be positive definite at all real places apart from the identity real place, this forces $k = \mathbb{Q}$. On the other hand, if (V, q) is defined over $\mathbb{Q}$, since it has signature $(n, 1)$, it is necessarily isotropic over $\mathbb{R}$. For $n \geq 4$, the dimension of V is greater than or equal to 5 and then (V, q) is isotropic over all p-adic fields. Thus by the Hasse–Minkowski theorem, (V, q) will then be isotropic over $\mathbb{Q}$.

Theorem 8.1. *For $n \geq 4$, discrete arithmetic groups in $\mathrm{Isom}(\mathbb{H}^n)$ are non-cocompact if and only they are of the simplest type and the corresponding spaces (V, q) are defined over $\mathbb{Q}$. Their commensurability classes are then as described in Theorems 7.2 and 7.4 for $k = \mathbb{Q}$.*

Corollary 8.2. *There are infinitely many commensurability classes of non-cocompact discrete arithmetic subgroups of* $\mathrm{O}^+(n,1)$ *for each dimension* $n \geq 4$.

We note, for example, that when $n = 4$, the commensurability classes of discrete non-cocompact arithmetic subgroups of $\mathrm{Isom}(\mathbb{H}^4)$ are parametrised by those quaternion algebras over $\mathbb{Q}$ which are ramified at the real place and so by finite sets of odd cardinality of primes in $\mathbb{Z}$.

For completeness, we indicate how the cases $n = 2, 3$ develop, although, as already pointed out, the results here are known. For $\mathrm{Isom}(\mathbb{H}^2)$, the defining field must be $\mathbb{Q}$. By Theorem 7.1, let B be a quaternion algebra over $\mathbb{Q}$, unramified at $\mathbb{R}$, with Hilbert symbol $\left(\frac{c,d}{\mathbb{Q}}\right)$. In the proof, we can take $\delta = 1$ thus giving a quadratic form with diagonalisation $\{c, d, -cd\}$. This is the negative of the norm form restricted to the space of pure quaternion B^o and that space is isotropic precisely when $B \cong M_2(\mathbb{Q})$ [19], [13]. Thus there is just one commensurability class of non-cocompact discrete arithmetic subgroups of $\mathrm{Isom}(\mathbb{H}^2)$.

For $n = 3$, take $\delta \in \mathbb{Q}^*$, $\delta < 0$, and a set of primes $p_1, p_2, \ldots, p_s$ which split in $\mathbb{Q}(\sqrt{\delta})|\mathbb{Q}$. Let B be a quaternion algebra over $\mathbb{Q}$, unramified at $\mathbb{R}$, having $\{p_1, \ldots, p_s\} \subset \mathrm{Ram}_f(B)$ and such that all other primes in $\mathrm{Ram}_f(B)$ are inert or ramified in $\mathbb{Q}(\sqrt{\delta})$. By the construction of Theorem 7.1, we have $V = V_0 \perp V_1$ with $C_0(V_0) \cong B$, $C_0(V) \cong B \otimes \mathbb{Q}(\sqrt{\delta})$ and V has diagonalisation $\{1, -\delta c, -\delta d, \delta cd\}$. Now if V is isotropic and p_i splits in $\mathbb{Q}(\sqrt{\delta})$, then $\{1, -c, -d, cd\}$ is isotropic over $\mathbb{Q}_{p_i}$. But that implies that B splits in $\mathbb{Q}_{p_i}$. Thus $s = 0$ and $B \otimes \mathbb{Q}(\sqrt{\delta}) \cong M_2(\mathbb{Q}(\sqrt{\delta}))$ so that the commensurability classes of non-cocompact arithmetic groups of the simplest type in $\mathrm{Isom}(\mathbb{H}^n)$ are parametrised by the quaternion algebras $M_2(\mathbb{Q}(\sqrt{\delta}))$, $\delta < 0$.

9. Application

Groups generated by reflections in Coxeter polytopes furnish examples of discrete groups of finite covolume in $\mathrm{Isom}(\mathbb{H}^n)$ for $n \geq 4$ and many such examples have been obtained (see [14], [29], [4], [5], [6], [12], [10], [7], [26]). It is also known that such polytopes do not exist when n is sufficiently large [29], [24]. Necessary and sufficient conditions for such Coxeter groups to be arithmetic are due to Vinberg and these groups are always of the simplest type [28]. Furthermore, it is now known that there are only finitely many commensurability classes of hyperbolic arithmetic Coxeter groups in all dimensions [21], [22], [16], [1], [2], [23]. We show here that Vinberg's methods determine the quadratic spaces and hence the commensurability class parameters, thus partitioning the arithmetic Coxeter groups into commensurability classes. This has already been accomplished for all Coxeter groups where the related polytope is a simplex [11].

We briefly recall Vinberg's method and, in particular, the rôle of Gram matrices. The Lobachevski model of $\mathbb{H}^n$ as described §2, is identified with $PC^+ = C^+/\mathbb{R}^+$.

Geodesic subspaces of dimension r in $\mathbb{H}^n$ are then the projective image of the intersection of a linear subspace of dimension $r+1$ with C^+. In particular, hyperplanes are determined by the orthogonal complement of vectors $\boldsymbol{e} \in \mathbb{R}^{n+1}$ such that $Q(\boldsymbol{e}) > 0$. A polytope P is bounded by a finite number of hyperplanes and we refer to those bounding hyperplanes F such that $P \cap F$ has non-empty interior as a subset of F, as the *facets* of P. If P has facets $F_1, F_2, \ldots, F_r$, choose outward-pointing normals $\boldsymbol{e}_i$ normalised such that $Q(\boldsymbol{e}_i) = 1$. It is convenient to take the associated bilinear form B on $\mathbb{R}^{n+1}$ to be defined by

$$B(\boldsymbol{x}, \boldsymbol{y}) = Q(\boldsymbol{x} + \boldsymbol{y}) - Q(\boldsymbol{x}) - Q(\boldsymbol{y})$$

so that $B(\boldsymbol{e}_i, \boldsymbol{e}_i) = 2$. The polytope P is the projective image of

$$\{\boldsymbol{x} \in \mathbb{R}^{n+1} \mid B(\boldsymbol{x}, \boldsymbol{e}_i) \leq 0 \quad \text{for } i = 1, 2, \ldots, r\} \cap C^+.$$

The Gram matrix $G(P)$ of P is the $(r \times r)$-matrix $[a_{ij}]$ where $a_{ij} = B(\boldsymbol{e}_i, \boldsymbol{e}_j)$. The Coxeter polytope is such that intersecting facets meet at a dihedral angle of the form π/m for some integer $m \geq 2$. The related Gram matrix has rank $n+1$ and signature $(n, 1, r-(n+1))$, giving, respectively, the number of positive, negative and zero eigenvalues. It is convenient to denote a Coxeter polytope by its Coxeter symbol. This has a node for each facet and two nodes are joined either by an edge of weight m or by $m-2$ edges if the facets meets at an angle π/m. In all cases, we adopt the useful geometrical convention that facets which meet orthogonally are not joined by an edge. If two facets are parallel, the joining edge is labelled with ∞, while, if they are ultraparallel, the nodes are joined by a broken line (see tables below).

The arithmeticity or otherwise of the Coxeter group generated by reflections in a Coxeter polytope is read off from the Gram matrix. For any subset

$$\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, r\} \tag{7}$$

define the cyclic product

$$b_{i_1 i_2 \ldots i_k} = a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_k i_1}.$$

The non-zero cyclic products correspond to the closed paths in the Coxeter symbol. Let $K = \mathbb{Q}(\{a_{ij}\})$ and $k = \mathbb{Q}(\{b_{i_1 i_2 \ldots i_k}\})$.

Theorem 9.1 (Vinberg). *Let P be a Coxeter polytope of finite volume in $\mathbb{H}^n$ and let $\Gamma(P)$ be the group generated by reflections in the facets of P. Then $\Gamma(P)$ is arithmetic if and only if the following three conditions hold:*

- *K is totally real,*
- *all $b_{i_1 i_2 \ldots i_k}$ are algebraic integers,*
- *for all $\sigma : K \to \mathbb{R}$ such that $\sigma|_k \neq \mathrm{Id}$, the matrix ${}^{\sigma}G(P) = [\sigma(a_{ij})]$ is positive semi-definite.*

In these circumstances, for $\{i_1, i_2, \ldots, i_k\}$ as at (7), let

$$\boldsymbol{v}_{i_1 i_2 \ldots i_k} = a_{1i_1} a_{i_1 i_2} \ldots a_{i_{k-1} i_k} \boldsymbol{e}_{i_k}.$$

Let V be the k-span of all $\boldsymbol{v}_{i_1 i_2 \ldots i_k}$. Then the restriction of Q to V defines V as an $(n+1)$-dimensional quadratic space over k with the appropriate signatures. Furthermore, $\Gamma(P)$ is commensurable with a group $O(L)$ where L is a complete R_k-lattice in (V, Q).

Thus to determine the commensurability class parameters for an arithmetic Coxeter group $\Gamma(P)$, first obtain the Gram matrix $G(P)$. From this determine the field $k = \mathbb{Q}(\{b_{i_1 i_2 \ldots i_k}\})$ and a basis for V over k. From this obtain a diagonalisation. From the diagonalisation, calculate the Hasse invariant, $s(V)$, of V (see §5). In determining the quaternion algebra which represents the Hasse invariant we make use of the following elementary relations on Hilbert symbols (see [13]), together with (1) and (2):

$$\left(\frac{a, 1}{k}\right) \cong \left(\frac{a, -a}{k}\right) \cong \left(\frac{a, 1-a}{k}\right) \cong M_2(k),$$

and

$$\left(\frac{a, b}{k}\right) \otimes \left(\frac{a, c}{k}\right) \cong \left(\frac{a, bc}{k}\right) \otimes M_2(k).$$

From $s(V)$, we obtain $c(V)$ (see (3)) and hence the parameters as described in Corollaries 7.3 and 7.5.

Figure 1.

We tabulate the results for some known groups by way of illustration. Note that two copies of any polytope with a Coxeter symbol of the form shown in Figure 1 on the left, where a is either an even integer, ∞ or facets 1 and 2 are ultraparallel, adjoined along facet 1 yields a Coxeter polytope with symbol as shown in Figure 1 on the right. The groups are obviously commensurable and, in the tables below, we always omit the second of these when the situation arises.

Note that, from Table 1, we deduce that there are exactly two commensurability classes of arithmetic Coxeter groups in $\mathbb{H}^4$ whose compact Coxeter polytopes have at most 6 facets.

Table 1. All 4-dimensional compact Coxeter polytopes with at most 6 facets (see [29], [6], [12]).

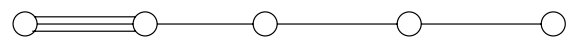
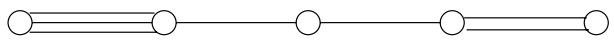
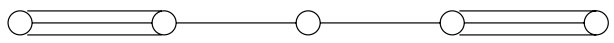
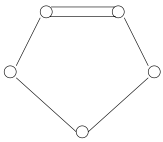
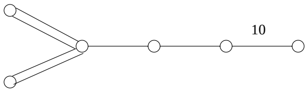
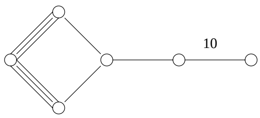
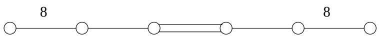
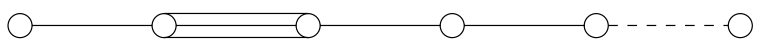
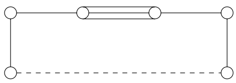
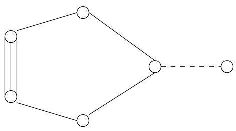
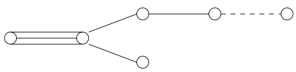
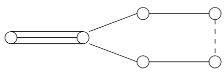
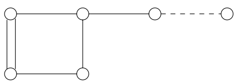
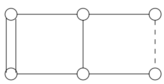
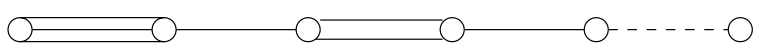
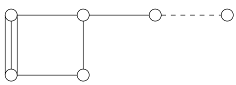
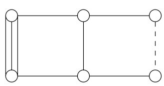
Symbol	Arith	Field	Parameter
	Yes	$\mathbb{Q}(\sqrt{5})$	$\emptyset$
	Yes	$\mathbb{Q}(\sqrt{5})$	$\emptyset$
	Yes	$\mathbb{Q}(\sqrt{5})$	$\emptyset$
	Yes	$\mathbb{Q}(\sqrt{2})$	$\emptyset$
	Yes	$\mathbb{Q}(\sqrt{5})$	$\emptyset$
	Yes	$\mathbb{Q}(\sqrt{5})$	$\emptyset$
	Yes	$\mathbb{Q}(\sqrt{2})$	$\emptyset$
	No		
	Yes	$\mathbb{Q}(\sqrt{5})$	$\emptyset$
	Yes	$\mathbb{Q}(\sqrt{5})$	$\emptyset$
	Yes	$\mathbb{Q}(\sqrt{5})$	$\emptyset$
	No		
	No		
	Yes	$\mathbb{Q}(\sqrt{2})$	$\emptyset$
	Yes	$\mathbb{Q}(\sqrt{5})$	$\emptyset$
	No		
	Yes	$\mathbb{Q}(\sqrt{5})$	$\emptyset$

Table 2. Some arithmetic 4-dimensional non-compact Coxeter polytopes.

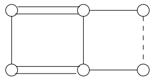
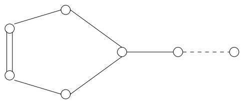
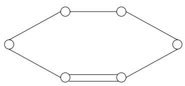
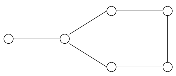
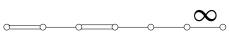
Symbol	No. of cusps	Parameter
All simplices		$\{2\}$
	2	$\{2\}$
	1	$\{3\}$

Table 3. Some 5-dimensional Coxeter polytopes.

Symbol	cusps	Arith	Field	Parameter
	0	Yes	$\mathbb{Q}(\sqrt{5})$	$((-1+3\sqrt{5})/2, \emptyset)$
	0	Yes	$\mathbb{Q}(\sqrt{5})$	$((1+\sqrt{5})/2, \emptyset)$
	0	Yes	$\mathbb{Q}(\sqrt{2})$	$((-1+2\sqrt{2}), \emptyset)$
	1	Yes	$\mathbb{Q}$	$\{2\}$
	2	No		
	1	Yes	$\mathbb{Q}$	$(5, \emptyset)$
	1	No		
	2	Yes	$\mathbb{Q}$	$\{2\}$

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