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Groups, Geometry, and Dynamics

Analyticity of the entropy for some random walks

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Abstract. We consider non-degenerate, finitely supported random walks on a free group. We show that the entropy and the linear drift vary analytically with the probability of constant support.

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1. Introduction

Let F be a finitely generated group and for $x \in F$, denote $|x|$ the word length of x. Let p be a finitely supported probability measure on F and define inductively, with $p^{(0)}$ being the Dirac measure at the identity e,

$$p^{(n)}(x) = [p^{(n-1)} \star p](x) = \sum_{y \in F} p^{(n-1)}(xy^{-1})p(y).$$

Some of the asymptotic properties of the probabilities $p^{(n)}$ as $n \to \infty$ are reflected in two non-negative numbers, the entropy h_p and the linear drift ℓ_p:

$$h_p := \lim_n -\frac{1}{n} \sum_{x \in F} p^{(n)}(x) \ln p^{(n)}(x), \quad \ell_p := \lim_n \frac{1}{n} \sum_{x \in F} |x| p^{(n)}(x).$$

Erschler asks whether h_p and ℓ_p depend continuously on p ([Er]). In this note, we fix a finite set $B \subset F$ such that $\bigcup_n B^n = F$ and we consider probability measures in $\mathcal{P}(B)$, where $\mathcal{P}(B)$ is the set of probability measures p such that $p(x) > 0$ if, and only if, $x \in B$. The set $\mathcal{P}(B)$ is naturally identified with an open subset of the probabilities on B which is an open bounded convex domain in $\mathbb{R}^{|B|-1}$. We show:

Theorem 1.1. *Assume that $F = \mathbb{F}_d$ is the free group with d generators, B is a finite subset of F such that $\bigcup_n B^n = F$. Then, with the above notation, the functions $p \mapsto h_p$ and $p \mapsto \ell_p$ are real analytic on $\mathcal{P}(B)$.*

Continuity of the entropy and of the linear drift is known for probabilities with first moment on a Gromov-hyperbolic group ([EK]). Also in the case when B is a set of free generators, there are formulas for the entropy and the linear drift which show that they are real analytic functions of the directing probability (see [De2] or imbed [DM] in the formulas (1) and (2) below). Similar formulas have been found for braid groups ([M]) and free products of finite groups or graphs ([MM], [G1], [G2]), but as soon as the set B is not reduced to the natural generating set, there is no direct formula for h_p or ℓ_p in terms of p.

The ratio h_p/ℓ_p has a geometric interpretation as the Hausdorff dimension D_p of the unique stationary measure for the action of F on the space ∂F of infinite reduced words. It follows from Theorem 1.1 that this dimension D_p is also real analytic in p, see Corollary 2.1 below for a more precise statement. Ruelle ([R3]) proved that the Hausdorff dimension of the Julia set of a rational function, as long as it is hyperbolic, depends real analytically of the parameters and our approach is inspired by [R3]. We first review properties of the random walk on F directed by a probability p. In particular, we can express h_p and ℓ_p in terms of the exit measure p^∞ of the random walk on the boundary ∂F (see [Le] and Section 2 for background and notation). We then express this exit measure using thermodynamical formalism: if one views ∂F as a one-sided subshift of finite type, the exit measure p^∞ is the isolated eigenvector of maximal eigenvalue for a dual transfer operator $\mathcal{L}_p^*$ involving the Martin kernel of the random walk. Finally, from the description of the Martin kernel by Derriennic ([De1]), we prove that the mapping $p \mapsto \mathcal{L}_p$ is real analytic. The proof uses contractions in projective metric on complex cones ([Ru], [Du1]), and I want to thank Loïc Dubois for useful comments. Regularity of $p \mapsto p^\infty$ and Theorem 1.1 follow.

Our argument may apply to other similar settings. For instance, let $\pi : \mathbb{F}_d \to \mathrm{SO}(k,1)$ be a faithful Schottky representation of the free group $\mathbb{F}_d$ as a convex cocompact group of $\mathrm{SO}(k,1)$. Namely, $\mathrm{SO}(k,1)$ is considered as a group of isometries of the hyperbolic space $\mathbb{H}^k$ and there are $2d$ disjoint open halfspaces H_a associated to the generators and their inverses in such a way that $\pi(a)$ sends the complement of $H_{a^{-1}}$ onto the closure of H_a in $\mathbb{H}^k$. Then another natural asymptotic quantity is the Lyapunov exponent

$$\gamma_p := \lim_n \frac{1}{n} \sum_{x \in F} p^{(n)}(x) \ln \|\pi(x)\|,$$

where $\|\cdot\|$ is some norm on matrices.

Theorem 1.2. *Assume that $\mathbb{F}_d$ is represented as a convex cocompact subgroup of* $\mathrm{SO}(k,1)$ *as above, and B is a finite subset $B \subset F$ such that $\bigcup_n B^n = F$. Then the function $p \mapsto \gamma_p$ is a real analytic function on $\mathcal{P}(B)$.*

Analyticity of the exponent of an independent random product of matrices is known for *positive* matrices ([R2], [P], [H]). Here we show it for matrices in some *discrete* subgroup. It is possible that our approach yield similar results for more general discrete subgroups of $\mathrm{SO}(k,1)$ or even for all Gromov-hyperbolic groups.

In the note, the letter C stands for a real number independent of the other variables, but which may vary from line to line. In the same way, the letter $\mathcal{O}_p$ stands for a neighborhood of $p \in \mathcal{P}(B)$ in $\mathbb{C}^B$ which may vary from line to line.

2. Convolutions of p

We recall in this section the properties of the convolutions $p^{(n)}$ of a finitely supported probability measure p on the free group $\mathbb{F}_d = F$. We follow the notation from [Le]. Any element of F has a unique reduced word representation in generators $\{a_1, \dots, a_d, a_{-1}, \dots, a_{-d}\}$. Set $\delta(x,x) = 0$ and, for $x \neq x'$, $\delta(x,x') = \exp -(x \wedge x')$, where $(x \wedge x')$ is the number of common letters at the beginning of the reduced word representations of x and x'. Then δ defines a metric on F and extends to the completion $F \cup \partial F$ with respect to δ. The boundary ∂F is a compact space which can be represented as the space of infinite reduced words. Then the distance between two distinct infinite reduced words ξ and ξ' is given by

$$\delta(\xi,\xi') = \exp -(\xi \wedge \xi'),$$

where $(\xi \wedge \xi')$ is the length of the initial common part of ξ and ξ'.

There is a natural continuous action of F over ∂F which extends the left action of F on itself: one concatenates the reduced word representation of $x \in F$ at the beginning of the infinite word ξ and one obtains a reduced word by making the necessary reductions. A probability measure μ on ∂F is called stationary if it satisfies

$$\mu = \sum_{x \in F} p(x) x_* \mu.$$

There is a unique stationary probability measure on ∂F, denoted by p^∞, and the entropy h_p and the linear drift ℓ_p are given by

$$h_p = -\sum_{x \in F} \left(\int_{\partial F} \ln \frac{d x_*^{-1} p^\infty}{d p^\infty}(\xi) \, d p^\infty(\xi) \right) p(x), \tag{1}$$

$$\ell_p = \sum_{x \in F} \left(\int_{\partial F} \theta_\xi(x^{-1}) \, d p^\infty(\xi) \right) p(x), \tag{2}$$

where $\theta_\xi(x) = |x| - 2(\xi \wedge x) = \lim_{y \to \xi} (|x^{-1} y| - |y|)$ is the Busemann function.

Observe that in both expressions, the sum is a finite sum over $x \in B$. In the case of a finitely supported random walk on a general group, formula (1) holds, but with $(\partial F, p^\infty)$ replaced by the Poisson boundary of the random walk (see [Fu], [Ka]); formula (2) also holds, but with $(\partial F, p^\infty)$ replaced by some stationary measure on the Busemann boundary of the group ([KL]).

Recall that in the case of the free group the Hausdorff dimension of the measure p^∞ on $(\partial F, \delta)$ is given by h_p/ℓ_p ([Le], Theorem 4.15). So we have the following corollary of Theorem 1.1:

Corollary 2.1. *Assume that $F = \mathbb{F}_d$ is the free group with d generators, B is a finite subset of F such that $\bigcup_n B^n = F$. Then, with the above notation, the Hausdorff dimension of the stationary measure on $(\partial F, \delta)$ is a real analytic function of p in $\mathcal{P}(B)$.*

The Green function $G(x)$ associated to (F, p) is defined by

$$G(x) = \sum_{n=0}^{\infty} p^{(n)}(x)$$

(see Proposition 3.2 below for the convergence of the series). For $y \in F$, the Martin kernel K_y is defined by

$$K_y(x) = \frac{G(x^{-1}y)}{G(y)}.$$

Derriennic ([De1]) showed that $y_n \to \xi \in \partial F$ if, and only if, the Martin kernels K_{y_n} converge towards a function K_ξ, called the Martin kernel at ξ. We have (see e.g. [Le] (3.11)):

$$\frac{dx_* p^\infty}{dp^\infty}(\xi) = K_\xi(x).$$

3. Random walk on F

The quantities introduced in Section 2 can be associated with the trajectories of a random walk on F. In this section, we recall the corresponding notation and properties. Let $\Omega = F^{\mathbb{N}}$ be the space of sequences of elements of F, M the product probability $p^{\mathbb{N}}$. The random walk is described by the probability $\mathbb{P}$ on the space of paths Ω, the image of M by the mapping

$$(\omega_n)_{n\in\mathbb{Z}} \mapsto (X_n)_{n\geq 0},$$

where $X_0 = e$ and $X_n = X_{n-1}\omega_n$ for $n > 0$. In particular, the distribution of X_n is the convolution $p^{(n)}$. The notation p^∞ reflects the following.

Theorem 3.1 (Furstenberg, [Le], Theorem 1.12). *There is a mapping $X_\infty \colon \Omega \to \partial F$ such that*

$$\lim_n X_n(\omega) = X_\infty(\omega)$$

for M-a.e. ω. The image measure p^∞ is the only stationary probability measure on ∂F.

For $x, y \in F$, let $u(x, y)$ be the probability of eventually reaching y when starting from x. By left invariance, $u(x, y) = u(e, x^{-1}y)$. Moreover, by the strong Markov property, $G(x) = u(e, x)G(e)$ so that we have

$$K_y(x) = \frac{u(x, y)}{u(e, y)}. \tag{3}$$

By definition, we have $0 < u(x, y) \leq 1$. The number $u(x, y)$ is given by the sum of the probabilities of the paths going from x to y which do not visit y before arriving at y.

Proposition 3.2. *Let $p \in \mathcal{P}(B)$. There are numbers C and ζ, $0 < \zeta < 1$, and a neighborhood $\mathcal{O}_p$ of p in $\mathbb{C}^B$ such that for all $q \in \mathcal{O}_p$, all $x \in F$ and all $n \geq 0$,*

$$|q|^{(n)}(x) \leq C\zeta^n.$$

Proof. Let $q \in \mathbb{C}^B$. Consider the convolution operator P_q in $\ell_2(F, \mathbb{R})$ defined by

$$P_q f(x) = \sum_{y \in F} f(xy^{-1})|q|(y).$$

Derriennic and Guivarc'h ([DG]) showed that, for $p \in \mathcal{P}(B)$, P_p has spectral radius smaller than one. In particular, there exists n_0 such that the operator norm of $P_p^{n_0}$ in $\ell_2(F)$ is smaller than one. Since B and B^{n_0} are finite, there is a neighborhood $\mathcal{O}_p$ of p in $\mathbb{C}^B$ such that for all $q \in \mathcal{O}_p$, $\|P_q^{n_0}\|_2 < \lambda$ for some $\lambda < 1$ and $\|P_q^k\|_2 \leq C$ for $1 \leq k \leq n_0$. It follows that for all $q \in \mathcal{O}_p$, all $n \geq 0$,

$$\|P_q^n\|_2 \leq C\lambda^{[n/n_0]}.$$

In particular, $|q|^{(n)}(x) = |[P_q^n \delta_e](x)| \leq |P_q^n \delta_e|_2 \leq C\lambda^{[n/n_0]}|\delta_e|_2 \leq C\lambda^{[n/n_0]}$ for all $x \in F$. $\square$

Fix $p \in \mathcal{P}(B)$. For $x \in F$, V a finite subset of F, and $v \in V$, let $\alpha_x^V(v)$ be the probability that the first visit in V of the random walk starting from x occurs at v. We have $0 < \sum_{v \in V} \alpha_x^V(v) \leq 1$ and

Proposition 3.3. *Fix x, V, and v. The mapping $p \mapsto \alpha_x^V(v)$ extends to an analytic function on a neighborhood of $\mathcal{P}(B)$ in $\mathbb{C}^B$.*

Proof. The number $\alpha_x^V(v)$ can be written as the sum of the probabilities $\alpha_x^{n,V}(v)$ of entering V at v in exactly n steps. The function $p \mapsto \alpha_x^{n,V}(v)$ is a polynomial of degree n on $\mathcal{P}(B)$:

$$\alpha_x^{n,V}(v) = \sum_{\mathcal{E}} q_{i_1} q_{i_2} \dots q_{i_n},$$

where $\mathcal{E}$ is the set of paths $\{x, xi_1, xi_1i_2, \dots, xi_1i_2\dots i_n = v\}$ of length n made of steps in B which start from x and enter V in v. By Proposition 3.2, there is a neighbourhood $\mathcal{O}_p$ of p in $\mathcal{P}(B)$ and numbers $C, \zeta, 0 < \zeta < 1$, such that for $q \in \mathcal{O}_p$ and for all $y \in F$,

$$|q|^{(n)}(y) \leq C\zeta^n.$$

It follows that for $q \in \mathcal{O}_p$,

$$|\alpha_x^{n,V}(v)| \leq C|q|^{(n)}(x^{-1}v) \leq C\zeta^n.$$

Therefore, $q \mapsto \alpha_x^V(v)$ is given locally by a uniformly converging series of polynomials, it is an analytic function on $\mathcal{O} := \bigcup_p \mathcal{O}_p$. □

4. Barriers and Hölder property of the Martin kernel

Set $r = \max\{|x| \mid x \in B\}$. A set V is called a barrier between x and y if $\delta(x, y) > r$ and if there exist two points z and z' of the geodesic between x and y, distinct from x and y such that $\delta(z, z') = r - 1$ and V is the intersection of the two balls of radius $r - 1$ centered at z and at z'. The basic geometric lemma is the following:

Lemma 4.1 ([De1], Lemme 1). *If x and y admit a barrier V, then every trajectory of the random walk starting from x and reaching y has to visit V before arriving at y.*

For V, W finite subsets of F, denote by A_V^W the matrix such that the row vectors are the $\alpha_v^W(w), w \in W$. In particular, if $W = \{y\}$, set u_V^y equal to the (column) vector

$$u_V^y = A_V^{\{y\}} = (\alpha_v^{\{y\}}(y))_{v\in V} = (u(v, y))_{v\in V}.$$

With this notation, Lemma 4.1 and the strong Markov property yield that if x and y admit V as a barrier, then

$$u(x, y) = \sum \alpha_x^V(v)u(v, y) = \langle \alpha_x^V, u_V^y\rangle,$$

with the natural scalar product on $\mathbb{R}^V$. Then Derriennic makes two observations: first, this formula iterates when one has k successive disjoint barriers between x and y, and secondly there are only a finite number of possible matrices A_V^W when V and W are successive disjoint barriers with $\delta(V, W) = 1$. This gives the following formula for $u(x, y)$:

Lemma 4.2 ([De1], Lemme 2). *Let $p \in \mathcal{P}(B)$. There are N square matrices with the same dimension $A_1, \dots, A_N$, depending on p, such that for any $x, y \in F$: if $V_1, V_2, \dots, V_k$ are disjoint successive barriers between x and y such that*

$\delta(V_i, V_{i+1}) = 1$ *for* $i = 1, \dots, k-1$, *then there are* $(k-1)$ *indices* $j_1, \dots j_{k-1}$, *depending only on the sequence* V_i, *such that*

$$u(x,y) = \langle \alpha_x^{V_1}, A_{j_1} \dots A_{j_{k-1}} u_{V_k}^y \rangle. \tag{4}$$

By construction, the matrices A_j have non-negative entries and $\sum_w A_j(v,w) \leq 1$. Moreover, we have the following properties:

Proposition 4.3 ([De1], Corollaire 1). *Assume that the set* B *contains the generators and their inverses. Then for each* $p \in \mathcal{P}(B)$, *for each* $j = 1, \dots, N$, *the matrix* A_j *has all its* 0 *entries in full columns.*

From the proof of Proposition 4.3, if the set B contains the generators and their inverses and $A_j = A_{V_j}^{V_j+1}$, columns of 0's correspond to the subset W_{j+1} of points in V_{j+1} which cannot be entry points from paths starting in V_j. In particular, they depend only of the geometry of B and are the same for all $p \in \mathcal{P}(B)$.

We may – and we shall from now on – assume that the set B contains the generators and their inverses. Indeed, since $h_{p^{(k)}} = kh_p$ and $\ell_{p^{(k)}} = k\ell_p$, we can replace in Theorem 1.1 the probability p by a convolution of order high enough that the generators and their inverses have positive probability. Then, by Proposition 4.3, the matrices $A_j(q)$ have the same columns of zeros for all $q \in \mathcal{P}(B)$.

Proposition 4.4. *For each* $j = 1, \dots, N$, *the mapping* $p \mapsto A_j$ *extends to an analytic function on a neighborhood of* $\mathcal{P}(B)$ *in* $\mathbb{C}^B$ *into the set of complex matrices with the same configuration of zeros as* A_j.

Proof. The proof is completely analogous to the proof of Proposition 3.3; one may have to take a smaller neighborhood for the sake of avoiding introducing new zeros. □

We are interested in the function $\Phi \colon \partial F \to \mathbb{R}$, $\Phi(\xi) = -\ln K_\xi(\xi_1)$. By (3), (4) and Deriennic's theorem, we have

$$\begin{aligned}
\Phi(\xi) &= -\ln \lim_{n\to\infty} K_{\xi_1\xi_2\dots\xi_n}(\xi_1) \\
&= -\ln \lim_{n\to\infty} \frac{u(\xi_1, \xi_1\xi_2\dots\xi_n)}{u(e, \xi_1\xi_2\dots\xi_n)} \\
&= -\ln \lim_{k\to\infty} \frac{\langle \alpha_{\xi_1}^{V_1(\xi)}, A_{j_1}(\xi)\dots A_{j_{k-1}}(\xi) u_{V_k(\xi)}^{y_k}\rangle}{\langle \alpha_e^{V_1(\xi)}, A_{j_1}(\xi)\dots A_{j_{k-1}}(\xi) u_{V_k(\xi)}^{y_k}\rangle},
\end{aligned}$$

where $A_{j_s}(\xi) = A_{V_s(\xi)}^{V_{s+1}(\xi)}$, the $V_s(\xi)$ are successive disjoint barriers between ξ_1 and ξ with $\delta(V_s(\xi), V_{s+1}(\xi)) = 1$ for all $s > 1$, $\delta(\xi_1, V_1) = 1$, and y_k is the closest point beyond V_k on the geodesic from ξ_1 to ξ.

Define on the non-negative convex cone C_0 in $\mathbb{R}^m$ the projective distance between half lines as

$$\vartheta(f,g) := |\ln[f,g,h,h']|,$$

where h, h' are the intersections of the boundaries of the cone with the plane (f,g) and $[f,g,h,h']$ is the cross ratio of the four directions in the same plane. Represent the space of directions as the sector of the unit sphere $D = C_0 \cup S^{m-1}$; then ϑ defines a distance on D. Let A be a $m \times m$ matrix with non-positive entries, and let $T\colon D \to D$ be the projective action of A. Then, by [Bi],

$$\vartheta(Tf,Tg) \leq \beta\vartheta(f,g), \quad \text{where } \beta = \tanh(\tfrac{1}{4}\operatorname{Diam} T(D)). \tag{5}$$

When A_j is one of the matrices of Lemma 4.2, it acts on $\mathbb{R}^V$ and the image $T_j(D)$ has finite diameter so that $\beta_j := \tanh(\frac{1}{4}\operatorname{Diam} T_j(D)) < 1$. Set $\beta_0 := \max_{j=1,\dots,N}\beta_j$. Then $\beta_0 < 1$.

Set $f_k(\xi) := \frac{u^{y_k}_{V_k(\xi)}}{\|u^{y_k}_{V_k(\xi)}\|}$, $\alpha(\xi) := \alpha_e^{V_1(\xi)}$, $\alpha_1(\xi) := \alpha_{\xi_1}^{V_1(\xi)}$. For all ξ, $f_k(\xi) \in D$ and $\alpha(\xi), \alpha_1(\xi) \in C_0 - \{0\}$. The above formula for $\Phi(\xi)$ becomes

$$\Phi(\xi) = -\ln \lim_{k\to\infty} \frac{\langle \alpha_1(\xi), T_{j_1}(\xi)\dots T_{j_{k-1}}(\xi) f_k(\xi)\rangle}{\langle \alpha(\xi), T_{j_1}(\xi)\dots T_{j_{k-1}}(\xi) f_k(\xi)\rangle}. \tag{6}$$

Proposition 4.5. *Fix $p \in \mathcal{P}$. The function $\xi \mapsto \Phi(\xi)$ is Hölder continuous on ∂F.*

Proof. Let ξ, ξ' be two points of ∂F with $\delta(\xi,\xi') \leq \exp(-((n+1)r+1))$. The points ξ and ξ' have the same first $(n+1)r+1$ coordinates. In particular, $V_s(\xi) = V_s(\xi')$ for $1 \leq s \leq n$. By using (6), we see that $\Phi(\xi') - \Phi(\xi)$ is given by the limit, as k goes to infinity, of

$$\ln \frac{\langle \alpha_1(\xi), T_{j_1}(\xi)\dots T_{j_{k-1}}(\xi) f_k(\xi)\rangle}{\langle \alpha_1(\xi'), T_{j_1}(\xi')\dots T_{j_{k-1}}(\xi') f_k(\xi')\rangle} \frac{\langle \alpha(\xi'), T_{j_1}(\xi')\dots T_{j_{k-1}}(\xi') f_k(\xi')\rangle}{\langle \alpha(\xi), T_{j_1}(\xi)\dots T_{j_{k-1}}(\xi) f_k(\xi)\rangle}.$$

We have $\alpha_1(\xi) = \alpha_1(\xi') =: \alpha_1, \alpha(\xi) = \alpha(\xi') =: \alpha$ and $T_{j_s}(\xi) = T_{j_s}(\xi') =: T_{j_s}$ for $s = 1,\dots,n$. Moreover, for any $f, f' \in D$,

$$\begin{aligned}\vartheta(T_{j_1}(\xi)\dots T_{j_{k-1}}(\xi)f, T_{j_1}(\xi')\dots T_{j_{k-1}}(\xi')f') \\ = \vartheta(T_{j_1}\dots T_{j_{n-1}}g_k, T_{j_1}\dots T_{j_{n-1}}g'_k)\end{aligned}$$

for $g_k = T_{j_n}T_{j_{n+1}}(\xi)\dots T_{j_{k-1}}(\xi)f, g'_k = T_{j_n}T_{j_{n+1}}(\xi')\dots T_{j_{k-1}}(\xi')f'$.

We have $\vartheta(g_k, g'_k) \leq \operatorname{Diam} T_{j_n} D < \infty$ and, by repeated application of (5),

$$\vartheta(T_{j_1}\dots T_{j_{n-1}}g_k, T_{j_1}\dots T_{j_{n-1}}g'_k) \leq \beta_0^{n-1}\vartheta\left(g_k, g'_k\right) \leq C\beta_0^n. \tag{7}$$

Using all the above notation, we get

$$\Phi(\xi) - \Phi(\xi') = \ln \lim_k \frac{\langle \alpha_1, T_{j_1}\dots T_{j_{n-1}}g'_k\rangle}{\langle \alpha_1, T_{j_1}\dots T_{j_{n-1}}g_k\rangle} \frac{\langle \alpha, T_{j_1}\dots T_{j_{n-1}}g_k\rangle}{\langle \alpha, T_{j_1}\dots T_{j_{n-1}}g'_k\rangle}.$$

As ξ varies, α and α_1 belong to a finite family of vectors of $C_0 - \{0\}$. It then follows from (7) that $|\Phi(\xi) - \Phi(\xi')| \leq C\beta_0^n$ as soon as $\delta(\xi, \xi') \leq \exp(-((n+1)r+1))$. □

Let us choose β, $\beta_0^{1/r} < \beta < 1$, and consider the space Γ_β of functions ϕ on ∂F such that there is a constant C_β with the property that if the points ξ and ξ' have the same first n coordinates, then $|\phi(\xi) - \phi(\xi')| < C_\beta \beta^n$. For $\phi \in \Gamma_\beta$, denote $\|\phi\|_\beta$ the best constant C_β in this definition. The space Γ_β is a Banach space for the norm $\|\phi\| := \|\phi\|_\beta + \max_{\partial F} |\phi|$. Proposition 4.5 says that for $p \in \mathcal{P}(B)$, the function $\Phi_p(\xi) = -\ln K_\xi(\xi_1)$ belongs to Γ_β.

5. Regularity of the Martin kernel

We want to extend the mapping $p \mapsto \Phi_p$ to a neighborhood $\mathcal{O}_p$ of p in $\mathbb{C}^B$. Firstly, we redefine Γ_γ as the space of complex functions ϕ on ∂F such that there is a constant C_γ with the property that, for all $n \geq 0$, if the points ξ and ξ' have the same first n coordinates, then $|\phi(\xi) - \phi(\xi')| < C_\gamma \gamma^n$. The space Γ_γ is a complex Banach space for the norm $\|\phi\| := \|\phi\|_\gamma + \max_{\partial F} |\phi|$, where $\|\phi\|_\gamma$ the best possible constant C_γ. In this section, we find a neighborhood $\mathcal{O}_p$ and a $\gamma = \gamma(p)$, $0 < \gamma < 1$, such that formula (6) makes sense on $\mathcal{O}_p$ and defines a function in Γ_γ.

In recent papers, Rugh ([Ru]) and Dubois ([Du1]) show how to extend (5) to the complex setting. In a complex Banach space X, they define a $\mathbb{C}$-cone as a subset invariant by multiplication by $\mathbb{C}$, different from $\{0\}$ and not containing any complex 2-dimensional subspace in its closure. A $\mathbb{C}$-cone $\mathcal{C}$ is called linearly convex if each point in the complement of $\mathcal{C}$ is contain in a complex hyperplane not intersecting $\mathcal{C}$. Let $K < +\infty$. A $\mathbb{C}$-cone $\mathcal{C}$ is called K-regular if it has some interior and if, for each vector space P of complex dimension 2, there is some nonzero linear form $m \in X^*$ such that, for all $u \in \mathcal{C} \cap P$,

$$\|m\|\|u\| \leq K|\langle m, u\rangle|.$$

Let $\mathcal{C}$ be a linearly convex $\mathbb{C}$-cone. A projective distance $\vartheta_{\mathcal{C}}$ on $(\mathcal{C} - \{0\}) \times (\mathcal{C} - \{0\})$ is defined as follows ([Du1], Section 2): if f and g are colinear, set $\vartheta_{\mathcal{C}}(x, y) = 0$; otherwise, consider the set

$$E(f, g) := \{z, z \in \mathbb{C} \mid zf - g \notin \mathcal{C}\},$$

and define

$$\vartheta_{\mathcal{C}}(f, g) = \ln \frac{b}{a},$$

where $b = \sup |E(f, g)| \in (0, +\infty]$, $a = \inf |E(f, g)| \in [0, +\infty)$.

Proposition 5.1 ([Du1], Theorem 2.7). *Let X_1, X_2 be complex Banach spaces, and let $\mathcal{C}_1 \subset X_1$, $\mathcal{C}_2 \subset X_2$ be complex cones. Let $A\colon X_1 \to X_2$ be a linear map with $A(\mathcal{C}_1 - \{0\}) \subset (\mathcal{C}_2 - \{0\})$ and assume that*

$$\Delta := \sup_{f,g\in(\mathcal{C}_1-\{0\})} \vartheta_{\mathcal{C}_2}(Af, Ag) < +\infty.$$

Then, for all $f, g \in \mathcal{C}_1$,

$$\vartheta_{\mathcal{C}_2}(Af, Ag) \le \tanh(\tfrac{\Delta}{4})\vartheta_{\mathcal{C}_1}(f, g). \tag{8}$$

Proposition 5.2 ([Du1], Lemma 2.6). *Let $\mathcal{C}$ be a K-regular, linearly convex $\mathbb{C}$-cone and let $f \sim g$ if, and only if, there is $\lambda, \lambda \neq 0$ such that $\lambda f = g$. Then $\vartheta_{\mathcal{C}}$ defines a complete projective metric on $\mathcal{C}/\sim$. Moreover, if $f, g \in \mathcal{C}$ and $\|f\| = \|g\| = 1$, then there is a complex number ρ of modulus 1, $\rho = \rho(f, g)$, such that*

$$\|\rho f - g\| \le K\vartheta_{\mathcal{C}}(f, g). \tag{9}$$

Proposition 5.3 ([Ru], Corollary 5.6, [Du1], Remark 3.6). *For $m \ge 1$, the set*

$$\begin{aligned}\mathbb{C}^m_+ &= \{u \in \mathbb{C}^m \mid \mathrm{Re}(u_i\overline{u_j}) \ge 0 \textit{ for all } i, j\}\\ &= \{u \in \mathbb{C}^m \mid |u_i + u_j| \ge |u_i - u_j| \textit{ for all } i, j\}\end{aligned}$$

is a regular linearly convex $\mathbb{C}$-cone. The inclusion

$$\pi\colon (C_0 - \{0\}, \vartheta) \to (\mathbb{C}^m_+ - \{0\}, \vartheta_{\mathbb{C}^m_+})$$

is an isometric embedding.

Moreover, [Du1] studies and characterizes the $m \times m$ matrices which preserve $\mathbb{C}^m_+$. We need the following properties. Let A be a $m \times m$ matrix with all 0 entries in m' full columns and $\lambda_1, \dots, \lambda_m$ the $(m - m')$-row vectors made up of the nonzeros entries of the row vectors of A. Set:

$$\delta_{k,l} := \vartheta_{\mathbb{C}^{m-m'}_+}(\lambda_k, \lambda_l), \quad \Delta_{k,l} := \mathrm{Diam}_{\mathrm{RHP}}\left\{\frac{\langle\lambda_k, x\rangle}{\langle\lambda_l, x\rangle} \mid x \in (\mathbb{C}^{m-m'}_+)^*, x \neq 0\right\},$$

where $\mathrm{Diam}_{\mathrm{RHP}}$ denotes the diameter with respect to the Poincaré metric of the right half-plane. Observe that $\mathrm{Diam}_{\vartheta_{\mathbb{C}^m_+}}(A(\mathbb{C}^m_+ - \{0\})) = \mathrm{Diam}_{\vartheta_{\mathbb{C}^m_+}}(A(\mathbb{C}^{m-m'}_+ - \{0\}))$. Then we have ([Du1], Proposition 3.5):

$$\mathrm{Diam}_{\vartheta_{\mathbb{C}^m_+}}(A(\mathbb{C}^m_+ - \{0\})) \le \max_{k,l} \delta_{k,l} + 2\max_{k,l} \Delta_{k,l} \le 3\,\mathrm{Diam}_{\vartheta_{\mathbb{C}^m_+}}(A(\mathbb{C}^m_+ - \{0\})). \tag{10}$$

From the proof of Proposition 3.5 in [Du1], in particular from equation (3.12), it also follows that for a real matrix A:

$$\mathrm{Diam}_{\vartheta_{\mathbb{C}^m_+}}(A(\mathbb{C}^m_+ - \{0\})) \le 3\,\mathrm{Diam}_{\vartheta}(A(\mathbb{R}^m_+ - \{0\})).$$

Fix $p \in \mathcal{P}(B)$. We choose $\gamma = \gamma(p) < 1$ such that

$$9(\tanh)^{-1}\beta_0 < (\tanh)^{-1}(\gamma^{2r}).$$

Then for the real matrices $A = A_1(p), \ldots, A_N(p)$,

$$\begin{aligned} 3\,\mathrm{Diam}_{\vartheta_{\mathbb{C}_+^m}}(A(\mathbb{C}_+^m - \{0\})) &\le 9\,\mathrm{Diam}_{\vartheta}(A(\mathbb{R}_+^m - \{0\})) \\ &\le 36(\tanh)^{-1}\beta_0 < 4(\tanh)^{-1}(\gamma^{2r}). \end{aligned} \tag{11}$$

Proposition 5.4. *Fix $p \in \mathcal{P}(B)$. There is a neighborhood $\mathcal{O}_p$ of p in $\mathbb{C}^B$ such that the mapping $p \mapsto \Phi_p$ extends to an analytic mapping from $\mathcal{O}_p$ into $\Gamma_{\gamma(p)}$.*

Proof. We first extend A_j, $j = 1, \ldots, N$, analytically on a neighborhood $\mathcal{O}_p$ by Proposition 4.4. Set $S = S^{2m-1} = \{f \mid f \in \mathbb{C}_+^m, \|f\| = 1\}$. For each $A_j(q)$, $j = 1, \ldots N$, $q \in \mathcal{O}_p$, and each $f \in S$ such that $A_j(q)f \neq 0$, we define again $T_j(q)f$ by

$$T_j(q)f = \frac{A_j(q)f}{\|A_j(q)f\|}.$$

For $p \in \mathcal{P}(B)$, the function Φ_p is given by the limit from formula (6),

$$\Phi_p(\xi) = -\ln \lim_{k\to\infty} \frac{\langle \alpha_1(\xi), T_{j_1}(\xi)\ldots T_{j_{k-1}}(\xi) f_0\rangle}{\langle \alpha(\xi), T_{j_1}(\xi)\ldots T_{j_{k-1}}(\xi) f_0\rangle},$$

where $f_0 \in S$ the column vector $\{1/\sqrt{|B|}, \ldots, 1/\sqrt{|B|}\}$: we use the fact that the limit of $T_{j_1}(\xi)\ldots T_{j_{k-1}}(\xi)f$ does not depend on the initial point f.

We have to show that this limit extends on some neighborhood $\mathcal{O}_p$ of p to an analytic function into Γ_γ. Set

$$\Phi_{p,k}(\xi) \ := \ -\ln \frac{\langle \alpha_1(\xi), A_{j_1}(\xi)\ldots A_{j_{k-1}}(\xi) f_0\rangle}{\langle \alpha(\xi), A_{j_1}(\xi)\ldots A_{j_{k-1}}(\xi) f_0\rangle}.$$

We are going to find $\mathcal{O}_p$ and k_0 such that, for $k \ge k_0$, the functions $\Phi_{p,k}(\xi)$ extend to analytic functions from $\mathcal{O}_p$ into Γ_γ and, as $k \to \infty$, the functions $\Phi_{p,k}(\xi)$ converge in Γ_γ uniformly on $\mathcal{O}_p$. The functions $q \mapsto \langle \alpha_1(\xi), A_{j_1}(\xi)\ldots A_{j_{k-1}}(\xi) f_0\rangle$, $q \mapsto \langle \alpha(\xi), A_{j_1}(\xi)\ldots A_{j_{k-1}}(\xi) f_0\rangle$ are polynomials in q and depend only on a finite number of coordinates of ξ. Therefore, if we can find a neighborhood $\mathcal{O}_p$ and a k such that these two functions do not vanish, then $\Phi_{p,k}$ extends to an analytic function from $\mathcal{O}_p$ to Γ_γ.

Step 1: *Contraction.* By (10), (11) and Proposition 4.4, we can choose a neighborhood $\mathcal{O}_p$ such that for $q \in \mathcal{O}_p$, the diameter Δ of $A_j(q)\mathbb{C}_+^m$ is smaller than $4(\tanh)^{-1}(\gamma^{2r})$ for all $j = 1, \ldots, N$.[1] The set $\mathcal{D} := S \cap \left(\bigcup_j A_j(p)\mathbb{C}_+^m\right)$ is compactly contained in the interior of S. We choose a smaller neighborhood $\mathcal{O}_p$ such that if $q \in \mathcal{O}_p$, then

$$\Delta < 4(\tanh)^{-1}(\gamma^{2r}) \quad \text{and} \quad 0 \notin A_j(\mathcal{D} \cup \{f_0\}) \quad \text{for } j = 1, \ldots, N.$$

[1] One can also use directly [Du2], Theorem 4.5.

For $q \in \mathcal{O}_p$, the projective images $T_{j_1}(\xi)\dots T_{j_{k-1}}(\xi) f_0$ are all defined and we have, by repeated application of (8),

$$\vartheta_{\mathcal{C}}(T_{j_1}(\xi)\dots T_{j_{k-1}}(\xi) f_0, T_{j_1}(\xi)\dots T_{j_{k-1}}(\xi) f_{k,k'}(\xi)) \le \gamma^{2(k-1)r}\vartheta(f_0, f_{k,k'}(\xi)),$$

where $k' > k$ and $f_{k,k'}(\xi) := T_{j_k}(\xi)\dots T_{j_{k'-1}}(\xi) f_0$. The $f_{k,k'}(\xi)$ are all in $\mathcal{D}$. Then $\vartheta_{\mathcal{C}}(f_0, f_{k,k'}(\xi)) \le C$ for all $\xi \in \partial F$, all $k, k' \ge 1$. Set

$$g = T_{j_1}(\xi)\dots T_{j_{k-1}}(\xi) f_0, \quad g' = T_{j_1}(\xi)\dots T_{j_{k-1}}(\xi) f_{k,k'}(\xi).$$

For all $\xi \in \partial F$, all $k, k' \ge 1$, consider the number $\rho(\xi, k, k')$ associated to g and g' by Proposition 5.2. We have by (9)

$$|\rho(\xi, k, k')| = 1 \quad \text{and} \quad \|\rho(\xi, k, k')g - g'\| \le KC\gamma^{2kr}.$$

Since $\alpha(p, \xi)$ and $\alpha_1(p, \xi)$ take finite many values, it follows that

$$\begin{aligned}
&|\langle \alpha(p,\xi), g\rangle\langle \alpha_1(p,\xi), g'\rangle - \langle \alpha(p,\xi), g'\rangle\langle \alpha_1(p,\xi), g\rangle| \\
&\quad = |\langle \alpha(p,\xi), \rho(\xi,k,k')g\rangle\langle \alpha_1(p,\xi), g'\rangle - \langle \alpha(p,\xi), g'\rangle\langle \alpha_1(p,\xi), \rho(\xi,k,k')g\rangle| \\
&\quad \le KC\gamma^{2kr}.
\end{aligned}$$

Since g and g' are in the compact set $\mathcal{D} \cup \{f_0\}$, we can, by Proposition 3.3, choose a neighborhood $\mathcal{O}_p$ such that

$$\begin{aligned}
&|\langle \alpha(\xi), T_{j_1}(\xi)\dots T_{j_{k-1}}(\xi) f_0\rangle\langle \alpha_1(\xi), T_{j_1}(\xi)\dots T_{j_{k'-1}}(\xi) f_0\rangle \\
&\quad - \langle \alpha(\xi), T_{j_1}(\xi)\dots T_{j_{k'-1}}(\xi) f_0\rangle\langle \alpha_1(\xi), T_{j_1}(\xi)\dots T_{j_{k-1}}(\xi) f_0\rangle| \le KC\gamma^{2kr}
\end{aligned}$$

for all $q \in \mathcal{O}_p$, all $\xi \in \partial F$, and all $k < k'$.

Step 2: *The* $\Phi_{q,k}$ *extend.* Recall that D is the set of unit vectors in the positive quadrant. For $g, g' \in \bigcup_j T_j(p)(D) \cup \{f_0\}$, $\langle \alpha(p,\xi), g\rangle\langle \alpha_1(p,\xi), g'\rangle$ is real positive and bounded away from 0 uniformly in ξ, g and g'. Recall the isometric inclusion $\pi : D \to S$ of Proposition 5.3. There is a neighborhood $\mathcal{C}_0$ of $\pi\big(\bigcup_j T_j(p)(D) \cup \{f_0\}\big)$ in S and $\delta > 0$ such that $|\langle \alpha(p,\xi), g\rangle\langle \alpha_1(p,\xi), g'\rangle| > \delta$ for $g, g' \in \mathcal{C}_0$. Of course, we can take $\mathcal{C}_0$ invariant by multiplication by all z with $|z| = 1$. Moreover, there exists $\varepsilon > 0$ such that if $\vartheta_{C^m_+}(g, \pi(\bigcup_j T_j(p)(D) \cup \{f_0\})) < \varepsilon$ and $\vartheta_{C^m_+}(g', \pi(\bigcup_j T_j(p)(D) \cup \{f_0\})) < \varepsilon$, then $|\langle \alpha(p,\xi), g\rangle\langle \alpha_1(p,\xi), g'\rangle| > \delta/2$.

For $q \in \mathcal{O}_p$ and $k_0 > 1 + \ln(\varepsilon/2)/2r\ln\gamma$, the $\vartheta_{\mathbb{C}^m_+}$-diameter of each one of the sets $T_{j_1}(q,\xi)\dots T_{j_{k_0-1}}(q,\xi)S$ is smaller than $\varepsilon/2$, for all ξ. As ξ varies, there is only a finite number of mappings $T_{j_1}(q,\xi)\dots T_{j_{k_0-1}}(q,\xi)$. By continuity of $q \mapsto T_j$ (where the T_js now are considered as mappings from $\mathcal{C}/\!\sim$ into itself), there is a neighborhood $\mathcal{O}_p$ such that for $q \in \mathcal{O}_p$, the Hausdorff distance between $T_{j_1}(q,\xi)\dots T_{j_{k_0-1}}(q,\xi)S/\!\sim$ and $T_{j_1}(p,\xi)\dots T_{j_{k_0-1}}(p,\xi)S/\!\sim$ is smaller than $\varepsilon/2$. It follows that if $q \in \mathcal{O}_p$, and g, g' are in the same $T_{j_1}(q,\xi)\dots T_{j_{k_0-1}}(q,\xi)S$ for some ξ, then

$$|\langle \alpha(p,\xi), g\rangle\langle \alpha_1(p,\xi), g'\rangle| > \delta/2.$$

By taking a possibly smaller $\mathcal{O}_p$, we have that if $q \in \mathcal{O}_p$, and g, g' are in the same $T_{j_1}(q,\xi) \dots T_{j_{k_0-1}}(q,\xi)S$ for some ξ, then

$$|\langle \alpha(q,\xi), g\rangle \langle \alpha_1(q,\xi), g'\rangle| > \delta/4.$$

In particular this last expression does not vanish and $\Phi_{q,k}$ is an analytic function on $\mathcal{O}_p$ for $k \geq k_0$.

Step 3: *The* $\Phi_{q,k}$ *converge uniformly on* ∂F. Take a neighborhood $\mathcal{O}_p$ and k_0 such that for $q \in \mathcal{O}_p$ the conclusions of steps 1 and 2 hold. We claim that for all $\varepsilon > 0$, there is k_1 such that for $k, k' \geq k_1$, $q \in \mathcal{O}_p$, $\max_\xi |\Phi_{q,k}(\xi) - \Phi_{q,k'}(\xi)| < \varepsilon$. Suppose that $k_1 > k_0$. We have to estimate

$$\max_\xi \left| \ln \frac{\langle \alpha(\xi), T_{j_1}(\xi) \dots T_{j_{k-1}}(\xi) f_0\rangle}{\langle \alpha(\xi), T_{j_1}(\xi) \dots T_{j_{k'-1}}(\xi) f_0\rangle} \frac{\langle \alpha_1(\xi), T_{j_1}(\xi) \dots T_{j_{k'-1}}(\xi) f_0\rangle}{\langle \alpha_1(\xi), T_{j_1}(\xi) \dots T_{j_{k-1}}(\xi) f_0\rangle} \right|.$$

By the conclusions of steps 1 and 2, this quantity is smaller that $C \max\{\gamma^{2kr}, \gamma^{2k'r}\}$. This is smaller than ε if k_1 is large enough.

Step 4: *The* $\Phi_{q,k}$ *converge in norm* $\|\cdot\|_{\gamma(p)}$. With the same $\mathcal{O}_p$, k_0, we now claim that for all $\varepsilon > 0$, there is $k_2 = \max\{k_0, \ln\gamma/r \ln\varepsilon\}$ such that for $k, k' \geq k_2$ and $q \in \mathcal{O}_p$, $\|\Phi_{q,k}(\xi) - \Phi_{q,k'}(\xi)\|_\gamma < \varepsilon$. Let ξ, ξ' be two points of ∂F with $\delta(\xi,\xi') \leq \exp(-((n+1)r+1))$. We want to show that there is a constant C independent on n, such that

$$|\Phi_{q,k}(\xi) - \Phi_{q,k'}(\xi) - \Phi_{q,k}(\xi') + \Phi_{q,k'}(\xi')| \leq C\gamma^{(n+1)r+1}\varepsilon$$

for all $q \in \mathcal{O}_p$, all $k, k' \geq k_2$. Since $k, k' \geq k_0$, the difference $\Phi_{q,k}(\xi) - \Phi_{q,k'}(\xi)$ is given by

$$\Phi_{q,k}(\xi) - \Phi_{q,k'}(\xi) = \ln \frac{\langle \alpha_1, T_{j_1} \dots T_{j_{k'-1}} f_0\rangle}{\langle \alpha_1, T_{j_1} \dots T_{j_{k-1}} f_0\rangle} \frac{\langle \alpha, T_{j_1} \dots T_{j_{k-1}} f_0\rangle}{\langle \alpha, T_{j_1} \dots T_{j_{k'-1}} f_0\rangle}.$$

For $k, k' \leq n+1$, $\Phi_{q,k}(\xi) - \Phi_{q,k'}(\xi) = \Phi_{q,k}(\xi') - \Phi_{q,k'}(\xi')$, and there is nothing to prove.

Assume that $k' > k \geq n+1$. Step 3 shows that both $|\Phi_{q,k}(\xi) - \Phi_{q,k'}(\xi)|$ and $|\Phi_{q,k}(\xi') - \Phi_{q,k'}(\xi')|$ are smaller than $C\gamma^{2kr} \leq C\gamma^{nr}\gamma^{kr} \leq C\gamma^{nr}\varepsilon$.

The remaining case, when $k_0 \leq k \leq n+1 \leq k'$, clearly follows from the other two, and this shows step 4.

Finally we have that the functions $\Phi_{p,k}$ are analytic and converge uniformly in Γ_γ on a neighborhood $\mathcal{O}_p$ of p. The limit is an analytic function on $\mathcal{O}_p$. □

6. Proof of Theorem 1.1

In this section, we consider ∂F as a subshift of finite type and let τ be the shift transformation on ∂F:

$$\tau\xi = \eta_1\eta_2 \dots \quad \text{with } \eta_n = \xi_{n+1}.$$

For $\gamma < 1$ and $\phi \in \Gamma_\gamma$ with real values, we define the transfer operator $\mathcal{L}_\phi$ on Γ_γ by

$$\mathcal{L}_\phi \psi(\xi) := \sum_{\eta \in \tau^{-1}\xi} e^{\phi(\eta)} \psi(\eta).$$

Then $\mathcal{L}_\phi$ is a bounded operator in Γ_γ and, by Ruelle's transfer operator theorem (see e.g. [Bo]), there exists a number $P(\phi)$, a positive function $h_\phi \in \Gamma_\gamma$ and an unique linear functional ν_ϕ on Γ_γ such that

$$\mathcal{L}_\phi h_\phi = e^{P(\phi)} h_\phi, \quad \mathcal{L}_\phi^* \nu_\phi = e^{P(\phi)} \nu_\phi \quad \text{and} \quad \nu_\phi(1) = 1.$$

The functional ν_ϕ extends to probability measure on ∂F and is the only eigenvector of $\mathcal{L}_\phi^*$ with that property. Moreover, $\phi \mapsto \mathcal{L}_\phi$ is a real analytic map from Γ_γ to the space of linear operators on Γ_γ ([R1], p. 91). Consequently, the mapping $\phi \mapsto \nu_\phi$ is real analytic from Γ_γ into the dual space Γ_γ^* (see e.g. [Co], Corollary 4.6). By Proposition 5.4, the mapping $p \mapsto \nu_{\Phi_p}$ is real analytic from a neighborhood of p in $\mathcal{P}(B)$ into the space $\Gamma^*_{\gamma(p)}$.

The main observation is that $\mathcal{L}^*_{\Phi_p} p^\infty = p^\infty$ for all $p \in \mathcal{P}(B)$; this implies that $P(\Phi_p) = 0$ and that the distribution ν_{Φ_p} is the restriction of the measure p^∞ to any Γ_γ such that $\Phi_p \in \Gamma_\gamma$. Indeed, we have

$$\frac{d\tau_* p^\infty}{dp^\infty}(\xi) = \frac{d(\xi_1)_* p^\infty}{dp^\infty} = K_\xi(\xi_1) = e^{\Phi_p(\xi)}$$

so that, for all continuous ψ,

$$\int (\mathcal{L}_{\Phi_p} \psi) dp^\infty = \sum_a \int_{a\xi, \xi_1 \neq a^{-1}} \frac{dp^\infty(a\xi)}{dp^\infty(\xi)} \psi(a\xi) dp^\infty(\xi) = \int \psi dp^\infty.$$

Recall the equations (1) and (2) for h_p and ℓ_p. The linear drift ℓ_p is given by a finite sum (in x) of integrals with respect to p^∞ of the functions $\xi \mapsto \theta_\xi(x)$. Since these functions only depend on a finite number of coordinates in ∂F, they belong to Γ_γ for all $\gamma < 1$. Since $p \mapsto \nu_{\Phi_p}$ is real analytic from a neighborhood of p into $\Gamma^*_{\gamma(p)}$, $p \mapsto \ell_p$ is real analytic on a neighborhood of p. Since this is true for all $p \in \mathcal{P}(B)$, the function $p \mapsto \ell_p$ is real analytic on $\mathcal{P}(B)$.

The argument is the same for h_p, since the function $\ln \frac{dx_*^{-1} p^\infty}{dp^\infty}(\xi) = \ln K_\xi(x^{-1}) \in \Gamma_\gamma$ for all x and for all $\gamma, \beta < \gamma < 1$ and the mappings $p \mapsto \ln K_\xi(x^{-1})$ are real analytic from a neighborhood of p into $\Gamma_{\gamma(p)}$. Indeed, $\ln K_\xi(\xi_1) \in \Gamma_\beta$ by Proposition 4.5 and $p \mapsto \ln K_\xi(\xi_1)$ is real analytic into $\Gamma_{\gamma(p)}$ by Proposition 5.4. Moreover, if a is a generator different from ξ_1, then $\ln K_\xi(a) = -\ln K_{a^{-1}\xi}(a^{-1})$ also lies in Γ_β and $p \mapsto \ln K_\xi(a)$ is real analytic into $\Gamma_{\gamma(p)}$ as well. For a general $x \in F$, $x = a_1 \ldots a_t$, write

$$K_\xi(x^{-1}) = K_\xi(a_t^{-1} \ldots a_1^{-1}) = K_\xi(a_t^{-1}) K_{a_t\xi}(a_{t-1}^{-1}) \ldots K_{a_2 \ldots a_t \xi}(a_1^{-1}).$$

This completes the proof of Theorem 1.1. For the proof of Theorem 1.2, fix an origin $o \in \mathbb{H}^k$. Then $\pi(F)o$ accumulates to the boundary of $\mathbb{H}^k$ in a Cantor set Λ called the limit set of $\pi(F)$. The mapping $\pi_o\colon F \to \mathbb{H}^n$, $\pi_o(x) = x \cdot o$, extends to a Hölder continuous mapping π_o from ∂F to the limit set Λ of $\pi(F)$. We can express the exponent γ_p as

$$\gamma_p = \lim_n \frac{1}{2n} \sum_{x \in F} d(o, \pi_o(x)) p^{(n)}(x),$$

where the distance d is the hyperbolic distance in $\mathbb{H}^k$. We obtain, in the same way as for formula (2),

$$\begin{aligned}\gamma_p &= \frac{1}{2} \sum_{x \in F} \Big(\int_\Lambda \Theta_\zeta(\pi_o(x^{-1})) d((\pi_o)_* p^\infty)(\zeta) \Big) p(x) \\ &= \frac{1}{2} \sum_{x \in F} \Big(\int_{\partial F} \Theta_{\pi_o(\xi)}(\pi_o(x^{-1})) d(p^\infty)(\xi) \Big) p(x),\end{aligned}$$

where Θ_ζ is now the Busemann function of $\mathbb{H}^k$: $\Theta_\zeta(z) := \lim_{w \to \zeta} d(w, z) - d(w, o)$. Since, for all $x \in F$, the function $\xi \mapsto \Theta_{\pi_o(\xi)}(\pi_o(x))$ is a ρ-Hölder continuous function for some fixed ρ, we deduce as above that $p \mapsto \gamma_p$ is real analytic on $\mathcal{P}(B)$.

Note added in proof. Analyticity of the entropy in related circumstances is also obtained in [G2] and [HMP].

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