Groups Geom. Dyn. 6 (2012), 279–315
DOI 10.4171/GGD/158
Groups, Geometry, and Dynamics

The simultaneous conjugacy problem in groups of piecewise linear functions

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This work is dedicated to the memory of Will Davis

Abstract. Guba and Sapir asked if the simultaneous conjugacy problem is solvable in diagram groups or, at least, for Thompson's group F. We give a solution to the latter question using elementary techniques which rely purely on the description of F as the group of piecewise linear orientation-preserving homeomorphisms of the unit interval. The techniques we develop extend the ones used by Brin and Squier allowing us to compute roots and centralizers as well. Moreover, these techniques can be generalized to solve the same question in larger groups of piecewise-linear homeomorphisms.

Mathematics Subject Classification (2010). 20F10; 20E45, 37E05.

Keywords. Conjugacy problem, Thompson's groups.

1. Introduction

Richard Thompson's group F can be defined by the following presentation:

$$F = \langle x_0, x_1, x_2, \dots \mid x_n x_k = x_k x_{n+1} \text{ for all } k < n \rangle.$$

This group was introduced and studied by Thompson in the 1960s. The standard introduction to F is [6]. The group F can be regarded as a subgroup of the group of piecewise linear self-homeomorphisms of the unit interval and this is the point of view that we will adopt throughout the paper, and that we will introduce in detail in Section 2.

We say that a group G has *solvable ordinary conjugacy problem* if there is an algorithm such that, given any two elements $y, z \in G$, we can determine whether there is, or not, an element $g \in G$ such that $g^{-1} y g = z$. Similarly, for fixed $k \in \mathbb{N}$, we say that the group G has *solvable k-simultaneous conjugacy problem* if

*The first author was partially supported by grants from National Science Foundation DMS 0600244, 0635607 and 0900932.

there is an algorithm such that, given any two k-tuples $(y_1, \dots, y_k)$, $(z_1, \dots, z_k)$ of elements in G, one can determine whether there is, or not, an element $g \in G$ such that $g^{-1} y_i g = z_i$ for all $i = 1, \dots, k$. For both these problems, we say that there is an *effective solution* if the algorithm produces such an element g, in addition to proving its existence.

This problem was studied before for various classes of groups. The simultaneous conjugacy problem was proved to be solvable for the matrix groups $\mathrm{GL}_n(\mathbb{Z})$ and $\mathrm{SL}_n(\mathbb{Z})$ by Sarkisyan in 1979 in [14], and independently by Grunewald and Segal in 1980 in [9]. In 1984, Scott [15] constructed examples of finitely presented groups that have an unsolvable conjugacy problem. In 1976, Collins showed in [7] that the solvability of the conjugacy problem does not imply the solvability of the simultaneous conjugacy problem. More recently, in their 2005 paper [4], Bridson and Howie constructed examples of finitely presented groups where the ordinary conjugacy problem is solvable, but the k-simultaneous conjugacy problem is unsolvable for every $k \geq 2$.

The ordinary conjugacy problem for F was addressed by Guba and Sapir [10] in 1997, who reduced the solution of the conjugacy problem for diagram groups to the solution of the word problem in the corresponding semigroup, solving this last problem for F and many similar groups. Their solution, for general diagram groups, reduces the problem to the isomorphism problem of planar graphs. We mention here relevant related work: in 2001, Brin and Squier in [5] produced a criterion for describing conjugacy classes in $\mathrm{PL}_+(I)$, the group of all piecewise-linear orientation preserving self-homeomorphisms of the unit interval with only finitely many breakpoints, that contains F as a proper subgroup. In 2007, Gill and Short [8] extended this criterion to work in F, thus finding another way to characterize conjugacy classes from a piecewise linear point of view. Using an approach similar to Guba and Sapir's original solution, in 2007 Belk and Matucci [2] produced a unified solution of the conjugacy problem for all three Thompson groups F, T and V.

In 1999, Guba and Sapir [11] posed the question of whether or not the simultaneous conjugacy problem was solvable for diagram groups. Some of the results of the present paper are already known, but we deduce all of them using our tools. We will show that our techniques can be used on a large class of groups of piecewise linear homeomorphisms.

Theorem 1.1. *Thompson's group F has a solvable k-simultaneous conjugacy problem for every $k \in \mathbb{N}$. There is an algorithm which produces an effective solution and enumerates all possible conjugators.*

The same algorithm also solves the k-simultaneous conjugacy problem in many "Thompson-like" subgroups of $\mathrm{PL}_+(I)$ (see Section 2.1 for the precise definition).

As an application of the proof of the above theorem we have the following corollaries:

Theorem 1.2. *For an element $x \in F$, we denote by $C_F(x)$ the centralizer of x in F. Then:*

(i) $C_F(x) \cong F^m \times \mathbb{Z}^n$ *for some numbers* $0 \leq m \leq n+1$.

(ii) *An element* $x \in F$ *has a finite number of roots, which can be effectively computed.*

(iii) *The centralizer of any finitely generated subgroup* $A \subset F$ *decomposes as the direct product of the groups* C_i*, where each* C_i *is either trivial, infinite cyclic or isomorphic to* F.

(iv) *The intersection of any number* $k \geq 2$ *centralizers of elements of* F *is equal to the intersection of two centralizers.*

Parts of the previous theorem were already proved either in the setting of F or in that of $\mathrm{PL}_+(I)$: in particular, parts (i) and (ii) were proved by Guba and Sapir in [10] for F, and by Brin and Squier for $\mathrm{PL}_+(I)$ in [5]. All the previous results can be suitably rephrased for a large class of subgroups of $\mathrm{PL}_+(I)$ (see Section 2.1 for the precise definition).

The paper is organized as follows. In Section 2 we will define the groups $\mathrm{PL}_{S,G}(I)$ that generalize Thompson's group F and give an outline of the solution of simultaneous conjugacy problem. In Section 4 we introduce the main algorithm to create candidate conjugators. In Section 5 we compute centralizers and roots. In Section 6 we show how to construct an approximate conjugator which makes the fixed point set of y and z coincide. In Section 7 we get the solution of the ordinary conjugacy problem and a variation of it, the *power conjugacy problem*. In Section 8 we describe how to reduce the simultaneous conjugacy problem to a special instance of the ordinary conjugacy problem. In Section 9 we show interesting instances where the simultaneous conjugacy problem can be solved.

2. The idea of the argument

In this section we describe the groups that we will study and outline the steps of our proof. It is intended to give a quick overview of the results that we will prove in the later sections.

2.1. Notations. We introduce here the notation that will be used across the paper. Let $I = [0,1]$ be the unit interval. We define $\mathrm{PL}_+(I)$ to be the group of piecewise linear[1] orientation-preserving homeomorphisms of the unit interval into itself, with finitely many breakpoints of the derivative function such that slopes are positive real numbers. The product of two elements is given by the composition of functions.

One can impose additional the requirements on the breakpoints and the slopes to define subgroups of $\mathrm{PL}_+(I)$. Let S be an additive subgroup of $\mathbb{R}$ containing 1, let $U(S)$ denote the multiplicative group $\{g \in \mathbb{R}^* \mid gS = S \text{ and } g > 0\}$, and let G be a subgroup of $U(S)$. Thus, S is a module over the ring $\mathbb{Z}[G]$. We define $\mathrm{PL}_{S,G}(I)$

[1] By piecewise linear we mean piecewise affine, although this abuse of language is now common.

to be the subgroup of $\mathrm{PL}_+(I)$ consisting of all functions f such that the breakpoints are in the subgroup S and the slopes are in the subgroup G. We observe that if the group G is trivial, then so is $\mathrm{PL}_{S,G}(I)$. Therefore in the rest of the paper we will assume that G is nontrivial, which implies that S is dense in $\mathbb{R}$ (with respect to the usual topology).

If $G = U(S)$, we write $\mathrm{PL}_S(I)$ instead of $\mathrm{PL}_{S,G}(I)$. If $S = \mathbb{R}$, then $\mathrm{PL}_S(I) = \mathrm{PL}_+(I)$. For the special case $S = \mathbb{Z}\left[\frac{1}{2}\right]$, we denote the group $\mathrm{PL}_{\mathbb{Z}[\frac{1}{2}]}(I)$ by $\mathrm{PL}_2(I)$. The group $\mathrm{PL}_2(I)$ is also known as *Thompson's group* F and is isomorphic to the group F defined in the introduction (see [6] for a proof).[2] We remark that in order to make some calculations possible inside the module S and its quotients, we need to ask for some requirements to be satisfied by S from the computability standpoint (like the existence of black box algorithms for performing the basic operations in S). These will be explicitly stated in Section 3 and will be assumed throughout this paper.

To attack the ordinary and the simultaneous conjugacy problems, we will split the study into that of some families of functions inside $\mathrm{PL}_+(I)$. The reduction to these subfamilies will come from the study of the fixed point subset of the interval I for a function f.

Remark 2.1. We would like to define the group $\mathrm{PL}_{S,G}(J)$, where $J = [\eta, \zeta]$ is any interval contained in I. We consider the group of restrictions of functions in $\mathrm{PL}_{S,G}(I)$ fixing the endpoints of J:

$$\mathrm{PL}_{S,G}^{\mathrm{Rest}}(J) := \{f|_J \mid f \in \mathrm{PL}_{S,G}(I),\ f(\eta) = \eta,\ f(\zeta) = \zeta\}.$$

In general, it is not true that $\mathrm{PL}_{S,G}^{\mathrm{Rest}}(J)$ is a subgroup of $\mathrm{PL}_{S,G}(I)$. Moreover, there is no natural embedding of $\mathrm{PL}_{S,G}^{\mathrm{Rest}}(J)$ into $\mathrm{PL}_{S,G}(I)$ such that the restriction of the image of a function is the initial function (see also Remark 9.5). If the endpoints of J are in S, we will denote the group $\mathrm{PL}_{S,G}^{\mathrm{Rest}}(J)$ by $\mathrm{PL}_{S,G}(J)$.[3]

Remark 2.2. Throughout the paper we will always assume the interval J to have endpoints in S (with the only exception of Lemma 6.5). For the special case $S = \mathbb{Z}\left[\frac{1}{2}\right]$, it is straightforward to verify that $\mathrm{PL}_2(J) \cong \mathrm{PL}_2(I)$. We observe that the analogous fact may not be true for the groups $\mathrm{PL}_{S,G}(I)$ (see Remark 9.5).

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[2] The family of groups $\mathrm{PL}_{S,G}(I)$ was first introduced by Bieri and Strebel in [3] and was later popularized through the work of Stein [16].

[3] There is another natural way to define $\mathrm{PL}_{S,G}(J)$: consider the subgroup of functions of $\mathrm{PL}_{S,G}(I)$ which fix the endpoints of J and are the identity on $I \setminus J$:

$$\mathrm{PL}_{S,G}^{\mathrm{Fix}=I\setminus J}(J) := \{f \in \mathrm{PL}_{S,G}(I) \mid f(t) = t \text{ for all } t \in I \setminus J\}.$$

We observe that by definition $\mathrm{PL}_{S,G}^{\mathrm{Fix}=I\setminus J}(J)$ is a subgroup of $\mathrm{PL}_{S,G}(I)$. In the case where the endpoints of J are contained in S, the two definitions coincide, i.e., $\mathrm{PL}_{S,G}^{\mathrm{Fix}=I\setminus J}(J) \cong \mathrm{PL}_{S,G}^{\mathrm{Rest}}(J)$, and thus the group $\mathrm{PL}_{S,G}^{\mathrm{Rest}}(J)$ can be regarded as a subgroup of $\mathrm{PL}_{S,G}(I)$.

For a function $f \in \mathrm{PL}_{S,G}(J)$ we define the fixed point set of the interval J by

$$\mathrm{Fix}_J(f) := \{t \in J \mid f(t) = t\},$$

which is a closed set. It follows from the definition that $\mathrm{Fix}_J(f)$ is a union of finitely many intervals with endpoints in S and finitely many "isolated" points. We will often simplify the notation by dropping the subscript J. The motivation for introducing this subset is easily explained: if $y, z \in \mathrm{PL}_+(J)$ are conjugate through $g \in \mathrm{PL}_+(I)$ and $t \in (\eta, \zeta)$ is such that $y(t) = t$, then $z(g^{-1}(t)) = (g^{-1}yg)(g^{-1}(t)) = g^{-1}(t)$, that is, if y has a fixed point, then z must have a fixed point.

Definition 2.3. We define $\mathrm{PL}^{<}_{S,G}(J)$ and $\mathrm{PL}^{>}_{S,G}(J)$ to be the set of all functions in $\mathrm{PL}_{S,G}(J)$ with graph below the diagonal, respectively above the diagonal. Following Brin and Squier [5], we define a function in $x \in \mathrm{PL}_{S,G}(J)$ to be a *one-bump function* if either $x \in \mathrm{PL}^{>}_{S,G}(J)$ or $x \in \mathrm{PL}^{<}_{S,G}(J)$.

In general it is not true that if $f \in \mathrm{PL}_{S,G}(I)$ then $\mathrm{Fix}(f) \subseteq S$, but $\mathrm{Fix}(f)$ is always a subset of the "field of fractions" of S. The example in Figure 1 shows a function in $\mathrm{PL}_2(I)$ with a non-dyadic rational fixed point. In order to avoid working

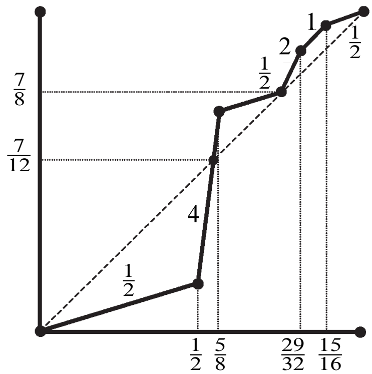

Figure 1. A function in $\mathrm{PL}_2(I)$ with a non-dyadic fixed point.

in intervals J where the endpoints may not be in S, we introduce a new definition of boundary which deals with this situation: for a subset $X \subseteq [0,1]$, we define

$$\partial_S X := \partial X \cap S,$$

where ∂X denotes the usual topological boundary of X inside $\mathbb{R}$. For the special case $S = \mathbb{Z}\left[\frac{1}{2}\right]$ we write $\partial_2 X$. We are going to apply this definition to the set $X = \mathrm{Fix}(f)$ so that $\partial \,\mathrm{Fix}(f)$ and $\partial_S \mathrm{Fix}(f)$ will always be finite.

Definition 2.4. The $\mathrm{PL}^0_{S,G}(J) \subseteq \mathrm{PL}_{S,G}(J)$ will denote set of functions $f \in \mathrm{PL}_{S,G}(J)$ such that the set $\mathrm{Fix}(f)$ does not contain elements of S other than the endpoints of J, i.e., $\mathrm{Fix}(f)$ is discrete and $\partial_S \mathrm{Fix}(f) = \partial_S J = \partial J$. The elements in $\mathrm{PL}^0_{S,G}(J)$ will be called *almost one-bump function*, although their graphs have several bumps in general.

2.2. Outline of the strategy. We are now going to describe the general steps and reductions of the algorithm to solve the simultaneous conjugacy problem in the groups $\mathrm{PL}_{S,G}(I)$. Most of the time we work in the larger group $\mathrm{PL}_+(I)$ and we then say what is necessary to generalize the argument to $\mathrm{PL}_{S,G}(I)$. The following outline describes the correct order of the steps needed to solve the problem, however we will start Section 4 by describing the central tool of the paper (the "stair algorithm") which is used in Step 2. Let $x, y, z \in \mathrm{PL}_{S,G}(I)$.

Step 1. *Find a* $g \in \mathrm{PL}_{S,G}(I)$ *such that* $\mathrm{Fix}(y) = g(\mathrm{Fix}(z))$. The set $\mathrm{Fix}(x)$ consists of a disjoint union of a finite number of closed intervals and isolated points, because every $x \in \mathrm{PL}_{S,G}(I)$ has only finitely many breakpoints. As mentioned before, if $g^{-1}yg = z$, then $\mathrm{Fix}(y) = g(\mathrm{Fix}(z))$. Thus, as a first step we need to know if, given y and z, there exists an element $g \in \mathrm{PL}_{S,G}(I)$ such that $\mathrm{Fix}(g^{-1}yg) = g(\mathrm{Fix}(y)) = \mathrm{Fix}(z)$. In Section 6 we show an algorithm which determines whether or not there exists a "candidate" conjugator g_* such that $\mathrm{Fix}(g_*^{-1}yg_*) = \mathrm{Fix}(z)$. We then study the conjugacy problem for $g_*^{-1}yg_*$ and z.

Step 2. *Solve the conjugacy problem if* $\mathrm{Fix}(y) = \mathrm{Fix}(z)$. In this case $\partial_S \mathrm{Fix}(y) = \partial_S \mathrm{Fix}(y) = \{\alpha_1, \dots, \alpha_n\}$. It is easy to see that any conjugator fixes the points α_i; for this we need to look for conjugators in $\mathrm{PL}_{S,G}([\alpha_i, \alpha_{i+1}])$ of the restrictions of y and z to $[\alpha_i, \alpha_{i+1}]$. Thus, we can reduce the conjugacy problem to the intervals where y and z are "almost one-bump functions", more precisely they are either in $\mathrm{PL}^0_{S,G}(J)$ or equal to the identity. The case $y = z = \mathrm{id}$ is trivial, this we can assume that y and z are "almost one-bump" functions.[4] This case will be dealt with through a procedure called the "stair algorithm" that we provide in Section 4.2.

Step 3. *Describe the intersection of centralizers of elements and derive a solution to the conjugacy problem.* Finding centralizers g of an element y is equivalent to find all elements g such that $g^{-1}yg = y$. Using similar techniques we can also classify the structure of intersection of centralizers, which will be useful for the last step. Since the set of all conjugators for y and z is given by a particular conjugator times an element in the centralizer of y, steps 1, 2 and 3 give us a solution to the conjugacy problem.

Step 4. *Reduce the simultaneous conjugacy problem to a "restricted" conjugacy problem.* It can be seen that the simultaneous conjugacy problem is equivalent to solving the conjugacy problem for two elements y and z with the restriction that the conjugator g must lie in the intersection of centralizers of some elements $x_1, \dots, x_k$. In Section 8 we will show how to construct such a conjugator if it exists, following the previous steps.

[4] One needs to be a bit more careful since y and z can have fixed points in the interval which do not lie in S.

3. Computational requirements

In order to effectively solve to conjugacy and the simultaneous conjugacy problem in the groups $\mathrm{PL}_{S,G}$ we need to assume that the additive group S and the multiplicative group G satisfy some computational requirements. First, we will assume that there is some representation of the elements in S and G in some data structure M.[5] Then we need to be able to perform the basic operations in S and G, thus we require that we are given some "oracles" which perform the following operations:

- determine if $m \in M$ represents an element in S and/or G;
- determine if $m, m' \in M$ represent the same element in S and/or G;
- perform the basic operations (additions and substraction) in S;
- given two elements in S, determine which one is bigger;
- given an element of S and a rational number, determine which one is bigger;
- construct an element in S in any given non-empty open interval;
- perform the basic operations (multiplication, division) in G;
- perform multiplication between the elements in G and S.

Using these oracles, one can construct a data structure which represents the elements in the group $\mathrm{PL}_{S,G}$ and new oracles which perform the group operations.

The following additional oracles are needed for the algorithms described in Section 6 (here $\mathcal{I}$ denote the subgroup of S generated by $(g-1)s$ for $s \in S$ and $g \in G$):

- given $g \in G$ and $s \in S$, determine if $s/(g-1)$ is an element in S;
- an effective solution of the membership problem in the submodule $\mathcal{I}$, i.e., given $s \in S$, an oracle determines if $s \in I$ or not, and if $s \in \mathcal{I}$ it produces elements s_i and g_i such that $s = \sum (g_i - 1)s_i$;
- an effective solution of the congruence $sG = s'G \pmod{(t-1)\mathcal{I}}$, i.e., given s, s' and t an oracle constructs a solution of the congruence or determines that it has no solutions.

These oracles allow us to effectively solve the conjugacy problem in the group $\mathrm{PL}_{S,G}(J)$, but for an effective solution of the simultaneous conjugacy problem we need another oracle:

- an effective solution of the equation $a^k = bc^i$, where $k, i \in \mathbb{Z}$, i.e., given $a, b, c \in G$ construct an integer solution of the equation or determine that there cannot be any.

[5] Usually the elements are represented by some finite strings over a given alphabet. If this is the case we require the sets S and G be countable. But our algorithms do not depend on the data structure M.

4. The stair algorithm

In this section we carry out the second step of the strategy described in Section 2.2 by restricting our study to a square where the given functions have "no relevant" intersection with the diagonal, and showing how to construct possible candidates for conjugator. Our goal for this section is, essentially, to solve the conjugacy problem in $\mathrm{PL}_+^<(J)$, where we do not pay attention to the intersection with the diagonal. Our methods extend the results of Brin and Squier [5], who develop a technique similar to our algorithm. In this section we develop an algorithm which allows us to recover Brin and Squier's analysis and to extend it to the case of $\mathrm{PL}_{S,G}(J)$, together with a description of the intersection of centralizers.

4.1. The linearity boxes. This and the following section deal with functions in $\mathrm{PL}_+(J)$ for an interval $J = [\eta, \zeta]$: we will reuse them in the discussion on $\mathrm{PL}_{S,G}(I)$. We start by making the following observation: the map $\mathrm{PL}_+(J) \to \mathbb{R}_+$ which sends a function f to $f'(\eta^+)$ is a group homomorphism. The very first thing to check, if y and z are to be conjugate through a $g \in \mathrm{PL}_+(J)$ in neighborhoods of the endpoints of J, the following trivial lemma says that this can happen only if the graphs of y and z coincide near the endpoints of J.

Lemma 4.1. *Given three functions $y, z, g \in \mathrm{PL}_+(J)$ such that $g^{-1}yg = z$, there exist $\alpha, \beta \in (\eta, \zeta)$ such that $y(t) = z(t)$ for all $t \in [\eta, \alpha] \cup [\beta, \zeta]$ (see Figure 2).*

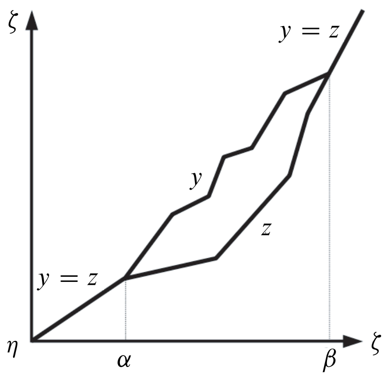

Figure 2. y and z coincide around the endpoints.

The next lemma gives us that any function $g \in \mathrm{PL}_+(J)$ which conjugates y to z needs to be linear in a specific neighborhood of each endpoint of J, which depends only on y and z. This lemma is the main ingredient, which allows us to extend the methods of Brin and Squier [5] to get a constructive solution of the conjugacy problem.

Lemma 4.2. *Suppose that* $y, z, g \in \mathrm{PL}_+(J)$ *and* $g^{-1}yg = z$. *Let* $\varepsilon > 0$ *and* $y'(\eta^+) = z'(\eta^+) = c > 1$ *satisfy*

$$y(t) - \eta = z(t) - \eta = c(t - \eta) \quad \textit{for } t \in [\eta, \eta + \varepsilon].$$

Then the graph of g is linear inside the square $[\eta, \eta + \varepsilon] \times [\eta, \eta + \varepsilon]$ (*see Figure* 3).

Proof. We can rewrite the conclusion of this lemma by saying that if we define

$$\tilde{\varepsilon} = \sup\{r \mid g \text{ is linear on } [\eta, \eta + r]\},$$

then $\eta + \tilde{\varepsilon} \geq \min\{g^{-1}(\eta + \varepsilon), \eta + \varepsilon\}$. Assume the contrary, let $\tilde{\varepsilon} < \varepsilon$ and $\eta + \tilde{\varepsilon} < g^{-1}(\eta + \varepsilon)$, and write $g(t) - \eta = \gamma(t - \eta)$ for $t \in [\eta, \eta + \varepsilon]$ and some constant $\gamma > 0$. Let $0 \leq \sigma < 1$ be any number. Since $\tilde{\varepsilon} < \varepsilon$, we have $\eta + \sigma\tilde{\varepsilon} < \eta + \varepsilon$ and so y is linear around $\eta + \sigma\tilde{\varepsilon}$:

$$g(y(\eta + \sigma\tilde{\varepsilon})) = g(\eta + c\sigma\tilde{\varepsilon}).$$

On the other hand, since $\eta + \tilde{\varepsilon} < g^{-1}(\eta + \varepsilon)$, it follows that $g(\eta + \sigma\tilde{\varepsilon}) < g(\eta + \tilde{\varepsilon}) < \eta + \varepsilon$ and so z is linear around the point $g(\eta + \sigma\tilde{\varepsilon}) = \eta + \gamma\sigma\tilde{\varepsilon}$:

$$z(g(\eta + \sigma\tilde{\varepsilon})) = z(\eta + \gamma\sigma\tilde{\varepsilon}) = \eta + c\gamma\sigma\tilde{\varepsilon}.$$

Since $gy = zg$, we can equate the previous two equations and write $g(\eta + c\sigma\tilde{\varepsilon}) = \eta + \gamma c\sigma\tilde{\varepsilon}$, for any number $0 \leq \sigma < 1$. If we choose $1/c < \sigma < 1$, we see that g must be linear on the interval $[0, c\sigma\tilde{\varepsilon}]$, where $c\sigma\tilde{\varepsilon} > \tilde{\varepsilon}$. This is a contradiction to the definition of $\tilde{\varepsilon}$. ☐

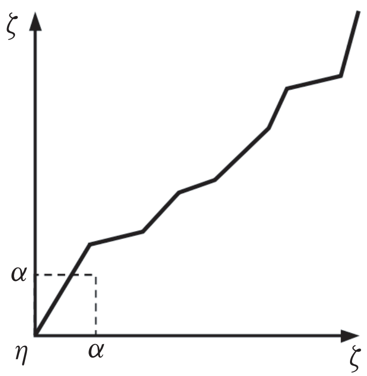

Figure 3. Initial linearity box.

Observe that the lemma also holds when $z'(\eta^+) = y'(\eta^+) = c < 1$ by applying it to the homeomorphisms y^{-1}, z^{-1}. Thus we can replace the condition $z'(\eta^+) = y'(\eta^+) = c > 1$ with $z'(\eta^+) = y'(\eta^+) \neq 1$. Note that Lemma 4.2 has an analogue for the points close to other endpoint of J:

Remark 4.3. Let $y, z, g \in \mathrm{PL}_+(J)$. Suppose that $g^{-1}yg = y$. If there exist $\beta \in J$ and $c < 1$ such that $y(t) = z(t) = c \cdot (t - \zeta) + \zeta$ on $[\beta, \zeta]$, then the graph of g is linear inside the square $[\beta, \zeta] \times [\beta, \zeta]$.

Lemma 4.2 does not hold when the initial slopes of y and z are equal to 1 because any function g with a support sufficiently close to the endpoints will conjugate y to itself.

4.2. The stair algorithm for $\mathrm{PL}_+^<(J)$. This section deals with the main construction of this paper. We show that if, under certain hypotheses, there is a conjugator, then it is unique. On the other hand, we give a construction of such a conjugator if it exists. Given two elements y, z the set of their conjugators is a coset of the centralizer of one of them, thus it makes sense to start by deriving properties of centralizers.

The first lemmas show that if y and z are one-bump functions in $\mathrm{PL}_+(J)$, then the graphs of the conjugators do not intersect.

Lemma 4.4. *Let $z \in \mathrm{PL}_+(J)$. Suppose that there exist $\eta \le \lambda \le \mu \le \zeta$ such that $z(t) \le \lambda$ for every $t \in [\eta, \mu]$. Suppose further that $g \in \mathrm{PL}_+(J)$ is such that $g(t) = t$ for all $t \in [\eta, \lambda]$ and $g^{-1}zg(t) = z(t)$ for all $t \in [\eta, \mu]$. Then $g(t) = t$ for all $t \in [\eta, \mu]$.*

Proof. The equation $g^{-1}zg(t) = z(t)$ implies that $g(t) = z^{-1}gz(t)$ for all $t \in [\eta, \mu]$. Since $z(t) \le \lambda$ and $g(x) = x$ for all $x \le \lambda$, we have

$$g(t) = z^{-1}(g(z(t))) = z^{-1}(z(t)) = t. \qquad \square$$

Corollary 4.5. *Let $z \in \mathrm{PL}_+^<(J)$ and $g \in \mathrm{PL}_+(J)$ be such that $g'(\eta^+) = 1$ and $g^{-1}zg = z$. Then $g(t) = \mathrm{id}$, the identity map.*

Proof. Since $g'(\eta^+) = 1$, we have $g(t) = t$ for all $t \in [\eta, \eta + \varepsilon]$. Applying the previous lemma several times we obtain $g(t) = t$ for all $t \in [\eta, z^{-k}(\eta + \varepsilon)]$. Since $z \in \mathrm{PL}_+^<(J)$, we have $\lim_{k\to\infty} z^{-k}(\eta + \varepsilon) = \zeta$, therefore $g(t) = t$ for $t \in J$. $\square$

Lemma 4.6. *Let $z \in \mathrm{PL}_+^<(J)$. Let $C_{\mathrm{PL}_+(J)}(z)$ be the centralizer of z in $\mathrm{PL}_+(J)$. Then*

$$\varphi_z : C_{\mathrm{PL}_+(J)}(z) \to \mathbb{R}_+, \quad g \mapsto g'(\eta^+),$$

is an injective group homomorphism.

Proof. Clearly φ_z is a group homomorphism. Suppose that there exist two elements $g_1, g_2 \in C_{\mathrm{PL}_+(J)}(z)$ such that $\varphi_z(g_1) = \varphi_z(g_2)$. Then $g_1^{-1}g_2$ has a slope 1 near η and is equal to the identity by the previous lemma. Therefore $g_1 = g_2$, which proves the injectivity of φ_z. $\square$

Lemma 4.7. *Let $y, z \in \mathrm{PL}_+^<(J)$, let $C_{\mathrm{PL}_+(J)}(y,z) = \{g \in \mathrm{PL}_+(J) \mid y^g = z\}$ be the set of all conjugators, and let λ be an interior point of J. Then the two maps $\varphi_{y,z}$ and $\psi_{y,z,\lambda}$ satisfy*

$$\begin{aligned} \varphi_{y,z} \colon C_{\mathrm{PL}_+(J)}(y,z) &\to \mathbb{R}_+, \quad g \mapsto g'(\eta^+), \\ \psi_{y,z,\lambda} \colon C_{\mathrm{PL}_+(J)}(y,z) &\to J, \qquad g \mapsto g(\lambda). \end{aligned}$$

(i) *$\varphi_{y,z}$ is an injective map.*

(ii) *There is a map $\rho_\lambda \colon J \to \mathbb{R}_+$ such that the diagram*

$$\begin{tikzcd} & J \arrow[d, "\rho_\lambda"] \\ C_{\mathrm{PL}_+(J)}(y,z) \arrow[ur, "\psi_{y,z,\lambda}"] \arrow[r, "\varphi_{y,z}"] & \mathbb{R}_+ \end{tikzcd}$$

commutes.

(iii) *$\psi_{y,z,\lambda}$ is injective.*

Proof. (i) is an immediate corollary of Lemma 4.6.

(ii) Without loss of generality we can assume that the initial slopes of y, z are the same (otherwise the set $C_{\mathrm{PL}_+(J)}(y,z)$ is obviously empty and any map will do). We define the map $\rho_\lambda \colon J \to \mathbb{R}_+$ by

$$\rho_\lambda(\mu) = \lim_{n\to\infty} \frac{y^n(\mu) - \eta}{z^n(\lambda) - \eta}.$$

The above limit exists, because the sequence stabilizes under the assumptions $y, z \in \mathrm{PL}_+^<(J)$ and $y'(\eta) = z'(\eta)$.

To prove that the diagram commutes we define $\mu = g(\lambda)$ and observe that $y^n(\mu) \xrightarrow[n\to\infty]{} \eta$ and $z^n(\lambda) \xrightarrow[n\to\infty]{} \eta$. By hypothesis $y(\mu) = g(z(\lambda))$ so that $g(z^n(\lambda)) = y^n(\mu)$, for every $n \in \mathbb{Z}$. Since g fixes η we have

$$g(t) = g'(\eta^+)(t - \eta) + \eta \text{ on a small interval } [\eta, \eta + \varepsilon],$$

where ε depends only on g. Let $N = N(g) \in \mathbb{N}$ be large enough, so that the numbers $y^N(\lambda), z^N(\lambda) \in (\eta, \eta + \varepsilon)$. This implies that

$$y^n(\mu) = g(z^n(\lambda)) = g'(\eta^+)(z^n(\lambda) - \eta) + \eta$$

for any $n \geq N$, and so

$$\varphi_{y,z}(g) = g'(\eta^+) = \frac{y^n(\mu) - \eta}{z^n(\lambda) - \eta} = \rho_\lambda(\psi_{y,z,\lambda}(g)).$$

(iii) Since $\varphi_{y,z} = \rho_\lambda \psi_{y,z,\lambda}$ is injective by part (i), $\psi_{y,z,\lambda}$ is injective as well. $\square$

Remark 4.8. Lemma 4.7 shows that for $y \in \mathrm{PL}_+^<(J)$ the graphs of the elements in the centralizer $C_{\mathrm{PL}_+^<(J)}(y)$ do not intersect; see Figure 4.

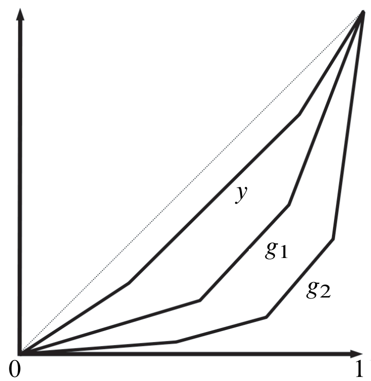

Figure 4. Two elements g_1, g_2 centralizing a map $y \in \mathrm{PL}_+^<(J)$.

The main tool of this section is the *stair algorithm*. This procedure constructs a conjugator (if it exists) with a given fixed initial slope. In order for y and z to be conjugate, they must have the same initial slope; by Lemma 4.2 this determines uniquely the first piece of a possible conjugator given the initial slope. Then we "walk up the first step of the stair" (Lemma 4.9): we identify y and z inside a rectangle next to the linearity box by taking a suitable conjugator. We then repeat and walk up more rectangles until we "reach the door" (represented by the final linearity box), and this happens when a rectangle that we are constructing crosses the final linearity box. This algorithm finishes in finitely many steps because the interval $J = [\eta, \zeta]$ is bounded. In other words, we will construct a "section" for the map $\varphi_{y,z}$ of Lemma 4.7. As a consequence we will also construct a "section" of the map $\psi_{y,z,\lambda}$.

Lemma 4.9. *Let $y, z \in \mathrm{PL}_+^<(J)$ and $g \in \mathrm{PL}_+(J)$ be functions such that $z = y^g$ and let $\alpha \in (\eta, \zeta)$. Then the functions y, z and the restriction of g to the interval (η, α) uniquely determines the restriction of g to the interval $(\eta, z^{-1}(\alpha))$.*

Proof. We can rewrite the equation $z = y^g$ as $g = y^{-1}gz$. The value of the right side of this equation at points inside the interval $(\eta, z^{-1}(\alpha))$ depends only on y, z and restriction of g to the interval (η, α). Therefore they determine uniquely the restriction of g to the interval $(\eta, z^{-1}(\alpha))$. □

Proposition 4.10. *Let $y, z \in \mathrm{PL}_+^<(J)$ and $g \in \mathrm{PL}_+(J)$ be functions such that $z = y^g$. Then the conjugator g is uniquely determined by its initial slope $g'(\eta)$.*

Proof. By Lemma 4.2, the graph of the conjugator g is linear in the box $[\eta, \eta + \varepsilon] \times [\eta, \eta + \varepsilon]$. Therefore the slope of $g'(\eta)$ uniquely determines the restriction of g to the interval (η, α), for some $\alpha \leq \zeta$. Applying the previous lemma several times we see that this also determines the restriction of g to the interval $(\eta, z^{-n}(\alpha))$ for any integer $n \geq 0$. However the function z is in $\mathrm{PL}_+^<(J)$, thus $\lim_{n\to\infty} z^{-n}(\alpha) = \zeta$, and so these restrictions determine the function g. □

Remark 4.11. Lemma 4.9 also holds for any (even non-piecewise linear) function on the interval J. The argument in the previous proposition gives that for any piecewise linear functions y and z in $\mathrm{PL}^+(J)$ and any initial slope there exists a unique conjugating function g which is linear in a neighborhood of the point η. Although this function g is piecewise linear on any interval (η, α) for any $\alpha < \zeta$, it may not be linear in a neighborhood of the point ζ and may not be piecewise linear on the entire interval J.

Using the final linearity box, it is very easy to algorithmically determine whether the function g is a piecewise linear function. It suffices to construct the restriction of g to the interval (η, γ) such that the point $(\gamma, g(\gamma))$ is inside the finial linearity box $[\beta, \zeta] \times [\beta, \zeta]$. It follows from Remark 4.3 that if there exists a conjugator, then it has to be linear in this box, thus we can determine the rest of the graph of g and then verify that it is indeed a conjugator.

Corollary 4.12. *Let $y, z \in \mathrm{PL}_+^<(J)$, let $[\eta, \alpha]$ be the initial linearity box and let q be a positive real number. There is an $r \in \mathbb{N}$ such that the unique candidate conjugator with initial slope $q < 1$ is given by*

$$g(t) = y^{-r} g_0 z^r(t) \quad \textit{for all } t \in [\eta, z^{-r}(\alpha)]$$

and is linear otherwise, where g_0 is any map in $\mathrm{PL}_+(J)$ which is linear in the initial box with $g_0'(\eta^+) = q$.

Lemma 4.13. *Let $y, z \in \mathrm{PL}_+^<(J)$, $g \in \mathrm{PL}_+(J)$ and $n \in \mathbb{N}$. Then $g^{-1}yg = z$ if and only if $g^{-1}y^n g = z^n$.*

Proof. The "only if" part is obvious. The "if" part follows from the injectivity of φ_x by Lemma 4.6 since $g^{-1}yg$ and z both centralize the element z^n and have the same initial slope. □

Corollary 4.14. *Let $y, z \in \mathrm{PL}_+^<(J)$, and let λ be in the interior of J. The map*

$$\psi_{y,z,\lambda} \colon C_{\mathrm{PL}_+(J)}(y, z) \to J, \quad g \mapsto g(\lambda),$$

admits a section, *i.e., if $\psi_{y,z,\lambda}(g) = \mu \in J$, then g is unique and can be constructed.*

Remark 4.15. Suppose that $y, z \in \mathrm{PL}_+^<(J) \cup \mathrm{PL}_+^>(J)$. Then in order to be conjugate, they have to be both in $\mathrm{PL}_+^<(J)$ or both in $\mathrm{PL}_+^>(J)$, because by Lemma 4.1 they must coincide in a small interval $[\eta, \alpha]$. Moreover, $g^{-1}yg = z$ if and only if $g^{-1}y^{-1}g = z^{-1}$, and so, up to working with y^{-1}, z^{-1}, we may reduce to studying the case where they are both in $\mathrm{PL}_+^<(J)$.

Remark 4.16. The stair algorithm for $\mathrm{PL}_+^<(J)$ can be reversed. This is to say that, given q a positive real number, we can determine whether or not there is a conjugator g with final slope $g'(\zeta^-) = q$. The proof is the same: we simply start constructing g from the final linearity box.

Remark 4.17. We mention here that all results of Sections 4.1 and 4.2 can be extended to the case of $\mathrm{PL}_{S,G}(J)$. All the statements can be reformulated and proved by replacing every appearance of $\mathrm{PL}_+(J)$ and $\mathrm{PL}_+^<(J)$ with the symbols $\mathrm{PL}_{S,G}(J)$ and $\mathrm{PL}_{S,G}^<(J)$, respectively.

The stair algorithm gives a practical way to find conjugators if they exist and we have chosen a possible initial slope. By analyzing the stair algorithm we can see that if two elements are in $\mathrm{PL}_{S,G}^<(J)$ and they are conjugate through an element with initial slope in G, then the conjugator is an element of $\mathrm{PL}_{S,G}(J)$.

Corollary 4.18. *Let $y, z \in \mathrm{PL}_{S,G}^<(J)$, $g \in \mathrm{PL}_+(J)$ such that $y^g = z$ and $g'(\eta^+) \in G$. Then $g \in \mathrm{PL}_{S,G}(J)$.*

We conclude this section with a lemma which will be used later on.

Lemma 4.19. *Let $\tau, \mu \in J$, $h \in \mathrm{PL}_+(J)$. Then:*

(i) *The limit $\varphi_\pm = \lim_{n\to\infty} h^{\pm n}(\tau)$ exists and $h(\varphi_\pm) = \varphi_\pm$.*

(ii) *We can determine whether or not there is an $n \in \mathbb{Z}$ such that $h^n(\tau) = \mu$.*

Proof. The two sequences $\{h^{\pm n}(\tau)\}_{n\in\mathbb{N}}$ are strictly monotone and they have a limit $\lim_{n\to\infty} h^{\pm n}(\tau) = \varphi_\pm \in J$. Thus, by continuity of h,

$$\varphi_\pm = \lim_{n\to\infty} h^{n+1}(\tau) = \lim_{n\to\infty} h(h^n(\tau)) = h(\varphi_\pm).$$

Hence we have $\{h^n(\tau)\}_{n\in\mathbb{Z}} \subseteq (\varphi_-, \varphi_+)$, and $\varphi_\pm$ are the closest intersections of the graph of h with the diagonal on the point τ. It is possible to compute φ_+, φ_- directly, since the graph of h is piecewise linear. As a first check, we must see if μ is between the points φ_- and φ_+. Then, since the two sequences $\{h^{\pm n}(\tau)\}_{n\in\mathbb{N}}$ are monotone, after a finite number of steps we find $n_1, n_2 \in \mathbb{Z}$ such that $h^{-n_1}(\tau) < \mu < h^{n_2}(\tau)$, which means that either there is an integer $-n_1 \le n \le n_2$ with $h^n(\tau) = \mu$ or not, but this is a finite check. $\square$

4.3. The stair algorithm for $\mathrm{PL}_{S,G}^0(J)$. In Section 6 it will be proved that we can reduce our study to y and z such that $\mathrm{Fix}(y) = \mathrm{Fix}(z)$. Recall that an intersection point α of the graph of z with the diagonal may not be a point in S (for instance, a dyadic rational in the case of $\mathrm{PL}_2(I)$; see again Figure 1). If this is the case, then α cannot be a breakpoint for y, z and more importantly for g. Recall that, by Definition 2.4, a function z is in $\mathrm{PL}_{S,G}^0(J)$ if $\mathrm{Fix}(z)$ does not contain any point of S, except for the endpoints of J.

Proposition 4.20. *Let $y, z \in \mathrm{PL}^0_{S,G}(J)$ and q be a fixed element in G. Suppose that* $\mathrm{Fix}(y) = \mathrm{Fix}(z)$. *We can decide whether or not there is a map $g \in \mathrm{PL}_{S,G}(J)$ with initial slope $g'(\eta^+) = q$ such that y is conjugate to z through g. If g exists it is unique. Moreover, there is an algorithm for constructing this conjugator.*

Proof. This proof will be essentially the same as the previous stair algorithm with a few more remarks. We assume therefore that such a conjugator exists and construct it. Let $\mathrm{Fix}(y) = \mathrm{Fix}(z) = \{\eta = \alpha_0 < \alpha_1 < \cdots < \alpha_s < \alpha_{s+1} = \zeta\}$. We restrict our attention to $\mathrm{PL}_{S,G}([\alpha_i, \alpha_{i+1}])$ (as defined in Remark 2.1), for each $i = 0, \ldots, s$. If y and z are conjugate on $[\alpha_i, \alpha_{i+1}]$, then we can speak of linearity boxes: let $\Gamma_i := [\alpha_i, \gamma_i] \times [\alpha_i, \gamma_i]$ be the initial linearity box and $\Delta_i := [\delta_i, \alpha_{i+1}] \times [\delta_i, \alpha_{i+1}]$ the final one for $\mathrm{PL}_{S,G}([\alpha_i, \alpha_{i+1}])$. Now what is left to do is to repeat the procedure of the stair algorithm for elements in $\mathrm{PL}^<_{S,G}(U)$ for some interval U. We construct a conjugator g on $[\alpha_0, \alpha_1]$ by means of the stair algorithm. We observe that α_1 is not a breakpoint, hence $g'(\alpha_1^+) = g'(\alpha_1^-)$. Thus we are given an initial slope for g in $[\alpha_1, \alpha_2]$, then we can repeat the same procedure and repeat the stair algorithm on $[\alpha_1, \alpha_2]$. We keep repeating the same procedure until we reach $\alpha_{s+1} = \zeta$. Then we check whether the g we have found conjugates y to z. Finally, we observe that in each square $[\alpha_i, \alpha_{i+1}] \times [\alpha_i, \alpha_{i+1}]$ the determined function is unique, since we can apply Lemma 4.7 to it. □

An immediate consequence of the previous result is the following lemma.

Lemma 4.21. *Suppose $z \in \mathrm{PL}^0_{S,G}(J)$ and $g \in \mathrm{PL}_{S,G}(J)$ are such that $g'(\eta^+) = 1$ and $(g^{-1}zg)(t) = z(t)$ for all $t \in J$. Then $g(t) = t$ for all $t \in J$.*

Remark 4.22. It is possible to run a backwards version of the stair algorithm also for $\mathrm{PL}^0_{S,G}(J)$. Moreover, in this case it also possible to run a midpoint version of it: if we are given a point λ in the interior of J fixed by y and z and $q \in G$, then, by running the stair algorithm at the left and at the right of λ we determine whether or not there is a conjugator g such that $g'(\lambda) = q$.

Notation 4.23. *We recall that, given $y \in \mathrm{PL}_{S,G}(J)$, we denote the centralizer of y in $\mathrm{PL}_{S,G}(J)$ by*

$$C_{\mathrm{PL}_{S,G}(J)}(y) = \{g \in \mathrm{PL}_{S,G}(J) \mid y^g = y\}.$$

From Lemma 4.21 and Remark 4.22 we have:

Corollary 4.24. *Let $y, z \in \mathrm{PL}^0_{S,G}(J)$ such that* $\mathrm{Fix}(y) = \mathrm{Fix}(z)$ *and let*

$$C_{\mathrm{PL}_{S,G}(J)}(y, z) = \{g \in \mathrm{PL}_{S,G}(J) \mid y^g = z\}$$

be the set of all conjugators. For any $\tau \in \mathrm{Fix}(y)$ define the map

$$\varphi_{y,z,\tau} \colon C_{\mathrm{PL}_{S,G}(J)}(y, z) \to \mathbb{R}_+, \quad g \mapsto g'(\tau),$$

where if τ is an endpoint of J we take only a one-sided derivative. Then

(i) *$\varphi_{y,z,\tau}$ is an injective map.*

(i) *If $\varphi_{y,z,\tau}$ admits a section, i.e., if there is a partially defined map $\mathbb{R}_+ \to C_{\mathrm{PL}_{S,G}(J)}(y,z)$, $\mu \to g_\mu$, with $\varphi_{y,z,\tau}(g_\mu) = \mu$, then g_μ is unique and can be constructed.*

Proposition 4.25. *Let $y, z \in \mathrm{PL}^0_{S,G}(J)$ such that* $\mathrm{Fix}(y) = \mathrm{Fix}(z)$ *and let λ be in the interior of J such that $y(\lambda) \neq \lambda$. Define*

$$\psi_{y,z,\lambda} \colon C_{\mathrm{PL}_{S,G}(J)}(y,z) \to J, \quad g \mapsto g(\lambda).$$

Suppose $y^n(\lambda) \xrightarrow[n\to\infty]{} \tau$. Then:

(i) *There is a map $\rho_\lambda \colon J \to \mathbb{R}_+$ such that the diagram commutes:*

$$\begin{tikzcd} & J \arrow[d, "\rho_\lambda"] \\ C_{\mathrm{PL}_{S,G}(J)}(y,z) \arrow[ur, "\psi_{y,z,\lambda}"] \arrow[r, "\varphi_{y,z,\tau}"] & \mathbb{R}_+ \end{tikzcd}$$

commutes.

(ii) *$\psi_{y,z,\lambda}$ is injective.*

(iii) *If $\psi_{y,z,\lambda}$ admits a section, that is, if there is a partially defined map $J \to C_{\mathrm{PL}_{S,G}(I)}(y,z)$, $\mu \to g_\mu$, with $\psi_{y,z,\lambda}(g_\mu) = \mu$, then g_μ is unique and can be constructed.*

Proof. Let $\mathrm{Fix}(y) = \mathrm{Fix}(z) = \{\eta = \mu_0 < \mu_1 < \cdots < \mu_k < \mu_{k+1} = \zeta\}$ and suppose $\mu_i < \lambda < \mu_{i+1}$ for some i. We define the partial map $\rho_\lambda \colon J \to \mathbb{R}_+$ by

$$\rho_\lambda(\mu) = \begin{cases} \lim_{n\to\infty} \frac{y^n(\mu)-\tau}{z^n(\lambda)-\tau} & \text{if } \mu \in [\mu_i, \mu_{i+1}], \\ 1 & \text{otherwise.} \end{cases}$$

Since $\mathrm{Fix}(y) = \mathrm{Fix}(z)$, $z^n(\lambda) \xrightarrow[n\to\infty]{} \tau$ and τ is fixed by g. Thus if $\mu = g(\lambda)$, then $y^n(\mu) = g(z^n(\lambda)) \xrightarrow[n\to\infty]{} \tau$. With this definition, the proof follows closely that of Lemma 4.7 (ii), Proposition 4.14 and by applying Corollary 4.24 and Remark 4.22. $\square$

Geometrically this says that if $y \in \mathrm{PL}^0_{S,G}$, then the graphs of the centralizers of y inside $\mathrm{PL}^0_{S,G}$ intersect only at the fixed points of y (see Figure 5), which justifies the terminology "almost one-bump" functions.

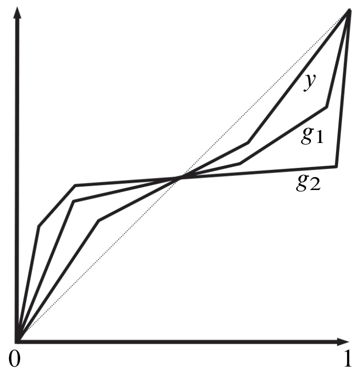

Figure 5. Two centralizers g_1, g_2 of a function $y \in \mathrm{PL}^0_{S,G}(J)$.

5. Centralizers in subgroups of $\mathrm{PL}_+(I)$

In this section we use the stair algorithm to derive several results about centralizers of elements in $\mathrm{PL}_+(J)$ and $\mathrm{PL}_{S,G}(J)$. Although most of these results are already known, our approach is new. The main result of Section 5.1 was first obtained by Brin and Squier [5]. We will provide a new proof, which generalizes to the case $\mathrm{PL}_{S,G}(J)$. The tools of Section 5.1 and the results and proofs in the remaining sections are new and constructive (except for the results on the special case of Thompson's group F), giving a procedure to solve the simultaneous conjugacy problem. We start by giving an easy application of the stair algorithm before getting into the conjugacy problem.

5.1. Centralizers of elements in $\mathrm{PL}^0_+(I)$ and $\mathrm{PL}^0_{S,G}(I)$. The stair algorithm from Section 4 does not tell us anything about the image of the homomorphism $\varphi_z\colon C_{\mathrm{PL}_+(J)}(z) \to \mathbb{R}_+$. In this section we will show that if z is in $\mathrm{PL}^<_+(J)$, then the image of φ_z is a discrete subgroup of $\mathbb{R}^+$, thus the centralizer of z is an infinite cyclic group. Let $A_z = \varphi_z(C_{\mathrm{PL}_+(J)}(z)) \subset \mathbb{R}_+$ be the set of all possible initial slopes of centralizers. The set A_z is infinite since $\langle z\rangle \subseteq C_{\mathrm{PL}_+(J)}(z)$. Using the injectivity of φ_z, we can define ψ_z to be the inverse of the function φ_z on A_z,

$$\psi_z\colon A_z \to C_{\mathrm{PL}_+(J)}(z), \quad \alpha \mapsto g_\alpha,$$

which is clearly a group isomorphism. In the previous section an algorithm is provided to determine whether $c \in \mathbb{R}$ is an element in A_z and the piecewise linear function $\psi_z(c)$, which sends an initial slope α to its associated conjugating function g_α, is constructed if it is defined.

The main result of this section is the following.

Theorem 5.1. *Let $J \subseteq [0,1]$ be a closed interval and let* $\mathrm{id} \neq z \in \mathrm{PL}^<_+(J)$. *Then $C_{\mathrm{PL}_+(J)}(z)$ is isomorphic to $\mathbb{Z}$. Moreover, there is an algorithm that constructs a generator w of this group and w is a root of z.*

We remark that Theorem 5.1 had originally been proved by Brin and Squier (Theorem 5.5 in [5]). The connection between our proof and the one of Brin and Squier was described in a paper by the second author [13]. We also observe that Altinel and Muranov gave another proof of this result using different methods (Lemma 4.2 in [1]). The tools that we will use in our version of the proof that we are about to give are relevant for Lemma 5.4, which is central in our construction of candidate conjugators.

Proof of Theorem 5.1. By the discussion above we have that the group

$$A_z = \{g'(\eta^+) \mid g \in C_{\mathrm{PL}_+(J)}(z)\}$$

is isomorphic to $C_{\mathrm{PL}_+(J)}(z)$. We start by assuming that $z \in \mathrm{PL}_+^<(J)$ and we want to prove that A_z is discrete, since any discrete subgroup of $\mathbb{R}_+$ is isomorphic to $\mathbb{Z}$. The argument below not only proves that A_z is discrete but also provides an algorithm to find a generator of this group.

The proof relies on the following key lemmas.

Lemma 5.2. *Let r be a positive integer such that $z^r(\beta) < \alpha$, where $[\eta,\alpha]^2$ and $[\beta,\zeta]^2$ are initial and final linearity boxes for the element z. Either z^r is not linear on the interval $[\beta, z^{-r}(\alpha)]$ or z^{2r} is not linear on the interval $[\beta, z^{-2r}(\alpha)]$.*

Proof. Assume that both z^r and z^{2r} are linear on these intervals and denote their slopes by s_1 and s_2, respectively. Using the linearity boxes for z it can be seen that z^r is linear on $[\eta,\alpha]$ with slope a^r, where $a = z'(\eta^+)$, and z^r is linear on $[z^{-r}(\beta),\zeta]$ with slope b^r, where $b = z'(\zeta^-)$. Since $z^{2r} = z^r \circ z^r$ we get that z^{2r} is linear on $[\beta, z^{-r}(\alpha)]$ with slope $a^r s_1$ and is also linear on $[z^{-r}(\beta), z^{-2r}(\alpha)]$ with slope $b^r s_1$. Thus we have

$$a^r s_1 = s_2 = b^r s_1.$$

However, this is a contradiction because $a < 1 < b$ and $s_1 \neq 0$. □

Lemma 5.3. *Let s be a positive integer such that $z^s(\beta) < \alpha$ and z^s is not linear on the interval $[\beta, z^{-s}(\alpha)]$. Then there exists $\varepsilon > 0$ such that there are only finitely many $g \in C_{\mathrm{PL}_+(J)}(z)$ with $1 \geq g'(\eta^+) \geq 1-\varepsilon$ and $1 \leq g'(\zeta^-) \leq 1+\varepsilon$, and they can be constructed.*

Proof. Since z^s is not linear on $[\beta, z^{-s}(\alpha)]$, there exists $\varepsilon > 0$ such that z^s has breakpoints on $I_\varepsilon = [(\beta+\varepsilon\zeta)/(1+\varepsilon), z^{-s}(\varepsilon\eta + (1-\varepsilon)\alpha)]$. Let $\{\mu_1 < \cdots < \mu_k\}$ be the breakpoints of z^s in this interval.

For any $g \in C_{\mathrm{PL}_+(J)}(z)$ with $1 \geq g'(\eta^+) \geq 1-\varepsilon$ and $1 \leq g'(\zeta^-) \leq 1+\varepsilon$ the linearity boxes give us that g is linear on $[\eta,\alpha]$ and $[(\beta+\varepsilon\zeta)/(1+\varepsilon),\zeta]$, and if $\varepsilon > 0$ is chosen small enough, the sets I_ε and $g^{-1}(I_\varepsilon)$ are not disjoint. By construction, the breakpoints of $g \circ z^s$ on I_ε are $\{\mu_1 < \cdots < \mu_k\}$ and the breakpoints of $z^s \circ g$ on $g^{-1}(I_\varepsilon)$ are $\{g^{-1}(\mu_1) < \cdots < g^{-1}(\mu_k)\}$. However for all but finitely many choices for $g'(\zeta^-)$ the sets $\{\mu_1 < \dots < \mu_k\}$ and $\{g^{-1}(\mu_1) < \dots < g^{-1}(\mu_k)\}$ are disjoint. Therefore $g \circ z^s \neq z^s \circ g$, which contradicts the assumption that $g \in C_{\mathrm{PL}_+(J)}(z)$. Let $V \subseteq [1-\varepsilon, 1]$ be the finite set of admissible final slopes $g'(\zeta^-)$ found before. We run the backwards stair algorithm on each slope in V and determine which element centralizes z. □

Lemmas 5.2 and 5.3 immediately give that A_z is discrete, which completes the proof of the first part of Theorem 5.1. To construct a generator v for $C_{\mathrm{PL}_+(J)}(z)$, we observe that $v^k = z$ for some integer k, hence v is a root of z and so $v'(\zeta^-) \in [z'(\zeta^-), 1]$. Let $V \subseteq [1-\varepsilon, 1]$ be the set of admissible final slopes for $g \in C_{\mathrm{PL}_+(J)}(z)$ given by Lemma 5.3. We run the backwards stair algorithm on the finite set

$$([z'(\zeta^-), 1-\varepsilon] \cup V) \cap \{\sqrt[m]{z'(\zeta^-)}\}_{m\in\mathbb{Z}}$$

of admissible final slopes and pick the centralizing element w with initial slope closest to 1. By injectivity of the map φ_z in Lemma 4.6, the map w is a generator for $C_{\mathrm{PL}_+(J)}(z)$. □

We finish with a generalization of Lemma 5.3: The following result is central in solving the simultaneous conjugacy problem in $\mathrm{PL}_+(I)$ (together with the stair algorithm (Corollary 4.12) and Lemma 4.2). It provides an algorithm for restricting the initial slopes of the conjugators. Not only this allows us to effectively solve the conjugacy problem in $\mathrm{PL}_+(I)$ but also to extend this solution to the groups $\mathrm{PL}_{S,G}(J)$, provided that the additive group S satisfies some mild computational requirements.

Lemma 5.4. *Let $J = [\eta, \zeta]$ be a closed interval with endpoints in S and let $c > 1$. Then the set*

$$N = \{g \mid g \in C_{\mathrm{PL}_+(J)}(y, z),\ g'(\eta) \in [c^{-1}, c],\ g'(\zeta) \in [c^{-1}, c]\}$$

is finite and can be constructed.

Proof. Let $\alpha' = \eta + c^{-1}(\alpha - \eta)$ and $\beta' = \zeta - c^{-1}(\zeta - \beta)$. Using Lemma 4.2 we can see that $g \in N$ is linear on the intervals $[\eta, \alpha']$ and $[\beta', \zeta]$. By Lemma 5.2 there exists a sufficiently large integer s such that $z^{-s}(\alpha') \geq \beta'$ and z^s is not linear on the interval $[\beta', z^{-s}(\alpha')]$. Let μ_i denote the set of breakpoints of z^s in this interval. The function gz^s has μ_i as breakpoints since g is linear in the first linearity box. If g is a conjugator we have $gz^s = y^s g$, therefore the μ_i are breakpoints of $y^s g$, which means that $g(\mu_i)$ are breakpoints of y^s. This condition leaves finitely many possibilities for the final slope of g, which shows that the set $\{g'(\eta) \mid g \in N\}$ is finite. For each of the slopes in $\{g'(\eta) \mid g \in N\}$ we can construct a candidate conjugator and test it. □

Theorem 5.5. *Let $J \subseteq [0, 1]$ be a closed interval with endpoints in S and let* $\mathrm{id} \neq z \in \mathrm{PL}^0_{S,G}(J)$*. Then $C_{\mathrm{PL}_{S,G}(J)}(z)$ is isomorphic to $\mathbb{Z}$. Moreover, there is an algorithm that constructs a generator w of this group and w is a root of z.*

Proof. Let $\partial\,\mathrm{Fix}(z) = \{\eta < \gamma < \cdots < \zeta\}$ and consider the injective homomorphism λ defined by sending each element of $C_{\mathrm{PL}_{S,G}(J)}(z)$ to its restriction in the interval $[\eta, \gamma]$. By construction, the image of λ is contained in $C_{\mathrm{PL}_+(J)}(z) \cong \mathbb{Z} = \langle w \rangle$, hence $C_{\mathrm{PL}_{S,G}(J)}(z)$ is also infinite cyclic and contains z. By Lemma 5.3 there are only finitely many admissible initial slopes to be tested, so to find a generator we follow the same procedure as in the proof of Theorem 5.1. □

5.2. Centralizers of elements in $\mathbf{PL}_+(J)$ and $\mathbf{PL}_{S,G}(I)$. The results about centralizers of elements in $\mathrm{PL}^0_+(J)$ and $\mathrm{PL}^0_{S,G}(I)$ can be extended to arbitrary elements by observing that any centralizer of y need to fix all points in $\partial_S\,\mathrm{Fix}(y)$.

Theorem 5.6. *Let $J = [\eta, \zeta] \subseteq [0, 1]$ be a closed interval and $z \in \mathrm{PL}_+(J)$. Then:*

(i) *$C_{\mathrm{PL}_+(I)}(z)$ is isomorphic to a direct product of copies of $\mathbb{Z}$ and $\mathrm{PL}_+(J_i)$ for some suitable intervals $J_i \subseteq I$.*

(ii) *For every positive integer n we can decide whether or not $\sqrt[n]{z}$ exists. The map z has only a finite number of roots and every root is constructible, i.e., there is an algorithm to compute it.*

Proof. (i) Consider the conjugacy problem with $y = z$ and let

$$\partial\,\mathrm{Fix}(z) = \{\eta = \alpha_0 < \alpha_1 < \cdots < \alpha_s < \alpha_{s+1} = \zeta\}.$$

Any centralizer g of z must fix the set $\partial\,\mathrm{Fix}(z)$ and thus each of the α_i's. Therefore we compute the centralizer of the restrictions z_i of z in each of the subgroups $\mathrm{PL}_+(J_i)$, where $J_i = [\alpha_i, \alpha_{i+1}]$ and so we can assume that $z_i \in \mathrm{PL}^<_+(J_i)$ or $z_i \in \mathrm{PL}^<_+(J_i)$ or $z_i = \mathrm{id}$. If $z_i = \mathrm{id}$, then it is immediate that $C_{\mathrm{PL}_+(J_i)}(z_i) = \mathrm{PL}_+(J_i)$. Suppose $z \neq \mathrm{id}$ on $[0, 1]$, then, by Theorem 5.1, we have $C_{\mathrm{PL}_+(J_i)}(z_i) \cong \mathbb{Z}$.

(ii) Again we suppose that $\partial\,\mathrm{Fix}(z) = \{0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r < \alpha_{s+1} = 1\}$ and we restrict to an interval $[\alpha_i, \alpha_{i+1}]$. Let $m = z'(0^+)$. We want to determine whether or not there is an element $g \in \mathrm{PL}_+([\alpha_i, \alpha_{i+1}])$ such that $g^{-1}zg = z$ and $g'(0^+) = \sqrt[n]{m}$. Suppose that there is such a g, then $g^{-n}zg^n = z$ and $(g^n)'(0^+) = m$. By injectivity of the map φ_{z,z,α_i} (Corollary 4.24), we have that $g^n = z$. Conversely, if we have h such that $h^n = z$, then $h'(0^+) = \sqrt[n]{m}$. But $h^{-1}zh = h^{-1}h^nh = h^n = z$. Thus an element h is a n-th root of z if and only if it is the solution the "differential type" equation with a given initial condition

$$\begin{cases} h^{-1}zh = z, \\ h'(0^+) = \sqrt[n]{m}. \end{cases}$$

So we can decide, by solving the equivalent conjugacy problem with a given initial slope, whether or not there is a n-th root. Moreover, if the n-th root of g exists, it is computable by Proposition 4.20 and unique by Corollary 4.24. Moreover, only finitely many roots are possible: the sequence $\sqrt[n]{z'(\eta^+)}$ converges to 1, but

Lemma 5.3 implies that only finitely many elements of this sequence can be candidate slopes for a root. □

Proposition 5.7. *Let $x \in \mathrm{PL}_+(J)$ and α be a point in J. If $g \in C_{\mathrm{PL}_+(I)}(x)$ and $g(\alpha) = \alpha$, then the functions*

$$g_{<,\alpha} = \begin{cases} t & \text{if } t \le \alpha, \\ g(t) & \text{if } t \ge \alpha, \end{cases} \qquad g_{>,\alpha} = \begin{cases} g(t) & \text{if } t \le \alpha, \\ t & \text{if } t \ge \alpha, \end{cases}$$

are also in the centralizer $C_{\mathrm{PL}_+(J)}(x)$ and g is equal to the product of $g_{<,\alpha}$ and $g_{>,\alpha}$.

Proof. If $x(\alpha) = \alpha$, this follows from Theorem 5.6. Assume now that $x(\alpha) \neq \alpha$ and let $[c, d]$ be the largest interval containing α on which x is a one-bump function. Since g centralizes x, the points c and d are fixed by both x and g and, in particular, the proposition follows for the maps $g_{<,c}$ and $g_{>,c}$. The conclusion will then follow if we can prove that $g_{<,\alpha} = g_{<,c}$ and $g_{>,\alpha} = g_{>,c}$. The restriction $g|_{[c,d]}$ centralizes $x|_{[c,d]}$ and so, by Theorem 5.1, we have $g|_{[c,d]} = (\sqrt[m]{x})^k$ for suitable integers m, k. Since $x(\alpha) \neq \alpha$, it follows that $k = 0$ and $g|_{[c,d]} = \mathrm{id}$. It is now straightforward to verify that $g_{<,\alpha} = g_{<,c}$ and $g_{>,\alpha} = g_{>,c}$. □

We will see that solving the simultaneous conjugacy problem is equivalent to detect whether or not a given candidate function lies in the intersection of finitely many centralizers. The next results shows that the intersection of centralizers has a structure similar to a single centralizer, which allows us to modify the solution of the conjugacy problem in $\mathrm{PL}_+(J)$ and $\mathrm{PL}_{S,G}(J)$, and to verify that it is possible to find a conjugator in the intersection of several centralizers.

Proposition 5.8. *Let $x_1, \dots, x_k \in \mathrm{PL}_+(J)$ and define $C := C_{\mathrm{PL}_+(J)}(x_1) \cap \dots \cap C_{\mathrm{PL}_+(J)}(x_k)$. If the interval J is divided by the points in the union $\partial\,\mathrm{Fix}(x_1) \cup \dots \cup \partial\,\mathrm{Fix}(x_k)$ into intervals J_i, then*

$$C = C_{J_1} \cdot C_{J_2} \cdots C_{J_r},$$

where $C_{J_i} := \{f \in C \mid f(t) = t \text{ for all } t \notin J_i\} = C \cap \mathrm{PL}_+(J_i)$. Moreover, each C_{J_i} is isomorphic to either $\mathbb{Z}$ or $\mathrm{PL}_+(J_i)$, or is the trivial group.

Proof. The set $\partial\,\mathrm{Fix}(x_i)$ is fixed by all elements in $C_{\mathrm{PL}_+(J)}(x_i)$. Therefore all elements in C fix the endpoints of the intervals J_i, since, for $\alpha \in \bigcup \partial\,\mathrm{Fix}(x_i)$ and any $g \in C$, the function $g_{<,\alpha}$ and $g_{>,\alpha}$ are in C by Proposition 5.7. Any element $z \in C$ can be written as the product $z_1 \dots z_r$, where $z_i \in \mathrm{PL}_+(J)$ is trivial outside of J_i and $z_i|_{J_i} \in C_{\mathrm{PL}_+(J_i)}(x_n|_{J_i})$ for all $n = 1, \dots, r$. Hence $z_i \in C_{J_i}$. □

Corollary 5.9. *The intersection of any number $k \ge 2$ centralizers of elements $x_1, \dots, x_k$ in $\mathrm{PL}_+(J)$ is equal to the intersection of centralizers of two elements $w_1, w_2 \in \mathrm{PL}_+(J)$ which are not necessarily part of the initial set $\{x_1, \dots, x_k\}$.*

Proof. Let $C = C_{\mathrm{PL}_+(I)}(x_1) \cap \cdots \cap C_{\mathrm{PL}_+(I)}(x_k)$ be the intersection of $k \geq 2$ centralizers of elements of $\mathrm{PL}_+(J)$. By the previous proposition we have $I = J_1 \cup \cdots \cup J_r$ and $C = C_{J_1} \cdots C_{J_r}$. We want to define $w_1, w_2 \in \mathrm{PL}_+(I)$ such that $C = C_{\mathrm{PL}_+(I)}(w_1) \cap C_{\mathrm{PL}_+(I)}(w_2)$. We define them on each interval $J_i := [\alpha_i, \alpha_{i+1}]$, depending on C_{J_i}. *Case* 1: If $C_{J_i} = \mathrm{id}$, then we define w_1, w_2 to be any two elements in $\mathrm{PL}_+^<(J_i)$ so that one is not a power of the other. *Case* 2: If $C_{J_i} \cong \langle x \rangle$ for some $\mathrm{id} \neq x \in \mathrm{PL}_+(J_i)$, then we define $w_1 = w_2 = x$. *Case* 3: If $C_{J_i} = \mathrm{PL}_+(J_i)$, then we define $w_1 = w_2 = \mathrm{id}$. □

Using Theorem 5.5 one can easily generalize the results in the previous section to the groups $\mathrm{PL}_{S,G}(J)$.

Theorem 5.10. *Let $J = [\eta, \zeta] \subseteq [0, 1]$ be a closed interval with endpoints in S and $z \in \mathrm{PL}_{S,G}(J)$. Then:*

(i) *$C_{\mathrm{PL}_{S,G}(I)}(z)$ is isomorphic to a direct product of copies of the group $\mathbb{Z}$'s and $\mathrm{PL}_{S,G}(J_i)$'s for some suitable intervals $J_i \subseteq I$.*

(ii) *For every positive integer n we can decide whether or not $\sqrt[n]{z}$ exists. The map z has only a finite number of roots and every root is constructible, i.e., there is an algorithm to compute it.*

Proof. (i) We consider the conjugacy problem with $y = z$ and let

$$\partial_S \operatorname{Fix}(z) = \{\eta = \alpha_0 < \alpha_1 < \cdots < \alpha_s < \alpha_{s+1} = \zeta\}.$$

Any centralizer g of z must fix $\partial_S \operatorname{Fix}(z)$ pointwise. We thus compute the centralizer of the restrictions z_i of z in each of the groups $\mathrm{PL}_{S,G}(J_i)$, where $J_i = [\alpha_i, \alpha_{i+1}]$, and assume that $z_i \in \mathrm{PL}^0_{S,G}(J_i)$ or $z = \mathrm{id}$. The rest of the proof follows as in Theorem 5.6 (i) by means of Theorem 5.5.

(ii) This is a consequence of Theorem 5.6 (ii). □

Knowing the structure of a centralizer in $\mathrm{PL}_{S,G}(I)$ allows us to extend the results about intersections of centralizers.

Proposition 5.11. *Let $J = [\eta, \zeta] \subseteq [0, 1]$ be a closed interval with endpoints in S, let $z_1, \ldots, z_k \in \mathrm{PL}_{S,G}(J)$ and define the subgroup $C := C_{\mathrm{PL}_{S,G}(I)}(z_1) \cap \cdots \cap C_{\mathrm{PL}_{S,G}(I)}(z_k)$. If the interval J is divided by the points in the union $\partial_S \operatorname{Fix}(z_1) \cup \cdots \cup \partial_S \operatorname{Fix}(z_k)$ into intervals J_i, then*

$$C = C_{J_1} \cdot C_{J_2} \cdots C_{J_r},$$

where $C_{J_i} := \{f \in C \mid f(t) = t \text{ for all } t \notin J_i\} = C \cap \mathrm{PL}_{S,G}(J_i)$. Moreover, each C_{J_i} is isomorphic to either $\mathbb{Z}$ or $\mathrm{PL}_{S,G}(J_i)$, or to the trivial group.

Corollary 5.12. *The intersection of any number $k \geq 2$ of centralizers of elements $x_1, \ldots, x_k$ in $\mathrm{PL}_{S,G}(J)$ is equal to the intersection of centralizers of two elements $w_1, w_2 \in \mathrm{PL}_{S,G}(J)$ which are not necessarily part of the initial set $\{x_1, \ldots, x_k\}$.*

Question 5.13. Corollary 5.12 shows that any intersection of $k \geq 2$ centralizers of elements $x_1, \dots, x_k$ in $\mathrm{PL}_{S,G}(J)$ can be expressed as the intersection $C_{\mathrm{PL}_{S,G}(J)}(w_1) \cap C_{\mathrm{PL}_{S,G}(J)}(w_2)$ for two suitable elements $w_1, w_2 \in \mathrm{PL}_{S,G}(J)$. Is it possible to construct the two elements w_1, w_2 inside the subgroup $\langle x_1, \dots, x_k \rangle$?

The groups $\mathrm{PL}_{S,G}(J_i)$ may not be isomorphic to each other (see Remark 9.5). However, in the special case of $S = \mathbb{Z}[\frac{1}{2}]$ it is true that $\mathrm{PL}_{S,G}(J_i) \cong F$ for all i (see Remark 2.2). This simplifies the statement of Theorem 5.10 in the case of Thompson's group F. Also the proof can be simplified because one can use the discreteness of the group G instead of Lemmas 5.2 and 5.3, and Theorem 5.5. As we have already mentioned, this result is well known and was first proved by Guba and Sapir [10] using different techniques.

Theorem 5.14. *Let $z \in F \cong \mathrm{PL}_2(I)$. Then:*

(i) *Its centralizer is $C_F(z) \cong F^m \times \mathbb{Z}^n$ for some positive integers m, n such that $0 \leq m \leq n + 1$ (see Figure 6).*

(ii) *If $z \neq \mathrm{id}$, the function z has only a finite number of roots and every root is constructible, i.e., there is an algorithm to compute it.*

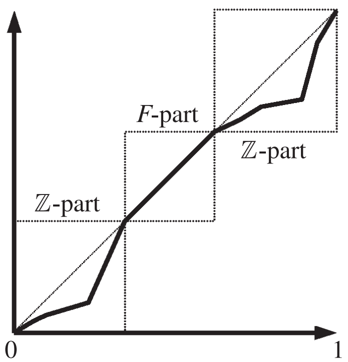

Figure 6. The structure of centralizers in F.

6. Moving fixed points

In this section we describe step 1 of the outline in Section 2.2. If two maps y, z are conjugate via g, then $g(\mathrm{Fix}(y)) = \mathrm{Fix}(z)$. Thus, moving fixed points is an intermediate step towards the conjugacy problem. We begin our proofs for the easier case of $\mathrm{PL}_+(J)$ and then move on to study the case of the groups $\mathrm{PL}_{S,G}(J)$.

6.1. Moving fixed points in $\mathrm{PL}_+(J)$. This case is the easiest one – essentially, in the case of $\mathrm{PL}_+(J)$, the only necessary thing to check is whether $\mathrm{Fix}(y)$ and $\mathrm{Fix}(z)$ have the same number and "type" of components and whether they have the "same order"[6]. We state without proof the following results.

Theorem 6.1. *Let $y_1 < y_2 < \cdots < y_n$ and $z_1 < z_2 < \cdots < z_n$ be points in the interval J. Then there exists a $g \in \mathrm{PL}_+(J)$ such that $g(y_i) = z_i$ for all $i = 1, \ldots, n$.*

Theorem 6.2. *Let $y, z \in \mathrm{PL}_+(J)$. There is an algorithm which constructs an element $g \in \mathrm{PL}_+(J)$ such that $g(\mathrm{Fix}(y)) = \mathrm{Fix}(g^{-1}yg) = \mathrm{Fix}(z)$, or shows that such an element does not exist.*

6.2. Moving fixed points in $\mathrm{PL}_{S,G}(J)$. The main difference between the groups $\mathrm{PL}_{S,G}(J)$ and $\mathrm{PL}_+(J)$ is that (in general) $\mathrm{PL}_{S,G}(J)$ does not act transitively on the interior points in the interval J. Our first step it to describe the orbits. Let us define an equivalence relation $\sim_{S,G,J}$ in J. If $x, y \in J$ we say that $x \sim y$ if and only if there exists $g \in \mathrm{PL}_{S,G}(J)$ such that $g(x) = y$. Unless otherwise stated, we always assume that the endpoints of J are in S.

Definition 6.3. Let $\mathcal{I}_{S,G}$ denote the submodule of the $\mathbb{Z}[G]$-module S generated by $(g-1)$ for $g \in G$. We denote by $\pi_{S,G} \colon S \to S/\mathcal{I}_{S,G}$ the natural quotient map. Unless otherwise stated, we will drop the subscript and write $\mathcal{I}$ and π instead of $\mathcal{I}_{S,G}$ and $\pi_{S,G}$.

We remark that the natural map π is a homomorphism. The next theorem plays central role in understanding the orbits of points in J under the action of $\mathrm{PL}_{S,G}(J)$ by detecting when two points of S are in the same $\mathrm{PL}_{S,G}$-orbit.

Theorem 6.4. *Let J be an interval with endpoints in S and let $x, y \in S \cap J$. Then $x \sim y$ if and only if $x - y \in \mathcal{I}$.*

The proof follows from the next two results.

Lemma 6.5. *Let $J \subseteq [0,1]$ be a closed interval with at least one of the endpoints η in S and let $g \in \mathrm{PL}_{S,G}(J)$. Then $\pi(g(t)) = \pi(t)$ for every $t \in J \cap S$.*

Proof. We may assume that η is a left endpoint and we apply induction on the number of breakpoints preceding t. In case the endpoint in S is the right one, we apply induction on the breakpoints following t. Let $\{\eta_1, \ldots, \eta_r\}$ be the set of all breakpoints of g in the interval $[\eta, t)$. Then $g(t) = c_r(t - \eta_r) + g(\eta_r)$ for some suitable $c_i \in G$.

[6] This is exactly the invariant Σ_2 defined by Brin and Squier in [5].

By induction hypothesis, the number of breakpoints preceding η_r is $r-1$, and so we have $\pi(g(\eta_r)) = \pi(\eta_r)$. Now observe that

$$\begin{aligned}\pi(g(t)) &= \pi(c_r(t-\eta_r) + g(\eta_r)) \\ &= \pi(c_r - 1)\pi(t-\eta_r) + \pi(1)\pi(t-\eta_r) + \pi(g(\eta_r)) \\ &= \pi(t-\eta_r) + \pi(\eta_r) = \pi(t). \end{aligned}$$ □

Proposition 6.6. *Let $J \subseteq [0,1]$ be a closed interval with both endpoints in S and let $u, v \in J \cap S$. Then $\pi(u) = \pi(v)$ if and only if there is a map $g \in \mathrm{PL}_{S,G}(J)$ such that $g(u) = v$.*

Proof. Sufficiency od the condition follows from Lemma 6.5. Now suppose that $J = [\eta, \zeta]$ and let $L = \zeta - \eta$. We recenter the axis at (η, η), so that interval J is now $[0, L]$. For $\alpha \in G$, $\beta \in J \cap S$ such that $\alpha\beta < L - \beta$ define (see Figure 7)

$$g_{\alpha,\beta}(t) := \begin{cases} \alpha t & \text{if } t \in [0,\beta], \\ t - (1-\alpha)\beta & \text{if } t \in [\beta, L - \alpha\beta], \\ \frac{1}{\alpha}(t - L) + L & \text{if } t \in [L - \alpha\beta, L]. \end{cases}$$

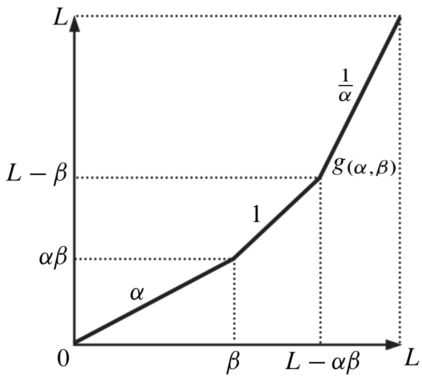

Figure 7. The basic function to get transitivity.

Using the maps $g_{(\alpha,\beta)}$ or $g^{-1}_{(\alpha,\beta)}$ we can send any number $\beta \le t \le L - \alpha\beta$ to $t - (1-\alpha)\beta$ and any number $\alpha\beta \le t \le L - \beta$ to $t + (1-\alpha)\beta$.

Since $\pi(u) = \pi(v)$, we have $v - u \in \mathcal{I}$ and so

$$v - u = (1-\alpha_1)\beta_1 + \cdots + (1-\alpha_k)\beta_k$$

for some $\alpha_i \in G$, $\beta_i \in J \cap S$. Adding extra terms if necessary we can assume that

$$u + (1-\alpha_1)\beta_1 + \cdots + (1-\alpha_i)\beta_i \in J$$

for any $1 \le i \le k$. Since S is a dense subgroup of $\mathbb{R}$, we can, for each β_i, find numbers $\beta_{i,j} \in J \cap S$ small enough such that

- $L - \beta_{i,j} > \alpha_i \beta_{i,j}$ so that the map $g_{(\alpha_i, \pm\beta_{i,j})}$ can be defined, and
- $\beta_i = \sum_j \beta_{i,j}$.

Finally we can see that the composition of the maps $g^{\pm 1}_{(\alpha_i, \beta_{i,j})}$ sends u to v, which finishes the proof. □

Corollary 6.7. *Any linear piece of the graph of an element $g \in \mathrm{PL}_{S,G}(J)$ has an equation of the form $x \to ax + b$ where $a \in G$ and $b \in \mathcal{I}$.*

Corollary 6.8. *Let J_1 and J_2 be two intervals containing x, y, then $x \sim_{S,G,J_1} y$ if and only if $x \sim_{S,G,J_2} y$.*

Theorem 6.9. *Let J be a closed interval with endpoints in S and suppose we have $u_1, v_1, \dots, u_k, v_k \in J \cap S$ such that $u_1 < \cdots < u_k$, $v_1 < \cdots < v_k$ and $u_i \sim v_i$ for all $i = 1, \dots, k$. Then there exists a map $g \in \mathrm{PL}_{S,G}(J)$ such that $g(u_i) = v_i$ for all $i = 1, \dots, k$.*

Proof. The proof is by induction. The base case $k = 1$ is just the definition of the equivalence relation $\sim$. Let $k > 1$. By the induction assumption, there exists $\hat{g} \in \mathrm{PL}_{S,G}(J)$ such that $\hat{g}(u_i) = v_i$ for $i = 1, \dots, k-1$. Using that $\sim_{S,G,J}$ is an equivalence relation we obtain that $\hat{g}(u_k) \sim_{S,G,J} u_k \sim_{S,G,J} v_k$. Let J' denote the interval $[v_{k-1}, \zeta]$ which contains the points $\hat{g}(u_k)$ and v_k. By Corollary 6.8 we have $\hat{g}(u_k) \sim_{S,G,J'} v_k$, therefore there exists $\bar{g} \in \mathrm{PL}_{S,G}(J')$ such that $\bar{g}(\hat{g}(u_k)) = v_k$, thus the element $g = \bar{g} \circ \hat{g}$ sends u_i to v_i for all i. □

Lemma 6.10. *Suppose that $I_1, \dots, I_k$ is a family of disjoint closed intervals $I_i = [a_i, b_i]$, with $b_i < a_{i+1}$ for all $i = 1, \dots, k$ and $a_i, b_i \in S$. Let $J_1, \dots, J_k \subseteq [0, 1]$, with $J_i = [c_i, d_i]$, be another family of intervals with the same property such that $a_i \sim c_i$ and $b_i \sim d_i$. Suppose that $g_i : I_i \to J_i$ is a piecewise-linear function with a finite number of breakpoints, occurring at S and such that all slopes are in G. Then there exists an element $\tilde{g} \in \mathrm{PL}_{S,G}(I)$ such that $\tilde{g}|_{I_i} = g_i$.*

Proof. By Theorem 6.9 there exists an $h \in \mathrm{PL}_{S,G}(J)$ with $h(a_i) = c_i$ and $h(b_i) = d_i$. Define

$$\tilde{g}(t) := \begin{cases} h(t) & \text{if } t \notin I_1 \cup \cdots \cup I_k, \\ g_i(t) & \text{if } t \in I_i. \end{cases}$$

By construction, it is clear that $\tilde{g} \in \mathrm{PL}_{S,G}(J)$ and $\tilde{g}|_{I_i} = g_i$. □

Corollary 6.11. *Any part of the graph of $x \to ax + b$, where $a \in G$ and $b \in \mathcal{I}$, inside the open square $J \times J$ can be extended to a graph of an element in $\mathrm{PL}_{S,G}$.*

Any isolated fixed point α of an element $g \in \mathrm{PL}_{S,G}(J)$ is of the form $\alpha = s/(t-1)$ for some $s \in S$ and $t \in G \setminus \{1\}$. Let Q_S denote the set of all points of the form $s/(t-1)$. The next step is to understand when two points in Q_S are in one and the same orbit under $\mathrm{PL}_{S,G}(J)$.

Theorem 6.12. *Let $J = [\eta, \zeta]$ be a closed interval with endpoints in S and let $\alpha, \beta \in J \cap Q_S$. The points α and β are equivalent under $\sim_{S,G,J}$ if and only if we can find $s, s' \in S$ and $t \in G$ such that $\alpha = s/(t-1)$, $\beta = s'/(t-1)$ and*

$$sG = s'G \pmod{(t-1)\mathcal{I}},$$

where $(t-1)\mathcal{I}$ denotes the image of the submodule $\mathcal{I}$ under the multiplication by $t - 1 \in \mathbb{Z}[G]$.

Proof. Suppose that there is a map $g \in \mathrm{PL}_{S,G}(J)$ with $g(\alpha) = \beta$ and let $g(x) = cx + d$ in a small neighborhood J_α of α. We can choose representatives $s \in S$ and $t \in G$ such that $\alpha = s/(t-1)$ and then, since $g \in \mathrm{PL}_{S,G}(J)$, we use Lemma 6.5 to get

$$\pi(x) = \pi(g(x)) = \pi(c-1)\pi(x) + \pi(x) + \pi(x),$$

for all $x \in J_\alpha \cap S$, and therefore $\pi(d) = 0$, which implies $d \in \mathcal{I}$. The equality $g(\alpha) = \beta$ implies that $\beta = s'/(t-1)$, where $s' = cs + d(t-1)$, and so $sG = s'G \pmod{(t-1)\mathcal{I}}$.

Conversely, suppose that we can write $\alpha = s/(t-1)$, $\beta = s'/(t-1)$ for some $s, s' \in S$ and $t \in G$ such that $sG = s'G \pmod{(t-1)\mathcal{I}}$. The second condition implies that there exist $c_1, c_2 \in G$, $d_2 \in \mathcal{I}$ such that

$$c_1 s = c_2 s' + (t-1)d_2.$$

Thus, if we set $c = c_2/c_1$ and $d = d_2/c_1$, we get $\alpha = c\beta + d$. Let $f(t) = ct + d$ be a line through the point (α, β) and let $[\gamma, \delta] \subseteq J$ be a small interval such that $\gamma, \delta \in S$. Since $\pi(d) = 0$, we have $\pi(f(\gamma)) = \pi(\gamma)$ and $\pi(f(\delta)) = \pi(\delta)$, and so by Lemma 6.10 there is a map $g \in \mathrm{PL}_{S,G}(J)$ with $g|_{[\gamma,\delta]} = f$. By construction $g(\alpha) = \beta$, as required. □

Using the previous two results one can easily generalize Theorem 6.2 to the groups $\mathrm{PL}_{S,G}(J)$. Of course this is only possible if the group S and the group G satisfy some mild computational requirements, which are described in Section 3.

Corollary 6.13. *Assume that S and G satisfy the computational requirements from Section 3. Then for any $\alpha, \beta \in Q_S \cap J$ there is an algorithm which constructs a $g \in \mathrm{PL}_{S,G}(J)$ such that $g(\alpha) = \beta$, or shows that such an element does not exist.*

We state the same result for a finite number of points. Its proof uses Lemma 6.10 on a number of disjoint intervals, one around each point.

Corollary 6.14. *Assume that S and G satisfy the computational requirements from Section 3. Let $\eta < \alpha_1 < \cdots < \alpha_r < \zeta$ and $\eta < \beta_1 < \cdots < \beta_r < \zeta$ be two partitions of J with elements of the set Q_S. Then there is an algorithm which constructs $g \in \mathrm{PL}_{S,G}(J)$ with $g(\alpha_i) = \beta_i$, or shows that such an element does not exist.*

Theorem 6.15. *Assume that S and G satisfy the computational requirements from Section 3. Then given any $y, z \in \mathrm{PL}_{S,G}(I)$, there is an algorithm which constructs $g \in \mathrm{PL}_{S,G}(I)$ such that $g(\mathrm{Fix}(y)) = \mathrm{Fix}(g^{-1}yg) = \mathrm{Fix}(z)$, or shows that such element does not exist.*

Proof. First we check if $\#\partial\,\mathrm{Fix}(y) = \#\partial\,\mathrm{Fix}(z)$. Then we use the previous corollary to find a $g \in \mathrm{PL}_2(I)$ with $g(\partial\,\mathrm{Fix}(y)) = \partial\,\mathrm{Fix}(z)$ if it exists. To finish we check whether $\mathrm{Fix}(g^{-1}yg)$ contains the same intervals as $\mathrm{Fix}(z)$. $\square$

6.3. The case of Thompson's group. Here are the analogues of previous results in the case of Thompson's groups F.[7]

Lemma 6.16. *If $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ and $0 = y_0 < y_1 < y_2 < \cdots < y_n = 1$ are two partitions of $[0, 1]$ consisting of dyadic rational numbers, then we can construct a map $g \in \mathrm{PL}_2(I)$ such that $g(x_i) = y_i$.*

An easy well-known consequence is the following extension lemma.

Lemma 6.17. *Suppose that $I_1, \dots, I_k \subseteq [0, 1]$ is a family of disjoint closed intervals $I_i = [a_i, b_i]$, with $b_i < a_{i+1}$ for all $i = 1, \dots, k$ and $a_i, b_i \in \mathbb{Z}[\frac{1}{2}]$. Let $J_1, \dots, J_k \subseteq [0, 1]$, with $J_i = [c_i, d_i]$, be another family of intervals with the same property. Suppose that $g_i \colon I_i \to J_i$ is a piecewise-linear function with a finite number of breakpoints, occurring at dyadic rational points, and such that all slopes are integral powers of 2. Then there exists a map $\tilde{g} \in \mathrm{PL}_2(I)$ such that $\tilde{g}|_{I_i} = g_i$.*

Proposition 6.18. *Let $\alpha = \frac{2^t m}{n}$ and $\beta = \frac{2^k p}{q}$ be rational numbers in $\mathbb{Q} \cap (0, 1)$, where $t, k \in \mathbb{Z}$, m, n, p, q odd integers such that $(m, n) = (p, q) = 1$. Then there is a map $g \in \mathrm{PL}_2(I)$ such that $g(\alpha) = \beta$ if and only if $n = q$ and*

$$p \equiv 2^R m \pmod{n} \tag{6.1}$$

for some $R \in \mathbb{Z}$. Equivalently, there exist integers t', k' such that $2^{t'}\alpha - 2^{k'}\beta$ is an integer. Moreover, there is an algorithm which constructs such element g if the above condition is satisfied.

Example 6.19. Let $\alpha = \frac{1}{17}$, $\beta = \frac{13}{17}$ and $\gamma = \frac{3}{17}$. It is easy to see that we can find a map $g \in \mathrm{PL}_2(I)$ with $g(\alpha) = \beta$, but there is no $h \in \mathrm{PL}_2(I)$ with $h(\alpha) = \gamma$.

We now state the analogue of Theorem 6.15 noticing that for Thompson's group the requirements Section 3 are satisfied.

Theorem 6.20. *Given $y, z \in \mathrm{PL}_2(I)$, there is an algorithm which constructs $g \in \mathrm{PL}_2(I)$ such that $g(\mathrm{Fix}(y)) = \mathrm{Fix}(g^{-1}yg) = \mathrm{Fix}(z)$, or shows that such an element does not exist.*

[7] The first two results are well known; see [6].

7. The conjugacy problem and the power conjugacy problem in $\mathrm{PL}_+(J)$ and $\mathrm{PL}_{S,G}(J)$

The results of Section 6, together with the assumption that S, G satisfy the computational requirements of Section 3, allow us to reduce the problem to the case where $\partial_S \operatorname{Fix}(y) = \partial_S \operatorname{Fix}(z)$.

7.1. Characterizing conjugacy in $\mathrm{PL}_+(J)$. To study conjugacy between two elements y and z we can assume that $\partial \operatorname{Fix}(y) = \partial \operatorname{Fix}(z) = \{\alpha_1, \ldots, \alpha_n\}$, and we look for conjugators in $\mathrm{PL}_+([\alpha_i, \alpha_{i+1}])$ of the restrictions of y and z to $[\alpha_i, \alpha_{i+1}]$. We reduce the study of the conjugacy problem to smaller intervals. If $y = z = \mathrm{id}$ on the interval $[\alpha_i, \alpha_{i+1}]$ there is nothing to prove, otherwise y and z are one-bump functions. Given two elements $f, g \in \mathrm{PL}_+(J)$ we say that they are *y-equivalent* if $f = y^n g$ for some integer n.

Lemma 7.1. *If g is a conjugator for y and z, then any y-equivalent map $y^n g$ is a conjugator as well.*

Proof. We observe that

$$(y^n g)^{-1} y (y^n g) = g^{-1} y g = z. \qquad \square$$

Lemma 7.2. *If $y, z \in \mathrm{PL}_+^<(J)$ are conjugate, there exists a y-equivalent conjugator $g \in \mathrm{PL}_+(J)$ such that $y(\lambda) < g(\lambda) < \lambda$ for any fixed λ in the interior of J.*

Proof. Let $h \in \mathrm{PL}_+(J)$ be a conjugator for y and z. Since $y \in \mathrm{PL}_+^<(J)$, there exists an integer n such that $y^n h(\lambda) < y(\lambda) \leq y^{n-1} h(\lambda)$. By applying y^{-1} to the inequality $y^n h(\lambda) < y(\lambda)$ we obtain

$$y^n h(\lambda) < y(\lambda) \leq y^{n-1} h(\lambda) < \lambda.$$

We define $g = y^{n-1} h$, and we are done by Lemma 7.1. $\square$

Proposition 7.3. *To detect whether or not two elements $y, z \in \mathrm{PL}_+(J)$ are conjugate, only finitely many functions need to be tested as possible candidate conjugators and they can be constructed. Moreover, we can enumerate all possible conjugators.*

Proof. By the discussion at the beginning of this section, we can assume that $y, z \in \mathrm{PL}_+^<(J)$. Let $\lambda \in J$ be a fixed interior point of J contained in the initial linearity box. For any conjugator of y and z, Lemma 7.2 implies that there is a y-equivalent conjugator $g \in \mathrm{PL}_+(J)$ such that $y(\lambda) < g(\lambda) < \lambda$. Now, since the map ρ_λ defined in Lemma 4.7 is increasing, it is immediate to see from its definition that

$$y'(\eta^+) = \rho_\lambda(y(\lambda)) \leq g'(\eta^+) = \rho_\lambda(g(\lambda)) \leq 1 = \rho_\lambda(\lambda) \leq y'(\eta^+)^{-1}.$$

Choosing another interior point μ in the final linearity box, we can use the analogous version of ρ_μ at the final slope to obtain $y'(\zeta^+)^{-1} \leq g'(\zeta^+) \leq y'(\zeta^+)$. Hence, the set of all conjugators g such that $y(\lambda) < g(\lambda) < \lambda$ is contained in the set

$$N := \{h \mid h \in C_{\mathrm{PL}_+(J)}(y, z), h'(\eta) \in [y'(\eta^+), y'(\eta^+)^{-1}],\\ h'(\zeta) \in [y'(\zeta^+)^{-1}, y'(\zeta^+)]\},$$

which by Lemma 5.4 is finite and can be constructed. If the set N is non-empty then, by the uniqueness of conjugators with a given initial slope (Lemma 4.7) and by Lemma 7.2, the set of all conjugators for y and z is given by $\{y^r g \mid g \in N,\ r \in \mathbb{Z}\}$. □

7.2. Conjugacy problem in $\mathrm{PL}_{S,G}(J)$. We can now solve the conjugacy problem for elements in $\mathrm{PL}^0_{S,G}(J)$. We recall that $\mathrm{PL}^0_{S,G}(J) \subseteq \mathrm{PL}_{S,G}(J)$ is the set of functions $f \in \mathrm{PL}_{S,G}(J)$ such that the set $\mathrm{Fix}(f)$ does not contain elements of S other than the endpoints of J.

Lemma 7.4. *For any $y, z \in \mathrm{PL}^0_{S,G}(J)$ such that $y \neq z$ and $\mathrm{Fix}(y) = \mathrm{Fix}(z)$, we can decide whether there is (or not) a map $g \in \mathrm{PL}_{S,G}(J)$ with $y^g = z$. Moreover, we can construct and enumerate all possible conjugators.*

Proof. In order to be conjugate, we must have $y'(\eta^+) = z'(\eta^+)$ and $y'(\zeta^-) = z'(\zeta^-)$. Up to taking inverses of y and z, we can assume that $y'(\eta^+) = z'(\eta^+) < 1$. Let α be the first interior fixed point of y. Since we are looking for conjugators fixing $\mathrm{Fix}(y)$ pointwise, we can restrict ourselves to find a conjugator for y and z in $\mathrm{PL}_{S,G}([\eta, \alpha])$. Since $y, z \in \mathrm{PL}^<_{S,G}([\eta, \alpha])$, by Proposition 7.3 there are only finitely many candidate conjugators. We test them, and if any of them is a conjugator in $\mathrm{PL}_+([\eta, \alpha])$, we extend it to J through the stair algorithm and test it on J. By the straightforward analogues for $\mathrm{PL}^0_{S,G}(J)$ of Lemma 7.2 and Proposition 7.3, we can enumerate all possible conjugators. □

Theorem 7.5. *The group $\mathrm{PL}_{S,G}(J)$ has solvable conjugacy problem. Moreover, we can construct and enumerate all possible conjugators.*

Proof. We use Theorem 6.15 and suppose that $\partial_S \mathrm{Fix}(y) = \partial_S \mathrm{Fix}(z) = \{\eta = \alpha_0 < \alpha_1 < \cdots < \alpha_r < \alpha_{r+1} = \zeta\}$. Now we restrict ourselves to an interval $[\alpha_i, \alpha_{i+1}]$ and consider $y, z \in \mathrm{PL}^0_{S,G}([\alpha_i, \alpha_{i+1}])$. If $\mathrm{Fix}(y)$ contains a subinterval of $[\alpha_i, \alpha_{i+1}]$, then we must have $y = z = \mathrm{id}$ on the whole interval $[\alpha_i, \alpha_{i+1}]$ and so any function $g \in \mathrm{PL}_{S,G}([\alpha_i, \alpha_{i+1}])$ will be a conjugator. Otherwise, $\mathrm{Fix}(y)$ does not contain any subinterval of $[\alpha_i, \alpha_{i+1}]$ and so we can apply the Lemma 7.4. If we find a solution in each such interval, then the conjugacy problem is solvable. Otherwise, it is not. □

Remark 7.6. For the case of Thompson's group $\mathrm{PL}_2(I)$ there is no need to use Lemma 5.4, because all possible initial slopes of g must be powers of 2. Hence, there are only finitely many conjugators with initial slope in $[y'(0), y'(0)^{-1}]$. We test all candidate conjugators with initial slope in $[y'(0), y'(0)^{-1}]$ to conclude the procedure.

The argument given to solve the conjugacy problem in $\mathrm{PL}_{S,G}(J)$ also works, in much the same way, to solve the power conjugacy problem. We say that a group G has *solvable power conjugacy problem* if there is an algorithm such that, given any two elements $y, z \in G$, we can determine whether there is, or not, an element $g \in G$ and two non-zero integers m, n such that $g^{-1}y^m g = z^n$, that is, there are some powers of y and z that are conjugate.

Theorem 7.7. *The group $\mathrm{PL}_{S,G}(J)$ has solvable power conjugacy problem.*

Proof. Again, we can use Theorem 6.15, the identity $\partial_S \operatorname{Fix}(y) = \partial_S \operatorname{Fix}(z)$, and restrict ourselves to a smaller interval $J = [\eta, \zeta]$ with endpoints in S and such that $y, z \in \mathrm{PL}^0_{S,G}(J)$. If $g \in \mathrm{PL}_{S,G}(J)$ and m, n exist, then the initial slopes of y^m and z^n must coincide. A simple argument on the exponent of these slopes implies that this can happen if and only if y^m and z^n are both powers of a common minimal power $(y^\alpha)'(\eta) = (z^\beta)'(\eta)$. Hence the problem can be reduced to finding whether there is a map $g \in \mathrm{PL}_{S,G}(J)$ and an integer k such that $g^{-1}y^{k\alpha}g = z^{k\beta}$. By Lemma 4.13 (that can be naturally generalized to $\mathrm{PL}^0_{S,G}(J)$; see Remark 4.17), we have that this is equivalent to finding a map $g \in \mathrm{PL}_{S,G}(J)$ such that $g^{-1}y^\alpha g = z^\beta$. Hence solving the power conjugacy problem is equivalent to solving the conjugacy problem for y^α and z^β. □

8. The k-simultaneous conjugacy problem

We will make a sequence of reductions to solve the simultaneous conjugacy problem in $\mathrm{PL}_+(J)$ and $\mathrm{PL}_{S,G}(J)$. Let M denote the group $\mathrm{PL}_+(J)$ or $\mathrm{PL}_{S,G}(J)$, which will allow us to treat both cases together. These reductions closely follow the solution of the ordinary conjugacy problem. First we notice that since we know how to solve the ordinary conjugacy problem, solving the $(k+1)$-simultaneous conjugacy problem is equivalent to finding a positive answer to the following problem:

Problem 8.1. Is there an algorithm such that given $(x_1, \dots, x_k, y)$ and $(x_1, \dots, x_k, z)$ it can decide whether there is a function $g \in C_M(x_1) \cap \cdots \cap C_M(x_k)$ such that $g^{-1}yg = z$?

Since we understand the structure of the intersection of centralizers, we are going to work on solving this last question. Our strategy now is to reduce the problem to the ordinary conjugacy problem and to isolate a very special case that must be dealt with.

As in the case of the ordinary conjugacy problem, the first step is to determine whether the set of fixed points can be made the same.

Lemma 8.2. *Let $x_1, \dots, x_k, y, z \in M$. We can determine whether there is, or not, an element $g \in C = C_M(x_1) \cap \dots \cap C_M(x_k)$ such that $g(\mathrm{Fix}(y)) = \mathrm{Fix}(z)$.*

Proof. The proof is essentially the same as that of Corollary 6.14 for each of the intervals between two fixed points of y and z that are in S. The only new tool required is Lemma 4.19 for the intervals where C is isomorphic to $\mathbb{Z}$. We omit the details of this proof. □

Lemma 8.3. *Let $x_1, \dots, x_k, y, z \in M$. The subgroup C' of elements g in $C_M(x_1) \cap \dots \cap C_M(x_k)$ such that $g(\mathrm{Fix}(y)) = \mathrm{Fix}(y)$ splits as a product*

$$C' = C'_{J_1} \cdot C'_{J_2} \cdots C'_{J_k}$$

for some disjoint intervals J_i with $\bigcup J_i = J$, where $C'_{J_i} := \{f \in C' \mid f(t) = t \text{ for all } t \notin J_i\} = C' \cap \mathrm{PL}_+(J_i)$. Moreover, each C'_{J_i} is isomorphic to either $\mathbb{Z}$ or $\mathrm{PL}_+(J_i) \cap M$, or is the trivial group.

Proof. Similar to the proof of Proposition 5.11. □

Using the two results we reduce the simultaneous conjugacy problem to the case when $\mathrm{Fix}(y) = \mathrm{Fix}(g)$. Again we can further reduce to the case when both y and z are in $\mathrm{PL}^0_{S,G}(J)$, but we are restricted to use only conjugating elements from the subgroup C'. By Lemma 8.3 the group C' splits as a product of several subgroups C'_{J_i}, which lead to several cases:

Case 1. *The number of intervals J_i is more than* 1: There is an interior point λ in J which is fixed by all elements in C' (since $\bigcup(\partial J_i) \not\subset \partial J$). By Lemma 4.7 (which can be naturally adapted to $\mathrm{PL}_{S,G}(J)$; see Remark 4.17) there is at most one element in $C_{\mathrm{PL}_{S,G}(J)}(y, z)$ which fixes λ, and we only need to verify if this element is inside C'.

Case 2. *The number of intervals J_i is exactly* 1: This case breaks further into three subcases depending on the subgroup C'.

Case 2a. *The group C' is trivial*: The elements y and z are conjugate by an element in C' if and only if they are the same.

Case 2b. *The group C' is isomorphic to* $\mathrm{PL}_{S,G}(J)$: If C' is the whole group, we can simply apply the algorithm which gives the solution of the ordinary conjugacy problem.

Case 2c. *The group C' is isomorphic to $\mathbb{Z}$*: We want to see if we can solve the ordinary conjugacy problem when we have a restriction on the possible conjugators. Let x denotes the generator of C', thus we want to check if there exists integer k such that $x^{-k} y x^k = z$. By assumption both y and z are in $\mathrm{PL}^0_{S,G}(J)$, solving the ordinary conjugacy problem we find that the set $C_{\mathrm{PL}_{S,G}(J)}(y, z)$ is either empty or is equal to

$$\{\hat{y}^i g \mid i \in \mathbb{Z}\},$$

where $\hat{y}$ is the generator of $C_{\mathrm{PL}_{S,G}(J)}(y)$ and g is some element which conjugates y to z. Thus we need to find integer solutions (or show that they do not exist) of the equation

$$x^k = \hat{y}^i g. \tag{8.1}$$

This equation can be solved using the following lemma (the proof is in Section 8.1):

Lemma 8.4. *For any $x, \hat{y}, g \in \mathrm{PL}_{S,G}(J)$ there is an algorithm which finds all solutions of equation (8.1).*

Thus in all cases we can check if there exists a conjugating element in the subgroup C', which finishes the solution of the simultaneous conjugacy problem.

The previous argument proves the following theorem:

Theorem 8.5. *The k-simultaneous conjugacy problem is solvable for the group $\mathrm{PL}_{S,G}(J)$. Moreover, we can construct and enumerate all possible conjugators.*

8.1. Proof of Lemma 8.4. We start by proving the lemma for the case of $\mathrm{PL}_2(J)$. We will then explain what is required to generalize the proof to the case of $\mathrm{PL}_{S,G}(J)$.[8] We observe that both x and $\hat{y}$ are in $\mathrm{PL}_2^0(J)$, therefore their initial slopes are not equal to 1. Comparing the slopes at η and taking logarithms we obtain

$$k \log_2 x'(\eta^+) = \log_2 g'(\eta^+) + i \log_2 \hat{y}'(\eta^+). \tag{8.2}$$

This equation does not have any solution unless $\log_2 g'(\eta)$ is divisible by the greatest common divisor of $\log_2 x'(\eta^+)$ and $\log_2 \hat{y}'(\eta^+)$. If this is the case, an elementary number theory argument tells us that all solutions are of the form

$$k = p_1 j + q_1 \quad \text{and} \quad i = p_2 j + q_2$$

for some integers p_1, p_2, q_1 and q_2, which reduces equation (8.1) to

$$\bar{x}^j = \bar{y}^j \bar{g}, \tag{8.3}$$

where $\bar{x}$ and $\bar{y}$ are powers of x and $\hat{y}$, respectively, and $\bar{g}'(\eta^+) = 1$.

If $\mathrm{Fix}(\bar{x}) \neq \mathrm{Fix}(\bar{y})$ we can use Lemma 4.19 to solve equation (8.3). We can also compare the derivatives at all fixed points and this will give us a unique solution (or there does not exist a solution at all) for j unless

$$\bar{x}'(\mu) = \bar{y}'(\mu) \quad \text{and} \quad \bar{g}'(\mu)$$

hold for any $\mu \in \mathrm{Fix}(\bar{x})$. Equation (8.3) can be written as

$$\bar{g} = \bar{x}^j \bar{y}^{-j}. \tag{8.4}$$

[8] The generalization to $\mathrm{PL}_{S,G}(J)$ is explained in the last paragraph of the current section.

If $\bar{x} = \bar{y}$, then equation (8.4) has solutions if and only if $\bar{g} = \mathrm{id}$, and in this case any integer j is a solution. Thus the only non-trivial case when $\bar{x} \neq \bar{y}$.

Without loss of generality we may assume that $\bar{x}, \bar{y} \in \mathrm{PL}_+^<([\mu_1, \mu_2])$ for some consecutive μ_1 and μ_2 in $\partial\,\mathrm{Fix}(\bar{x}) = \partial\,\mathrm{Fix}(\bar{y})$. Let p denote the function $\bar{x}\bar{y}^{-1}$ and let λ be the closest breakpoint of p to μ_1, i.e., $p(t) = t$ for all $\mu_1 \leq t \leq \lambda$ and $p(\lambda + \varepsilon) \neq \lambda + \varepsilon$ if $\varepsilon > 0$ is sufficiently small. For any $j > 0$ we can write

$$\bar{g} = \bar{x}^j \bar{y}^{-j} = p p^{\bar{y}^{-1}} \dots p^{\bar{y}^{-j+1}}. \tag{8.5}$$

It is clear that the first breakpoint for $p^{\bar{y}^r}$, for any integer r, is given by $\bar{y}^{-r}(\lambda)$. Since $\bar{y} \in \mathrm{PL}_+^<([\mu_1, \mu_2])$, formula (8.5) gives that the first breakpoint of $\bar{g}$ is at $\bar{y}^{j-1}(\lambda)$. There can be at most one positive j such that the number $\bar{y}^{j-1}(\lambda)$ coincides with the actual first breakpoint of $\bar{g}$. Therefore, we can convince ourselves whether equation (8.3) has solutions for positive j. If j is negative we can similarly write

$$\bar{g}^{-1} = \bar{y}^{-j} \bar{x}^j = \bar{p}\bar{p}^{\bar{x}^{-1}} \dots \bar{p}^{\bar{x}^{j+1}}, \tag{8.6}$$

where $\bar{p} := p^{-1}$. Since $\bar{x} \in \mathrm{PL}_+^<([\mu_1, \mu_2])$, formula (8.6) gives that the first breakpoint of $\bar{g}^{-1}$ is at $\bar{x}^{-j-1}(\lambda)$. Therefore, we can check whether equation (8.3) has solutions for negative j.

This completes the proof of Lemma 8.4 for $\mathrm{PL}_2(J)$. To generalize this proof to the groups $\mathrm{PL}_{S,G}(J)$, we observe that all of the previous proof has been carried out in $\mathrm{PL}_+(J)$, save for the first step, that is, taking logarithms to get an argument to pass from equation (8.1) to equation (8.3). To do this step in $\mathrm{PL}_{S,G}(J)$, we appeal to the last of the requirements in Section 3.

9. Interesting examples

Now that we have developed the general theory, we are going to see a few interesting examples where the simultaneous conjugacy problem is solvable. We will not dwell too much on the details here, sketching only why it is possible to verify the requirements.

Example 9.1. $S = \mathbb{Q}$ and $G = \mathbb{Q}^*_{\geq 0} = \mathbb{Q} \cap (0, \infty)$.

There are many structures that can be used to represent the rational numbers, which comes with algorithms for performing the arithmetic operations that give us the oracles in the first group. The oracles in the second group are very easy to implement since $\mathbb{Q}$ is a field and the quotients $S/\mathcal{I} = \{0\}$ and $S/(t-1)\mathcal{I} = \{0\}$ consist of just one element. The last oracle which is needed for solving the simultaneous conjugacy problem is slightly more complicated – we need to factor a, b, c as product of prime numbers and then reduce the problem to solving several congruences in integers.

Example 9.2. S finite real algebraic extension over $\mathbb{Q}$ and $G = S^* := S \cap (0, \infty)$. This is the same as the previous example, we only need to "implement" the field S.

Example 9.3. $S = \mathbb{Z}\big[\frac{1}{n_1}, \dots, \frac{1}{n_k}\big]$ and $G = \langle n_1, \dots, n_k\rangle$ for $n_1, \dots, n_k \in \mathbb{Z}$.

As in Example 9.1 there are many data structures to represent S and G, which provide the oracles in the first group. For the oracles in the second group one observes that $S/\mathcal{I} \cong \mathbb{Z}/d\mathbb{Z}$, where $d := \mathrm{GCD}(n_1 - 1, \dots, n_k - 1)$. This reduces an effective solution of the membership problem in $\mathcal{I}$ to expressing a given element in $d\mathbb{Z}$ as a sum of multiples of $n_i - 1$, which can be done using the Euclidian algorithm. As in the previous example, the implementation of the last oracle relies on the factorization of integers as a product of primes. For $k = 1$, we recall that the groups $\mathrm{PL}_{S,G}(I)$ are known as *generalized Thompson groups*.

Example 9.4. $S = \mathbb{Z}\big[\frac{1}{n_1}, \dots, \frac{1}{n_k}, \dots\big]$ with $G = \langle \{n_i\}_{i\in\mathbb{N}}\rangle$, where $\{n_i\}_{i\in\mathbb{N}} \subseteq \mathbb{Z}$.

This example can be reduced to the previous one. If we are given a finite set E of elements in $\mathrm{PL}_{S,G}(I)$, we can consider the set $\{n_{i_1}^{\alpha_{i_1}}, \dots, n_{i_v}^{\alpha_{i_v}}\}$ of all slopes of elements of E. Then $E \subseteq \mathrm{PL}_{S',G'}(I)$, where $S' := \mathbb{Z}\big[\frac{1}{n_{i_1}}, \dots, \frac{1}{n_{i_v}}\big]$ and $G' := \langle n_{i_1}, \dots, n_{i_v}\rangle$. By Corollary 4.18, we know that if there is a conjugator, it must be in $\mathrm{PL}_{S',G'}(I)$.

Remark 9.5. In general, given two intervals J_1, J_2 with endpoints in S, it is not clear whether or not the groups $\mathrm{PL}_{S,G}(J_1)$ and $\mathrm{PL}_{S,G}(J_2)$ are isomorphic. Proposition 6.6 tells us that two elements in S are in the same $\mathrm{PL}_{S,G}$-orbit if their image under the map π is the same. For example in the cases $S = \mathbb{R}$, $G = \mathbb{R}_+$, $S = \mathbb{Q}$, $G = \mathbb{Q}^*$, and $S = \mathbb{Z}\big[\frac{1}{2}\big]$, $G = \langle 2\rangle$, it is not difficult to see that every two points in S have the same image under π (the case of F is treated in Lemma 6.16) and that any two groups $\mathrm{PL}_{S,G}(J_1)$ and $\mathrm{PL}_{S,G}(J_2)$ are thus isomorphic, for any two intervals J_1, J_2 with endpoints in S. In fact, if there is a $\mathrm{PL}_{S,G}$-map $\varphi\colon J_1 \to J_2$, then conjugation by φ yields an isomorphism between $\mathrm{PL}_{S,G}(J_1)$ and $\mathrm{PL}_{S,G}(J_2)$.

On the other hand, if we consider generalized Thompson groups (see Example 9.3) and use the map π, it is straightforward to show that the number of orbits of elements is finite, but more than one, for certain choices of $n_1, \dots, n_k$ (see Example 9.3 for a proof of this). Hence there are only finitely many inequivalent types of intervals J with endpoints in S. This implies that there can be at most only finitely many isomorphism classes for the groups $\mathrm{PL}_{S,G}(J)$, for $S = \mathbb{Z}\big[\frac{1}{n_1}, \dots, \frac{1}{n_k}\big]$ and $G = \langle n_1, \dots, n_k\rangle$ for $n_1, \dots, n_k \in \mathbb{Z}$. We note that the generalized Thompson groups which are most often studied are those with $\mathrm{GCD}(n_1 - 1, \dots, n_k - 1) = 1$, which implies that $S/\mathcal{I}$ is trivial. In general, it seems likely that if two elements $\alpha, \beta \in S$ have different images under π, then the groups $\mathrm{PL}_{S,G}([0,\alpha])$ and $\mathrm{PL}_{S,G}([0,\beta])$ are not isomorphic, but this is not easy to prove it.

Acknowledgments. The first author was partially supported by AMS Centennial Fellowship. This work is part of the second author's PhD thesis at Cornell University. The second author gratefully acknowledges the Centre de Recerca Matemàtica (CRM) and its staff for the support received during the completion of this work. The authors would like to thank Collin Bleak, Martin Bridson, Ken Brown, Kai-Uwe Bux and Mark Sapir for the useful discussions and comments on earlier drafts of this paper.

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Received December 17, 2009; revised December 7, 2010

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