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Hereditary conjugacy separability of right-angled Artin groups and its applications

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Abstract. We prove that finite-index subgroups of right-angled Artin groups are conjugacy separable. We then apply this result to establish various properties of other classes of groups. In particular, we show that any word hyperbolic Coxeter group contains a conjugacy separable subgroup of finite index and has a residually finite outer automorphism group. Another consequence of the main result is that Bestvina–Brady groups are conjugacy separable and have solvable conjugacy problem.

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Contents

1. Introduction

If G is a group, the *profinite topology* $\mathcal{PT}(G)$ on G is the topology whose basic open sets are cosets to finite-index normal subgroups in G. It follows that every finite-index subgroup $K \le G$ is both closed and open in $\mathcal{PT}(G)$, and G, equipped with $\mathcal{PT}(G)$, is a topological group (that is, the group operations are continuous with respect to this topology). This topology is Hausdorff if and only if the intersection of all finite-index normal subgroups is trivial in G. In this case G is said to be *residually finite*.

We will say that a subset $A \subseteq G$ is *separable in* G if A is closed in $\mathcal{PT}(G)$. Suppose that for every element $g \in G$, its conjugacy class $g^G := \{hgh^{-1} \mid h \in G\} \subseteq G$ is closed in $\mathcal{PT}(G)$. Then G is called *conjugacy separable*. In other words, G is conjugacy separable if and only if for any two non-conjugate elements $x, y \in G$ there exists a homomorphism φ from G to a finite group Q such that $\varphi(x)$ is not conjugate to $\varphi(y)$ in Q.

Conjugacy separability is evidently stronger than residual finiteness, and is (usually) much harder to establish. The following classes of groups are known to be conjugacy separable: virtually free groups (J. Dyer [24]); virtually surface groups (A. Martino [40]); virtually polycyclic groups (V. Remeslennikov [52]; E. Formanek [27]); limit groups (S. Chagas and P. Zalesskii [11]) and, more generally, finitely presented residually free groups (S. Chagas and P. Zalesskii [12]).

Unfortunately, conjugacy separability does not behave very well under free constructions. V. Remeslennikov [53] and P. Stebe [58] showed that the free product of two conjugacy separable groups is conjugacy separable. But so far we do not know of any global criteria which tell when an amalgamated product (or an HNN-extension) of conjugacy separable groups is conjugacy separable. Perhaps the most general of local results can be found in [54], where L. Ribes, D. Segal and P. Zalesskii define a new class of conjugacy separable groups $\mathcal{X}$, which is closed under taking free products with amalgamation along cyclic subgroups and contains all virtually free and virtually polycyclic groups. Note that there is no analogue of this result for HNN-extensions with associated cyclic subgroups, because an HNN-extension of the infinite cyclic group may fail to be residually finite, as it happens for many Baumslag–Solitar groups.

Let Γ be a finite simplicial graph, and let $\mathcal{V}$ and $\mathcal{E}$ be the sets of vertices and edges of Γ respectively. The *right-angled Artin group* G, associated to Γ, is given by the presentation

$$G := \langle \mathcal{V} \mid uv = vu \text{ whenever } u, v \in \mathcal{V} \text{ and } (u, v) \in \mathcal{E} \rangle. \tag{1.1}$$

In the literature, right-angled Artin groups are also called *graph groups* or *partially commutative groups*. These groups received a lot of attention in the recent years: they seem to be interesting from both combinatorial and geometric viewpoints (they are fundamental groups of compact non-positively curved cube complexes). A good overview of the current results concerning right-angled Artin groups can be found in R. Charney's paper [13].

In the case when the finite graph Γ is a simplicial tree, conjugacy separability of the associated right-angled Artin group was proved by E. Green [29]. It also follows from the result of Ribes, Segal and Zalesskii [54] mentioned above, because such *tree groups* are easily seen to belong to the class $\mathfrak{X}$.

Following [12], we will say that a group G is *hereditarily conjugacy separable* if every finite-index subgroup of G is conjugacy separable.

Note that all of the classes of conjugacy separable groups that we mentioned above (possibly, with the exception of class $\mathfrak{X}$) consist, in fact, of hereditarily conjugacy separable groups due to the obvious reason: these classes are closed under taking subgroups of finite index. However, there exist conjugacy separable, but not hereditarily separable groups. The first (infinitely generated) example, demonstrating this, was constructed by Chagas and Zalesskii in [12]. It is also possible to find finitely generated and finitely presented examples of this sort even among subgroups of right-angled Artin groups (see [41]).

The main result of this work is the following theorem:

Theorem 1.1. *Right-angled Artin groups are hereditarily conjugacy separable.*

Remark that a finite-index subgroup of a right-angled Artin group may not be a right-angled Artin group itself. The following example was suggested to the author by M. Bridson:

Example 1.2. Let S be a finite group with a (finite) non-trivial second homology group $H_2(S)$ (for instance, on can take S to be the alternating group A_5, since $H_2(A_5) \cong \mathbb{Z}/2\mathbb{Z}$). As we know, there is an epimorphism $\psi\colon F \to S$ for some finitely generated free group F. Let $K := \{(x, y) \in F \times F \mid \psi(x) = \psi(y)\}$ be the fibre product associated to ψ. Observe that $F \times F$ is a right-angled Artin group (associated to some finite complete bipartite graph) and K is a finite-index subgroup in it. By [8], Theorem A, $H_2(S)$ embeds into $H_1(K) \cong K/[K, K]$. Therefore K is not isomorphic to any right-angled Artin group, because the abelianization of a right-angled Artin group is always free abelian and thus is torsion-free.

Generally speaking, we think that hereditary conjugacy separability is a lot stronger than simply conjugacy separability. Corollaries in the next section can be viewed as a confirmation of this.

Our proof of Theorem 1.1 is purely combinatorial and mostly self-contained (we use basic properties of right-angled Artin groups and HNN-extensions). The basic idea is to approximate right-angled Artin groups by HNN-extensions of finite groups (which are, of course, virtually free). This is the main step of the proof. Once this is done, we can use known properties of virtually free groups to obtain the desired results.

In Section 3 we study the *Centralizer Condition*, which, among other things, shows that a given conjugacy separable group is hereditarily conjugacy separable. This condition was originally introduced by Chagas and Zalesskii in [12], but in a different form. In Sections 4, 5 and 7 we develop machineries of *commuting retractions* and *special HNN-extensions* which are the two basic tools behind the proof of Theorem 1.1.

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2. Consequences of the main theorem

Recall that a subgroup H of a group G is said to be a *virtual retract* of G, if there is a finite-index subgroup $K \leq G$ such that $H \leq K$ and H is a retract of K (see Section 4 for the definition).

It is not difficult to show (see Lemma 9.5) that a virtual retract of a hereditarily conjugacy separable group is itself hereditarily conjugacy separable. Therefore, Theorem 1.1 immediately yields

Corollary 2.1. *If G is a right-angled Artin group and H is a virtual retract of G, then H is hereditarily conjugacy separable.*

In view of the above corollary, it makes sense to define two classes of groups: the class $\mathcal{VR}$ will consist of all groups which are virtual retracts of finitely generated right-angled Artin groups, and the class $\mathcal{AVR}$ will consist of groups that contain finite-index subgroups from the class $\mathcal{VR}$.

Looking at the definition, it might seem that the class of right-angled Artin groups is not very large. However, the class of subgroups and virtual retracts of right-angled Artin groups is quite rich and includes many interesting examples. For instance, in the famous paper [5] M. Bestvina and N. Brady constructed subgroups of right-angled Artin groups which have the property FP_2 but are not finitely presented.

On the other hand, in the recent work [34] F. Haglund and D. Wise introduced a new class of *special* (or *A-special*, in the terminology of [34]) cube complexes, that admit a combinatorial local isometry to the Salvetti cube complex (see [13]) of a right-angled Artin group (possibly, infinitely generated). They proved that the fundamental group of every special complex $\mathcal{X}$ embeds into some right-angled Artin group G (if $\mathcal{X}$ is not compact and has infinitely many hyperplanes, then the corresponding right-angled Artin group G will be associated to an infinite graph Γ, and, hence, will not be finitely generated).

An important property established by Haglund and Wise in [34], states that if $\mathcal{X}$ is a compact A-special cube complex, then $\pi_1(\mathcal{X})$ is a virtual retract of some right-angled Artin group, i.e., $\pi_1(\mathcal{X}) \in \mathcal{VR}$. Therefore, using Corollary 2.1, we immediately obtain

Corollary 2.2. *If H is the fundamental group of a compact A-special cube complex, then H is hereditarily conjugacy separable.*

Moreover, many other groups are *virtually special*, i.e., they possess finite-index subgroups that are fundamental groups of special cube complexes. Among virtually special groups are all Coxeter groups (see [35]), fundamental groups of compact surfaces (see [18]), fundamental groups of compact virtually clean square $\mathcal{VH}$-complexes, introduced by Wise in [59], (see [34]), graph braid groups, introduced by A. Abrams in [1], (see [18]), and some hyperbolic 3-manifold groups (see [16]).

In this paper we will mainly discuss applications of Theorem 1.1 to Coxeter groups, even though similar corollaries can be derived for the other classes of virtually special groups listed above.

Recall that a *Coxeter group* is a group G given by the presentation

$$G = \langle s_1, \dots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \text{ for all } i, j \text{ with } m_{ij} \in \mathbb{N} \rangle, \tag{2.1}$$

where $M := (m_{ij})$ is a symmetric $n \times n$ matrix, whose entries satisfy the following conditions: $m_{ii} = 1$ for every $i = 1, \dots, n$, $m_{ij} \in \mathbb{N} \cup \{\infty\}$, and $m_{ij} \geq 2$ whenever $1 \leq i < j \leq n$. In the case, when $m_{ij} \in \{2, \infty\}$ for all $i \neq j$, G is said to be a *right-angled Coxeter group*.

For any Coxeter group G, G. Niblo and L. Reeves [48] constructed a locally finite, finite dimensional CAT(0) cube complex $\mathcal{C}$ on which G acts properly discontinuously. In [35] Haglund and Wise show that G has a finite-index subgroup F such that F acts freely on $\mathcal{C}$ and the quotient $F \backslash \mathcal{C}$ is an A-special cube complex. In the case when G is right-angled or word hyperbolic (in Gromov's sense [30]), Niblo and Reeves proved that the action of G on $\mathcal{C}$ is cocompact (see [48]). These results, combined with the virtual retraction theorem of Haglund and Wise mentioned above, imply that word hyperbolic (or right-angled) Coxeter groups belong to the class $\mathcal{AVR}$. Therefore, using Corollary 2.1 we achieve

Corollary 2.3. *Every word hyperbolic* (*or right-angled*) *Coxeter group G contains a finite-index subgroup F which is hereditarily conjugacy separable.*

Actually, as the paragraph above Corollary 2.3 shows, the conclusion of this corollary holds for every finitely generated Coxeter group G, whose action on the corresponding Niblo-Reeves cube complex is cocompact. Such Coxeter groups were completely characterized by P.-E. Caprace and B. Mühlherr in [10].

The sole fact of existence of a conjugacy separable finite-index subgroup F in G may seem somewhat unsatisfactory. However, every Coxeter group is virtually torsion-free, and in a given Coxeter group G it is usually easy to find some torsion-free subgroup of finite index (for instance, if G is a right-angled Coxeter group (2.1), then the kernel of the natural homomorphism from G onto $\langle s_1\rangle_2 \times \cdots \times \langle s_n\rangle_2 \cong (\mathbb{Z}/2\mathbb{Z})^n$ is torsion-free). The following statement is proved in Section 9:

Corollary 2.4. *If G is a word hyperbolic Coxeter group, then every torsion-free finite-index subgroup $H \le G$ is hereditarily conjugacy separable.*

The above corollary produces a lot of new examples of conjugacy separable groups. More generally, in Corollary 9.11 we show that every torsion-free word hyperbolic group from the class $\mathcal{AVR}$ is hereditarily conjugacy separable.

Now, let us discuss some other consequences of the main result. Besides being a classical subject of group theory, conjugacy separability has two main applications. One of the applications was found by E. Grossman in [31], where she showed that the outer automorphism group $\mathrm{Out}(G)$ of a finitely generated conjugacy separable group G is residually finite, provided that every pointwise inner automorphism of G is inner (an automorphism $\phi \in \mathrm{Aut}(G)$ is called *pointwise inner* if for every $g \in G$, $\phi(g)$ is conjugate to g in G). Thereafter, Grossman used this observation to prove that the mapping class group of a compact orientable surface is residually finite.

Note that for a finitely generated residually finite group G, the group of outer automorphisms $\mathrm{Out}(G)$ need not be residually finite (this should be compared with the classical theorem of G. Baumslag [4] claiming that the automorphism group $\mathrm{Aut}(G)$ of a finitely generated residually finite group G is residually finite). This is a consequence of the result of I. Bumagina and D. Wise ([9]) which asserts that for every finitely presented group S there exists a finitely generated residually finite group G such that $\mathrm{Out}(G) \cong S$.

In Section 6 we prove that pointwise inner automorphisms of right-angled Artin groups are inner (see Proposition 6.9). Thus Grossman's result, combined with Theorem 1.1, gives

Theorem 2.5. *For any right-angled Artin group G, the group of outer automorphisms* $\mathrm{Out}(G)$ *is residually finite.*

Presently not much is yet known about the outer automorphisms of an arbitrary right-angled Artin group G. M. Laurence [37] showed that $\mathrm{Aut}(G)$ (and, hence, $\mathrm{Out}(G)$) is finitely generated. More recently, M. Day [19] proved that $\mathrm{Aut}(G)$ (and, hence, $\mathrm{Out}(G)$) is finitely presented. In [14] R. Charney and K. Vogtmann showed that $\mathrm{Out}(G)$ is virtually torsion-free and has finite virtual cohomological dimension. Imposing additional conditions on the finite graph Γ, corresponding to G, M. Gutierrez, A. Piggott and K. Ruane were able to extract more information about the structure of $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ in [33]. After finishing this article the author learned that Charney and Vogtmann gave a different proof of Theorem 2.5 in [15].

On the other hand, in Section 10 we use a recent result of the author with D. Osin from [44] to prove the following theorem:

Theorem 2.6. *If $G \in \mathcal{AVR}$ is a relatively hyperbolic group, then* $\operatorname{Out}(G)$ *is residually finite.*

Note that Theorem 2.5 is not a consequence of Theorem 2.6: it is not difficult to show that a (non-cyclic) right-angled Artin group G is relatively hyperbolic if and only if the graph Γ, corresponding to G, is disconnected.

Applying Theorem 2.6 to our favorite class of groups from $\mathcal{AVR}$, we achieve

Corollary 2.7. *For any word hyperbolic Coxeter group G,* $\operatorname{Out}(G)$ *is residually finite.*

Unlike automorphisms of right-angled Artin groups, automorphism groups of Coxeter groups have already attracted a lot of attention. In many particular cases the structure of the (outer) automorphism group is known: see, for instance, P. Bahls's paper [3] and references therein. However, because of its generality, the statement of Corollary 2.7 seems to be new.

The second classical application of conjugacy separability was found by A. Mal'cev. As Mal'cev proved in [39] (see also [45]), a finitely presented conjugacy separable group G has solvable conjugacy problem. Observe that finite presentability of G is important here, because the set of finite quotients of an infinitely presented group does not need to be recursively enumerable.

It follows that the conjugacy problem is solvable for every group $G \in \mathcal{VR}$: G is finitely presented as a retract of a finitely presented group, and G is conjugacy separable by Corollary 2.1. However, most of the groups from the class $\mathcal{VR}$ that we discussed above are already known to have solvable conjugacy problem. Moreover, J. Crisp, E. Godelle and B. Wiest [17] showed that the conjugacy problem for fundamental groups of A-special cube complexes can be solved in linear time.

Nevertheless, the property of hereditary conjugacy separability for a group G turns out to be powerful enough to yield conjugacy separability and solvability of the conjugacy problem for many subgroups which are not virtual retracts of G – see Corollary 11.2.

Recall that a group G is called *subgroup separable* (or LERF) if every finitely generated subgroup $H \leq G$ is separable in G. In Section 11 we prove

Theorem 2.8. *Let N be a normal subgroup of a right-angled Artin group G such that the quotient G/N is subgroup separable. Then N is conjugacy separable. If, in addition, N is finitely generated, then N has solvable conjugacy problem.*

Note that requiring G/N to be subgroup separable cannot be dropped in the above statement: in [43] C. Miller gives an example of a finitely generated subgroup of $F_2 \times F_2$ that has unsolvable conjugacy problem (here F_2 denotes the free group of rank 2, and so $F_2 \times F_2$ is the right-angled Artin group associated to a square).

The second claim of Theorem 2.8 may seem surprising: in general we cannot use Mal'cev's result, mentioned above, to reach the needed conclusion, because the conditions imposed on N do not constrain it to be finitely presented. Indeed, let G be the right-angled Artin group associated to a finite graph Γ and given by (1.1). Let N_Γ be the kernel of the homomorphism $\psi: G \to \mathbb{Z}$ satisfying $\psi(v) = 1$ for each $v \in \mathcal{V}$, and let $\mathcal{L}_\Gamma$ be the simplicial flag complex, whose 1-skeleton is Γ.

J. Meier and L. VanWyk [42] proved that the group N_Γ is finitely generated if and only if the graph Γ is connected. And in [5] Bestvina and Brady showed that N_Γ is finitely presented if and only if the complex $\mathcal{L}_\Gamma$ is simply connected. In the case when Γ is connected, we will say that N_Γ is the *Bestvina–Brady group* associated to Γ.

For example, if the graph Γ is a cycle of length at least 4, then N_Γ is finitely generated, but not finitely presented. Obviously, the quotient $G/N_\Gamma \cong \mathbb{Z}$ is subgroup separable, hence, by Theorem 2.8, N_Γ is conjugacy separable and has solvable conjugacy problem. More generally, we have the following corollary:

Corollary 2.9. *If N is a finitely generated normal subgroup of a right-angled Artin group G such that G/N is abelian, then N is hereditarily conjugacy separable and has solvable conjugacy problem. In particular, Bestvina–Brady groups are hereditarily conjugacy separable and have solvable conjugacy problem.*

Corollary 2.9 is a direct consequence of Corollary 11.3 (proved at the end of Section 11), that covers the more general case when G/N is polycyclic. We have chosen to mention the particular situation when the quotient G/N abelian in Corollary 2.9, because in this case one can tell whether or not the given normal subgroup N is finitely generated, using Bieri–Neumann–Strebel invariants, which were studied for right-angled Artin groups by Meier and VanWyk in [42].

After finishing this paper the author learned that the positive solution of the conjugacy problem for the group N from Theorem 2.8 (or Corollary 2.9) has been known before. This follows from a more general result of M. Bridson [7], claiming that a normal subgroup N of a bicombable group G has solvable conjugacy problem, provided G/N has solvable generalized word problem (see [7] for the definitions). Indeed, any right-angled Artin group G acts properly and cocompactly on a CAT(0) space (which is the universal cover of the corresponding compact non-positively curved Salvetti cube complex), therefore G is bicombable by a theorem of J. Alonso and M. Bridson [2]. And if $N \lhd G$, the subgroup separability of G/N implies that G/N has solvable generalized word problem, by Mal'cev's result [39].

Nevertheless, the statement claiming that N is conjugacy separable in Theorem 2.8 (resp. hereditarily conjugacy separable in Corollary 2.9) is new. Our solution of the conjugacy problem for N uses a Mal'cev-type argument and can be viewed as another application of hereditary conjugacy separability of G.

3. Hereditary conjugacy separability and Centralizer Conditions

First, let us specify some notations. If G is a group, $H \le G$ is a subgroup and $g \in G$, then the H*-conjugacy class* of the element $g \in G$ is the subset $g^H := \{hgh^{-1} \mid h \in H\} \subseteq G$. For any $A \subseteq G$, we denote $A^H := \{hah^{-1} \mid a \in A,\, h \in H\}$. The H*-centralizer* of $g \in G$ is the subgroup $C_H(g) := \{h \in H \mid hg = gh\} \le G$. For an epimorphism $\psi : G \to F$ from G onto a group F, $\psi^{-1} : 2^F \to 2^G$ will denote the corresponding *full preimage map*. If A, B are two subsets of G then their *product* AB is defined as the subset $\{ab \mid a \in A,\, b \in B\} \subseteq G$. Note that if either A or B is empty, then the product AB is empty as well.

Definition 3.1. Suppose that G is a group. We will say that G satisfies the *Centralizer Condition* (briefly, CC), if for every finite-index normal subgroup $K \lhd G$ and every $g \in G$ there is a finite-index normal subgroup $L \lhd G$ such that $L \le K$ and

$$C_{G/L}(\bar{g}) \subseteq \psi(C_G(g)K) \tag{3.1}$$

in G/L (where $\psi\colon G \to G/L$ is the natural epimorphism and $\bar{g} := \psi(g)$).

Note that (3.1) is equivalent to $\psi^{-1}(C_{G/L}(\bar{g})) \subseteq C_G(g)K$ in G, since $\ker(\psi) = L \le K$.

The idea behind this condition is to provide control over the growth of centralizers in finite quotients of G. If the group G is residually finite, the Centralizer Condition CC can be reformulated in terms of the topology on the *profinite completion* $\widehat{G}$ of the group G. In the Appendix to this paper (see Corollary 12.2) we prove that the condition CC from Definition 3.1 is equivalent to

$$\overline{C_G(g)} = C_{\widehat{G}}(g) \tag{3.2}$$

in $\widehat{G}$ for every $g \in G$ (where $\overline{C_G(g)}$ denotes the closure of $C_G(g)$ in $\widehat{G}$).

Originally the condition (3.2) appeared in the recent work of Chagas and Zalesskii [12], where they proved that a conjugacy separable group G satisfying (3.2) is hereditarily conjugacy separable (see [12], Proposition 3.1). We will actually show that, provided G is conjugacy separable, this condition is equivalent to hereditary conjugacy separability:

Proposition 3.2. *Let G be a group. Then the following are equivalent:*

(a) *G is hereditarily conjugacy separable;*

(b) *G is conjugacy separable and satisfies* CC.

Before proving Proposition 3.2, let us define two more conditions.

Definition 3.3. Let G be a group, $H \le G$ and $g \in G$. We will say that the pair (H, g) satisfies the *Centralizer Condition in G* (briefly, CC_G), if for every finite-index normal subgroup $K \lhd G$ there is a finite-index normal subgroup $L \lhd G$ such that $L \le K$ and $C_{\bar{H}}(\bar{g}) \subseteq \psi(C_H(g)K)$ in G/L, where $\psi\colon G \to G/L$ is the natural homomorphism, $\bar{H} := \psi(H) \le G/L$, $\bar{g} := \psi(g) \in G/L$.

The subgroup H will be said to satisfy the *Centralizer Condition in G* (briefly, CC_G) if for each $g \in G$, the pair (H, g) has CC_G.

Now let us demonstrate why the Centralizer Conditions are useful.

Lemma 3.4. *Suppose that G is a group, $H \le G$ and $g \in G$. Assume that the pair (G, g) satisfies CC_G and the conjugacy class g^G is separable in G. If the double coset $C_G(g)H$ is separable in G, then the H-conjugacy class g^H is also separable in G.*

Proof. Consider any element $y \in G$ with $y \notin g^H$.

If $y \notin g^G$, then, using the separability of g^G, we can find a finite quotient Q of G and a homomorphism $\phi\colon G \to Q$ so that $\phi(y) \notin \phi(g)^Q$. Hence $\phi(y) \notin \phi(g)^{\phi(H)} = \phi(g^H)$, as required.

Therefore we can assume that $y = zgz^{-1}$ for some $z \in G$. If there existed an element $f \in C_G(g) \cap z^{-1}H$, then $zf \in H$ and $y = zgz^{-1} = (zf)g(zf)^{-1} \in g^H$, leading to a contradiction with our assumption on y. Hence $C_G(g) \cap z^{-1}H = \emptyset$, i.e., $z^{-1} \notin C_G(g)H$. Since $C_G(g)H$ is separable in G, there is $K \lhd G$ such that $|G : K| < \infty$ and $z^{-1} \notin C_G(g)HK$. Now, the condition CC_G implies that there exists a finite-index normal subgroup $L \lhd G$ such that $L \le K$ and $C_{G/L}(\psi(g)) \subseteq \psi(C_G(g)K)$, where $\psi\colon G \to G/L$ is the natural epimorphism.

We claim that $\psi(y) \notin \psi(g^H)$ in G/L. Indeed, if $\psi(y) = \psi(hgh^{-1})$ for some $h \in H$, then $\psi(z^{-1}h) \in C_{G/L}(\psi(g))$. Hence $\psi(z^{-1}) \in C_{G/L}(\psi(g))\psi(H) \subseteq \psi(C_G(g)KH)$, i.e., $z^{-1} \in C_G(g)KHL = C_G(g)HK$ because $L \le K \lhd G$. But this yields a contradiction with the construction of K.

Therefore we have found an epimorphism ψ from G to a finite group G/L such that the image of y does not belong to the image of g^H. Hence g^H is separable in G. □

Observe that for a subgroup H of a group G and any subset $S \subseteq H$, if S is closed in $\mathcal{PT}(G)$, then S is closed in $\mathcal{PT}(H)$. Therefore Lemma 3.4 immediately implies the following:

Corollary 3.5. *Let G be a conjugacy separable group satisfying* CC, *and let $H \le G$ be a subgroup such that $C_G(h)H$ is separable in G for every $h \in H$. Then H is conjugacy separable. Moreover, for each $h \in H$ the H-conjugacy class h^H is closed in $\mathcal{PT}(G)$.*

It is not difficult to see that Lemma 3.4 has a partial converse (we leave its proof as an exercise for the reader):

Remark 3.6. Assume that H is a subgroup of a group G and $g \in G$ is an arbitrary element. If g^H is separable in G then the double coset $C_G(g)H$ is separable in G.

In this paper we are going to use another converse to Lemma 3.4:

Lemma 3.7. *Let G be a group. Suppose that $H \le G$, $g \in G$, $K \lhd G$ and $|G : K| < \infty$. If the subset $g^{H \cap K}$ is separable in G, then there is a finite-index normal subgroup $L \lhd G$ such that $L \le K$ and $C_{\bar{H}}(\bar{g}) \subseteq \psi(C_H(g)K)$ in G/L (in the notations of Definition* 3.3*).*

Proof. Write $k := |H : (H \cap K)| \le |G : K| < \infty$. Then $H = \bigsqcup_{i=1}^{k} z_i(H \cap K)$ for some $z_1, \dots, z_k \in H$. Renumbering the elements z_i, if necessary, we can suppose that there is $l \in \{0, 1, \dots, k\}$ such that whenever $1 \le i \le l$, $z_i^{-1} g z_i \notin g^{H \cap K}$, and whenever $l + 1 \le j \le k$, $z_j^{-1} g z_j \in g^{H \cap K}$ in G.

By the assumptions, there exists a finite-index normal subgroup $L \lhd G$ such that $z_i^{-1} g z_i \notin g^{H \cap K} L$ whenever $1 \le i \le l$. Moreover, after replacing L with $L \cap K$, we can assume that $L \le K$.

Let ψ be the natural epimorphism from G to G/L and consider any element $\bar{x} \in C_{\bar{H}}(\bar{g})$. Then $\bar{x} = \psi(x)$ for some $x \in H$, and $\psi(x^{-1} g x) = \psi(g)$ in G/L, i.e., $x^{-1} g x \in gL$ in G. As we know, there is $i \in \{1, \dots, k\}$ and $y \in H \cap K$ such that $x = z_i y$. Consequently, $z_i^{-1} g z_i \in y g L y^{-1} = y g y^{-1} L \subseteq g^{H \cap K} L$. Hence, $i \ge l + 1$, that is, $z_i^{-1} g z_i = u g u^{-1}$ for some $u \in H \cap K$.

Thus $z_i u \in C_H(g)$ and $x = z_i y = (z_i u)(u^{-1} y) \in C_H(g)(H \cap K) \subseteq C_H(g)K$. Therefore we proved that $\bar{x} \in \psi(C_H(g)K)$ in G/L for every $\bar{x} \in C_{\bar{H}}(\bar{g})$. This yields the inclusion $C_{\bar{H}}(\bar{g}) \subseteq \psi(C_H(g)K)$ in G/L, as required. □

We are now ready to prove Proposition 3.2.

Proof of Proposition 3.2. First let us assume (b). Consider an arbitrary finite-index subgroup $H \le G$. For every $h \in H$ the double coset $C_G(h)H$ is a finite union of left cosets modulo H, hence it is separable in G. Therefore, by Corollary 3.5, H is conjugacy separable. That is, (b) implies (a).

Now, assume that G is hereditarily conjugacy separable. We need to show that G satisfies CC. Take any $g \in G$ and any $K \lhd G$ with $|G : K| < \infty$. Observe that the subgroup $H := K\langle g \rangle \le G$ has finite index in G, and $g^H = g^K = g^{H \cap K}$. Since H is conjugacy separable, g^H is closed in $\mathcal{PT}(H)$, but then it is also closed in $\mathcal{PT}(G)$ because any finite-index subgroup of H has finite index in G. Therefore $g^{H \cap K} = g^H$ is separable in G, and so we can apply Lemma 3.7 to find the finite-index normal subgroup $L \lhd G$ from its claim. Hence the group G satisfies CC. □

4. Commuting retractions

In this section we establish certain properties of commuting retractions that constitute the core of our approach to studying residual properties of right-angled Artin groups. This approach is based on a simple observation that canonical retractions of a right-angled Artin group onto its special subgroups pairwise commute (see Remark 6.1 in Section 6).

Let G be a group and let H be a subgroup of G. Recall that an endomorphism $\rho_H : G \to G$ is called a *retraction* of G onto H if $\rho_H(G) = H$ and $\rho_H(h) = h$ for every $h \in H$. In this case H is said to be a *retract* of G. Note that $\rho_H \circ \rho_H = \rho_H$. If H is a retract and $g \in G$, then the subgroup $F := gHg^{-1} \leq G$, conjugate to H in G, is also a retract. The corresponding retraction $\rho_F \in \mathrm{End}(G)$ is given by the formula $\rho_F(x) := g\rho_H(g^{-1}xg)g^{-1}$ for all $x \in G$.

The following observation is very useful.

Lemma 4.1. *Let H be a retract of a group G and let $\rho_H : G \to G$ be the corresponding retraction. Suppose that $M \lhd G$ satisfies $\rho_H(M) \subseteq M$. Then the retraction ρ_H canonically induces a retraction $\rho_{\bar{H}} : G/M \to G/M$ of G/M onto the natural image $\bar{H}$ of H in G/M, defined by the formula $\rho_{\bar{H}}(gM) = \rho_H(g)M$ for all $gM \in G/M$.*

Proof. Evidently, it is enough to check that $\rho_{\bar{H}}$ is well-defined. If $g_1M = g_2M$ for some $g_1, g_2 \in G$, then $f = g_2^{-1}g_1 \in M$, $g_1 = g_2 f$ and $\rho_H(f) \in M$. Hence

$$\rho_{\bar{H}}(g_1M) = \rho_H(g_1)M = \rho_H(g_2)\rho_H(f)M = \rho_H(g_2)M = \rho_{\bar{H}}(g_2M),$$

as required. □

Assume that H and F are two retracts of a group G and $\rho_H, \rho_F \in \mathrm{End}(G)$ are the corresponding retractions. We will say ρ_H *commutes* with ρ_F if they commute as elements of the monoid of endomorphisms $\mathrm{End}(G)$, i.e., if $\rho_H(\rho_F(g)) = \rho_F(\rho_H(g))$ for all $g \in G$.

Remark 4.2. If the retractions ρ_H and ρ_F commute then $\rho_H(F) = H \cap F = \rho_F(H)$ and the endomorphism $\rho_{H\cap F} := \rho_H \circ \rho_F = \rho_F \circ \rho_H$ is a retraction of G onto $H \cap F$.

Indeed, obviously the restriction of $\rho_{H\cap F}$ to $H \cap F$ is the identity map. And $\rho_{H\cap F}(G) \subseteq \rho_H(G) \cap \rho_F(G) = H \cap F$, hence $\rho_{H\cap F}(G) = H \cap F$. Consequently $\rho_H(F) = \rho_H(\rho_F(G)) = \rho_{H\cap F}(G) = H \cap F$. Similarly, $\rho_F(H) = H \cap F$.

In the next proposition we establish an important property of commuting retractions that could be of independent interest.

Proposition 4.3. *Let $H_1, \dots, H_m$ be retracts of a group G such that the corresponding retractions $\rho_{H_1}, \dots, \rho_{H_m}$ pairwise commute. Then for any finite-index normal subgroup $K \lhd G$ there is a finite-index normal subgroup $M \lhd G$ such that $M \leq K$*

and $\rho_{H_i}(M) \subseteq M$ for each $i = 1, \dots, m$. Consequently, for every $i = 1, \dots, m$, the retraction ρ_{H_i} canonically induces a retraction $\rho_{\overline{H}_i}$ of G/M onto the image $\overline{H}_i$ of H_i in G/M.

Proof. The second claim of the proposition follows from Lemma 4.1, so it suffices to construct the subgroup $M \lhd G$ with the needed properties.

If $J = \{i_1, \dots, i_k\}$ is a subset of the finite set $I := \{1, 2, \dots, m\}$, we define the retraction ρ_J of G onto $\bigcap_{j \in J} H_j$ by

$$\rho_J := \rho_{H_{i_1}} \circ \rho_{H_{i_2}} \circ \dots \circ \rho_{H_{i_k}}.$$

This makes sense since our retractions pairwise commute. When $J = \emptyset$, ρ_J will be the identity map of G.

Now, for every subset J of $I = \{1, 2, \dots, m\}$ we define the subgroup $D_J \leq G$ as follows. First we set $D_I := \bigcap_{i=1}^m H_i \cap K$ – a finite-index normal subgroup of $(H_1 \cap \dots \cap H_m)$. Next, if J is a proper subset of I, we define D_J recursively, according to the formula

$$D_J := \rho_J\Big(\bigcap_{i \in I \setminus J} \rho^{-1}_{J \cup \{i\}}(D_{J \cup \{i\}})\Big) \cap K, \tag{4.1}$$

where $\rho^{-1}_{J \cup \{i\}}(D_{J \cup \{i\}})$ denotes the full preimage (under $\rho_{J \cup \{i\}}$) of $D_{J \cup \{i\}}$ in G.

Since the intersection of a finite number of finite-index normal subgroups is again a finite-index normal subgroup, and images, as well as full preimages, of finite-index normal subgroups under homomorphisms are again normal and of finite index (in their respective groups), we see that D_J is normal and has finite index in $\rho_J(G) = \bigcap_{j \in J} H_j$. Thus, if we set $M := D_\emptyset = \bigcap_{i \in I} \rho^{-1}_{H_i}(D_{\{i\}}) \cap K$, we shall have $M \lhd G$, $|G : M| < \infty$ and $M \leq K$.

If $J \subset I$ and $i \in I \setminus J$, using (4.1) and the fact that $\rho_{\{i\}} \circ \rho_J = \rho_{J \cup \{i\}}$, we can observe that $\rho_{\{i\}}(D_J) \subseteq D_{J \cup \{i\}}$.

On the other hand, let us show that $D_{J \cup \{i\}} \subseteq D_J$. We will use induction on the cardinality $|I \setminus J|$. If $|I \setminus J| = 1$ then $I = J \sqcup \{i\}$. And if $g \in D_{J \cup \{i\}} = D_I = \bigcap_{i \in I} H_i \cap K$, then $\rho_I(g) = g$, therefore $g \in \rho_I^{-1}(D_I)$ and $g = \rho_J(g) \in \rho_J(\rho_I^{-1}(D_I))$. Thus $g \in D_J$.

Now suppose that the statement has been proved for all proper subsets J' of I with $|J'| > |J|$. Take any $i \in I \setminus J$ and consider an element $g \in D_{J \cup \{i\}} \leq \bigcap_{j \in J \cup \{i\}} H_j \cap K$. Then $\rho_{J \cup \{i\}}(g) = g$, therefore $g \in \rho^{-1}_{J \cup \{i\}}(D_{J \cup \{i\}})$. We need to show that for any $i' \in I \setminus (J \cup \{i\})$, $g \in \rho^{-1}_{J \cup \{i'\}}(D_{J \cup \{i'\}})$, or, equivalently, that $\rho_{J \cup \{i'\}}(g) \in D_{J \cup \{i'\}}$. But

$$\rho_{J \cup \{i'\}}(g) = \rho_{i'}(\rho_J(g)) = \rho_{\{i'\}}(g) \in \rho_{\{i'\}}(D_{J \cup \{i\}}) \subseteq D_{J \cup \{i,i'\}}.$$

And, since $D_{J \cup \{i,i'\}} \subseteq D_{J \cup \{i'\}}$ by the induction hypothesis, we can conclude that $g \in \bigcap_{i' \in I \setminus J} \rho^{-1}_{J \cup \{i'\}}(D_{J \cup \{i'\}})$. Recalling that $g \in \bigcap_{j \in J} H_j \cap K$, we achieve $g = \rho_J(g) \in D_J$. Thus $D_{J \cup \{i\}} \subseteq D_J$ and the inductive step is established.

We are now able to show that $\rho_{H_i}(M) \subseteq M$ for every $i \in I$. Indeed, since $\rho_{H_i}(M) \subseteq D_{\{i\}}$, it is enough to check that $D_{\{i\}} \subseteq M$. Take any $j \in I$. As we proved, $\rho_{H_j}(D_{\{i\}}) = \rho_{\{j\}}(D_{\{i\}}) \subseteq D_{\{i\}\cup\{j\}} \subseteq D_{\{j\}}$. Therefore, $D_{\{i\}} \subseteq \rho_{H_j}^{-1}(D_{\{j\}})$ for each $j \in J$. By definition, $D_{\{i\}} \leq K$, consequently, for any $i \in I$, we achieve

$$\rho_{H_i}(M) \subseteq D_{\{i\}} \subseteq \bigcap_{i \in I} \rho_{H_i}^{-1}(D_{\{i\}}) \cap K = M,$$

as required. □

The next observation is an easy consequence of the definition of M using the formula (4.1).

Remark 4.4. In Proposition 4.3, if G/K is a finite p-group for some prime number p, then so is G/M.

Given two subgroups H and F of a group G, it is usually difficult to find quotient-groups Q of G such that the image of the intersection of H and F in Q coincides with the intersection of the images of these subgroups in Q. However, in the case when H and F are retracts and the corresponding retractions commute this will be automatic for many quotients of G.

Lemma 4.5. *Suppose that the retractions $\rho_H, \rho_F \in \mathrm{End}(G)$ commute, and $M \lhd G$ is a normal subgroup satisfying $\rho_H(M) \subseteq M$ and $\rho_F(M) \subseteq M$. Then $\varphi(H \cap F) = \varphi(H) \cap \varphi(F)$ in G/M, where $\varphi\colon G \to G/M$ is the natural epimorphism.*

Proof. By Lemma 4.1, ρ_H and ρ_F canonically induce retractions $\rho_{\varphi(H)}$ and $\rho_{\varphi(F)}$ of G/M onto $\varphi(H)$ and $\varphi(F)$ respectively.

Clearly, $\varphi(H \cap F) \subseteq \varphi(H) \cap \varphi(F)$, and so we only need to establish the inverse inclusion. Consider an arbitrary $\bar{g} \in \varphi(H) \cap \varphi(F)$. Then $\bar{g} = \varphi(g)$ for some $g \in G$, and $\rho_{\varphi(F)}(\bar{g}) = \bar{g}$, $\rho_{\varphi(H)}(\bar{g}) = \bar{g}$. Therefore

$$\bar{g} = \rho_{\varphi(H)}(\rho_{\varphi(F)}(\varphi(g))) = \rho_{\varphi(H)}(\varphi(\rho_F(g))) = \varphi(\rho_H(\rho_F(g))) \in \varphi(H \cap F),$$

where the last inclusion follows from Remark 4.2. Thus $\varphi(H) \cap \varphi(F) \subseteq \varphi(H \cap F)$. □

Lemma 4.5 allows to obtain the first interesting application of Proposition 4.3.

Corollary 4.6. *Let $H_1, \ldots, H_m$ be retracts of a group G such that the corresponding retractions $\rho_{H_1}, \ldots, \rho_{H_m}$ pairwise commute. Then for any finite-index normal subgroup $K \lhd G$ there is a finite-index normal subgroup $M \lhd G$ such that $M \leq K$ and $\rho_{H_i}(M) \subseteq M$ for each $i = 1, \ldots, m$. Moreover, if $\varphi\colon G \to G/M$ denotes the natural epimorphism, then $\varphi(\bigcap_{i=1}^m H_i) = \bigcap_{i=1}^m \varphi(H_i)$.*

Proof. First we apply Proposition 4.3 to find the finite-index normal subgroup M from its claim. The last statement of the corollary will be proved by induction on m. If $m = 1$ there is nothing to prove. So let us assume that $m \geq 2$ and we have already shown that $\varphi(\bigcap_{i=1}^{m-1} H_i) = \bigcap_{i=1}^{m-1} \varphi(H_i)$. Using Remark 4.2 we see that the map $\rho_F := \rho_{H_1} \circ \cdots \circ \rho_{H_{m-1}} \in \mathrm{End}(G)$ is a retraction of G onto $F := \bigcap_{i=1}^{m-1} H_i$. By Proposition 4.3, $\rho_{H_i}(M) \subseteq M$ for each $i = 1, \ldots, m$, therefore

$$\begin{aligned}\rho_F(M) &= (\rho_{H_1} \circ \cdots \circ \rho_{H_{m-2}})(\rho_{H_{m-1}}(M)) \\ &\subseteq (\rho_{H_1} \circ \cdots \circ \rho_{H_{m-3}})(\rho_{H_{m-2}}(M)) \subseteq \cdots \subseteq \rho_{H_1}(M) \subseteq M.\end{aligned}$$

By the assumptions, the retractions ρ_F and ρ_{H_m} commute, hence we can apply Lemma 4.5 to conclude that $\varphi(F \cap H_m) = \varphi(F) \cap \varphi(H_m)$. But $\varphi(F) = \bigcap_{i=1}^{m-1} \varphi(H_i)$ by the induction hypothesis, consequently $\varphi(\bigcap_{i=1}^{m} H_i) = \varphi(F \cap H_m) = \bigcap_{i=1}^{m} \varphi(H_i)$, and the proof is finished. □

Let us now give an example which shows that the statements of Corollary 4.6 and Lemma 4.5 are no longer true if the retractions do not commute.

Example 4.7. Let S be any infinite simple group, and let H be an arbitrary group possessing non-trivial finite quotients. Set $G := H * S$, fix an element $s \in S \setminus \{1\}$ and denote $F := sHs^{-1} \leq G$. Evidently H is a retract of G, where the retraction $\rho_H : G \to G$ of G onto H is the identity on H and trivial on S. Clearly the endomorphism $\rho_F \in \mathrm{End}(G)$ defined by $\rho_F(g) := s\rho_H(s^{-1}gs)s^{-1}$ for every $g \in G$, is a retraction of G onto F.

It is not difficult to see that the retractions ρ_H and ρ_F do not commute (for instance, because $(\rho_H \circ \rho_F)(G) = H$, $(\rho_F \circ \rho_H)(G) = F$ and $H \cap F = \{1\}$).

If $K \lhd G$ is an arbitrary proper normal subgroup of finite index, then $S \subset K$ (because S has no non-trivial finite quotients), hence the kernel $\ker(\rho_H)$ (which is equal to the normal closure of S in G) is contained in K. Consequently, $\rho_H(K) \subseteq \rho_H^{-1}(\rho_H(K)) \subseteq K$. Similarly, $\rho_F(K) \subseteq K$.

Observe that $H \cap F = \{1\}$ by construction. Denote by Q the quotient G/K and let $\varphi : G \to Q$ be the natural epimorphism. Since $s \in S \leq \ker(\varphi)$ we see that

$$\varphi(H) \cap \varphi(F) = \varphi(H) = Q \neq \{1\} = \varphi(H \cap F).$$

That is, in any non-trivial finite quotient Q of G the intersection of the images of H and F is strictly larger than the image of $H \cap F$.

5. Implications for the profinite topology

Throughout this section we will assume that A and B are retracts of a group G such that the corresponding retractions $\rho_A \in \mathrm{End}(G)$ and $\rho_B \in \mathrm{End}(G)$ commute. Our goal here is to establish several consequences of these settings for the profinite topology on G.

Lemma 5.1. *For arbitrary elements $x, y \in G$ define $\alpha := \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1}) \in AxB \subseteq G$ and $\beta := \rho_A(\rho_B(y)y^{-1})y\rho_B(y^{-1}) \in AyB \subseteq G$. Then the following two conditions are equivalent:*

(i) $y \in AxB$;
(ii) $\beta \in \alpha^{A\cap B}$.

Proof. Observe that $y \in AxB$ if and only if $AyB = AxB$, which is equivalent to $A\beta B = A\alpha B$. Thus $y \in AxB$ if and only if $\beta \in A\alpha B$.

To show that (i) implies (ii), suppose that there are $a \in A$ and $b \in B$ such that

$$\beta = a\alpha b. \tag{5.1}$$

By definition, $\rho_A(\alpha) = 1 = \rho_A(\beta)$, hence $1 = \rho_A(a)\rho_A(b)$. Therefore, (5.1) implies that $a = \rho_A(a) = \rho_A(b^{-1}) \in \rho_A(B) = A \cap B$ (by Remark 4.2).

Now, since $\rho_B \circ \rho_A = \rho_A \circ \rho_B$, we have $\rho_B(\alpha) = 1 = \rho_B(\beta)$. Therefore, applying ρ_B to both sides of the equality (5.1), we get $1 = \rho_B(a)\rho_B(b) = ab$ because $a, b \in B$. Hence $b = a^{-1} \in A \cap B$ and $\beta = a\alpha a^{-1} \in \alpha^{A\cap B}$.

Now suppose that $\beta \in \alpha^{A\cap B}$. Then $\beta \in (A \cap B)\alpha(A \cap B) \subseteq A\alpha B$. Thus (ii) implies (i). □

Let us look at the proof of the above lemma in the particular case when $y = x$. Then we see that $\beta = \alpha$, and

$$A \cap xBx^{-1} = \gamma^{-1}(A \cap \alpha B\alpha^{-1})\gamma,$$

where $\gamma := \rho_A(\rho_B(x)x^{-1}) \in A$.

We also see that $a \in A\cap\alpha B\alpha^{-1}$ if and only if there is $b \in B$ such that $\alpha = a^{-1}\alpha b$. But, as we showed in the proof of Lemma 5.1, this can happen only if $b = a \in A\cap B$. I.e, $\alpha a = a\alpha$ and $a \in A \cap B$, which is equivalent to $a \in C_{A\cap B}(\alpha)$. Thus, in this particular case we obtain the following statement:

Lemma 5.2. *If $x \in G$ is an arbitrary element, then*

$$A \cap xBx^{-1} = \gamma^{-1}C_{A\cap B}(\alpha)\gamma$$

in G, where $\alpha := \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1}) \in AxB$ and $\gamma := \rho_A(\rho_B(x)x^{-1}) \in A$.

Combining Lemma 5.1 with Corollary 4.6 we achieve

Lemma 5.3. *Consider any $x \in G$ and denote $\alpha := \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1}) \in AxB \subseteq G$. If the conjugacy class $\alpha^{A\cap B}$ is separable in G, then the double coset AxB is also separable in G.*

Proof. Suppose that an element $y \in G$ satisfies $y \notin AxB$. By Lemma 5.1, this is equivalent to $\beta \notin \alpha^{A\cap B}$, where $\beta := \rho_A(\rho_B(y)y^{-1})y\rho_B(y^{-1})$. Since $\alpha^{A\cap B}$ is separable, there is a finite-index normal subgroup $K \lhd G$ such that $\psi(\beta) \notin \psi(\alpha^{A\cap B}) = \psi(\alpha)^{\psi(A\cap B)}$, where $\psi\colon G \to G/K$ is the canonical epimorphism.

By Corollary 4.6, there exists a finite-index normal subgroup $M \lhd G$ such that $M \le K$, $\rho_A(M) \subseteq M$, $\rho_B(M) \subseteq M$ and $\varphi(A\cap B) = \varphi(A) \cap \varphi(B)$, where φ is the natural epimorphism from G to G/M. Since ψ factors through φ, we can conclude that $\varphi(\beta) \notin \varphi(\alpha)^{\varphi(A\cap B)} = \varphi(\alpha)^{\varphi(A)\cap\varphi(B)}$. But by Lemma 4.1, there are canonically induced commuting retractions $\rho_{\bar{A}}$ and $\rho_{\bar{B}}$ of G/M onto $\bar{A} := \varphi(A)$ and $\bar{B} := \varphi(B)$ respectively. Moreover, letting $\bar{x} := \varphi(x)$, $\bar{y} := \varphi(y)$ and using the definition of the retractions $\rho_{\bar{A}}$ and $\rho_{\bar{B}}$, we obtain $\varphi(\alpha) = \rho_{\bar{A}}(\rho_{\bar{B}}(\bar{x})\bar{x}^{-1})\bar{x}\rho_{\bar{B}}(\bar{x}^{-1})$ and $\varphi(\beta) = \rho_{\bar{A}}(\rho_{\bar{B}}(\bar{y})\bar{y}^{-1})\bar{y}\rho_{\bar{B}}(\bar{y}^{-1})$. Therefore, by Lemma 5.1, applied to the retracts $\bar{A}$ and $\bar{B}$ in G/M, we have $\bar{y} \notin \bar{A}\bar{x}\bar{B}$. That is, $\varphi(y) \notin \varphi(AxB)$. Hence the double coset AxB is separable in G. □

Since the $1^{A\cap B} = \{1\}$ is separable in G whenever G is residually finite, we have the following immediate consequence of Lemma 5.3.

Corollary 5.4. *If A and B are retracts of a residually finite group G such that the corresponding retractions commute, then the double coset AB is separable in G.*

The statement of Corollary 5.4 has been known before – see, for example, [34], Lemma 9.3, but the proof that we have presented here is new.

The following statement is well known.

Lemma 5.5. *Suppose that G is a residually finite group and $A \le G$ is a retract of G. If a subset $S \subseteq A$ is closed in $\mathcal{PT}(A)$, then S is closed in $\mathcal{PT}(G)$.*

Proof. We will show that S coincides with its closure $cl_G(S)$ in the profinite topology on G. By Corollary 5.4 the subgroup $A = AA$ is closed in $\mathcal{PT}(G)$, hence $cl_G(S) \subseteq A$. Now, if $a \in A \setminus S$, then there is a homomorphism $\phi\colon A \to Q$ from A to a finite group Q such that $\phi(a) \notin \phi(S)$. Since A is a retract of G, we have a homomorphism $\psi\colon G \to Q$ defined by $\psi := \phi \circ \rho_A$. Evidently $\psi(a) = \phi(a) \notin \phi(S) = \psi(S)$, hence $a \notin cl_G(S)$. Thus $S = cl_G(S)$, as required. □

Now the reason why we need Lemma 5.2 is because it tells us that if one can control the $A\cap B$-centralizers in G, then one can also control the intersections of conjugates of the retracts A and B. As it can be seen from Example 4.7, in general we may not be able to find a finite quotient Q of G, in which the image of the intersection of two particular conjugates of A and B is equal to the intersection of their images. However, provided that a certain Centralizer Condition is satisfied, we can find many finite quotients Q of G where these two sets are very close to each other.

Lemma 5.6. *Let x be an element of G and let $\alpha := \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1}) \in G$. Suppose that the pair $(A \cap B, \alpha)$ satisfies the Centralizer Condition in G. Then for any finite-index normal subgroup $K \lhd G$ there exists a finite-index normal subgroup $M \lhd G$ such that $M \le K$, $\rho_A(M) \subseteq M$, $\rho_B(M) \subseteq M$ and $\varphi(A) \cap \varphi(xBx^{-1}) \subseteq \varphi(A \cap xBx^{-1})\varphi(K)$ in G/M, where $\varphi\colon G \to G/M$ is the natural epimorphism.*

Proof. By Lemma 5.2, $A \cap xBx^{-1} = \gamma^{-1}C_{A\cap B}(\alpha)\gamma$, where $\gamma := \rho_A(\rho_B(x)x^{-1}) \in A$. Since the pair $(A \cap B, \alpha)$ has CC_G, there is a subgroup $L \lhd G$ of finite index in G, such that $L \le K$ and $\psi^{-1}(C_{\psi(A\cap B)}(\psi(\alpha))) \subseteq C_{A\cap B}(\alpha)K$ in G, where $\psi\colon G \to G/L$ is the natural epimorphism. Applying Corollary 4.6 to A, B and L we find a finite-index normal subgroup $M \lhd G$, together with the epimorphism $\varphi\colon G \to G/M$, such that $M \le L \le K$, $\rho_A(M) \subseteq M$, $\rho_B(M) \subseteq M$ and $\varphi(A) \cap \varphi(B) = \varphi(A \cap B)$.

By Lemma 4.1, ρ_A and ρ_B canonically induce retractions $\rho_{\overline{A}}$ and $\rho_{\overline{B}}$ of G/M onto $\overline{A} := \varphi(A)$ and $\overline{B} := \varphi(B)$ respectively. Obviously $\rho_{\overline{A}}$ commutes with $\rho_{\overline{B}}$ in $\mathrm{End}(G/M)$, because ρ_A commutes with ρ_B in $\mathrm{End}(G)$. Put $\bar{x} := \varphi(x)$, $\bar{\alpha} = \rho_{\overline{A}}(\rho_{\overline{B}}(\bar{x})\bar{x}^{-1})\bar{x}\rho_{\overline{B}}(\bar{x}^{-1}) \in G/M$ and $\bar{\gamma} := \rho_{\overline{A}}(\rho_{\overline{B}}(\bar{x})\bar{x}^{-1}) \in \overline{A}$. Observe that $\bar{\alpha} = \varphi(\alpha)$ and $\bar{\gamma} = \varphi(\gamma)$ by the definitions of $\rho_{\overline{A}}$ and $\rho_{\overline{B}}$. Then by Lemma 5.2, $\overline{A} \cap \bar{x}\overline{B}\bar{x}^{-1} = \bar{\gamma}^{-1}C_{\overline{A}\cap\overline{B}}(\bar{\alpha})\bar{\gamma}$ in G/M. Therefore, recalling that $\overline{A} \cap \overline{B} = \varphi(A \cap B)$, we get

$$\varphi^{-1}(\overline{A} \cap \bar{x}\overline{B}\bar{x}^{-1}) = \varphi^{-1}(\bar{\gamma}^{-1}C_{\overline{A}\cap\overline{B}}(\bar{\alpha})\bar{\gamma}) = \gamma^{-1}\varphi^{-1}(C_{\varphi(A\cap B)}(\bar{\alpha}))\gamma.$$

But since ψ factors through φ (as $\ker(\varphi) = M \le L = \ker(\psi)$), we obviously have

$$\varphi^{-1}(C_{\varphi(A\cap B)}(\bar{\alpha})) \subseteq \psi^{-1}(C_{\psi(A\cap B)}(\psi(\alpha))) \subseteq C_{A\cap B}(\alpha)K.$$

Hence we can conclude that $\varphi^{-1}(\overline{A} \cap \bar{x}\overline{B}\bar{x}^{-1}) \subseteq \gamma^{-1}C_{A\cap B}(\alpha)\gamma K = (A \cap xBx^{-1})K$. Consequently, $\varphi(A) \cap \varphi(xBx^{-1}) = \overline{A} \cap \bar{x}\overline{B}\bar{x}^{-1} \subseteq \varphi(A \cap xBx^{-1})\varphi(K)$, and the lemma is proved. □

In this paper we will need one more criterion for separability of specific double cosets in G. In a certain sense it generalizes Remark 3.6.

Lemma 5.7. *Consider arbitrary elements $x, g \in G$. Write $D := xBx^{-1} \le G$ and $\alpha := \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1}) \in G$. Suppose that the conjugacy classes $\alpha^{A\cap B}$ and $g^{A\cap D}$ are separable in G, and the pair $(A \cap B, \alpha)$ satisfies CC_G. Then the double coset $C_A(g)D$ is separable in G.*

Proof. Consider any $z \in G$ with $z \notin C_A(g)D$. First, suppose that $z \notin AD$. Since $\alpha^{A\cap B}$ is separable in G, Lemma 5.3 implies that AxB is separable, hence $AD = (AxB)x^{-1}$ is separable as well (because multiplication by a fixed group element on the right is a homeomorphisms of G with respect to the profinite topology). Therefore there is a finite-index normal subgroup $N \lhd G$ such that $z \notin ADN$, hence $z \notin C_A(g)DN$ because $C_A(g) \le A$.

Thus we can assume that $z \in AD$, i.e., there exist $a_0 \in A$ and $d_0 \in D$ such that $z = a_0 d_0$. Since $z \notin C_A(g)D$, $y := z d_0^{-1} \notin C_A(g)(A \cap D)$. Consequently, for every $h \in A \cap D$, $(yh)g(yh)^{-1} \neq g$, i.e., $y^{-1}gy \neq hgh^{-1}$, implying that $y^{-1}gy \notin g^{A\cap D}$ in G.

Now the separability of $g^{A\cap D}$ in G implies that there is a finite-index normal subgroup $K \lhd G$ such that $y^{-1}gy \notin g^{A\cap D}K$. And, by Lemma 5.6, we can find a finite-index normal subgroup $M \lhd G$ such that $M \leq K$ and $\varphi(A) \cap \varphi(D) \subseteq \varphi(A \cap D)\varphi(K)$, where $\varphi\colon G \to G/M$ is the natural epimorphism. Let $\bar{K}$, $\bar{A}$, $\bar{D}$, $\bar{y}$ and $\bar{g}$ denote the φ-images of K, A, D, y and g respectively.

Since $\bar{K} \lhd G/M$, we have $\bar{g}^{\bar{A}\cap\bar{D}} \subseteq \bar{g}^{\varphi(A\cap D)\bar{K}} \subseteq \bar{g}^{\varphi(A\cap D)}\bar{K}$ and $\bar{y}^{-1}\bar{g}\bar{y} \notin \bar{g}^{\varphi(A\cap D)}\bar{K}$ as $M \leq K$. Hence $\bar{y}^{-1}\bar{g}\bar{y} \notin \bar{g}^{\bar{A}\cap\bar{D}}$.

To finish the proof, it remains to show that $\varphi(z) \notin \varphi(C_A(g)D)$. Suppose, on the contrary, that there exist $a \in C_A(g)$ and $d \in D$ such that $\varphi(z) = \varphi(ad)$. Then $\varphi(a_0 d_0) = \varphi(ad)$, thus $\bar{h} := \varphi(a^{-1}a_0) = \varphi(d d_0^{-1}) \in \bar{A} \cap \bar{D}$, and $\varphi(z) = \varphi(a)\bar{h}\varphi(d_0)$. Consequently, $\bar{y} = \varphi(z)\varphi(d_0^{-1}) = \varphi(a)\bar{h}$ and

$$\bar{y}^{-1}\bar{g}\bar{y} = \bar{h}^{-1}\varphi(a^{-1}ga)\bar{h} = \bar{h}^{-1}\bar{g}\bar{h} \in \bar{g}^{\bar{A}\cap\bar{D}},$$

contradicting to our construction.

Thus, for every $z \notin C_A(g)D$ we found $M \lhd G$ with $|G : M| < \infty$ such that $z \notin C_A(g)DM$. Therefore the double coset $C_A(g)D$ is separable in G. □

6. Some properties of right-angled Artin groups

In this section we recall a few properties of right-angled Artin groups, which will be used in the proof of the main result. At the end of the section we prove that every pointwise inner automorphism of a right-angled Artin group is inner.

Let Γ be a finite graph (without loops or multiple edges) with the set of vertices $\mathcal{V}$. For any vertex $v \in \mathcal{V}$ its *star* $\mathrm{star}(v)$ consists of all vertices (including v itself) that are adjacent to v in Γ. If $\mathcal{S} \subseteq \mathcal{V}$, then $\mathrm{star}(\mathcal{S}) := \bigcap_{v\in\mathcal{S}} \mathrm{star}(v)$. Observe that for two subsets $\mathcal{S}, \mathcal{T} \subseteq \mathcal{V}$, $\mathcal{T} \subseteq \mathrm{star}(\mathcal{S})$ happens if and only if $\mathcal{S} \subseteq \mathrm{star}(\mathcal{T})$.

Let $G = G(\Gamma)$ be the associated right-angled Artin group. To simplify notation, we will identify elements of $\mathcal{V}$ with the corresponding generators of G. Then for each $v \in \mathcal{V}$, $\mathrm{star}(v)$ contains precisely those elements from $\mathcal{V}$ that commute with v in G. For any subset A of G, $A^{\pm1}$ will denote the union $A \cup A^{-1} \subseteq G$. Thus every element $g \in G$ can be represented as a word W in letters from $\mathcal{V}^{\pm1}$. The *support* $\mathrm{supp}(W)$ is the set of all $v \in \mathcal{V}$ such that $v^{\pm1}$ appears as a letter in W. A word W is said to be *graphically reduced* if it has no subwords of the form vUv^{-1} or $v^{-1}Uv$, where $v \in \mathcal{V}$ and $\mathrm{supp}(U) \subseteq \mathrm{star}(v)$. Evidently, if the word W is not graphically reduced, then one can find a shorter word representing the same element of the group G. This process will eventually terminate (because the length of W is finite), hence for each element $g \in G$ there exists a graphically reduced word representing it in G.

E. Green [29] proved that if two graphically reduced words W and W' represent the same element $g \in G$, then W and W' have the same length and $\mathrm{supp}(W) = \mathrm{supp}(W')$. Moreover, for any given $g \in G$, graphically reduced words are precisely the shortest possible words representing g in G (proofs of these facts using re-writing systems can also be found in [25], Section 2.2). Therefore, for any element g we can define its *length* $|g|$ as the length of any graphically reduced word W representing g in G, and its *support* $\mathrm{supp}(g)$ as $\mathrm{supp}(W)$.

Finally, for any $g \in G$ define $\mathrm{FL}(g)$ – *the set of first letters of* g – as the set of all letters $a \in \mathcal{V}^{\pm 1}$ such that a appears as the first letter of some graphically reduced word W representing g in G. Similarly, we define *the set of last letters* $\mathrm{LL}(g)$ of g as those $a \in \mathcal{V}^{\pm 1}$ that appear as a last letter of some graphically reduced word representing g in G. A useful fact observed by Green in [29] states that for any $g \in G$ the letters in $\mathrm{FL}(g)$ pairwise commute (in G). Evidently, $\mathrm{LL}(g) = (\mathrm{FL}(g^{-1}))^{-1}$.

Consider any subset $\mathcal{S}$ of $\mathcal{V}$ and let Δ be the full subgraph of Γ on the vertices from $\mathcal{S}$. Let H denote the right-angled Artin group corresponding to Δ. The identity map on $\mathcal{S}$ can be regarded as a map from the generating set of H into G. Since Δ is a subgraph of Γ all the relations between these generators of H hold between their images in G. Therefore, by von Dyck's Theorem, there is a homomorphism $\xi\colon H \to G$ extending the identity map on $\mathcal{S}$.

On the other hand, since Δ is a full subgraph of Γ, by von Dyck's Theorem, the map $\rho_{\mathcal{S}}\colon \mathcal{V} \cup \{1\} \to \mathcal{V} \cup \{1\}$ defined by $\rho_{\mathcal{S}}(1) := 1$ and

$$\rho_{\mathcal{S}}(v) := \begin{cases} v & \text{if } v \in \mathcal{S}, \\ 1 & \text{if } v \in \mathcal{V} \setminus \mathcal{S}, \end{cases} \tag{6.1}$$

can be extended to a homomorphism $\rho_H\colon G \to H$. Obviously, the composition $\rho_H \circ \xi\colon H \to H$ is the identity map on H. Therefore ξ is injective, hence it is an isomorphism between H and the subgroup of G generated by $\mathcal{S}$. Consequently, ρ_H, regarded as an endomorphism of G, becomes a (canonical) retraction of G onto $\langle \mathcal{S} \rangle \leq G$.

For any $\mathcal{S} \subseteq \mathcal{V}$ the subgroup $H := \langle \mathcal{S} \rangle \leq G$ is called *special* (or *full*, or *canonically parabolic*, depending on the source). Note that the trivial subgroup $\{1\} \leq G$ is also special and corresponds to the empty subset of $\mathcal{V}$. As we saw above, any special subgroup is a right-angled Artin group itself, and is a retract of G. It is easy to see that if $\mathcal{S}$, $\mathcal{T}$ are two subsets of $\mathcal{V}$ then the corresponding maps $\rho_{\mathcal{S}}$ and $\rho_{\mathcal{T}}$ defined by (6.1) commute with each other. This leads to the following important observation.

Remark 6.1. If H and F are special subgroups of a right-angled Artin group G, then H and F are retracts of G and the corresponding canonical retractions $\rho_H, \rho_F \in \mathrm{End}(G)$ commute.

Remark 6.2. If $\mathcal{S}, \mathcal{T} \subseteq \mathcal{V}$ then $\langle \mathcal{S} \rangle \cap \langle \mathcal{T} \rangle = \langle \mathcal{S} \cap \mathcal{T} \rangle$.

Indeed, by Remarks 6.1 and 4.2, we have

$$\langle \mathcal{S} \rangle \cap \langle \mathcal{T} \rangle = \rho_{\langle \mathcal{T} \rangle}(\langle \mathcal{S} \rangle) = \langle \rho_{\langle \mathcal{T} \rangle}(\mathcal{S}) \rangle = \langle \rho_{\mathcal{T}}(\mathcal{S}) \rangle = \langle \mathcal{S} \cap \mathcal{T} \rangle.$$

Recall that a group G is said to have the *Unique Root Property* if for any positive integer n and arbitrary elements $x, y \in G$ the equality $x^n = y^n$ implies that $x = y$ in G. The group G is called *bi-orderable* if G can be endowed with a total order $\preceq$, which is *bi-invariant*, i.e., for any $x, y, z \in G$, if $x \preceq y$, then $zx \preceq zy$ and $xz \preceq yz$.

Lemma 6.3. *Right-angled Artin groups have the Unique Root Property.*

Proof. G. Duchamp and D. Krob [21] (see also [22]) proved that right-angled Artin groups are bi-orderable. Let G be a right-angled Artin group, and let $\preceq$ be a total bi-invariant order on G.

Suppose that $x^n = y^n$ for some $x, y \in G$, $n \in \mathbb{N}$, and $x \neq y$. Without loss of generality we can assume that $x \prec y$. Let us show that $x^k \prec y^k$ for every $k \in \mathbb{N}$. This is true for $k = 1$, so, proceeding by induction on k, suppose that $k \geq 2$ and $x^{k-1} \prec y^{k-1}$ has already been shown. Then $x^k = x^{k-1}x \prec x^{k-1}y \prec y^{k-1}y = y^k$, where we used the induction hypothesis together with the bi-invariance of the order.

Hence, we have proved that $x^n \prec y^n$, contradicting to $x^n = y^n$. Thus $x = y$. □

The Unique Root Property for right-angled Artin groups can also be easily established using the fact that these groups are residually torsion-free nilpotent, which was also proved in [21].

Lemma 6.4. *Let H be a conjugate of a special subgroup in a right-angled Artin group G. If $K \leq G$ is a subgroup such that $|K : (K \cap H)| < \infty$ then $K \subseteq H$.*

Proof. By the assumptions, H is a retract of G. Let $\rho_H \in \mathrm{End}(G)$ denote the corresponding retraction. Take any $x \in K$. Since $|K : (K \cap H)| < \infty$, there is $n \in \mathbb{N}$ such that $x^n \in H$. Therefore, setting $y := \rho_H(x) \in H$, we obtain $x^n = \rho_H(x^n) = y^n$. And the Unique Root Property for G implies that $x = y \in H$. Thus $K \subseteq H$. □

After writing down the proof of the next technical result (Lemma 6.5), the author learned that it has already been established by A. Duncan, I. Kazachkov and V. Remeslennikov in their recent paper [23], Proposition 2.6. However, the proof presented here is somewhat different, and the author decided to keep it in this work for completeness.

Lemma 6.5. *Let G be a right-angled Artin group associated to a finite graph Γ with vertex set $\mathcal{V}$. Suppose that $\mathcal{S}, \mathcal{T} \subseteq \mathcal{V}$ and $g \in G$. Then there are $\mathcal{P} \subseteq \mathcal{T}$ and $h \in \langle \mathcal{T} \rangle$ such that $g\langle \mathcal{S} \rangle g^{-1} \cap \langle \mathcal{T} \rangle = h\langle \mathcal{P} \rangle h^{-1}$ in G. Thus, the intersection of conjugates of two special subgroups in G is a conjugate of a special subgroup of G.*

Proof. We will use induction on $(|\mathcal{S}| + |g|)$, where $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$. If $\mathcal{S} = \emptyset$, then $\langle \mathcal{S} \rangle = \{1\}$ and the statement is trivial. If $|g| = 0$, i.e., $g = 1$ in G, then $g\langle \mathcal{S} \rangle g^{-1} \cap \langle \mathcal{T} \rangle = \langle \mathcal{S} \rangle \cap \langle \mathcal{T} \rangle = \langle \mathcal{S} \cap \mathcal{T} \rangle$ by Remark 6.2.

Thus we can assume that $\mathcal{S} \neq \emptyset$ and $g \neq 1$, $n := |\mathcal{S}| + |g| \geq 2$, and the claim has been proved for all $\mathcal{S}$ and g with $|\mathcal{S}| + |g| < n$.

If there is $a \in \mathrm{FL}(g) \cap \mathcal{T}^{\pm 1}$, set $f := a^{-1}g$. Then $|f| < |g|$, $a\langle \mathcal{T} \rangle a^{-1} = \langle \mathcal{T} \rangle$ and

$$g\langle \mathcal{S} \rangle g^{-1} \cap \langle \mathcal{T} \rangle = a(f\langle \mathcal{S} \rangle f^{-1} \cap \langle \mathcal{T} \rangle)a^{-1} = (ah)\langle \mathcal{P} \rangle (ah)^{-1}$$

for some $h \in \langle \mathcal{T} \rangle$ and some $\mathcal{P} \subseteq \mathcal{T}$ by the induction hypothesis.

If, on the other hand, there is $b \in \mathrm{LL}(g) \cap \big(\mathcal{S} \cup \mathrm{star}(\mathcal{S})\big)^{\pm 1}$, set $f := gb^{-1}$. Then $|f| < |g|$, $b\langle \mathcal{S} \rangle b^{-1} = \langle \mathcal{S} \rangle$ and $g\langle \mathcal{S} \rangle g^{-1} \cap \langle \mathcal{T} \rangle = f\langle \mathcal{S} \rangle f^{-1} \cap \langle \mathcal{T} \rangle$ and we can apply the induction hypothesis once again.

Therefore, we can suppose that $\mathrm{FL}(g) \cap \mathcal{T}^{\pm 1} = \emptyset$ and $\mathrm{LL}(g) \cap \big(\mathcal{S} \cup \mathrm{star}(\mathcal{S})\big)^{\pm 1} = \emptyset$. We assert that in this case

$$g\langle \mathcal{S} \rangle g^{-1} \cap \langle \mathcal{T} \rangle = \bigcup_{s \in \mathcal{S}} (g\langle \mathcal{S} \setminus \{s\} \rangle g^{-1} \cap \langle \mathcal{T} \rangle). \tag{6.2}$$

Indeed, if (6.2) is false, then there exist $x \in \langle \mathcal{S} \rangle$ and $y \in \langle \mathcal{T} \rangle$ such that $\mathrm{supp}(x) = \mathcal{S}$ and $gxg^{-1} = y$ in G.

Choose graphically reduced words W, X and Y representing in G the elements g, x and y respectively, so that $\mathrm{supp}(X) = \mathcal{S}$ and $\mathrm{supp}(Y) \subseteq \mathcal{T}$. Let a be the first letter of W, then $a = v^{\pm 1}$ for some $v \in \mathcal{V}$. According to our assumptions, $v \in \mathrm{supp}(WXW^{-1}) \setminus \mathrm{supp}(Y)$ and $WXW^{-1} = Y$ in G. Hence the left-hand side of the latter equality cannot be graphically reduced.

Note that no letter a of W can be cancelled with a letter of X in the word WXW^{-1}, because this would mean that $a \in \mathrm{supp}(X)^{\pm 1} = \mathcal{S}^{\pm 1}$ and a commutes with the suffix of W after it, hence $a \in \mathrm{LL}(g) \cap \mathcal{S}^{\pm 1} = \emptyset$. Similarly, no letter from X can cancel with a letter from W^{-1}, therefore a reduction in WXW^{-1} can occur only from the presence of a subword cUc^{-1}, where c is a letter from the initial copy of W and U contains X as a subword. Thus, $c = w^{\pm 1}$ for some $w \in \mathcal{V}$, $\mathrm{supp}(W) \subseteq \mathrm{supp}(U) \subseteq \mathrm{star}(w)$. Consequently, $\mathcal{S} \subseteq \mathrm{star}(w)$, and c is a last letter of g, because it commutes with the suffix of W after it. This implies that $c \in \mathrm{LL}(g) \cap \mathrm{star}(\mathcal{S})^{\pm 1} \neq \emptyset$ contradicting to our assumption.

Therefore (6.2) is true, implying that the group $K := g\langle \mathcal{S} \rangle g^{-1} \cap \langle \mathcal{T} \rangle \leq G$ is covered by a finite union of its subgroups. A classical theorem of B. Neumann [46] claims that in this case one of these subgroups must have finite index in K. Thus there is $s_0 \in \mathcal{S}$ such that $|K : (g\langle \mathcal{S} \setminus \{s_0\} \rangle g^{-1} \cap \langle \mathcal{T} \rangle)| < \infty$. Using the induction hypothesis, we can find $\mathcal{P} \subseteq \mathcal{T}$ and $h \in \langle \mathcal{T} \rangle$ such that $g\langle \mathcal{S} \setminus \{s_0\} \rangle g^{-1} \cap \langle \mathcal{T} \rangle = h\langle \mathcal{P} \rangle h^{-1}$. Therefore $h\langle \mathcal{P} \rangle h^{-1} \leq K$ and $|K : h\langle \mathcal{P} \rangle h^{-1}| < \infty$. Hence Lemma 6.4 can be applied to achieve the required equality $K = h\langle \mathcal{P} \rangle h^{-1}$. □

Let us recall a few more facts about right-angled Artin groups.

An element $g \in G$ is said to be *A-cyclically reduced* if it cannot be written as $g = aha^{-1}$, where $a \in \mathcal{V}^{\pm 1}$ and $|h| = |g| - 2$ (we have added the letter “A” to avoid confusion with a similar notion for special HNN-extensions introduced in Section 7). In the paper [57] H. Servatius proved that for every element g of a right-angled Artin group G there exists a unique A-cyclically reduced element h such that $g = fhf^{-1}$ for some $f \in G$ with $|g| = |h| + 2|f|$. In particular, $\mathrm{supp}(h) \subseteq \mathrm{supp}(g)$. Therefore, if $g \in G$ is not A-cyclically reduced then $|g^2| = |fh^2f^{-1}| \leq 2|h| + 2|f| < 2|g|$. Thus we obtain the following

Remark 6.6. If an element $g \in G$ is not A-cyclically reduced, then for any graphically reduced word W representing g in G, the word $W^2 \equiv WW$ cannot be graphically reduced.

Another consequence of the above theorem of Servatius is that every given element of G is conjugate to a unique (up to a cyclic permutation) A-cyclically reduced element. In particular, we can make

Remark 6.7. If the elements $g, h \in G$ are A-cyclically reduced and conjugate in G, then $\mathrm{supp}(g) = \mathrm{supp}(h)$.

A special subgroup A of the right-angled Artin group G is said to be *maximal* if $A = \langle \mathcal{S} \rangle$ for some maximal proper subset $\mathcal{S}$ of $\mathcal{V}$ (i.e., if $|\mathcal{S}| = |\mathcal{V}| - 1$).

Lemma 6.8. *For any non-trivial element $g \in G$ there is a maximal special subgroup A in G such that $g \notin A^G$.*

Proof. Arguing by contradiction, suppose that there are $f_1, \dots, f_n \in G$ such that $g \in \bigcap_{i=1}^n f_i A_i f_i^{-1}$ in G, where $A_1, \dots, A_n$ is the list of all maximal special subgroups of G. Then for each $i \in \{1, \dots, n\}$, there is an A-cyclically reduced element $a_i \in A_i \setminus \{1\}$ such that g is conjugate to a_i in G. Choose any letter $v \in \mathrm{supp}(a_1)$ and take $j \in \{1, \dots, n\}$ such that $A_j = \langle \mathcal{V} \setminus \{v\} \rangle$. Then a_1 must be conjugate to a_j in G, which is impossible by Remark 6.7, because $v \in \mathrm{supp}(a_1) \setminus \mathrm{supp}(a_j)$ (as $\mathrm{supp}(a_j) \subseteq \mathcal{V} \setminus \{v\}$). This contradiction proves the lemma. □

Recall that an automorphism ϕ of a group G is called *pointwise inner* if for each $g \in G$ there exists $f = f(g) \in G$, such that $\phi(G) = fgf^{-1}$ in G. Let $\mathrm{Aut}_{pi}(G)$ denote the set of all pointwise inner automorphisms of G. It is easy to see that $\mathrm{Aut}_{pi}(G)$ is a normal subgroup of the full automorphism group $\mathrm{Aut}(G)$, containing the subgroup of all inner automorphisms $\mathrm{Inn}(G)$.

We are now going to prove that the group of pointwise inner automorphisms of a right-angled Artin group G coincides with the group of inner automorphisms of G.

Proposition 6.9. *For any right-angled Artin group G we have* $\mathrm{Aut}_{pi}(G) = \mathrm{Inn}(G)$.

Proof. Suppose there exists $\phi \in \mathrm{Aut}_{pi}(G) \setminus \mathrm{Inn}(G)$. Since ϕ maps every generator of G to its conjugate, we can replace ϕ by its composition with an inner automorphism of G to assume that $\phi(v) = v$ for some $v \in \mathcal{V}$.

Then there is an automorphism μ lying in the right coset $\mathrm{Inn}(G)\phi \subset \mathrm{Aut}_{pi}(G)$ such that the set $\mathrm{Fix}(\mu)$ is maximal, where $\mathrm{Fix}(\mu)$ denotes the subset of all elements in $\mathcal{V}$ fixed by μ. Note that $\mathrm{Fix}(\mu) \subsetneqq \mathcal{V}$ because $\mu \neq \mathrm{id}_G$ (the identity map id_G of G does not belong to the coset $\mathrm{Inn}(G)\phi$ by our assumption). And $\mathrm{Fix}(\mu) \neq \emptyset$ since $v \in \mathrm{Fix}(\phi) \neq \emptyset$.

Let $\mathrm{Fix}(\mu) = \{v_1, \dots, v_k\} \subset \mathcal{V}$ and pick any $w \in \mathcal{V} \setminus \mathrm{Fix}(\mu)$. By the assumptions, there is $f \in G$ such that $\mu(w) = fwf^{-1}$. Choose a shortest element f with this property.

First suppose that there exists $a \in \mathrm{FL}(f) \cap (\bigcap_{i=1}^k \mathrm{star}(v_i))^{\pm 1}$. Then $a \in \bigcap_{i=1}^k C_G(v_i)$, hence after defining a new automorphism $\lambda \in \mathrm{Inn}(G)\phi$ by $\lambda(g) := a^{-1}\mu(g)a$ for all $g \in G$, we have $\mathrm{Fix}(\mu) \subseteq \mathrm{Fix}(\lambda)$, $\lambda(w) = (a^{-1}f)w(a^{-1}f)^{-1}$ and $|a^{-1}f| < |f|$. Note that $a^{-1}f \neq 1$ because, otherwise, we would have $w \in \mathrm{Fix}(\lambda)$ contradicting to the maximality of $\mathrm{Fix}(\mu)$. Thus we can replace μ with λ, making f shorter. We can continue doing the same for λ, and so on. Eventually we will end up with an automorphism $\mu \in \mathrm{Inn}(G)\phi$ (we keep the same notation for it, although the actual automorphism may be different) such that $\mu(w) = fwf^{-1}$, f is a shortest element with this Property, $f \neq 1$, and

$$\mathrm{FL}(f) \cap \mathrm{star}(\mathrm{Fix}(\mu))^{\pm 1} = \mathrm{FL}(f) \cap \Big(\bigcap_{i=1}^{k} \mathrm{star}(v_i)\Big)^{\pm 1} = \emptyset. \tag{6.3}$$

Put $\mathcal{S} := \mathrm{Fix}(\mu) \cap \mathrm{FL}(f)^{\pm 1} \subset \mathcal{V}$. Using (6.3), we see that for every $s \in \mathcal{S}$ there is $v(s) \in \mathrm{Fix}(\mu)$ such that $v(s) \notin \mathrm{star}(s)$. Note that in this case $v(s) \notin \mathrm{FL}(f)^{\pm 1}$ because $s \in \mathrm{FL}(f)^{\pm 1}$ and any two elements of $\mathrm{FL}(f)^{\pm 1}$ commute. Set $\mathcal{T} := \{v(s) \mid s \in \mathcal{S}\} \subseteq \mathrm{Fix}(\mu) \setminus \mathrm{FL}(f)^{\pm 1}$, and write $\mathrm{Fix}(\mu) = \{t_1, \dots, t_l\} \sqcup \{s_1, \dots, s_m\}$, where $\mathcal{T} = \{t_1, \dots, t_l\}$ (consequently, $\mathcal{S} \subseteq \{s_1, \dots, s_m\}$). Finally, define the words $T := t_1 \dots t_l$, $S := s_1 \dots s_m$ and let g be the element represented by the word by $TSTw$ in G.

Since $\mu \in \mathrm{Aut}_{pi}(G)$, there exists $x \in G$ such that $\mu(g) = xgx^{-1}$. On the other hand, $\mu(g) = TSTUwU^{-1}$, for some graphically reduced non-empty word U representing f in G. Note that the word UwU^{-1} is graphically reduced (otherwise, we could make U, and hence f, shorter) and $w \notin \mathrm{supp}(TST) = \mathrm{Fix}(\mu)$. Therefore the only possible reduction which could occur in the word $W \equiv TSTUwU^{-1}$ would arise from cancellation of a letter from TST with a letter from U or U^{-1}. However, no letter t from the first copy of T could cancel with a letter from U or U^{-1} in W, because $\mathrm{supp}(S) \not\subset \mathrm{star}(t)$ (as $t = v(s)$ for some $s \in \mathcal{S}$ and $s \in \mathrm{supp}(S) \setminus \mathrm{star}(v(s))$). On the other hand, if some letter t from the second copy of T in W cancelled with some letter a from U, then $a \in \mathrm{FL}(f)$, but this would contradict to $t \in \mathcal{T}$ and $\mathcal{T} \cap \mathrm{FL}(f)^{\pm 1} = \emptyset$. If this letter t cancelled with a letter a from U^{-1} in W, then we would have $\mathrm{supp}(U) \subset \mathrm{star}(t)$, which is impossible as $t = v(s)$ for some

$s \in \mathrm{supp}(U)$ and $s \notin \mathrm{star}(v(s))$. Therefore, if W is not graphically reduced, then a letter s from S must cancel with a letter $a = s^{-1}$ from U or U^{-1}, in particular, $\mathcal{T} = \mathrm{supp}(T) \subseteq \mathrm{star}(s)$, implying that $s \notin \mathcal{S}$. However, if s cancels with a letter from U, we see that $a \in \mathrm{FL}(f)$, hence $s \in \mathcal{S}$, which is false. And if s cancelled with a letter a from U^{-1}, then we would get a contradiction with the fact that UwU^{-1} is graphically reduced, because $a^{-1} = s$ is a letter of U lying between s and a in W.

Therefore $W \equiv TSTUwU^{-1}$ is a graphically reduced word representing $\mu(g)$ in G. Consequently, $|\mu(g)| = 2\|T\| + \|S\| + 2\|U\| + 1 > 2\|T\| + \|S\| + 1 = |g|$ since $\|U\| > 0$ ($\|U\|$ denotes the length of the word U). But $\mu(g) = xgx^{-1}$, hence $\mu(g)$ is not A-cyclically reduced. By Remark 6.6, a reduction can be made in the word $W^2 \equiv TSTUwU^{-1}TSTUwU^{-1}$. But an argument similar to the above shows that this is impossible.

Thus we have arrived to a contradiction, which proves that $\mathrm{Aut}_{pi}(G) = \mathrm{Inn}(G)$, as needed. □

Remark 6.10. The reader could have noticed that in the proof of Proposition 6.9 we have actually shown more than it claims. In fact, we have proved that any endomorphism ϕ of a right-angled Artin group G, which maps each conjugacy class of G into itself, is an inner automorphism of G.

7. Special HNN-extensions

The purpose of this section is to develop necessary tools for dealing with special HNN-extensions.

Let A be a group and let $H \leq A$ be a subgroup.

Definition 7.1. The *special HNN-extension of A with respect to H* is the group G given by the presentation

$$G = \langle A, t \mid tht^{-1} = h \text{ for every } h \in H \rangle. \qquad (7.1)$$

In other words, the special HNN-extension G is a particularly simple HNN-extension of A, where both of the associated subgroups are equal to H and the isomorphism between these subgroups is the identity map on H.

Let Γ be a finite graph with the set of vertices $\mathcal{V}$ of cardinality $n \in N$. The reason why we are interested in special HNN-extensions is the the observation below.

Remark 7.2. Let G be the right-angled Artin group associated to Γ. Then G can be obtained from the trivial group via a sequence of special HNN-extensions. More precisely, there are right-angled Artin groups $\{1\} = G_0, G_1, \ldots, G_n = G$ such that G_{i+1} is a special HNN-extension of G_i with respect to some special subgroup $H_i \leq G_i$ for every $i = 0, \ldots, n-1$.

The groups G_i can be constructed as follows. Let $\mathcal{V} = \{v_1, \dots, v_n\}$ and denote $\mathcal{S}_{i+1} := \{v_1, \dots, v_i\} \cap \text{star}(v_{i+1}) \subset \mathcal{V}$ for $i = 1, \dots, n-1$. Set $G_0 := \{1\}$, $G_1 := \langle v_1 \rangle$ (the infinite cyclic group generated by v_1), and

$$G_{i+1} := \langle G_i, v_{i+1} \mid v_{i+1} s v_{i+1}^{-1} = s \text{ for every } s \in \mathcal{S}_{i+1} \rangle, \quad i = 1, \dots, n-1.$$

Clearly, $G = G_n$ and for each i, G_{i+1} is a special HNN-extension of G_i with respect to the special subgroup $\langle \mathcal{S}_{i+1} \rangle$ of G_i, and G_i is a special subgroup of G generated by $\{v_1, \dots, v_i\}$.

Remark 7.3. If G is a right-angled Artin group associated to Γ, then for any maximal special subgroup $A \le G$, G splits as a special HNN-extension (7.1) of A with respect to a certain special subgroup H of A.

Indeed, if $A = \langle \mathcal{S} \rangle$, where $\mathcal{S} \subset \mathcal{V}$ and $\mathcal{V} = \mathcal{S} \sqcup \{t\}$, set $\mathcal{U} := \text{star}(t) \setminus \{t\} \subseteq \mathcal{S}$. Then $G = \langle A, t \mid tut^{-1} = u \text{ for all } u \in \mathcal{U} \rangle$ is a special HNN-extension of A with respect to the subgroup $H := \langle \mathcal{U} \rangle \le A$.

Special HNN-extensions are usually much easier to deal with than general HNN-extensions. Throughout this section we will limit ourselves to considering only the former ones, even though most of the statements can be re-formulated in the general situation.

Let G be the special HNN-extension given by (7.1). von Dyck's Theorem yields the following *Universal Property* of special HNN-extensions, which will be important for us:

Remark 7.4. For any group B, every homomorphism $\psi \colon A \to B$ can be naturally extended to a homomorphism $\widetilde{\psi} \colon G \to P$, where $P := \langle B, s \mid sxs^{-1} = x \text{ for all } x \in \psi(H) \rangle$ is the special HNN-extension of B with respect to $\psi(H)$, so that $\widetilde{\psi}|_A = \psi$ and $\widetilde{\psi}(t) = s$.

Lemma 7.5. *In the notations of Remark* 7.4, $\ker(\widetilde{\psi}) = N$, *where* $N \lhd G$ *is the normal closure of* $\ker(\psi) \le A \le G$ *in* G.

Proof. Obviously, $N \le \ker(\widetilde{\psi})$, and hence $N \cap A = \ker(\widetilde{\psi}) \cap A = \ker(\psi)$. Let $\phi \colon G \to Q := G/N$ be the natural epimorphism with $\ker(\phi) = N$. Consequently, if we define $\theta \colon Q \to P$ to be the natural epimorphism with the kernel $\phi(\ker(\widetilde{\psi}))$, then we will have $\widetilde{\psi} = \theta \circ \phi$.

Observe that $\phi(t)\phi(x)\phi(t)^{-1} = \phi(txt^{-1}) = \phi(x)$ in Q for every $x \in H$, and the map $\xi \colon \psi(A) \to \phi(A)$ defined by $\xi(\psi(a)) := \phi(a)$ for all $a \in A$, is an isomorphism, since $\ker(\psi) = \ker(\phi) \cap A$. Therefore, by von Dyck's Theorem, there is a homomorphism $\tilde{\xi} \colon P \to Q$ such that $\tilde{\xi}(\psi(a)) = \phi(a)$ for every $a \in A$ and $\tilde{\xi}(s) = \phi(t)$. It is easy to see that $\tilde{\xi} \circ \theta \colon Q \to Q$ is the identity map on Q. Hence, θ is injective, that is $\{1\} = \ker(\theta) = \phi(\ker(\widetilde{\psi}))$ in Q, implying that $\ker(\widetilde{\psi}) \le \ker(\phi) = N$. Thus $\ker(\widetilde{\psi}) = N$. □

Lemma 7.6. *Suppose that $\rho \in \mathrm{End}(A)$ is a retraction of A onto its subgroup B. Then there are retractions $\tilde{\rho}_1, \tilde{\rho}_2 \in \mathrm{End}(G)$ of G onto subgroups $B \le G$ and $C := \langle B, t\rangle \le G$ respectively, such that $\tilde{\rho}_i|_A = \rho$ for $i = 1, 2$, $\tilde{\rho}_1(t) = 1$ and $\tilde{\rho}_2(t) = t$.*

Proof. Define the maps $\rho_i\colon A \sqcup \{t\} \to A \sqcup \{t\}$ by $\rho_i(a) := \rho(a)$ for each $a \in A$, $i = 1, 2$, and $\rho_1(t) := 1$ and $\rho_2(t) := t$.

The map ρ_1 can be extended to an endomorphism $\tilde{\rho}_1\colon G \to G$ by von Dyck's Theorem, and the map ρ_2 can be extended to an endomorphism $\tilde{\rho}_2\colon G \to G$ by Remark 7.4. Obviously, $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are retractions of G onto B and C respectively. □

Every element of the special HNN-extension G, given by (7.1), is a product of the form

$$x_0 t^{\varepsilon_1} x_1 t^{\varepsilon_2} \dots t^{\varepsilon_n} x_n \tag{7.2}$$

for some $n \in \mathbb{N} \cup \{0\}$, $x_i \in A$, $i = 0, \dots, n$, and $\varepsilon_j \in \mathbb{Z} \setminus \{0\}$, $j = 1, \dots, n$.

The product (7.2) is said to be *reduced* if $x_i \notin H$ for every $i \in \{1, 2, \dots, n-1\}$. Since t commutes with every element of H, it is easy to see that any $g \in G$ is equal to some reduced product in G. By Britton's Lemma (see [38, IV.2]) a non-empty reduced product represents a non-trivial element in G. It follows, that if two reduced products $x_0 t^{\varepsilon_1} x_1 t^{\varepsilon_2} \dots t^{\varepsilon_n} x_n$ and $y_0 t^{\zeta_1} y_1 t^{\zeta_2} \dots t^{\zeta_m} y_m$ are equal in G, then $m = n$ and $\varepsilon_i = \zeta_i$ for every $i = 1, \dots, n$ (see [38], IV.2.3).

Suppose that an element $g \in G$ is equal to a product $t^{\varepsilon_1} x_1 t^{\varepsilon_2} \dots t^{\varepsilon_n} x_n$. Let us fix this presentation for g. Any product $t^{\varepsilon_k} x_k t^{\varepsilon_{k+1}} \dots t^{\varepsilon_n} x_n t^{\varepsilon_1} x_1 \dots t^{\varepsilon_{k-1}} x_{k-1} \in G$, for some $k \in \{1, \dots, n\}$, is said to be a *cyclic permutation* of g. The element $g = t^{\varepsilon_1} x_1 t^{\varepsilon_2} \dots t^{\varepsilon_n} x_n$ is called *cyclically reduced* if each of its cyclic permutations is reduced. A *prefix* of g is an element of the form $t^{\varepsilon_1} x_1 t^{\varepsilon_2} \dots t^{\varepsilon_k} x_k$ for some $k \in \{0, 1, \dots, n\}$ (if $k = 0$ then we have the empty prefix, corresponding to the trivial element of G). Similarly, a *suffix* of g is an element of the form $t^{\varepsilon_l} x_l \dots t^{\varepsilon_n} x_n$ for some $l \in \{1, 2, \dots, n+1\}$.

It is not difficult to see that every element $f \in G$ either belongs to A^G in G or is conjugate to some non-trivial cyclically reduced element in G.

Below is the statement of Collins's Lemma (see [38], IV.2.5) in the case of special HNN-extensions.

Lemma 7.7. *Suppose that $g = t^{\varepsilon_1} x_1 t^{\varepsilon_2} \dots t^{\varepsilon_n} x_n$ and $f = t^{\zeta_1} y_1 t^{\zeta_2} \dots t^{\zeta_m} y_m$ are cyclically reduced in G, with $n \ge 1$. Then $g \notin A^G$. And if f is conjugate to g in G then $m = n$ and there exist $h \in H$ and a cyclic permutation f' of f such that $f' = hgh^{-1}$ in G.*

We will also use the following description of centralizers in special HNN-extensions:

Proposition 7.8. *Let G be the special HNN-extension given by* (7.1). *Suppose that the element $g = t^{\varepsilon_1} x_1 t^{\varepsilon_2} \dots t^{\varepsilon_n} x_n \in G$ is cyclically reduced and $n \geq 1$.*

If $x_n \in H$ then $n = 1$ and $C_G(g) = \langle t \rangle \times C_H(x_1) = \langle t \rangle \times C_H(g) \leq G$.

If $x_n \in A \setminus H$, let $\{p_1, \dots, p_k\}$, $1 \leq k \leq n+1$, be the set all of prefixes of g satisfying $p_i^{-1} g p_i \in g^H$ in G. For each $i = 1, \dots, k$, choose $h_i \in H$ such that $h_i p_i^{-1} g p_i h_i^{-1} = g$, and define the finite subset $\Omega \subset G$ by $\Omega := \{h_i p_i^{-1} \mid i = 1, \dots, k\}$. Then $C_G(g) = C_H(g) \langle g \rangle \Omega$.

In order to prove Proposition 7.8 we will need two lemmas below. The proof of the next statement is similar to the proof of Collins's Lemma.

Lemma 7.9. *Let $g = t^{\varepsilon_1} x_1 t^{\varepsilon_2} \dots t^{\varepsilon_n} x_n$ and $f = t^{\zeta_1} y_1 t^{\zeta_2} \dots t^{\zeta_n} y_n$ be cyclically reduced elements of G, with $n \geq 1$ and $x_n \notin H$. Assume that $cgc^{-1} = f$ in G, where the element c is equal to the reduced product $z_0 t^{\eta_1} z_1 t^{\eta_2} \dots t^{\eta_m} z_m$. Then there are the following three mutually exclusive possibilities.*

a) *$m = 0$ and $c \in H$;*
b) *$m \geq 1$, $z_m \in H$ and there is a prefix p of g such that $c = hp^{-1} g^l$ in G for some $h \in H$ and $l \in \mathbb{Z}$, $l \leq 0$;*
c) *$m \geq 1$, $x_n z_m^{-1} \in H$ and there is a suffix s of g such that $c = hsg^l$ in G for some $h \in H$ and $l \in \mathbb{Z}$, $l \geq 0$.*

Proof. First suppose that $m = 0$, i.e., $c = z_0 \in A$. Then $f^{-1} cgc^{-1} = 1$ in G, yielding

$$y_n^{-1} t^{-\zeta_n} \dots y_1^{-1} t^{-\zeta_1} z_0 t^{\varepsilon_1} x_1 t^{\varepsilon_2} \dots t^{\varepsilon_n} x_n z_0^{-1} = 1.$$

Therefore the left-hand side is not reduced, hence $c = z_0 \in H$.

Now assume that $m \geq 1$. The equality $cgc^{-1} = f$ in G gives rise to the equation

$$z_0 t^{\eta_1} \dots t^{\eta_m} z_m t^{\varepsilon_1} x_1 t^{\varepsilon_2} \dots t^{\varepsilon_n} x_n z_m^{-1} t^{-\eta_m} \dots t^{-\eta_1} z_0^{-1} = t^{\zeta_1} y_1 \dots t^{\zeta_n} y_n.$$

The left-hand side cannot be reduced because it contains more t-letters than the right-hand side. Hence either $z_m \in H$ or $x_n z_m^{-1} \in H$ (note that both of these inclusions cannot happen simultaneously since $x_n \notin H$). Let us consider the case when $z_m \in H$, as the second case is similar. Then $t^{\eta_m} z_m t^{\varepsilon_1} = z_m t^{\eta_m + \varepsilon_1}$, and thus we get

$$\begin{aligned} z_0 t^{\eta_1} \dots t^{\eta_{m-1}} (z_{m-1} z_m) t^{\eta_m + \varepsilon_1} x_1 t^{\varepsilon_2} \dots t^{\varepsilon_n} x_n z_m^{-1} t^{-\eta_m} z_{m-1}^{-1} \dots t^{-\eta_1} z_0^{-1} \\ = t^{\zeta_1} y_1 \dots t^{\zeta_n} y_n. \end{aligned} \tag{7.3}$$

Once again we see that the left-hand side of the above equation cannot be reduced. In the case when $m = 1$, this implies that $\eta_m + \varepsilon_1 = 0$. On the other hand, if $m > 1$, then $z_{m-1} \in A \setminus H$, hence $z_{m-1} z_m \notin H$, and again, in order for a reduction to be possible, we must have $\eta_m + \varepsilon_1 = 0$. Hence $\varepsilon_1 = -\eta_m$ and $z_m^{-1} t^{-\eta_m} z_{m-1}^{-1} = t^{\varepsilon_1} x_1 h_1^{-1}$, where $h_1 := z_{m-1} z_m x_1 \in A$. Now (7.3) becomes

$$\begin{aligned} (z_0 t^{\eta_1} \dots z_{m-2} t^{\eta_{m-1}} h_1)(t^{\varepsilon_2} x_2 \dots t^{\varepsilon_n} x_n t^{\varepsilon_1} x_1)(z_0 t^{\eta_1} \dots t^{\eta_{m-1}} h_1)^{-1} \\ = t^{\zeta_1} y_1 \dots t^{\zeta_n} y_n. \end{aligned} \tag{7.4}$$

If $m = 1$, i.e., $c = z_0 t^{-\varepsilon_1} z_1 = z_0 z_1 t^{-\varepsilon_1}$, then we have

$$1 = f^{-1} c g c^{-1} = (y_n^{-1} t^{-\zeta_n} \dots y_1^{-1} t^{-\zeta_1}) h_1 (t^{\varepsilon_2} x_2 \dots t^{\varepsilon_n} x_n t^{\varepsilon_1} x_1) h_1^{-1}.$$

Hence $h_1 \in H$ and $c = h_1 (t^{\varepsilon_1} x_1)^{-1}$, where $t^{\varepsilon_1} x_1$ is a prefix of g.

Otherwise, if $m = M \geq 2$ we will use induction on m to prove the claim b) of the lemma. Thus, we will assume that b) has been established for all elements c with $1 \leq m \leq M - 1$.

Note that $x_1 h_1^{-1} = z_m^{-1} z_{m-1}^{-1} \notin H$ since $z_m \in H$ and $z_{m-1} \in A \setminus H$ (because $m \geq 2$ and the product $z_0 t^{\eta_1} \dots t^{\eta_{m-1}} z_{m-1} t^{\eta_m} z_m$ was assumed to be reduced).

Let us look at the equation (7.4). Since $m - 1 \geq 1$, the left-hand side cannot be reduced. And a reduction in it can only occur if $h_1 \in H$ because $x_1 h_1^{-1} \notin H$. Hence we are in the case b) of the lemma, and can apply the induction hypothesis to (7.4). Thus there is a prefix p of the element $g_1 := t^{\varepsilon_2} x_2 \dots t^{\varepsilon_n} x_n t^{\varepsilon_1} x_1$, $h \in H$ and $l \in \mathbb{Z}$, $l \leq 0$, such that $z_0 t^{\eta_1} \dots z_{m-2} t^{\eta_{m-1}} h_1 = h p^{-1} g_1^l$. Consequently,

$$\begin{aligned} c &= z_0 t^{\eta_1} \dots t^{\eta_{m-1}} z_{m-1} t^{\eta_m} z_m \\ &= z_0 t^{\eta_1} \dots t^{\eta_{m-1}} h_1 x_1^{-1} t^{-\varepsilon_1} = h p^{-1} g_1^l x_1^{-1} t^{-\varepsilon_1} = h q^{-1} g^l, \end{aligned}$$

where $q = t^{\varepsilon_1} x_1 p$. It is easy to see that either q is a prefix of g, or $q = g t^{\varepsilon_1} x_1$. In the latter case, $c = h (t^{\varepsilon_1} x_1)^{-1} g^{l-1}$ and $t^{\varepsilon_1} x_1$ is a prefix of g. Thus the step of induction is established, and the proof of the lemma is finished. □

The next lemma treats the case which was not covered by Lemma 7.9.

Lemma 7.10. *Suppose that $g = t^\varepsilon x \in G$, where $\varepsilon \in \mathbb{Z} \setminus \{0\}$ and $x \in H$. Then $C_G(g) = \langle t \rangle \times C_H(x) = \langle t \rangle \times C_H(g)$. In particular, $C_G(t^\varepsilon) = \langle t \rangle \times H \leq G$.*

Proof. For any $c \in C_G(g)$ we have $c g c^{-1} = g$. Let $z_0 t^{\eta_1} z_1 t^{\eta_2} \dots t^{\eta_m} z_m$ be a reduced product representing c in G. Then we have

$$z_0 t^{\eta_1} z_1 t^{\eta_2} \dots t^{\eta_m} z_m t^\varepsilon x z_m^{-1} t^{-\eta_m} \dots t^{-\eta_2} z_1^{-1} t^{-\eta_1} z_0^{-1} = t^\varepsilon x. \tag{7.5}$$

Arguing as in the proof of Lemma 7.9, we see that if $m = 0$ then $c = z_0 \in H$ and $t^\varepsilon x = z_0 t^\varepsilon x z_0^{-1} = t^\varepsilon z_0 x z_0^{-1}$, thus $1 = z_0 x z_0^{-1}$, i.e., $c = z_0 \in C_H(x)$.

So, assume now that $m \geq 1$. Then the equation (7.5) implies that either $z_m \in H$ or $x z_m^{-1} \in H$. But either of these inclusions leads to $z_m \in H$ because $x \in H$ by the assumptions. Therefore $t^{\eta_m} z_m t^\varepsilon x z_m^{-1} t^{-\eta_m} = z_m x z_m^{-1} t^\varepsilon$ in G and (7.5) becomes

$$z_0 t^{\eta_1} \dots z_{m-2} t^{\eta_{m-1}} (z_{m-1} z_m x z_m^{-1}) t^\varepsilon z_{m-1}^{-1} t^{\eta_{m-1}} z_{m-2}^{-1} \dots t^{-\eta_1} z_0^{-1} = t^\varepsilon x.$$

If $m \geq 2$, then $z_{m-1} \in A \setminus H$, hence $z_{m-1} z_m x z_m^{-1} \notin H$, and the above equation contradicts to Britton's Lemma: the left-hand side is reduced, but contains more t-letters than the right-hand side.

Therefore $m = 1$, i.e., $c = z_0 t^{\eta_1} z_1$ and $z_1 \in H$. Consequently,

$$1 = g^{-1} c g c^{-1} = x^{-1} t^{-\varepsilon} z_0 t^{\eta_1} z_1 t^{\varepsilon} x z_1^{-1} t^{-\eta_1} z_0^{-1} = x^{-1} t^{-\varepsilon} (z_0 z_1 x z_1^{-1}) t^{\varepsilon} z_0^{-1}.$$

Applying Britton's Lemma again, we achieve $z_0 z_1 x z_1^{-1} \in H$, implying that $z_0 \in H$ and $1 = x^{-1} z_0 z_1 x z_1^{-1} z_0^{-1}$ in G. That is, $z_0 z_1 \in C_H(x)$, and $c = z_0 t^{\eta_1} z_1 = t^{\eta_1} z_0 z_1 \in \langle t \rangle C_H(x)$.

Thus we proved that $C_G(g) \subseteq \langle t \rangle C_H(x)$. Finally, since t commutes with every element from H, it is clear that $t \in C_G(g)$ and $C_H(x) \subset C_H(g) \subset C_G(g)$. Hence $C_G(g) = \langle t \rangle C_H(x) = \langle t \rangle \times C_H(x) \le G$. The equality $C_H(g) = C_H(x)$ is immediate. □

Proof of Proposition 7.8. If $x_n \in H$, g can be cyclically reduced only when $n = 1$, and the claim follows from Lemma 7.10.

So, we can assume that $x_n \in A \setminus H$. Therefore we are able to apply Lemma 7.9 to g and $f := g$, which tells us that for any $c \in C_G(g)$ there exist $h \in H$ and $l \in \mathbb{Z}$ such that either there is a prefix p of g with $c = hp^{-1}g^l$, or there is a suffix s of g with $c = hsg^l$. Note that in the latter case there is a prefix p of g such that $s = p^{-1}g$. Hence we can assume that $c = hp^{-1}g^l$ for some prefix p of g, $h \in H$ and $l \in \mathbb{Z}$.

But then $g = cgc^{-1} = hp^{-1}gph^{-1}$, hence $p = p_i$ for some $i \in \{1, \dots, k\}$. The equalities $hp_i^{-1}gp_ih^{-1} = g = h_i p_i^{-1} g p_i h_i^{-1}$ yield $h^{-1}gh = p_i^{-1}gp_i = h_i^{-1}gh_i$. Consequently $hh_i^{-1} \in C_H(g)$ and $h \in C_H(g)h_i$. Thus $c = hp_i^{-1}g^l \in C_H(g)h_i p_i^{-1} g^l \subseteq C_H(g)\Omega\langle g \rangle$.

We have shown that $C_G(g) \subseteq C_H(g)\Omega\langle g \rangle$. Observe that $\Omega \subset C_G(g)$ by definition, hence $C_G(g) = C_H(g)\Omega\langle g \rangle = C_H(g)\langle g \rangle \Omega$. □

Now we formulate a criterion for conjugacy in special HNN-extensions.

Lemma 7.11. *Denote by G be the special HNN-extension* (7.1). *Moreover suppose that $B \le A$ is a subgroup and $g, f \in G$ are elements represented by reduced products $x_0 t^{\varepsilon_1} x_1 t^{\varepsilon_2} \dots t^{\varepsilon_n} x_n$ and $y_0 t^{\zeta_1} y_1 t^{\zeta_2} \dots t^{\zeta_m} y_m$, respectively, with $n \ge 1$. Then $f \in g^B$ in G if and only if all of the following conditions hold:*

(i) *$m = n$ and $\varepsilon_i = \zeta_i$ for all $i = 1, \dots, n$;*

(ii) *$y_0 y_1 \dots y_n \in (x_0 x_1 \dots x_n)^B$ in A;*

(iii) *for every $b_0 \in B$ with $y_0 y_1 \dots y_n = b_0 (x_0 x_1 \dots x_n) b_0^{-1}$ in A, the intersection $I \subseteq A$ is non-empty, where*

$$I := b_0 C_B(x_0 \dots x_n) \cap y_0 H x_0^{-1} \cap (y_0 y_1) H (x_0 x_1)^{-1} \cap \cdots$$
$$\cdots \cap (y_0 \dots y_{n-1}) H (x_0 \dots x_{n-1})^{-1}.$$

Proof. First we establish the sufficiency. Assume that the conditions (i), (ii) and (iii) hold. Take any $b_0 \in B$ satisfying (iii) (it exists by (ii)) and let I be the corresponding intersection. Then there exists an element $b \in I$. The inclusion $b \in b_0 C_B(x_0 \dots x_n)$

implies that $y_0y_1 \dots y_n = b(x_0x_1 \dots x_n)b^{-1}$. We will show that $f^{-1}bgb^{-1} = 1$, which is equivalent (in view of (i)) to

$$y_n^{-1}t^{-\varepsilon_n} \dots t^{-\varepsilon_2}y_1^{-1}t^{-\varepsilon_1}y_0^{-1}bx_0t^{\varepsilon_1}x_1t^{\varepsilon_2} \dots t^{\varepsilon_n}x_nb^{-1} = 1. \tag{7.6}$$

Note that $y_0^{-1}bx_0 \in H$ since $b \in I$, hence $y_1^{-1}t^{-\varepsilon_1}y_0^{-1}bx_0t^{\varepsilon_1}x_1 = y_1^{-1}y_0^{-1}bx_0x_1 \in H$. Therefore we can continue reducing the left-hand side of (7.6), until it becomes $y_n^{-1} \dots y_1^{-1}y_0^{-1}bx_0x_1 \dots x_nb^{-1}$, which is equal to 1 in G. Thus the sufficiency is proved.

To obtain the necessity, assume that $f = bgb^{-1}$ for some $b \in B$, that is,

$$bx_0t^{\varepsilon_1}x_1t^{\varepsilon_2} \dots t^{\varepsilon_n}x_nb^{-1} = y_0t^{\zeta_1}y_1t^{\zeta_2} \dots t^{\zeta_m}y_m.$$

Since both of the sides of the above equality are reduced, applying Britton's Lemma we obtain (i). Therefore the equation (7.6) holds in G.

Note that there is a canonical retraction $\rho_A \in \operatorname{End}(G)$ of G onto A, such that $\rho_A(t) = 1$ (apply Lemma 7.6 to the identity map on A). Hence, $y_0 \dots y_n = \rho_A(f) = \rho_A(bgb^{-1}) = b(x_0 \dots x_n)b^{-1}$, yielding (ii).

To achieve (iii), take any $b_0 \in B$ satisfying $y_0y_1 \dots y_n = b_0(x_0x_1 \dots x_n)b_0^{-1}$. Then $b_0^{-1}b \in C_B(x_0 \dots x_n)$, i.e., $b \in b_0C_B(x_0 \dots x_n)$. By Britton's Lemma the left-hand side of (7.6) cannot be reduced, therefore $y_0^{-1}bx_0 \in H$ and $b \in y_0Hx_0^{-1}$. Consequently, if $n \geq 2$, $y_1^{-1}t^{-\varepsilon_1}y_0^{-1}bx_0t^{\varepsilon_1}x_1 = y_1^{-1}y_0^{-1}bx_0x_1$ and (7.6) becomes

$$y_n^{-1}t^{-\varepsilon_n} \dots t^{-\varepsilon_2}(y_1^{-1}y_0^{-1}bx_0x_1)t^{\varepsilon_2} \dots t^{\varepsilon_n}x_nb^{-1} = 1.$$

Applying Britton's Lemma to the above equation, we see again that $y_1^{-1}y_0^{-1}bx_0x_1 \in H$, hence $b \in (y_0y_1)H(x_0x_1)^{-1}$. Clearly, we can continue this process, showing that $b \in (y_0 \dots y_i)H(x_0 \dots x_i)^{-1}$ for every $i \in \{0, \dots, n-1\}$. Thus, $b \in I \neq \emptyset$ and the condition (iii) is satisfied. □

Next comes a similar statement about centralizers.

Lemma 7.12. *Denote by G be the special HNN-extension* (7.1). *Moreover suppose that $B \leq A$ is a subgroup and an element $g \in G$ is represented by a reduced product $x_0t^{\varepsilon_1}x_1t^{\varepsilon_2} \dots t^{\varepsilon_n}x_n$ in G, with $n \geq 1$. Then the equality $C_B(g) = I$ holds in G, where*

$$I := C_B(x_0 \dots x_n) \cap x_0Hx_0^{-1} \cap (x_0x_1)H(x_0x_1)^{-1} \cap \cdots \cdots \cap (x_0 \dots x_{n-1})H(x_0 \dots x_{n-1})^{-1}.$$

Proof. Basically, we have already shown this while proving Lemma 7.11. Indeed, denote $f := g$. For any $b \in I$, if we take $b_0 = 1$, the proof of sufficiency in Lemma 7.11 asserts that $bgb^{-1} = g$, i.e., $b \in C_B(g)$. On the other hand, if $bgb^{-1} = g$ for some $b \in B$, then the proof of necessity in Lemma 7.11 shows that $b \in I$. Therefore $C_B(g) = I$. □

8. Proof of the main result

Throughout this section we assume that G is a right-angled Artin group associated to some fixed finite graph Γ with the set of vertices $\mathcal{V}$. The *rank* $\mathrm{rank}(G)$ of G is, by definition, the number of elements in $\mathcal{V}$.

Our proof of the main result will make use of the following two statements below. The next Lemma 8.1 was proved by Green in [29]. It can be easily demonstrated by induction on the number of vertices in the graph associated to a right-angled Artin group using Remark 7.2 and Britton's Lemma. On the other hand, it also follows from the linearity of such groups, which was established by S. Humphries in [36].

Lemma 8.1. *Right-angled Artin groups are residually finite.*

We are also going to use the following important fact proved by Dyer in [24].

Lemma 8.2. *Virtually free groups are conjugacy separable.*

The main result, Theorem 1.1, will be proved by induction on the rank of the right Artin group G. The lemma below is used to establish the inductive step.

Lemma 8.3. *Assume that every special subgroup B of G satisfies the condition* CC_G *from Definition* 3.3*, and that, for each $g \in G$, the B-conjugacy class g^B is separable in G.*

Suppose that $A_1, \dots, A_n$ are special subgroups of G, A_0 is a conjugate of a special subgroup of G, and b, x_0, ..., x_n, y_1, ..., $y_n \in G$ are arbitrary elements. Then for any finite-index normal subgroup $K \lhd G$ there exists a finite-index normal subgroup $L \lhd G$ such that $L \le K$ and

$$\bar{b} C_{\bar{A}_0}(\bar{x}_0) \cap \bigcap_{i=1}^{n} \bar{x}_i \bar{A}_i \bar{y}_i \subseteq \psi([b C_{A_0}(x_0) \cap \bigcap_{i=1}^{n} x_i A_i y_i] K) \tag{8.1}$$

in G/L, where $\psi \colon G \to G/L$ is the natural epimorphism, $\bar{b} := \psi(b)$, $\bar{A}_i := \psi(A_i)$, $\bar{x}_i := \psi(x_i)$, $i = 0, \dots, n$, and $\bar{y}_j := \psi(y_j)$, $j = 1, \dots, n$.

Proof. By the assumptions, $A_0 = hAh^{-1}$ for some special subgroup A of G and some $h \in G$. The proof will proceed by induction on n.

If $n = 0$, then the existence of a finite-index normal subgroup $L \lhd G$, $L \le K$, enjoying (8.1), follows from the fact that the pair (A_0, x_0) satisfies the Centralizer Condition CC_G, because the pair (A, g) has CC_G, where $g := h^{-1} x_0 h \in G$, and $b C_{A_0}(x_0) = bh C_A(g) h^{-1}$ in G.

Base of induction: Assume $n = 1$.

Case 1: Suppose that $b C_{A_0}(x_0) \cap x_1 A_1 y_1 = \emptyset$, which is equivalent to the condition $x_1 \notin b C_{A_0}(x_0) y_1^{-1} A_1 = bh[C_A(g) D] h^{-1} y_1^{-1}$, where $D := (y_1 h)^{-1} A_1 (y_1 h)$.

In this case, by Lemma 6.5, the intersections $A \cap A_1$ and $A \cap D$ are conjugates of special subgroups in G, hence for any $f \in G$, the conjugacy classes $f^{A \cap A_1}$ and $f^{A \cap D}$ are separable in G ($A \cap D = cSc^{-1}$ for some special subgroup S of G, hence $f^{A\cap D} = c[(c^{-1}fc)^S]c^{-1}$), and the pair $(A \cap A_1, f)$ satisfies CC_G. Therefore, Lemma 5.7 allows us to conclude that the double coset $C_A(g)D$ is separable in G. Thus the double coset $bC_{A_0}(x_0)y_1^{-1}A_1$ is separable as well, implying that there is a finite-index normal subgroup $N \lhd G$ such that $x_1 \notin bC_{A_0}(x_0)y_1^{-1}A_1N$. After replacing N with $N \cap K$ we can assume that $N \leq K$.

Now, since the pair (A_0, x_0) has CC_G, there exists a finite-index normal subgroup $L \lhd G$ such that $L \leq N \leq K$ and $\psi^{-1}(C_{\bar{A}_0}(\bar{x}_0)) \subseteq C_{A_0}(x_0)N$ in G (using the same notations as in the formulation of the lemma). Therefore, $\psi^{-1}(\bar{b}C_{\bar{A}_0}(\bar{x}_0)\bar{y}_1^{-1}\bar{A}_1) \subseteq bC_{A_0}(x_0)y_1^{-1}A_1N$, yielding that $x_1 \notin \psi^{-1}(\bar{b}C_{\bar{A}_0}(\bar{x}_0)\bar{y}_1^{-1}\bar{A}_1)$. Hence $\bar{b}C_{\bar{A}_0}(\bar{x}_0) \cap \bar{x}_1\bar{A}_1\bar{y}_1 = \emptyset$ in G/L, implying that (8.1) holds in this first case.

Case 2: $bC_{A_0}(x_0) \cap x_1A_1y_1 \neq \emptyset$ in G.

Let us make the following general observation.

Remark 8.4. Let H, F be subgroups of a group G such that $bH \cap xFy \neq \emptyset$ in G for some elements $b, x, y \in G$. Then for any $a \in bH \cap xFy$ we have $bH \cap xFy = a(H \cap y^{-1}Fy)$.

Thus we can pick any $a \in bC_{A_0}(x_0) \cap x_1A_1y_1$, and according to Remark 8.4, we will have $bC_{A_0}(x_0) \cap x_1A_1y_1 = a(C_{A_0}(x_0) \cap y_1^{-1}A_1y_1) = aC_E(x_0)$ in G, where $E := A_0 \cap y_1^{-1}A_1y_1 \leq G$. By Lemma 6.5, $E = cSc^{-1}$ for some special subgroup S of G and some $c \in A_0$.

As we saw in the beginning of the proof, it follows from our assumptions that the pair (E, x_0) satisfies CC_G. Hence there must exist a finite-index normal subgroup $M \lhd G$ such that $M \leq K$ and $C_{\varphi(E)}(\varphi(x_0)) \subseteq \varphi(C_E(x_0)K)$ in G, where $\varphi\colon G \to G/M$ denotes the natural epimorphism. On the other hand, by Lemma 5.6, there is $L \lhd G$ such that $|G : L| < \infty$, $L \leq M \leq K$ and $\psi(A) \cap \psi\big((y_1h)^{-1}A_1y_1h_1\big) \subseteq \psi\big(A \cap (y_1h)^{-1}A_1y_1h_1\big)\psi(M)$ in G/L. Therefore

$$\begin{aligned}\bar{A}_0 \cap \bar{y}_1^{-1}\bar{A}_1\bar{y}_1 &= \psi(h)[\psi(A) \cap \psi((y_1h)^{-1}A_1y_1h_1)]\psi(h^{-1})\\ &\subseteq \psi(h)[\psi\big(A \cap (y_1h)^{-1}A_1y_1h_1\big)\psi(M)]\psi(h^{-1})\\ &= \psi(A_0 \cap y_1^{-1}A_1y_1)\psi(M)\\ &= \psi(E)\psi(M)\end{aligned} \tag{8.2}$$

(in the notations from the formulation of the lemma). Observe that $\bar{a} := \psi(a) \in \bar{b}C_{\bar{A}_0}(\bar{x}_0) \cap \bar{x}_1\bar{A}_1\bar{y}_1$, hence, by Remark 8.4, $\bar{b}C_{\bar{A}_0}(\bar{x}_0) \cap \bar{x}_1\bar{A}_1\bar{y}_1 = \bar{a}(C_{\bar{A}_0}(\bar{x}_0) \cap \bar{y}_1^{-1}\bar{A}_1\bar{y}_1)$, and applying (8.2) we achieve

$$\bar{b}C_{\bar{A}_0}(\bar{x}_0) \cap \bar{x}_1\bar{A}_1\bar{y}_1 \subseteq \bar{a}C_{\psi(EM)}(\bar{x}_0). \tag{8.3}$$

Since $L \leq M$, there is an epimorphism $\xi\colon G/L \to G/M$ such that $\ker(\xi) = \psi(M)$ and $\varphi = \xi \circ \psi$. Note that $\xi(\psi(EM)) = \xi(\psi(E)\psi(M)) = \varphi(E)$ and $\xi(\bar{x}_0) = \varphi(x_0)$. Consider any $z \in C_{\psi(EM)}(\bar{x}_0)$ in G/L. Then

$$\xi(z) \in C_{\varphi(E)}(\varphi(x_0)) \subseteq \varphi(C_E(x_0)K) = \xi(\psi(C_E(x_0)K)).$$

Hence $z \in \psi(C_E(x_0)K)\ker(\xi) = \psi(C_E(x_0)K)$ because $\ker(\xi) = \psi(M) \leq \psi(K)$. Thus we have shown that $C_{\psi(EM)}(\bar{x}_0) \subseteq \psi(C_E(x_0)K)$ in G/L. Finally, combining this with (8.3), we obtain

$$\begin{aligned}\bar{b}C_{\bar{A}_0}(\bar{x}_0) \cap \bar{x}_1\bar{A}_1\bar{y}_1 \subseteq \bar{a}\psi(C_E(x_0)K) &= \psi(aC_E(x_0)K)\\ &= \psi([bC_{A_0}(x_0) \cap x_1A_1y_1]K),\end{aligned}$$

therefore (8.1) holds in Case 2.

Thus we have established the base of induction.

Step of induction: Suppose that $n \geq 2$ and the statement of the lemma has been proved for $n-1$.

If $bC_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i = \emptyset$ in G, then, by the induction hypothesis, there is a finite-index normal subgroup $L \lhd G$ such that $L \leq K$ and

$$\bar{b}C_{\bar{A}_0}(\bar{x}_0) \cap \bigcap_{i=1}^{n-1} \bar{x}_i\bar{A}_i\bar{y}_i \subseteq \psi\big([bC_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i]K\big) = \emptyset$$

in G/L. Hence the left-hand side of (8.1) will also be empty, and thus (8.1) will be true.

Therefore, we can assume that $bC_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i \neq \emptyset$ in G. But in this case we can apply Remark 8.4 $(n-1)$ times to find some $a \in G$ such that $bC_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i = a(C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} y_i^{-1}A_iy_i) = aC_E(x_0)$, where $E := A_0 \cap \bigcap_{i=1}^{n-1} y_i^{-1}A_iy_i$ is a conjugate of a special subgroup in G by Lemma 6.5. Now we can use the base of induction $n = 1$, to find a finite-index normal subgroup $M \lhd G$ such that $M \leq K$ and for the natural epimorphism $\varphi\colon G \to G/M$ we have

$$\varphi^{-1}(\varphi(a)C_{\varphi(E)}(\varphi(x_0)) \cap \varphi(x_nA_ny_n)) \subseteq [aC_E(x_0) \cap x_nA_ny_n]K \tag{8.4}$$

in G. By the induction hypothesis, there exists a finite-index normal subgroup $L \lhd G$ such that $L \leq M \leq K$ and

$$\psi^{-1}\big(\bar{b}C_{\bar{A}_0}(\bar{x}_0) \cap \bigcap_{i=1}^{n-1} \bar{x}_i\bar{A}_i\bar{y}_i\big) \subseteq \big[bC_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i\big]M \tag{8.5}$$

in G. Combining (8.5) with (8.4) and recalling that $\ker(\psi) = L \leq M = \ker(\varphi)$ we obtain the following in G:

$$
\begin{aligned}
&\psi^{-1}\big(\bar{b}C_{\bar{A}_0}(\bar{x}_0) \cap \bigcap_{i=1}^{n} \bar{x}_i \bar{A}_i \bar{y}_i\big) \\
&\quad = \psi^{-1}\big(\bar{b}C_{\bar{A}_0}(\bar{x}_0) \cap \bigcap_{i=1}^{n-1} \bar{x}_i \bar{A}_i \bar{y}_i\big) \cap \psi^{-1}(\bar{x}_n \bar{A}_n \bar{y}_n) \\
&\quad \subseteq \big[bC_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_i A_i y_i\big]M \cap x_n A_n y_n L \subseteq aC_E(x_0)M \cap x_n A_n y_n M \\
&\quad \subseteq \varphi^{-1}(\varphi(a)C_{\varphi(E)}(\varphi(x_0))) \cap \varphi^{-1}(\varphi(x_n A_n y_n)) \\
&\quad = \varphi^{-1}(\varphi(a)C_{\varphi(E)}(\varphi(x_0)) \cap \varphi(x_n A_n y_n)) \\
&\quad \subseteq [aC_E(x_0) \cap x_n A_n y_n]K = \big[bC_{A_0}(x_0) \cap \bigcap_{i=1}^{n} x_i A_i y_i\big]K.
\end{aligned}
$$

Hence (8.1) holds in G/L and we have verified the inductive step, finishing the proof of the lemma. □

The next two statements basically establish the main result. These Lemmas 8.5 and 8.6 will be proved by simultaneous induction on $\operatorname{rank}(G)$. The proofs of each of these two lemmas when $\operatorname{rank}(G) = r$ will use both of their conclusions about right-angled Artin groups of ranks strictly less than r.

Lemma 8.5. *If G is a right-angled Artin group of rank r, then the B-conjugacy class g^B is separable in G for any special subgroup B of G and any element $g \in G$.*

Lemma 8.6. *Let G be a right-angled Artin group of rank r. Then every special subgroup B of G satisfies the Centralizer Condition* CC_G *from Definition* 3.3.

The base of induction for both lemmas is $r = 0$, that is, when G is the trivial group. In this case the two statements are trivial. Therefore the proofs of Lemmas 8.5 and 8.6 start with assuming that both of their claims have been established for all right-angled Artin groups of rank $< r$, and will aim to prove the inductive step by considering the case when $\operatorname{rank}(G) = r \geq 1$.

The proofs of Lemmas 8.5 and 8.6 make use of the four auxiliary statements below. These statements – Lemmas 8.7 through 8.10 – start with a right-angled Artin group G of rank r (presented as a special HNN-extension

$$G = \langle A, t \mid tht^{-1} = h \text{ for all } h \in H\rangle \tag{8.6}$$

of a maximal special subgroup $A \leq G$ with respect to some special subgroup $H \leq A$, see Remark 7.3), and assume that Lemmas 8.5 and 8.6 have already been established for A since $\operatorname{rank}(A) = r - 1 < r = \operatorname{rank}(G)$.

Lemma 8.7. *Suppose that B is a special subgroup of G contained in A, $g \in G \setminus A$ and $f \in G \setminus g^B$. Then there exists an epimorphism ψ from A onto a finite group Q such that for the corresponding extension $\widetilde{\psi}: G \to P$ from G onto the special HNN-extension P of Q (with respect to $\psi(H)$), obtained according to Remark* 7.4*, we have $\widetilde{\psi}(f) \notin \widetilde{\psi}(g)^{\widetilde{\psi}(B)}$ in P.*

Proof. Let $x_0 t^{\varepsilon_1} x_1 t^{\varepsilon_2} \dots t^{\varepsilon_n} x_n$ and $y_0 t^{\zeta_1} y_1 t^{\zeta_2} \dots t^{\zeta_m} y_m$ be reduced products representing g and f in G respectively. Since $g \notin A$ we have $n \geq 1$.

Case 1: First suppose that condition (i) from Lemma 7.11 does not hold. By Corollary 5.4 and Lemma 8.1, the special subgroup $H = HH$ of A is closed in $\mathcal{PT}(A)$, hence there is a finite-index normal subgroup $L \lhd A$ and the corresponding epimorphism $\psi: A \to Q := A/L$ such that such that $\psi(x_i) \notin \psi(H)$, $i = 1, \dots, n-1$, and $\psi(y_j) \notin \psi(H)$, $j = 1, \dots, m-1$. Let

$$P := \langle Q, s \mid sqs^{-1} = q \text{ for all } q \in \psi(H)\rangle \tag{8.7}$$

be the special HNN-extension of Q with respect to $\psi(H)$. By Remark 7.4, ψ can be extended to a homomorphism $\widetilde{\psi}: G \to P$ such that $\widetilde{\psi}|_A = \psi$ and $\widetilde{\psi}(t) = s$. Therefore, $\widetilde{\psi}(g) = \bar{x}_0 s^{\varepsilon_1} \bar{x}_1 s^{\varepsilon_2} \dots s^{\varepsilon_n} \bar{x}_n$, $\widetilde{\psi}(f) = \bar{y}_0 s^{\zeta_1} \bar{y}_1 s^{\zeta_2} \dots s^{\zeta_m} \bar{y}_m$ and these products are reduced in P, where $\bar{x}_i := \psi(x_i)$ and $\bar{y}_j := \psi(y_j)$ for all $i = 0, \dots, n$, $j = 0, \dots, m$. And since the condition (i) did not hold for g and f, this condition will not hold for $\widetilde{\psi}(g)$ and $\widetilde{\psi}(f)$. Therefore, $\widetilde{\psi}(f) \notin \widetilde{\psi}(g)^{\widetilde{\psi}(B)}$ by Lemma 7.11.

Thus we can now assume that $n = m$ and $\varepsilon_i = \zeta_i$ for $i = 1, \dots, n$. Set $x := x_0 \dots x_n \in A$ and $y := y_0 \dots y_n \in A$.

Case 2: Suppose that $y \notin x^B$ in A. Then, by the induction hypothesis, x^B is separable in A, hence there is a finite group Q and an epimorphism $\psi: A \to Q$ such that $\psi(y) \notin \psi(x)^{\psi(B)}$. Let P be the special HNN-extension of Q defined by (8.7), and let $\widetilde{\psi}: G \to P$ be the corresponding extension of ψ with $\widetilde{\psi}(t) = s$. By Lemma 7.6, there is a retraction $\tilde{\rho}_Q \in \operatorname{End}(P)$ of P onto Q (extending the identity map on Q) satisfying $\tilde{\rho}_Q(s) = 1$. Therefore, using the above notations, we have

$$\tilde{\rho}_Q(\widetilde{\psi}(f)) = \tilde{\rho}_Q(\bar{y}_0 s^{\varepsilon_1} \bar{y}_1 s^{\varepsilon_2} \dots s^{\varepsilon_n} \bar{y}_n) = \bar{y}_0 \bar{y}_1 \dots \bar{y}_n = \psi(y) \in Q,$$

similarly, $\tilde{\rho}_Q(\widetilde{\psi}(g)) = \psi(x) \in Q$. And since $\tilde{\rho}_Q(\widetilde{\psi}(B)) = \psi(B)$ and $\psi(y) \notin \psi(x)^{\psi(B)}$ we can conclude that $\widetilde{\psi}(f) \notin \widetilde{\psi}(x)^{\widetilde{\psi}(B)}$.

Case 3: We can now assume that both of the conditions (i) and (ii) of Lemma 7.11 are satisfied. Choose any $b_0 \in B$ such that $y = b_0 x b_0^{-1}$. As $f \notin g^B$ in G, according to Lemma 7.11 we must have $I = \emptyset$ in A, where

$$I := b_0 C_B(x) \cap y_0 H x_0^{-1} \cap (y_0 y_1) H (x_0 x_1)^{-1} \cap \dots \cap (y_0 \dots y_{n-1}) H (x_0 \dots x_{n-1})^{-1}.$$

As we saw earlier, H is separable in A, therefore there is a finite-index normal subgroup $K \lhd A$ such that $x_i \notin HK$ and $y_i \notin HK$ for $1 \leq i \leq n-1$. Now, since $\operatorname{rank}(A) = r - 1 < r$, the right-angled Artin group A satisfies the claims of Lemmas 8.5 and 8.6 by the induction hypothesis. Consequently, we can apply Lemma 8.3 to A and K, finding a finite-index normal subgroup $L \lhd A$ such that $L \leq K$ and

$$\bar{b}_0 C_{\bar{B}}(\bar{x}) \cap \bigcap_{i=1}^{n} \bar{x}_i \bar{H} \bar{y}_i \subseteq \psi(IK) = \emptyset \tag{8.8}$$

in $Q := A/L$, where $\bar{b}_0, \bar{B}, \bar{x}$, $\bar{x}$_i, $\bar{H}, \bar{y}_i$ denote the ψ-images of b_0, B, x, x_i, H, y_i in Q respectively. As before we can extend ψ to a homomorphism $\tilde{\psi}\colon G \to P$, where P is given by (8.7), and $\tilde{\psi}(t) = s$. Since $L \leq K$ we have $\bar{x}_i, \bar{y}_i \notin \psi(H)$ for $i = 1, \dots, n$, and so $\bar{x}_0 s^{\varepsilon_1} \bar{x}_1 s^{\varepsilon_2} \dots s^{\varepsilon_n} \bar{x}_n$ and $\bar{y}_0 s^{\varepsilon_1} \bar{y}_1 s^{\varepsilon_2} \dots s^{\varepsilon_n} \bar{y}_n$ are reduced products in P representing the elements $\tilde{\psi}(g)$ and $\tilde{\psi}(f)$ respectively. Thus, Lemma 7.11, in view of (8.8), implies that $\tilde{\psi}(f) \notin \tilde{\psi}(g)^{\tilde{\psi}(B)}$ in P. And Lemma 8.7 is proved. □

Lemma 8.8. *Suppose that $g_0, f_0, f_1, \dots, f_m \in G$, and that the elements $g_0 = t^{\varepsilon_1} x_1 \dots t^{\varepsilon_n} x_n$, $f_0 = t^{\zeta_1} y_1 \dots t^{\zeta_k} y_k$ are cyclically reduced in G, with $n \geq 1$. If $f_j \notin g_0^H$ for every $j = 1, \dots, m$, then there is a finite group Q and an epimorphism $\psi\colon A \to Q$ such that for the corresponding epimorphism $\tilde{\psi}\colon G \to P$, extending ψ, with $\tilde{\psi}(t) = s$ (where P is the special HNN-extension given by* (8.7)*), the following holds:*

- *$\tilde{\psi}(f_j) \notin \tilde{\psi}(g_0)^{\tilde{\psi}(H)}$ in P for each $j \in \{1, \dots, m\}$;*
- *the elements $\tilde{\psi}(g_0) = s^{\varepsilon_1} \bar{x}_1 \dots s^{\varepsilon_n} \bar{x}_n$ and $\tilde{\psi}(f_0) = s^{\zeta_1} \bar{y}_1 \dots s^{\zeta_k} \bar{y}_k$ are cyclically reduced in P, where $\bar{x}_i := \tilde{\psi}(x_i)$, $i = 1, \dots, n$, $\bar{y}_l := \tilde{\psi}(y_l)$, $l = 1, \dots, k$.*

Proof. For every $j = 1, \dots, m$, since $f_j \notin g_0^H$ in G, we can apply Lemma 8.7 (as $H \leq A$ is a special subgroup of G), to find a finite-index normal subgroup $L_j \lhd A$, such that $\tilde{\psi}_j(f_j) \notin \tilde{\psi}_j(g_0)^{\tilde{\psi}_j(H)}$ in P_j, where $\tilde{\psi}_j\colon G \to P_j$ is the homomorphism (obtained according to Remark 7.4) extending the natural epimorphism $\psi_j\colon A \to A/L_j$, and P_j is the special HNN-extension of A/L_j with respect to $\psi_j(H)$.

Now, since H is separable in A (by Lemma 8.1 and Corollary 5.4), there is a finite-index normal subgroup $K \lhd A$ such that $x_i \notin HK$ whenever $x_i \notin H$, for all $i = 1, \dots, n$, and $y_l \notin HK$ whenever $y_l \notin H$, for all $l = 1, \dots, k$. Define the finite-index normal subgroup L of A by $L := L_1 \cap \dots \cap L_m \cap K$, and let $\psi\colon A \to Q := A/L$ be the natural epimorphism. Observe that for each j, the map ψ_j factors through the map ψ. Hence, once we let $\tilde{\psi}\colon G \to P$ be the extension of ψ as in the formulation of Lemma 8.8, the Universal Property of special HNN-extensions (Remark 7.4) will imply that $\tilde{\psi}_j$ factors through $\tilde{\psi}$ for every $j = 1, \dots, m$. Consequently, $\tilde{\psi}(f_j) \notin \tilde{\psi}(g_0)^{\tilde{\psi}(H)}$ in P for each $j \in \{1, \dots, m\}$. The second assertion of Claim B holds due to the choice of K and because $L \leq K$. Thus Lemma 8.8 is proved. □

Lemma 8.9. *Let $K \lhd G$ be a normal subgroup of finite index, let B be a special subgroup of G with $B \leq A$, and let an element $g \in G \setminus A$ be represented by a* *reduced product* $x_0 t^{\varepsilon_1} x_1 t^{\varepsilon_2} \dots t^{\varepsilon_n} x_n$ *in* G*, with* $n \geq 1$*. Then there is a finite group* Q *and an epimorphism* $\psi \colon A \to Q$ *such that for the corresponding homomorphism* $\widetilde{\psi} \colon G \to P$*, extending* ψ *and obtained according to Remark* 7.4*, with* $\widetilde{\psi}(t) = s$ (*where* P *is the special HNN-extension given by* (8.7))*, the following holds:*

- $C_{\widetilde{\psi}(B)}(\widetilde{\psi}(g)) \subseteq \widetilde{\psi}(C_B(g)K)$ *in* P;
- $\ker(\psi) \leq A \cap K$ *and* $\ker(\widetilde{\psi}) \leq K$.

Proof. Since A is residually finite (Lemma 8.1), the special subgroup $H = HH$ is closed in $\mathcal{PT}(A)$ by Corollary 5.4. Therefore there exists a finite-index normal subgroup $M_1 \lhd A$ such that $x_i \notin HM_1$ in A for all $i = 1, \dots, n-1$. As usual, we can replace M_1 with $M_1 \cap K_1$, to make sure that $M_1 \leq K_1$, where $K_1 := A \cap K$.

Note that, according to Lemma 7.12, $C_B(g) = I$ in G, where

$$I := C_B(x) \cap x_0 H x_0^{-1} \cap (x_0 x_1) H (x_0 x_1)^{-1} \cap \dots \cap (x_0 \dots x_{n-1}) H (x_0 \dots x_{n-1})^{-1},$$

and $x := x_0 \dots x_n \in A$.

Since $\operatorname{rank}(A) = r-1 < r$, by the induction hypothesis the claims of Lemmas 8.5 and 8.6 hold for A. Hence, we can use Lemma 8.3 to find a finite-index normal subgroup $L_1 \lhd A$ such that $L_1 \leq M_1 \leq K_1$ and, for the corresponding epimorphism $\psi \colon A \to Q := A/L_1$, we have

$$J := C_{\bar{B}}(\bar{x}) \cap \bar{x}_0 \bar{H} \bar{x}_0^{-1} \cap (\bar{x}_0 \bar{x}_1) \bar{H} (\bar{x}_0 \bar{x}_1)^{-1} \cap \cdots \\ \cdots \cap (\bar{x}_0 \dots \bar{x}_{n-1}) \bar{H} (\bar{x}_0 \dots \bar{x}_{n-1})^{-1} \subseteq \psi(IM_1)$$

in Q, where $\bar{B}$, $\bar{x}$, $\bar{H}$ and $\bar{x}_i$ denote the ψ-images of B, x, H and x_i in Q, $i = 0, \dots, n$.

Let P be the special HNN-extension of Q given by (8.7), and let $\widetilde{\psi} \colon G \to P$ be the extension of ψ provided by Remark 7.4, with $\widetilde{\psi}(t) = s$. Since $x_i \notin HL_1$ in A for $i = 1, \dots, n-1$, the product $\bar{x}_0 s^{\varepsilon_1} \bar{x}_1 \dots s^{\varepsilon_n} \bar{x}_n$ is reduced and represents the element $\widetilde{\psi}(g)$ in P. Consequently, Lemma 7.12 tells us that $C_{\widetilde{\psi}(B)}(\widetilde{\psi}(g)) = J$ in P. And noting that $\psi(M_1) \leq \psi(K_1) = \widetilde{\psi}(K_1) \leq \widetilde{\psi}(K)$, we arrive at

$$C_{\widetilde{\psi}(B)}(\widetilde{\psi}(g)) = J \subseteq \psi(I)\psi(M_1) \subseteq \widetilde{\psi}(I)\widetilde{\psi}(K) = \widetilde{\psi}(C_B(g)K)$$

in P.

Finally, observe that $\ker(\psi) = L_1 \leq K_1 = A \cap K$ and $\ker(\widetilde{\psi})$ is the normal closure of L_1 in G (by Lemma 7.5). And since $L_1 \leq K \lhd G$, we see that $\ker(\widetilde{\psi}) \leq K$ in G. Thus Lemma 8.9 has been established. □

Lemma 8.10. *Let* $g_0 = t^{\varepsilon_1} x_1 \dots t^{\varepsilon_n} x_n$ *be a cyclically reduced element in* G*, with* $n \geq 1$*. Then there exists an epimorphism* ψ *from* A *onto a finite group* Q *such that for the corresponding extension* $\widetilde{\psi} \colon G \to P$ *from* G *onto the special HNN-extension* P *of* Q (*given by* (8.7))*, with* $\widetilde{\psi}|_A = \psi$ *and* $\widetilde{\psi}(t) = s$*, we have*

$$\ker(\psi) \leq A \cap K,\ \ker(\widetilde{\psi}) \leq K \text{ in } G, \text{ and } C_P(\widetilde{\psi}(g_0)) \subseteq \widetilde{\psi}(C_G(g_0)K) \text{ in } P.$$

Proof. Clearly there exists $m \in \{0, 1, \dots, n\}$ such that we can enumerate all the prefixes $p_1, \dots, p_{n+1}$ of g_0 so that $p_j^{-1} g_0 p_j \notin g_0^H$ in G whenever $1 \le j \le m$, and $p_j^{-1} g_0 p_j \in g_0^H$ in G whenever $m + 1 \le j \le n + 1$. For each $j \in \{m + 1, m + 2, \dots, n + 1\}$, choose $h_j \in H$ such that $h_j p_j^{-1} g_0 p_j h_j^{-1} = g_0$ in G, and set $\Omega := \{h_j p_j^{-1} \mid m + 1 \le j \le n + 1\} \subset G$.

Let $f_0 := g_0 = t^{\varepsilon_1} x_1 \dots t^{\varepsilon_n} x_n$ and $f_j := p_j^{-1} g_0 p_j$ for $j = 1, \dots, m$. Applying Lemma 8.8 to $g_0, f_0, \dots, f_m \in G$ we find a finite group Q_1, an epimorphism $\psi_1 \colon A \to Q_1$, the special HNN-extension P_1 of Q_1 with respect to $\psi_1(H)$, and the corresponding extension $\widetilde{\psi}_1 \colon G \to P_1$ of ψ_1 (obtained by Remark 7.4), such that $\widetilde{\psi}_1(f_j) \notin \widetilde{\psi}_1(g_0)^{\widetilde{\psi}_1(H)}$ in P_1, for each $j \in \{1, \dots, m\}$, and the element $\widetilde{\psi}_1(g_0)$ is cyclically reduced in P_1.

On the other hand, by Lemma 8.9, there exist a finite group Q_2, an epimorphism $\psi_2 \colon A \to Q_2$, the special HNN-extension P_2 of Q_2 with respect to $\psi_2(H)$, and the corresponding extension $\widetilde{\psi}_2 \colon G \to P_2$ of ψ_2, such that $C_{\widetilde{\psi}_2(H)}(\widetilde{\psi}_2(g_0)) \subseteq \widetilde{\psi}_2(C_H(g_0)K)$ in P_2, $\ker(\psi_2) \le A \cap K$ and $\ker(\widetilde{\psi}_2) \le K$.

Define a finite-index normal subgroup $L_0 \lhd A$ by $L_0 := \ker(\psi_1) \cap \ker(\psi_2) \le A \cap K$, and let $\psi \colon A \to Q := A/L_0$ be the natural epimorphism. By Remark 7.4, there is an epimorphism $\widetilde{\psi} \colon G \to P$, extending ψ so that $\widetilde{\psi}(t) = s$, where P is the special HNN-extension of Q given by (8.7). Since $\ker(\psi) = L_0 \le \ker(\psi_i)$, the maps $\psi_i \colon A \to Q_i$ factor through ψ for $i = 1, 2$. Consequently, according to the Universal Property of special HNN-extensions (Remark 7.4), the maps $\widetilde{\psi}_i \colon G \to P_i$ factor through $\widetilde{\psi} \colon G \to P$ for $i = 1, 2$. Therefore we have

$$\widetilde{\psi}(g_0) \text{ is cyclically reduced and } \widetilde{\psi}(f_j) \notin \widetilde{\psi}(g_0)^{\widetilde{\psi}(H)} \text{ in } P \tag{8.9}$$

for all $j \in \{1, \dots, m\}$.

On the other hand, since $\ker(\widetilde{\psi}) \le \ker(\widetilde{\psi}_2) \le K$ in G, we also have

$$\widetilde{\psi}^{-1}(C_{\widetilde{\psi}(H)}(\widetilde{\psi}(g_0))) \subseteq \widetilde{\psi}_2^{-1}(C_{\widetilde{\psi}_2(H)}(\widetilde{\psi}_2(g_0))) \subseteq C_H(g_0)K$$

in G, which implies that

$$C_{\widetilde{\psi}(H)}(\widetilde{\psi}(g_0)) \subseteq \widetilde{\psi}(C_H(g_0)K) \tag{8.10}$$

in P.

Case 1: $x_n \in H$. Then, according to Proposition 7.8, $n = 1$, $C_G(g_0) = \langle t \rangle C_H(g_0)$ in G, and $C_P(\widetilde{\psi}(g_0)) = \langle s \rangle C_{\widetilde{\psi}(H)}(\widetilde{\psi}(g_0))$ in P. Recalling (8.10), we see that

$$C_P(\widetilde{\psi}(g_0)) \subseteq \langle \widetilde{\psi}(t) \rangle \, \widetilde{\psi}(C_H(g_0)K) = \widetilde{\psi}(C_G(g_0)K)$$

in P.

Case 2: $x_n \in A \setminus H$. In this case (8.9) implies that $\widetilde{\psi}(p_{m+1}), \dots, \widetilde{\psi}(p_{n+1})$ is the list of all prefixes of $\widetilde{\psi}(g_0)$ satisfying $\widetilde{\psi}(p_j)^{-1} \widetilde{\psi}(g_0) \widetilde{\psi}(p_j) \in \widetilde{\psi}(g_0)^{\widetilde{\psi}(H)}$, because if $1 \le j \le m$, then $\widetilde{\psi}(p_j)^{-1}\widetilde{\psi}(g_0)\widetilde{\psi}(p_j) = \widetilde{\psi}(p_j^{-1}g_0p_j) = \widetilde{\psi}(f_j) \notin \widetilde{\psi}(g_0)^{\widetilde{\psi}(H)}$ in P.

Therefore, by Proposition 7.8, $C_P(\widetilde{\psi}(g_0)) = C_{\widetilde{\psi}(H)}(\widetilde{\psi}(g_0))\langle\widetilde{\psi}(g_0)\rangle\bar{\Omega}$, where $\bar{\Omega} := \{\widetilde{\psi}(h_j)\widetilde{\psi}(p_j)^{-1} \mid m+1 \le j \le n+1\} = \widetilde{\psi}(\Omega) \subset P$. Thus, recalling (8.10), we find

$$C_P(\widetilde{\psi}(g_0)) \subseteq \widetilde{\psi}(C_H(g_0)K\langle g_0\rangle\Omega) = \widetilde{\psi}(C_G(g_0)K)$$

in P. In either of the two cases we have shown that $C_P(\widetilde{\psi}(g_0)) \subseteq \widetilde{\psi}(C_G(g_0)K)$ in P. This completes the proof of Lemma 8.10. □

We are finally ready to prove the two main Lemmas 8.5 and 8.6 announced above.

Proof of Lemma 8.5. There are two separate cases to consider.

Case 1: $B \ne G$. Choose a maximal special subgroup A of G containing B. Then A is a right-angled Artin group of rank $r-1$, and, according to Remark 7.3, G splits as a special HNN-extension (8.6) of A with respect to some special subgroup H of A.

If $g \in A$, then g^B is closed in $\mathcal{PT}(A)$ by the induction hypothesis. Since G is residually finite (Lemma 8.1), g^B is separable in G by Lemma 5.5.

Thus we can suppose that $g \in G \setminus A$. Take any element $f \in G \setminus g^B$. Let Q, P, $\psi\colon A \to Q$ and $\widetilde{\psi}\colon G \to P$ be given by Lemma 8.7, so that $\widetilde{\psi}(f) \notin \widetilde{\psi}(g)^{\widetilde{\psi}(B)}$ in P.

Observe that P is a virtually free group as an HNN-extension of the finite group Q, hence P is residually finite. Since $\widetilde{\psi}(B) = \psi(B) \subseteq Q$ is finite, $\widetilde{\psi}(g)^{\widetilde{\psi}(B)}$ is a finite subset of P. Hence there is a homomorphism $\xi\colon P \to R$ from P to a finite group R such that $\xi(\widetilde{\psi}(f)) \notin \xi(\widetilde{\psi}(g)^{\widetilde{\psi}(B)})$ in R. Consequently, the homomorphism $\varphi\colon G \to R$, defined by $\varphi := \xi \circ \widetilde{\psi}$, satisfies the condition $\varphi(f) \notin \varphi(g^B)$. Therefore we have shown that g^B is separable in G in Case 1.

Case 2: $B = G$. If $g = 1$ then $g^G = \{1\}$ is separable in G because G is residually finite (Lemma 8.1). Hence we can suppose that $g \ne 1$. But then, by Lemma 6.8, there exists a maximal special subgroup A of G such that $g \notin A^G$. The group G is a special HNN-extension (8.6) of A with respect to a certain special subgroup $H \le A$ (by Remark 7.3). Obviously, g is conjugate in G to some cyclically reduced element $g_0 = t^{\varepsilon_1}x_1 \dots t^{\varepsilon_n}x_n$ with $n \ge 1$, because $g \notin A^G$. This implies that $g^G = g_0^G$ in G.

To show that g^G is closed in $\mathcal{PT}(G)$, consider any element $f \in G \setminus g^G$.

Subcase 2.1: First suppose that $f \notin A^G$. Then we can find a cyclically reduced element $f_0 = t^{\zeta_1}y_1 \dots t^{\zeta_m}y_m \in f^G$. Let $f_1, f_2, \dots, f_m$ be the list of all cyclic permutations of f_0 in G.

Observe that $f_j \notin g_0^H \subset g^G$ for every $j = 1, \dots, m$, because $f_0 \notin g^G$. Therefore we can apply Lemma 8.8 to find Q, P, $\psi\colon A \to Q$ and $\widetilde{\psi}\colon G \to P$ from its claim.

Since $\widetilde{\psi}(f_1), \dots, \widetilde{\psi}(f_m)$ is the list of all cyclic permutations of $\widetilde{\psi}(f_0)$ in P, Lemma 8.8, together with Lemma 7.7, imply that $\widetilde{\psi}(f_0) \notin \widetilde{\psi}(g_0)^P$ in P. Now, according to Lemma 8.2, there is a homomorphism $\xi\colon P \to R$ such that R is a finite group and $\xi(\widetilde{\psi}(f_0)) \notin \xi(\widetilde{\psi}(g_0))^R$. Therefore, defining the homomorphism $\varphi\colon G \to R$ by $\varphi := \xi \circ \widetilde{\psi}$ we achieve $\varphi(f_0) \notin \varphi(g_0)^R$ in R. But since $\varphi(f)$ is conjugate to $\varphi(f_0)$ and $\varphi(g_0)$ is conjugate to $\varphi(g)$ in R, we can conclude that $\varphi(f) \notin \varphi(g)^R = \varphi(g^G)$ in R.

To finish proving Case 2, it remains to consider

Subcase 2.2: $f \in A^G$. Set $m := 0$ and denote $f_0 = g_0 \in G$. Applying Lemma 8.8 to g_0 and f_0, we can find a homomorphism $\widetilde{\psi}$ from G to a special HNN-extension P of a finite group Q such that $\widetilde{\psi}(g_0) = s^{\varepsilon_1}\widetilde{\psi}(x_1)\dots s^{\varepsilon_n}\widetilde{\psi}(x_n)$ is cyclically reduced in P. Since $n \geq 1$, by Lemma 7.7 we have $\widetilde{\psi}(g_0) \notin \widetilde{\psi}(A)^P = \widetilde{\psi}(A^G)$ in P, hence $\widetilde{\psi}(f) \notin \widetilde{\psi}(g_0)^P = \widetilde{\psi}(g)^P$ in P. Arguing as above, we can find a finite quotient R of P (and, hence, of G) such that the images of f and g are not conjugate in R.

We can now conclude that the conjugacy class g^G is closed in $\mathcal{PT}(G)$. Thus Case 2 is completed. This finishes the proof of Lemma 8.5. □

Proof of Lemma 8.6. Take any element $g \in G$ and any finite-index normal subgroup $K \lhd G$. As in Lemma 8.5, the proof splits into two main cases.

Case 1: $B \neq G$. Choose a maximal special subgroup A of G containing B. Then A is a right-angled Artin group of rank $r-1 < r$, and G is the special HNN-extension (8.6) of A with respect to a certain special subgroup $H \leq A$ (by Remark 7.3). Define the finite-index normal subgroup K_1 of A by $K_1 := K \cap A$.

Subcase 1.1: $g \in A$. Then, according to the induction hypothesis, the pair (B, g) satisfies the Centralizer Condition CC_A in A, hence there exists $L_1 \lhd A$ such that $|A : L_1| < \infty$, $L_1 \leq K_1$, and the natural epimorphism $\psi\colon A \to Q := A/L_1$ satisfies

$$C_{\psi(B)}(\psi(g)) \subseteq \psi(C_B(g)K_1) \tag{8.11}$$

in Q. Let $\rho_A\colon G \to A$ be the canonical retraction and set $L := \rho_A^{-1}(L_1) \cap K$. Then $L \lhd G$, $|G : L| < \infty$, $L \leq K$ and $\rho_A(L) = L_1 \leq K_1$ (since $K_1 = K \cap A \subseteq \rho_A(K)$). Let $\varphi\colon G \to R := G/L$ be the natural epimorphism. Observe that $\ker(\psi) = \ker(\varphi) \cap A$ in G. Indeed, $\ker(\psi) = L_1$, $\ker(\varphi) = L$, and $L_1 \subseteq \rho_A^{-1}(L_1) \cap K \cap A = L \cap A$, $L \cap A \subseteq \rho_A(L) = L_1$.

Therefore, without loss of generality, we can assume that $Q \leq R$, and the restriction of φ to A coincides with ψ. Then we have $\psi(K_1) = \varphi(K_1) \subseteq \varphi(K)$ in R. Since $g \in A$ and $B \leq A$, (8.11) implies that

$$C_{\varphi(B)}(\varphi(g)) = C_{\psi(B)}(\psi(g)) \subseteq \psi(C_B(g))\psi(K_1) \subseteq \varphi(C_B(g))\varphi(K)$$

in R, which shows that the pair (B, g) has CC_G in Subcase 1.1.

Subcase 1.2: $g \in G \setminus A$. Then the element g can be represented as a reduced product $x_0 t^{\varepsilon_1} x_1 t^{\varepsilon_2} \dots t^{\varepsilon_n} x_n$ in G, with $n \geq 1$. Therefore we can find the groups Q, P and the maps $\psi\colon A \to Q$, $\widetilde{\psi}\colon G \to P$ from the claim Lemma 8.9, so that all of the assertions of that lemma hold.

Note that the subgroup $\widetilde{\psi}(B) \cap \widetilde{\psi}(K) \le Q \le P$ is finite, therefore, since P is residually finite (as a virtually free group), the finite set $\widetilde{\psi}(g)^{\widetilde{\psi}(B)\cap\widetilde{\psi}(K)}$ is separable in P. Consequently, by Lemma 3.7, there exists a finite group R and an epimorphism $\xi\colon P \to R$ such that $\ker(\xi) \le \widetilde{\psi}(K)$ and

$$C_{\xi(\widetilde{\psi}(B))}(\xi(\widetilde{\psi}(g))) \subseteq \xi(C_{\widetilde{\psi}(B)}(\widetilde{\psi}(g))\widetilde{\psi}(K))$$

in R. Define the epimorphism $\varphi\colon G \to R$ by $\varphi := \xi \circ \widetilde{\psi}$, and observe that $\ker(\varphi) = \widetilde{\psi}^{-1}(\ker(\xi)) \subseteq \widetilde{\psi}^{-1}(\widetilde{\psi}(K)) = K \ker(\widetilde{\psi})$. But $\ker(\widetilde{\psi}) \le K$ according to the second assertion of Lemma 8.9, hence $L := \ker(\varphi) \le K$ in G.

Finally, recalling the first assertion of Lemma 8.9, we see that

$$\begin{aligned} C_{\varphi(B)}(\varphi(g)) = C_{\xi(\widetilde{\psi}(B))}(\xi(\widetilde{\psi}(g))) &\subseteq \xi(C_{\widetilde{\psi}(B)}(\widetilde{\psi}(g))\widetilde{\psi}(K)) \\ &\subseteq \xi(\widetilde{\psi}(C_B(g)K)\widetilde{\psi}(K)) = \varphi(C_B(g)K) \end{aligned}$$

in R. Thus we have shown that the pair (B, g) has CC_G in Subcase 1.2. Therefore, B has CC_G in Case 1.

Case 2: $B = G$. The pair $(G, 1)$ evidently satisfies CC_G, therefore we can assume that $g \neq 1$. In this case, by Lemma 6.8, there exists a maximal special subgroup A of G such that $g \notin A^G$. By Remark 7.3, G splits as a special HNN-extension (8.6) of A with respect to a certain special subgroup $H \le A$. Obviously, there exists $z \in G$ such that $g = zg_0z^{-1}$ in G for some cyclically reduced element $g_0 = t^{\varepsilon_1}x_1 \dots t^{\varepsilon_n}x_n$, where $n \ge 1$ because $g \notin A^G$.

Now we apply Lemma 8.10 to find Q, P, $\psi\colon A \to Q$ and $\widetilde{\psi}\colon G \to P$ from its claim, so that $\ker(\widetilde{\psi}) \le K$ and $C_P(\widetilde{\psi}(g_0)) \subseteq \widetilde{\psi}(C_G(g_0)K)$ in P. Note that P is virtually free (being an HNN-extension of a finite group Q), hence every subgroup of P is virtually free as well. Therefore, by Lemma 8.2, P is hereditarily conjugacy separable, and thus, by Proposition 3.2, P satisfies the Centralizer Condition CC.

Consequently, there exists a finite group R and an epimorphism $\xi\colon P \to R$ such that $\ker(\xi) \le \widetilde{\psi}(K)$ and

$$C_R(\xi(\widetilde{\psi}(g_0))) \subseteq \xi(C_P(\widetilde{\psi}(g_0))\widetilde{\psi}(K))$$

in R. Defining the epimorphism $\varphi\colon G \to R$ by $\varphi := \xi \circ \widetilde{\psi}$, and arguing in the same manner as in Subcase 1.2, we can show that $L := \ker(\varphi) \le K$ and $C_R(\varphi(g_0)) \subseteq \varphi(C_G(g_0)K)$ in R. Conjugating both sides of the latter inclusion by $\varphi(z)$ (and recalling that $g = zg_0z^{-1}$ in G), we obtain $C_R(\varphi(g)) \subseteq \varphi(C_G(g)K)$.

Hence $B = G$ has CC_G in Case 2, and Lemma 8.6 is proved. □

Thus Lemmas 8.5 and 8.6 have been proved when $\mathrm{rank}(G) = r$. Therefore, by induction they are true for all $r \in \mathbb{N} \cup \{0\}$, and we are ready to prove the main result of this paper.

Proof of Theorem 1.1. Let G be a right-angled Artin group associated to a finite simplicial graph Γ. Then for every $g \in G$, the conjugacy class g^G is separable in

G by Lemma 8.5. Lemma 8.6 tells us that G satisfies the Centralizer Condition CC. Therefore, by Proposition 3.2, G is hereditarily conjugacy separable. □

9. Applications to separability properties

The first two applications that we mention do not directly follow from the statement of Theorem 1.1, but are consequences of its proof.

Corollary 9.1. *Let A and B be conjugates of special subgroups of a right-angled Artin group G. Then for any element $x \in G$, the double coset AxB is separable in G.*

Proof. Evidently, it is enough to consider the case when A and B are special subgroups of G. Then A and B are retracts of G and the corresponding retractions commute (Remark 6.1). By Remark 6.2, $A \cap B$ is also a special subgroup of G, hence Lemma 8.5 implies that the subset $\alpha^{A\cap B}$ is separable in G for every $\alpha \in G$. Therefore, AxB is separable in G by Lemma 5.3. □

In the case when $x = 1$ and A,B are special subgroups of G (not conjugates of them), Corollary 9.1 was proved by Haglund and Wise in [34], Corollary 9.4, using different arguments, based on Niblo's criterion for separability of double cosets (see [47]). Unfortunately, in general this criterion cannot be used to prove separability of double cosets of the form AxB if $x \in G$ is an arbitrary element (because the retractions onto A and xBx^{-1} may no longer commute).

Similarly, using Lemmas 8.5 and 8.6 together with Lemmas 6.5 and 5.7, we can obtain the following.

Corollary 9.2. *Suppose that A and D are conjugates of special subgroups in a right-angled Artin group G, and $g \in G$ is an arbitrary element. Then the double coset $C_A(g)D$ is separable in G.*

The rest of applications in this section discuss conjugacy separability of various groups. Let us start with the following well-known observation.

Lemma 9.3. *If H is a retract of a conjugacy separable group G, then H is conjugacy separable.*

Proof. Indeed, let $\rho_H \in \mathrm{End}(G)$ be a retraction of G onto H. Suppose that $x, y \in H$ and $y \notin x^H$ in H. If there existed $g \in G$ such that $y = gxg^{-1}$ in G, then we would have $y = \rho_H(y) = \rho_H(g)\rho_H(x)\rho_H(g)^{-1} = \rho_H(g)x\rho_H(g)^{-1}$ in H, contradicting to our assumption. Therefore, $y \notin x^G$, and since G is conjugacy separable, there is a finite group R and a homomorphism $\varphi\colon G \to R$ such that $\varphi(y) \notin \varphi(x)^R$. Let $Q := \varphi(H)$ and $\psi\colon H \to Q \le R$ be the restriction of φ to H. By construction, we have that $\psi(y) \notin \psi(x)^Q$ in Q. Therefore H is conjugacy separable. □

Remark 9.4. If F is a finite-index subgroup in a virtual retract H of a group G, then F itself is a virtual retract of G.

Indeed, let $K \le G$ be a finite-index subgroup containing H, and let ρ_H be a retraction of K onto H. Then $M := \rho_H^{-1}(F) \le K$ has finite index in K, and, hence, in G. Evidently the restriction of ρ_H to M is a retraction of M onto F.

Combining Remark 9.4 with Lemma 9.3 we obtain the following statement (cf. [12], Theorem 3.4):

Lemma 9.5. *A virtual retract of a hereditarily conjugacy separable group is hereditarily conjugacy separable itself.*

Next comes a classical fact about conjugacy separable groups:

Lemma 9.6. *Suppose G is a group satisfying the Unique Root Property. If G contains a conjugacy separable subgroup H of finite index, then G is conjugacy separable.*

Proof. Consider any element $x \in G$. We need to show that the conjugacy class x^G is separable in G.

Assume, first, that $x \in H$. Then x^H is closed in $\mathcal{PT}(H)$, and since $|G : H| < \infty$, it is also closed in $\mathcal{PT}(G)$. Choose $z_1, \dots, z_k \in G$ so that $G = \bigsqcup_{i=1}^k z_i H$. Then $x^G = \bigcup_{i=1}^k z_i x^H z_i^{-1}$ is a finite union of closed sets in $\mathcal{PT}(G)$, hence x^G is separable in G.

Now, if $x \in G$ is an arbitrary element, then there is $n \in \mathbb{N}$ such that $g := x^n \in H$, and, as we have shown above, g^G is separable in G. Take any $y \in G \setminus x^G$. Since G has the Unique Root Property, we see that $y^n \notin g^G$. Hence, there exists a finite-index normal subgroup $N \lhd G$ such that $y^n \notin g^G N$ in G. Consequently, $y \notin x^G N$ (because the inclusion $y \in x^G N$ implies the inclusion $y^n \in (x^n)^G N$). Thus G is conjugacy separable. □

It is easy to see that Lemma 9.6 can be generalized as follows:

Lemma 9.7. *If a group G has the Unique Root Property and contains a hereditarily conjugacy separable subgroup of finite index, then G is itself hereditarily conjugacy separable.*

Note that the assumption of Lemma 9.6 demanding G to satisfy the Unique Root Property is important: in [28] A. Goryaga constructed an example of a finitely generated group G which is not conjugacy separable, but contains a conjugacy separable subgroup of index 2.

Corollary 9.8. *If a group $G \in \mathcal{AVR}$ has the Unique Root Property, then G is hereditarily conjugacy separable.*

Proof. Let $K \leq G$ be a subgroup of finite index. By definition, G contains a finite-index subgroup H which is a virtual retract of some right-angled Artin group A. Since $|H : (K \cap H)| \leq |G : K| \leq \infty$, $K \cap H$ is a virtual retract of A by Remark 9.4. But the index $|K : (K \cap H)|$ is also finite, hence $K \in \mathcal{AVR}$.

Now Theorem 1.1 and Lemma 9.5 imply that $K \cap H$ is conjugacy separable. Hence K is conjugacy separable by Lemma 9.6. Thus G is hereditarily conjugacy separable. □

Recall that two groups G_1 and G_2 are said to be *commensurable*, if there exist finite-index subgroups $H_1 \leq G_1$ and $H_2 \leq G_2$ such that H_1 is isomorphic to H_2. The proof of Corollary 9.8 allows to conclude that the class $\mathcal{AVR}$ is closed under passing to subgroups of finite index. Therefore we can make

Remark 9.9. If G_1 is commensurable to G_2 and $G_1 \in \mathcal{AVR}$, then $G_2 \in \mathcal{AVR}$.

As we observed in Lemma 6.3, right-angled Artin groups have the Unique Root Property. Another well-known class of groups with this property is the class of torsion-free word hyperbolic groups.

Lemma 9.10. *Torsion-free word hyperbolic groups have the Unique Root Property.*

Proof. Let G be a torsion-free word hyperbolic group. Suppose that $x^n = y^n$ in G for some $x, y \in G$ and $n \in \mathbb{N}$. Since G is torsion-free, we can assume that the orders of x and y are infinite. It is well known that every element $g \in G$, of infinite order, belongs to a unique maximal virtually cyclic subgroup $E(g) \leq G$ (see, for instance, [49], Lemma 1.16).

Note that the element $g := x^n \in G$ has infinite order and $g \in E(x) \cap E(y)$. Therefore, $E(x) = E(y)$, thus $y \in E(x)$. But since G is torsion-free, the virtually cyclic subgroup $E(x) \leq G$ must be cyclic. That is, there exists $z \in G$ such that $x = z^k$ and $y = z^l$ for some $k, l \in \mathbb{Z}$. Obviously, the equality $x^n = y^n$ implies that $k = l$. Thus $x = y$, and hence G enjoys the Unique Root Property. □

Combining Lemma 9.10 with Corollary 9.8 we obtain

Corollary 9.11. *If $G \in \mathcal{AVR}$ is a torsion-free word hyperbolic group, then G is hereditarily conjugacy separable.*

Using Osin's results from [50], it is not difficult to generalize Lemma 9.10 as follows: if a group G is torsion-free and hyperbolic relative to a collection of proper subgroups, each of which has the Unique Root Property, then G has the Unique Root Property. As a result, Corollary 9.11 can also be restated for this kind of relatively hyperbolic groups.

We can also establish Corollary 2.4, mentioned in Section 2.

Proof of Corollary 2.4. Since H has finite index in G, it is also word hyperbolic ([30]), thus, according to Lemma 9.10, H enjoys the Unique Root Property.

By Corollary 2.3, G has a hereditarily conjugacy separable subgroup $F \leq G$ of finite index. Define $K := H \cap F \leq G$. Then K will be hereditarily conjugacy separable (because $|F : K| < \infty$). Since $|H : K| < \infty$, Lemma 9.7 implies that H is hereditarily conjugacy separable. □

10. Applications to outer automorphism groups

We have already discussed a few applications of Theorem 1.1 to outer automorphism groups in Section 2. This aim in this section is to prove Theorem 2.6.

We refer the reader to Osin's monograph [51] for the definition and basic properties of relatively hyperbolic groups. All relatively hyperbolic groups that we consider here are hyperbolic relative to families of *proper* subgroups. In the sense of B. Farb [26], this would correspond to *weak relative hyperbolicity* together with the *Bounded Coset Penetration Condition* (the equivalence of Osin's and Farb's definitions for finitely generated groups is proved in [51], Theorem 6.10).

The following lemma is not difficult to prove but its statement is very useful (see, for example, [32], Lemma 5.4).

Lemma 10.1. *Suppose that G is a finitely generated group and N is a centerless normal subgroup of finite index in G. Then some finite-index subgroup of* $\mathrm{Out}(G)$ *is isomorphic to a quotient of a subgroup of* $\mathrm{Out}(N)$ *by a finite normal subgroup. In particular, if* $\mathrm{Out}(N)$ *is residually finite, then* $\mathrm{Out}(G)$ *is residually finite.*

Recall that a group G is called *elementary* if it contains a cyclic subgroup of finite index.

Lemma 10.2. *If G is a non-elementary relatively hyperbolic group, then its center $Z(G)$ is finite.*

Proof. Suppose that G is hyperbolic relative to a family of proper non-trivial subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$.

First, if $|\Lambda| = \infty$, then G splits as a non-trivial free product by [51], Theorem 2.44, and, thus, $Z(G) = \{1\}$. If the set Λ is finite and each parabolic subgroup H_λ, $\lambda \in \Lambda$, is finite, then G is word hyperbolic (in the sense of Gromov) by [51], Corollary 2.41. And it is well known that the center of a non-elementary word hyperbolic group is finite.

Therefore we can assume that there is some $\mu \in \Lambda$ such that $|H_\mu| = \infty$. A theorem of Osin [51], Theorem 1.4, asserts that the intersection $H_\mu \cap gH_\mu g^{-1}$ is finite for every $g \in G \setminus H_\mu$. If $z \in Z(G)$, then $H_\mu = H_\mu \cap zH_\mu z^{-1}$ is infinite, hence $z \in H_\mu$, i.e., $Z(G) \subseteq H_\mu$. On the other hand, there exists $g \in G \setminus H_\mu$ because

H_μ is a proper subgroup of G. By Osin's theorem, $H_\mu \cap gH_\mu g^{-1}$ is finite. And since $Z(G) \subseteq gH_\mu g^{-1}$, we see that $Z(G) \subseteq H_\mu \cap gH_\mu g^{-1}$ must be finite. □

The proof of Theorem 2.6 will use the following fact, established in [44], Corollary 1.4:

Lemma 10.3. *If G is a torsion-free non-elementary relatively hyperbolic group, then* $\mathrm{Aut}_{pi}(G) = \mathrm{Inn}(G)$.

Proof of Theorem 2.6. Let G be a relatively hyperbolic group from the class $\mathcal{AVR}$. If G is virtually cyclic, then $\mathrm{Out}(G)$ is finite (cf. [44], Lemma 6.6). Hence we can further suppose that G is non-elementary. By the assumptions, G contains a finite-index subgroup $N \in \mathcal{VR}$, and, in view of Remark 9.4, we can assume that $N \lhd G$.

Note that N is finitely generated (and even finitely presented) as a virtual retract of a finitely presented group. Since N has finite index in G, it is non-elementary and relatively hyperbolic. The latter is an immediate consequence of Bowditch's definition of relatively hyperbolic groups given in [6], Definition 2 (which is equivalent to Osin's definition, as shown in [51], Theorem 6.10); this also follows from the powerful result of C. Druţu [20], Theorem 1.2, which claims that relative hyperbolicity is invariant under quasi-isometries.

By construction, N is a virtual retract of some right-angled Artin group A. And since A is torsion-free, $N \leq A$ is torsion-free as well. Therefore, according to Lemma 10.2, $Z(N) = \{1\}$. The group N is finitely generated and conjugacy separable by Corollary 2.1, and $\mathrm{Aut}_{pi}(N) = \mathrm{Inn}(N)$ by Lemma 10.3. Hence we can apply Grossman's theorem [31], Theorem 1, to conclude that $\mathrm{Out}(N)$ is residually finite. Consequently, $\mathrm{Out}(G)$ is residually finite by Lemma 10.1. □

11. Applications to the conjugacy problem

As it was shown by Mal'cev [39] and Mostowskii [45], a finitely presented conjugacy separable group has solvable conjugacy problem. This result can be generalized as follows:

Lemma 11.1. *Suppose that H is a finitely generated subgroup of a finitely presented group G, such that for every $h \in H$ the H-conjugacy class h^H is separable in G. Then the conjugacy problem for H is solvable.*

Proof. Without loss of generality we can assume that $G = \langle X \mid R \rangle$, for some finite set X and a finite set of words R over the alphabet $X^{\pm 1}$, and H is generated by a finite subset Y of X. Let $F(X)$ denote the free group on the set X and let $F(Y)$ be the subgroup of $F(X)$ generated by Y. Then the identity map on X gives rise to the epimorphism $\theta\colon F(X) \to G$, such that $\ker\theta = N$ is the normal closure of R in $F(X)$.

Since N is the normal closure of only finitely many words in $F(X)$ and Y is finite, a standard argument (cf. [45]) shows that there is a partial algorithm $\mathfrak{A}$, which, given two reduced words $U, W \in F(Y)$, terminates if and only if $U \in W^{F(Y)}N$ (i.e., if $\theta(U) \in \theta(W)^H$ in G). The algorithm $\mathfrak{A}$ lists every word from $W^{F(Y)}N$ in $F(X)$, freely reduces it and compares it with U; it stops once it finds a word in $W^{F(Y)}N$ that is equal to U in $F(X)$.

On the other hand, as $G \cong F(X)/N$ is finitely presented, there is an effective procedure listing all homomorphisms ψ from $F(X)$ to all finite groups Q, satisfying $N \subseteq \ker\psi$ (see [45]). Given such a homomorphism ψ and any two reduced words $U, W \in F(Y)$, one can decide in finitely many steps whether or not $\psi(U) \in \psi(W)^{\psi(F(Y))}$ in Q, because $\psi(F(Y)) = \langle \psi(Y) \rangle$ and Y is finite.

For any $U, W \in F(Y)$ denote $u := \theta(U)$ and $w := \theta(W)$. If $u \notin w^H$, the separability of w^H in G implies the existence of a finite group Q and a homomorphism $\phi\colon G \to Q$ such that $\phi(u) \notin \phi(w^H) = \phi(w)^{\phi(H)}$ in Q. Thus the homomorphism $\psi := \phi \circ \theta\colon F(X) \to Q$ satisfies $N \subseteq \ker\psi$ and $\psi(U) \notin \psi(W)^{\psi(F(Y))}$ in Q. And, of course, the existence of such a homomorphism ψ tells us that $u \notin w^H$ in G.

Hence, there is a partial algorithm $\mathfrak{B}$, which takes on input two words $U, W \in F(Y)$ and terminates if and only if $\theta(U) \in \theta(W)^H$ in G. This algorithm goes through all the homomorphisms ψ from $F(X)$ to finite groups Q with $N \subseteq \ker\psi$, and stops when it finds one such that $\psi(U) \notin \psi(W)^{\psi(F(Y))}$ in Q.

The solution of the conjugacy problem for H amounts to taking on input two reduced words $U, V \in F(Y)$ and running the two partial algorithms $\mathfrak{A}$ and $\mathfrak{B}$ simultaneously. One (and only one) of these two algorithms will eventually terminate, thus answering whether or not $\theta(U)$ is conjugate to $\theta(W)$ in H. □

Corollary 11.2. *Let G be a hereditarily conjugacy separable group. Suppose that H is a subgroup of G such that the double coset $C_G(h)H$ is separable in G for every $h \in H$. Then H is conjugacy separable. If, in addition, G is finitely presented and H is finitely generated, then H has solvable conjugacy problem.*

Proof. The first claim is a direct consequence of Proposition 3.2 and Corollary 3.5. They also imply that h^H is separable in G for every $h \in H$. Therefore, the second claim follows from Lemma 11.1. □

We are now in a position to prove Theorem 2.8, announced in Section 2.

Proof of Theorem 2.8. In [57] Servatius completely described centralizers of elements in right-angled Artin groups. In particular, it follows from his description that $C_G(h)$ is finitely generated for every $h \in G$.

Let $\psi\colon G \to G/N$ be the natural epimorphism and consider any $h \in N$. Then $E := \psi(C_G(h))$ is a finitely generated subgroup of G/N, hence E is closed in $\mathcal{PT}(G/N)$ by the assumptions. Since the map ψ is continuous (when G and G/N are considered as topological groups equipped with their profinite topologies), we can conclude that $C_G(h)N = \psi^{-1}(E)$ is closed in $\mathcal{PT}(G)$ for every $h \in N$.

So the claim of Theorem 2.8 follows from Theorem 1.1 and Corollary 11.2. □

Corollary 11.3. *Let N be a finitely generated normal subgroup of a right-angled Artin group G such that the quotient G/N is virtually polycyclic. Then every finite-index subgroup K of N is conjugacy separable and has solvable conjugacy problem.*

Proof. Since N is finitely generated, K contains a characteristic subgroup L of N with $|N : L| < \infty$. And since $N \lhd G$, we can conclude that $L \lhd G$, and the group G/L is an extension of the finite group N/L by the virtually polycyclic group G/N. An easy induction on the length of the series with cyclic quotients shows that every finite-by-polycyclic group is polycyclic-by-finite. Thus G/L is virtually polycyclic, hence it is subgroup separable – see [56], Exercise 11 in Chapter 1.C.

Arguing as in the proof of Theorem 2.8, we see that the double coset $C_G(h)L$ is separable in G for each $h \in G$. But $K = \bigcup_{i=1}^{k} Lx_i$ for some $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in K$. Therefore $C_G(h)K = \bigcup_{i=1}^{k} C_G(h)Lx_i$ is separable in G (as a finite union of separable subsets) for all $h \in K$.

Note that K is finitely generated as a finite-index subgroup of N, hence K is conjugacy separable and has solvable conjugacy problem by Theorem 1.1 and Corollary 11.2. □

12. Appendix: the Centralizer Condition in profinite terms

Our intention here is to prove that for residually finite groups the condition CC from the Definition 3.1 is equivalent to the condition (3.2) of Chagas and Zalesskii. We refer the reader to the book [55] for the background on profinite completions.

Proposition 12.1. *Let H be a subgroup of a residually finite group G and let $g \in G$. The following are equivalent:*

1) *the pair (H, g) satisfies the condition CC_G from Definition 3.3;*
2) *$\overline{C_H(g)} = C_{\overline{H}}(g)$, where $\overline{H} \leq \widehat{G}$ is the closure of H in the profinite completion $\widehat{G}$ of G.*

Proof. The profinite completion $\widehat{G}$ of G is the inverse limit of finite quotients of G. There is a canonical embedding of $\widehat{G}$ into the Cartesian product $\prod_{N \in \mathcal{N}} G/N$, where $\mathcal{N}$ is the set of all finite-index normal subgroups of G. Thus $\widehat{G}$ can be equipped with the product topology, making it a compact topological group (each finite group G/N is endowed with the discrete topology).

For each $N \in \mathcal{N}$ let ψ_N denote the natural epimorphism from G to G/N. Then the map $\psi\colon G \to \widehat{G}$, defined by $\psi(x) := (\psi_N(x))_{N \in \mathcal{N}}$ for every $x \in G$, is a homomorphism. And since G is residually finite, ψ is injective. Therefore we can assume that $G \leq \widehat{G}$, and so the condition 2) makes sense. Every homomorphism ψ_M,

$M \in \mathcal{N}$, can be uniquely extended to a continuous homomorphism $\hat{\psi}_M : \widehat{G} \to G/M$ ($\hat{\psi}_M$ can also be regarded as a restriction to $\widehat{G}$ of the canonical projection from $\prod_{N \in \mathcal{N}} G/N$ to G/M).

First, suppose that the pair (H, g) satisfies CC_G. Consider any $h \in \overline{H}$ such that $h \notin \overline{C_H(g)}$. Then there exists $K \in \mathcal{N}$ such that $\hat{\psi}_K(h) \notin \psi_K(C_H(g))$. Hence, by CC_G, there is $L \in \mathcal{N}$ satisfying $L \leq K$ and $\psi_L^{-1}(C_{\psi_L(H)}(\psi_L(g))) \subseteq C_H(g)K = \psi_K^{-1}(\psi_K(C_H(g)))$. Therefore $\hat{\psi}_K$ factors through $\hat{\psi}_L$, hence $\hat{\psi}_L(h) \notin \psi_L(C_H(g)K)$. Consequently, $\hat{\psi}_L(h) \notin \overline{C_{\psi_L(H)}(\psi_L(g))}$, and so $h \notin C_{\overline{H}}(g)$. Thus we established the inclusion $C_{\overline{H}}(g) \subseteq \overline{C_H(g)}$. Since the inverse inclusion is evident, we have proved that 1) implies 2).

Now let us assume that the condition 2) holds. Choose any $K \in \mathcal{N}$ and denote $\mathcal{L} := \{L \in \mathcal{N} \mid L \leq K\}$. Arguing by contradiction, suppose that for each $L \in \mathcal{L}$ there is $x_L \in H$ such that $\psi_L(x_L) \in C_{\psi_L(H)}(\psi_L(g)) \setminus (\psi_L(C_H(g)K))$. Note that $\mathcal{L}$ is a directed set (if $L_1, L_2 \in \mathcal{L}$ then $L_1 \preceq L_2$ if and only if $L_2 \subseteq L_1$), hence $(x_L)_{L \in \mathcal{L}}$ is a net in $\widehat{G}$. Since $\widehat{G}$ is compact, this net has a cluster point $h \in \overline{H} \leq \widehat{G}$.

Consider any $N \in \mathcal{N}$ and set $L = N \cap K \in \mathcal{L}$. Then, according to the definition of the topology on $\widehat{G}$, there is $M \in \mathcal{L}$ such that $M \subseteq L$ and $\psi_L(x_M) = \hat{\psi}_L(h)$. By construction, $\psi_L(x_M) \in C_{\psi_L(H)}(\psi_L(g))$, hence $\hat{\psi}_L(h) \in C_{\psi_L(H)}(\psi_L(g))$, implying that $\hat{\psi}_N(h) \in C_{\psi_N(H)}(\psi_N(g))$ because $L \leq N$. Since the latter holds for every $N \in \mathcal{N}$, we can conclude that $h \in C_{\overline{H}}(g)$.

On the other hand, since h is a cluster point of the net $(x_L)_{L \in \mathcal{L}}$ and $K \in \mathcal{L}$, there exists $M \in \mathcal{L}$ such that $\psi_K(x_M) = \hat{\psi}_K(h)$. But since $M \leq K$ we have $x_M \notin C_H(g)KM = C_H(g)K = \psi_K^{-1}(\psi_K(C_H(g)))$. Thus $\hat{\psi}_K(h) = \psi_K(x_M) \notin \psi_K(C_H(g))$, which implies that $h \notin \overline{C_H(g)}$.

Thus we found an element $h \in C_{\overline{H}}(g) \setminus \overline{C_H(g)}$, contradicting to the condition 2). Consequently, 2) implies 1). □

Proposition 12.1 implies that for residually finite groups the Centralizer Condition CC from Definition 3.1 is equivalent to the condition (3.2) introduced by Chagas and Zalesskii in [12]:

Corollary 12.2. *A is residually finite group G satisfies* CC *if and only if* $\overline{C_G(g)} = C_{\widehat{G}}(g)$ *for every* $g \in G$.

It is well known that conjugacy separability of a residually finite group G is equivalent to the condition

$$g^{\widehat{G}} \cap G = g^G \text{ in } \widehat{G} \text{ for all } g \in G. \tag{12.1}$$

In other words, condition (12.1) says that two elements g and g' of G are conjugate in $\widehat{G}$ if and only if they are conjugate in G.

We are now able to reformulate the hereditary conjugacy separability of G in purely profinite terms:

Corollary 12.3. *Suppose that G is a residually finite group. Then G is hereditarily conjugacy separable if and only if for every $g \in G$ both of the following hold in the profinite completion $\widehat{G}$ of G:*

- $g^{\widehat{G}} \cap G = g^G$;
- $\overline{C_G(g)} = C_{\widehat{G}}(g)$.

Proof. The necessity follows from Proposition 3.2 and Corollary 12.2.

The sufficiency is given by the result of Chagas and Zalesskii [12], Proposition 3.1. It can also be deduced by first applying Corollary 12.2 and then Proposition 3.2. □

References

[1] A. Abrams, Configuration spaces and braid groups of graphs. PhD thesis, University of California at Berkeley, 2000.

[2] J. M. Alonso and M. R. Bridson, Semihyperbolic groups. *Proc. London Math. Soc.* (3) **70** (1995), 56–114. Zbl 0823.20035 MR 1300841

[3] P. Bahls, Automorphisms of Coxeter groups. *Trans. Amer. Math. Soc.* **358** (2006), 1781–1796. Zbl 1133.20022 MR 2186996

[4] G. Baumslag, Automorphism groups of residually finite groups. *J. London Math. Soc.* **38** (1963), 117–118. Zbl 0124.26003 MR 0146271

[5] M. Bestvina and N. Brady, Morse theory and finiteness properties of groups. *Invent. Math.* **129** (1997), 445–470. Zbl 0888.20021 MR 1465330

[6] B. Bowditch, *Relatively hyperbolic groups*. First draft: July 1997, revised: September 2011. Available at http://www.warwick.ac.uk/~masgak/preprints.html

[7] M. R. Bridson, On the subgroups of semihyperbolic groups. In *Essays on geometry and related topics*, Vol. 1, Monogr. Enseign. Math. 38, Enseignement Math., Geneva 2001, 85–111. Zbl 1053.20038 MR 1929323

[8] M. R. Bridson and C. F. Miller, III, Structure and finiteness properties of subdirect products of groups. *Proc. Lond. Math. Soc.* (3) **98** (2009), 631–651. Zbl 1167.20016 MR 2500867 (2010g:20049)

[9] I. Bumagin and D. T. Wise, Every group is an outer automorphism group of a finitely generated group. *J. Pure Appl. Algebra* **200** (2005), 137–147. Zbl 1082.20021 MR 2142354

[10] P.-E. Caprace and B. Mühlherr, Reflection triangles in Coxeter groups and biautomaticity. *J. Group Theory* **8** (2005), 467–489. Zbl 1081.20049 MR 2152693

[11] S. C. Chagas and P. A. Zalesskii, Limit groups are conjugacy separable. *Internat. J. Algebra Comput.* **17** (2007), 851–857. Zbl 1172.20026 MR 2340819

[12] S. C. Chagas and P. A. Zalesskii, Finite index subgroups of conjugacy separable groups. *Forum Math.* **21** (2009), 347–353. Zbl 1171.20019 MR 2503309 (2010a:20063)

[13] R. Charney, An introduction to right-angled Artin groups. *Geom. Dedicata* **125** (2007), 141–158. Zbl 1152.20031 MR 2322545

[14] R. Charney and K. Vogtmann, Finiteness properties of automorphism groups of right-angled Artin groups. *Bull. Lond. Math. Soc.* **41** (2009), 94–102. Zbl 05530801 MR 2481994

[15] R. Charney and K. Vogtmann, Subgroups and quotients of automorphism groups of RAAGs. In *Low-dimensional and symplectic topology*, Proc. Sympos. Pure Math. 82, Amer. Math. Soc., Providence, RI, 2011, 9–27. Zbl 05995062 MR 2768650

[16] E. Chesebro, J. DeBlois, and H. Wilton, Some virtually special hyperbolic 3-manifold groups. *Comment. Math. Helv.* **87** (2012), no. 3.

[17] J. Crisp, E. Godelle, and B. Wiest, The conjugacy problem in subgroups of right-angled Artin groups. *J. Topol.* **2** (2009), 442–460. Zbl 1181.20030 MR 2546582

[18] J. Crisp and B. Wiest, Embeddings of graph braid and surface groups in right-angled Artin groups and braid groups. *Algebr. Geom. Topol.* **4** (2004), 439–472. Zbl 1057.20028 MR 2077673

[19] M. B. Day, Peak reduction and finite presentations for automorphism groups of right-angled Artin groups. *Geom. Topol.* **13** (2009), 817–855. Zbl 1226.20024 MR 2470964

[20] C. Druţu, Relatively hyperbolic groups: geometry and quasi-isometric invariance. *Comment. Math. Helv.* **84** (2009), 503–546. Zbl 1175.20032 MR 2507252

[21] G. Duchamp and D. Krob, The lower central series of the free partially commutative group. *Semigroup Forum* **45** (1992), 385–394. Zbl 0814.20025 MR 1179860

[22] G. Duchamp and J.-Y. Thibon, Simple orderings for free partially commutative groups. *Internat. J. Algebra Comput.* **2** (1992), 351–355. Zbl 0772.20017 MR 1189240

[23] A. J. Duncan, I. V. Kazachkov, and V. N. Remeslennikov, Parabolic and quasiparabolic subgroups of free partially commutative groups. *J. Algebra* **318** (2007), 918–932. Zbl 1183.20026 MR 2371978

[24] J. L. Dyer, Separating conjugates in free-by-finite groups. *J. London Math. Soc.* (2) **20** (1979), 215–221. Zbl 0424.20027 MR 551447

[25] E. S. Esyp, I. V. Kazatchkov, and V. N. Remeslennikov, Divisibility theory and complexity of algorithms for free partially commutative groups. In *Groups, languages, algorithms*, Contemp. Math. 378, Amer. Math. Soc., Providence, RI, 2005, 319–348. Zbl 1160.20306 MR 2159318

[26] B. Farb, Relatively hyperbolic groups. *Geom. Funct. Anal.* **8** (1998), 810–840. Zbl 0985.20027 MR 1650094

[27] E. Formanek, Conjugate separability in polycyclic groups. *J. Algebra* **42** (1976), 1–10. Zbl 0345.20031 MR 0419608

[28] A. V. Goryaga, Example of a finite extension of an FAC-group that is not an FAC-group (Russian). *Sibirsk. Mat. Zh.* **27** (1986), 203–205. Zbl 0601.20037 MR 853899

[29] E. R. Green, Graph products of groups. PhD thesis, The University of Leeds, 1990. http://etheses.whiterose.ac.uk/236/

[30] M. Gromov, Hyperbolic groups. In *Essays in group theory*, Math. Sci. Res. Inst. Publ. 8, Springer-Verlag, New York 1987, 75–263. Zbl 0634.20015 MR 0919829

[31] E. K. Grossman, On the residual finiteness of certain mapping class groups. *J. London Math. Soc.* (2) **9** (1974/75), 160–164. Zbl 0292.20032 MR 0405423 (53 #9216)

[32] V. Guirardel and G. Levitt, The outer space of a free product. *Proc. Lond. Math. Soc.* (3) **94** (2007), 695–714. Zbl 1168.20011 MR 2325317

[33] M. Gutierrez, A. Piggott, and K. Ruane, On the automorphisms of a graph product of abelian groups. *Groups Geom. Dyn.* **6** (2012), 125–153.

[34] F. Haglund and D. T. Wise, Special cube complexes. *Geom. Funct. Anal.* **17** (2008), 1551–1620. Zbl 1155.53025 MR 2377497

[35] F. Haglund and D. T. Wise, Coxeter groups are virtually special. *Adv. Math.* **224** (2010), 1890–1903. Zbl 1195.53055 MR 2646113

[36] S. P. Humphries, On representations of Artin groups and the Tits conjecture. *J. Algebra* **169** (1994), 847–862. Zbl 0822.20041 MR 1302120

[37] M. R. Laurence, A generating set for the automorphism group of a graph group. *J. London Math. Soc.* (2) **52** (1995), 318–334. Zbl 0836.20036 MR 1356145

[38] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*. Ergeb. Math. Grenzgeb. 89, Springer-Verlag, Berlin 1977. Zbl 0368.20023 MR 0577064

[39] A. I. Mal'cev, On homomorphisms onto finite groups. *Uchen. Zap. Ivanovskogo Gos. Ped. Inst.* **18** (1958), no. 5, 49–60; English transl. Amer. Math. Soc. Transl. (2) 119 (1983), 67–79. Zbl 0511.20026

[40] A. Martino, A proof that all Seifert 3-manifold groups and all virtual surface groups are conjugacy separable. *J. Algebra* **313** (2007), 773–781. Zbl 1129.57002 MR 2329568

[41] A. Martino and A. Minasyan, Conjugacy in normal subgroups of hyperbolic groups. *Forum Math.*, to appear. Doi 10.1515/form.2011.089

[42] J. Meier and L. VanWyk, The Bieri-Neumann-Strebel invariants for graph groups. *Proc. London Math. Soc.* (3) **71** (1995), 263–280. Zbl 0835.20037 MR 1337468

[43] C. F. Miller, III, *On group-theoretic decision problems and their classification*. Ann. of Math. Stud. 68, Princeton University Press, Princeton 1971. Zbl 0277.20054 MR 0310044

[44] A. Minasyan and D. Osin, Normal automorphisms of relatively hyperbolic groups. *Trans. Amer. Math. Soc.* **362** (2010), 6079–6103. Zbl 1227.20041 MR 2661509

[45] A. W. Mostowski, On the decidability of some problems in special classes of groups. *Fund. Math.* **59** (1966), 123–135. Zbl 0143.03701 MR 0224693

[46] B. H. Neumann, Groups covered by permutable subsets. *J. London Math. Soc.* **29** (1954), 236–248. Zbl 0055.01604 MR 0062122

[47] G. A. Niblo, Separability properties of free groups and surface groups. *J. Pure Appl. Algebra* **78** (1992), 77–84. Zbl 0766.57002 MR 1154898

[48] G. A. Niblo and L. D. Reeves, Coxeter groups act on CAT(0) cube complexes. *J. Group Theory* **6** (2003), 399–413. Zbl 1068.20040 MR 1983376

[49] A. Yu. Ol'shanskii, On residualing homomorphisms and G-subgroups of hyperbolic groups. *Internat. J. Algebra Comput.* **3** (1993), 365–409. Zbl 0830.20053 MR 1250244

[50] D. V. Osin, Elementary subgroups of relatively hyperbolic groups and bounded generation. *Internat. J. Algebra Comput.* **16** (2006), 99–118. Zbl 1100.20033 MR 2217644

[51] D. V. Osin, *Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems*. *Mem. Amer. Math. Soc.* **179** (2006), no. 843. Zbl 1093.20025 MR 2182268

[52] V. N. Remeslennikov, Conjugacy in polycyclic groups. *Algebra i Logika* **8** (1969), 712–725; English transl. *Algebra Logic* **8** (1969), 404–411. Zbl 0291.20037 MR 0280593

[53] V. N. Remeslennikov, Groups that are residually finite with respect to conjugacy. *Sibirsk. Mat. Zh.* **12** (1971), 1085–1099; English transl. *Siberian Math. J.* **12** (1971), 783–792. Zbl 0282.20022 MR 0294469

[54] L. Ribes, D. Segal, and P. A. Zalesskii, Conjugacy separability and free products of groups with cyclic amalgamation. *J. London Math. Soc.* (2) **57** (1998), 609–628. Zbl 0922.20031 MR 1659845

[55] L. Ribes and P. Zalesskii, *Profinite groups*. 2nd ed., Ergeb. Math. Grenzgeb. (3) 40, Springer-Verlag, Berlin 2010. Zbl 1197.20022 MR 2599132

[56] D. Segal, *Polycyclic groups*. Cambridge Tracts in Math. 82, Cambridge University Press, Cambridge 1983. Zbl 0516.20001 MR 713786

[57] H. Servatius, Automorphisms of graph groups. *J. Algebra* **126** (1989), 34–60. Zbl 0682.20022 MR 1023285

[58] P. F. Stebe, A residual property of certain groups. *Proc. Amer. Math. Soc.* **26** (1970), 37–42. Zbl 0205.03005 MR 0260874

[59] D. T. Wise, The residual finiteness of negatively curved polygons of finite groups. *Invent. Math.* **149** (2002), 579–617. Zbl 1040.20024 MR 1923477

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