

On the asymptotics of visible elements and homogeneous equations in surface groups

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Abstract. Let F be a group whose abelianization is \mathbb{Z}^k , $k \geq 2$. An element of F is called visible if its image in the abelianization is visible, that is, the greatest common divisor of its coordinates is 1.

In this paper we compute three types of densities, annular, even and odd spherical, of visible elements in surface groups. We then use our results to show that the probability of a homogeneous equation in a surface group to have solutions is neither 0 nor 1, as the lengths of the right- and left-hand side of the equation go to infinity.

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1. Introduction

Let F be a group whose abelianization is \mathbb{Z}^k , with $k \geq 2$. An element of F is called *visible* with respect to a basis of \mathbb{Z}^k if its image in the abelianization is visible, that is, the greatest common divisor of its coordinates is 1. Being visible is, in fact, independent of the basis of \mathbb{Z}^k (see Definition 2.3), and we therefore omit the references to the basis henceforth.

Let Σ be a compact connected orientable surface of genus r , $r \geq 2$. If Σ has no boundary, then a presentation for the fundamental group of Σ , which we call the *surface group of genus r* , is $\langle a_1, b_1, \dots, a_r, b_r \mid [a_1, b_1] \dots [a_r, b_r] \rangle$. If Σ has boundary, then the fundamental group of Σ is simply a free group of finite rank. For a group G , a positive integer n , and a fixed generating set A , one defines the *sphere of radius n* to be the set of elements in G of length n , with respect to A . Then the *spherical density* of a set S of elements in G measures the proportion of elements of length n in S in the sphere of radius n as n goes to infinity (see Section 2). The *annular density* of a set S records the proportions of S in two successive spheres.

While the spherical density of visible elements does not exist for the groups we consider, one can instead look at the ‘odd spherical density’ and ‘even spherical density’ of visible elements of odd and even length, respectively. In this paper we

compute the annular, odd and even spherical densities of visible elements in a class of groups containing the surface groups of compact connected orientable surfaces, with or without boundary. In [9] the annular density of visible elements was computed for all free groups of finite rank ([9], Theorem A), and odd and even spherical density values were also given for the free group of rank two ([9], Theorem 3.7). Since the limits we obtain are different from 0 and from 1, this shows that visible elements form a set of *intermediate* density in the groups we study. Intermediate density of sets in groups has been displayed for the first time in [9] for visible and test elements. Then Dunfield and Thurston ([6]) proved that for a random 2-generator one-relator group the property that this group splits as a mapping torus of a free group automorphism is ‘intermediate’. However, this tends to be a relatively rare behaviour for many combinatorial and algebraic properties encountered in group theory and most of the properties studied so far in the literature (see [10]) turn out to be *negligible* or *generic*, that is, with density equal to 0 or 1, respectively. For example, in a finitely generated free group the set of all powers is negligible, as is the set of primitive elements, any subgroup of finite index or the union of all proper free factors ([1], [14]). On the other hand, the set of all words whose symmetrizations satisfy the $C'(\frac{1}{6})$ small cancellation condition, or the set of words with nontrivial images in the abelianization are generic ([1], [14]).

We would also like to mention the results of [11], where densities of sets of conjugacy classes in free and surface groups are investigated. More precisely, the density considered in [11] is the asymptotic density of sets of root-free conjugacy classes of hyperbolic elements in surface groups, and for free groups, the density is similar to the annular density but records the proportion in two successive balls instead of two successive spheres.

A consequence of our results is the fact that the solvability of homogeneous equations in the class of groups we study is a non-negligible and non-generic property. Let G be a finitely generated group, A a fixed generating set, and $X = \{X_1, \dots, X_n\}$, $n \geq 1$, a set of variables. An *equation* in variables X_1, \dots, X_n with coefficients g_1, \dots, g_{m+1} in G is a formal expression given by

$$g_1 X_{i_1}^{\varepsilon_1} g_2 X_{i_2}^{\varepsilon_2} \dots X_{i_m}^{\varepsilon_m} g_{m+1} = 1,$$

where $m \geq 1$, $\varepsilon_j \in \{1, -1\}$ for all $1 \leq j \leq m$, and $i_j \in \{1, \dots, n\}$. An equation is *homogeneous* if the variables are on the left-hand side of the equation and the constants are on the right-hand side of the equation,

$$X_{i_1}^{\varepsilon_1} X_{i_2}^{\varepsilon_2} \dots X_{i_m}^{\varepsilon_m} = w, \tag{1.1}$$

where $w \in G$. We say that the equation (1.1) is a *homogeneous equation of type* $(m, |w|_A)$ or an $(m, |w|_A)$ -*homogeneous equation*, where $|w|_A$ denotes the length of w with respect to A .

We will be interested in the asymptotic behavior of $(m, |w|_A)$ -homogeneous equations when G is a surface or a free group and m and $|w|_A$ go to infinity. Our study of

the asymptotics of homogeneous equations was motivated by two related questions: firstly, how often does a homogeneous equation in a free or surface group have solutions, and secondly, how likely is it for two random words u and v in the group to have that v is an endomorphic image of u ? The second question asks what the ‘generic’ answer to the endomorphism problem in free groups is (see [5] for more on the endomorphism problem) and was also inspired by the work of Kapovich, Schupp and Shpilrain ([10]). They show that the probability of two elements u and v in F_k to be in the same automorphic orbit is 0 as the lengths of u and v go to infinity. The following paragraph clarifies the relation between the two questions.

Suppose that $z(X_1, \dots, X_n)$ is the word in X_1, \dots, X_n representing the left-hand side of (1.1), i.e., $z(X_1, \dots, X_n) = X_{i_1}^{\epsilon_1} X_{i_2}^{\epsilon_2} \dots X_{i_m}^{\epsilon_m}$. Let F_n be the free group of rank n on generators x_1, \dots, x_n . Notice that the equation (1.1) has solutions if and only if there exists an homomorphism $\phi: F_n \rightarrow G$ such that $\phi(z(x_1, \dots, x_n)) = w$, where z is written in the generators x_1, \dots, x_n . The following ratios quantify the pairs of elements of the form (z, w) .

Definitions 1.1. Let F, G be countable groups and $l_F: F \rightarrow \mathbb{N}$ and $l_G: G \rightarrow \mathbb{N}$ be length functions as defined in Definition 2.1.

(1) The (s, t) -mapping ratio $e_\rho(F, G, s, t)$ is the ratio of the pairs of elements $(f, g) \in F \times G$ such that $l_F(f) \leq s, l_G(g) \leq t$ and with the property that g is a homomorphic image of f among all pairs $(f, g) \in F \times G$ with $l_F(f) \leq s, l_G(g) \leq t$, that is,

$$e_\rho(F, G, s, t) = \frac{\#\{(f, g) \in F \times G : l_F(f) \leq s, l_G(g) \leq t, \phi(f) = g \text{ for some } \phi \in \text{Hom}(F, G)\}}{\#\{(f, g) \in F \times G : l_F(f) \leq s, l_G(g) \leq t\}}.$$

(2) The spherical (s, t) -mapping ratio $e_\gamma(F, G, s, t)$ is the ratio of the pairs of elements $(f, g) \in F \times G$ such that $l_F(f) = s, l_G(g) = t$ and with the property that g is a homomorphic image of f among all pairs $(f, g) \in F \times G$ with $l_F(f) = s, l_G(g) = t$, that is,

$$e_\gamma(F, G, s, t) = \frac{\#\{(f, g) \in F \times G : l_F(f) = s, l_G(g) = t, \phi(f) = g \text{ for some } \phi \in \text{Hom}(F, G)\}}{\#\{(f, g) \in F \times G : l_F(f) = s, l_G(g) = t\}}.$$

In Section 3 we will study the asymptotic behavior of the (s, t) -mapping ratio $e_\rho(F, G, s, t)$ for F and G free abelian groups with l_G and l_F being the restriction of the $\|\cdot\|_p$ norm, $1 \leq p \leq \infty$. We will show that the limit of $e_\rho(F, G, s, t)$, as s and t go to infinity, is neither 0 nor 1. The computation of the asymptotic behavior of this ratio is based on the densities of visible elements in a free abelian group. In Section 4 we study the annular, even and odd spherical densities of visible elements in free and surface groups:

Corollary 4.11. *Let $k \geq 2$ and let F be a free group of rank k or a surface group of genus k . Let A be the symmetric generating set of F , r the rank of the abelianization, and V_1 the set of visible elements in F . Then*

$$(i) \lim_{m \rightarrow \infty} \frac{\gamma_A(2m, V_1)}{\gamma_A(2m, F)} = \frac{2^r - 2}{(2^r - 1)\zeta(r)},$$

$$(ii) \lim_{m \rightarrow \infty} \frac{\gamma_A(2m-1, V_1)}{\gamma_A(2m-1, F)} = \frac{2^r}{(2^r - 1)\zeta(r)},$$

where $\gamma_A(n, S)$ is the number of elements of length n in the set $S \subseteq F$, and ζ is Euler's ζ function.

We obtain our main result in Theorem 4.10, which relates the densities of visible points in surface and free groups with the densities in the abelianization. In Section 5 we study the asymptotic behavior of the spherical (s, t) -mapping ratio $e_\gamma(F, G, s, t)$ when F and G are free or surface groups. We exploit the connection of $e_\gamma(F, G, s, t)$ with $e_\rho(F_{\text{ab}}, G_{\text{ab}}, s, t)$ to obtain upper and lower bounds for e_γ :

Theorem 5.1. *Let G_k and G_n be free or surface groups and let A, B be their respective generating sets. Let $r(k)$ and $r(n)$ denote the ranks of the abelianization of G_k and G_n , respectively. Let $\varepsilon, \delta \in \{0, 1\}$ and ζ be Euler's ζ function. Then the following inequalities hold:*

$$\frac{2^{r(n)} - 2(1 - \varepsilon)}{(2^{r(n)} - 1)\zeta(r(n))} \leq \liminf_{s \rightarrow \infty, t \rightarrow \infty} e_\gamma(G_n, G_k, 2s + \varepsilon, 2t + \delta),$$

$$\limsup_{s \rightarrow \infty, t \rightarrow \infty} e_\gamma(G_n, G_k, 2s + \varepsilon, 2t + \delta)$$

$$\leq 1 - \frac{2^{r(k)} - 2(1 - \delta)}{(2^{r(k)} - 1)\zeta(r(k))} \left(1 - \frac{2^{r(n)} - 2(1 - \varepsilon)}{(2^{r(n)} - 1)\zeta(r(n))} \right).$$

As a corollary, we obtain that the probability of an (s, t) -homogeneous equation in a surface group to be satisfiable is neither 0 nor 1 as s, t go to infinity (Corollary 5.2).

The results developed in this paper lend themselves to generalizations. For example, one can extend Theorem 4.10 to those strongly Markov groups whose abelianization is free abelian by using Sharp's Corollary 2.1 [13]. One may try to generalize our results to word-hyperbolic groups, which are strongly Markov. However, they could have finite abelianization, and hence no homomorphism onto \mathbb{Z} . In [3], Calegari and Fujiwara deal with this situation by considering quasimorphisms to \mathbb{Z} which are, informally, homomorphisms up to bounded error. The central limit theorem of Calegari and Fujiwara may be used to compute densities in hyperbolic groups of sets related to these quasimorphisms.

The asymptotic density of satisfiable homogeneous and one-variable equations in free groups has recently been considered by B. Gilman, A. Miasnikov and V. Roman'kov in [7], who also study the asymptotic behavior of equations in free abelian and free nilpotent groups in [8].

2. Notation

Definitions 2.1. Let F be a finitely generated group, and let A be a finite generating set of F . If $w \in F$, then $|w|_A$ denotes the length of the shortest word in $A^{\pm 1}$ representing w .

For $1 \leq p \leq \infty$, let $l_p : \mathbb{Z}^r \rightarrow \mathbb{R}$ denote the restriction to \mathbb{Z}^r of the $\|\cdot\|_p$ -norm from \mathbb{R}^r .

A length function for a set S is a function $l : S \rightarrow \mathbb{N}$ such that, for every $n \in \mathbb{N}$, the set $l^{-1}(\{0, 1, 2, \dots, n\})$ is finite. The functions $|\cdot|_A$ and l_p are examples of length functions in F and \mathbb{Z}^r .

Definitions 2.2. Let F be a group (or more generally, a set) and $l_F : F \rightarrow \mathbb{N}$ a length function.

(1) Let $S \subseteq F$ and $n \geq 0$. Then

$$\rho_{l_F}(n, S) = \#\{x \in S : l_F(x) \leq n\}$$

and

$$\gamma_{l_F}(n, S) = \#\{x \in S : l_F(x) = n\}$$

denote the cardinality of the intersection of S with the ball and sphere of radius n in F , respectively.

(2) Let $S \subseteq F$. The *asymptotic density* of S in F is

$$\bar{\rho}_{l_F}(S) = \limsup_{n \rightarrow \infty} \frac{\rho_{l_F}(n, S)}{\rho_{l_F}(n, F)}.$$

If the limit exists, then we denote it by $\rho_{l_F}(S)$ and we call it the *strict asymptotic density*.

(3) Let $S \subseteq F$. The *spherical density* of S in F is

$$\bar{\gamma}_{l_F}(S) = \limsup_{n \rightarrow \infty} \frac{\gamma_{l_F}(n, S)}{\gamma_{l_F}(n, F)}.$$

If the limit exists, then we denote it by $\gamma_{l_F}(S)$ and we call it the *strict spherical density*.

(4) Let $S \subseteq F$. The *annular density* of S in F is

$$\bar{\sigma}_{l_F}(S) = \limsup_{n \rightarrow \infty} \frac{1}{2} \left(\frac{\#\{x \in S : l_F(x) = n - 1\}}{\#\{x \in F : l_F(x) = n - 1\}} + \frac{\#\{x \in S : l_F(x) = n\}}{\#\{x \in F : l_F(x) = n\}} \right)$$

If the limit exists, then we denote it by $\sigma_{l_F}(S)$ and we call it the *strict annular density*.

When F is a group, finitely generated by A , and $l_F = |\cdot|_A$, the word length, we will just write ρ_A, γ_A and σ_A . Similarly if $F = \mathbb{Z}^r$ and $l_F = l_p$, the restriction of the p -norm, we will just write ρ_p, γ_p and σ_p .

Definitions 2.3. For a nonzero element $z \in \mathbb{Z}^r$ we denote by $\gcd(z)$ the greatest common divisor of its coordinates. If $z = (0, \dots, 0) \in \mathbb{Z}^r$ we set $\gcd(z) = \infty$. Note that \gcd is invariant under the action of $\text{Aut}(\mathbb{Z}^r) = \text{SL}(r, \mathbb{Z})$. Hence, for all $z \in \mathbb{Z}^r$, $\gcd(z)$ does not depend on the basis of \mathbb{Z}^r .

An element of $z \in \mathbb{Z}^r$ is called *visible* if $\gcd(z) = 1$. If $\gcd(z) = t$, then we call the element *t-visible*.

We denote by F_{ab} the abelianization of the group F , that is, $F_{\text{ab}} = F/[F, F]$. Suppose that F_{ab} is a free abelian group of finite rank and let $\text{ab}: F \rightarrow F_{\text{ab}}$ be the abelianization map. We say that an element $f \in F$ is *visible* (resp. *t-visible*) if $\text{ab}(f)$ is visible (resp. *t-visible*) in F_{ab} .

3. Densities of visible elements in \mathbb{Z}^r

Let $r \geq 2$ be an integer and let U_t denote the set of all *t-visible* elements in \mathbb{Z}^r . For a complex number k , recall that the Riemann zeta function is given by

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}, \quad \text{Re}(k) > 1.$$

A classical result in number theory provides the value for the strict asymptotic density of *t-visible* elements in \mathbb{Z}^r .

Proposition 3.1 ([4]). *For any integer $t \geq 1$,*

$$\rho_{\infty}(U_t) = \frac{1}{t^r \zeta(r)}.$$

By [9], Theorem A (1), one can substitute ρ_{∞} by ρ_p for the sets U_t :

Proposition 3.2 ([9], Theorem A (1)). *For any integer $t \geq 1$ and any p , $1 \leq p \leq \infty$,*

$$\rho_p(U_t) = \rho_{\infty}(U_t).$$

The following lemma shows that homomorphisms between groups with free abelian abelianization (of finite rank) send *t-visible* elements to (tm) -visible elements, where t, m are positive integers. The second part of the lemma shows that a visible element in a group can be mapped to any element via a homomorphism.

Lemma 3.3. *Let F, G be groups whose abelianization is free abelian of finite rank. Let $f \in F$.*

- (i) *Let $\phi: F \rightarrow G$ be a group homomorphism. Then $\gcd(\text{ab}(\phi(f)))$ is a multiple of $\gcd(\text{ab}(f))$. In particular, if $\gcd(\text{ab}(f)) = \infty$, then $\gcd(\text{ab}(\phi(f))) = \infty$.*

(ii) If, moreover $\gcd(\text{ab}(f)) = 1$, then for any element g in G there exists an homomorphism $\phi: F \rightarrow G$ such that $\phi(f) = g$.

Proof. Let n be the rank of F_{ab} and let $\{e_1, \dots, e_n\}$ be a basis of F_{ab} . For $f \in F_{\text{ab}}$, we denote by $(f)_i$ the i th coordinate of f with respect to the basis. That is, $f = (f)_1 e_1 + \dots + (f)_n e_n$.

(i) Let $g = \phi(f)$. Then $(\text{ab}(g))_j = \sum_{i=1}^n (\text{ab}(f))_i (\phi(e_i))_j$.

Thus each $(\text{ab}(g))_j$ is a multiple of $\gcd(\text{ab}(f))$ since each $(\text{ab}(f))_i$ is a multiple of $\gcd(\text{ab}(f))$.

(ii) Since $\gcd(\text{ab}(f)) = 1$, $\gcd((\text{ab}(f))_1, \dots, (\text{ab}(f))_n) = 1$ and therefore there exist integers p_1, \dots, p_n such that $\sum_{i=1}^n (\text{ab}(f))_i p_i = 1$. Consider the homomorphism $\psi_1: F_{\text{ab}} \rightarrow \langle x \mid \rangle$ which sends e_i to x^{p_i} for all $1 \leq i \leq n$. It follows that $\psi_1(\text{ab}(f)) = x$. Let $\psi_2: \langle x \mid \rangle \rightarrow G$ be any homomorphism sending x to g . This shows that the composition of ab , ψ_1 and ψ_2 produces a homomorphism $\phi: F \rightarrow G$ such that $\phi(f) = g$. □

Corollary 3.4. Let \mathbb{Z}^n and \mathbb{Z}^k be the free abelian groups of ranks n and k , respectively. Then the following inequalities hold with respect to l_p for $1 \leq p \leq \infty$:

$$\frac{1}{\zeta(n)} \leq \liminf_{s \rightarrow \infty, t \rightarrow \infty} e_\rho(\mathbb{Z}^n, \mathbb{Z}^k, s, t), \tag{3.1}$$

$$\limsup_{s \rightarrow \infty, t \rightarrow \infty} e_\rho(\mathbb{Z}^n, \mathbb{Z}^k, s, t) \leq 1 - \frac{1}{\zeta(k)} \left(1 - \frac{1}{\zeta(n)} \right). \tag{3.2}$$

Proof. We fix some p , $1 \leq p \leq \infty$. Let $e_{\text{ab}}(s, t) := e_\rho(\mathbb{Z}^n, \mathbb{Z}^k, s, t)$ with respect to the length l_p and let $|u| = l_p(u)$.

By Lemma 3.3 (ii)

$$\begin{aligned} e_{\text{ab}}(s, t) &\geq \frac{\#\{(u, v) \in \mathbb{Z}^n \times \mathbb{Z}^k : |u| \leq s, |v| \leq t, \gcd(u) = 1\}}{\rho_p(s, \mathbb{Z}^n) \rho_p(t, \mathbb{Z}^k)} \\ &= \frac{\#\{u \in \mathbb{Z}^n : |u| \leq s, \gcd(u) = 1\}}{\rho_p(s, \mathbb{Z}^n)}. \end{aligned}$$

Taking limits, we obtain (3.1) by Propositions 3.1 and 3.2.

By Lemma 3.3 (i),

$$\begin{aligned} e_{\text{ab}}(s, t) &\leq 1 - \frac{\#\{(u, v) \in \mathbb{Z}^n \times \mathbb{Z}^k : |u| \leq s, |v| \leq t, \gcd(u) \neq 1, \gcd(v) = 1\}}{\rho_p(s, \mathbb{Z}^n) \rho_p(t, \mathbb{Z}^k)} \\ &= 1 - \left(1 - \frac{\#\{u \in \mathbb{Z}^n : |u| \leq s, \gcd(u) = 1\}}{\rho_p(s, \mathbb{Z}^n)} \right) \\ &\quad \cdot \frac{\#\{v \in \mathbb{Z}^k : |v| \leq t, \gcd(v) = 1\}}{\rho_p(t, \mathbb{Z}^k)}. \end{aligned}$$

Taking limits, we obtain (3.2) by Propositions 3.1 and 3.2. □

One of the key ingredients needed to extend the previous result to the analogue for surface groups is determining the asymptotic density of elements of even length in \mathbb{Z}^k . This was done in [9], Proposition 3.6, for $k = 2$, and we now compute the value for a general k .

Proposition 3.5. *Let $k \geq 2$, and let $U_1^{\text{ev}} = \{z \in U_1 : l_1(z) \text{ is even}\}$ denote the set of visible elements of even length in \mathbb{Z}^k . Then*

$$\rho_\infty(U_1^{\text{ev}}) = \frac{2^{k-1} - 1}{2^k - 1} \rho_\infty(U_1) = \frac{2^{k-1} - 1}{(2^k - 1)\zeta(k)}.$$

Proof. Let n be a positive integer and let $[0, n] = \{0, 1, \dots, n\}$. For $X_1, \dots, X_k \in \{\mathcal{A}, \mathcal{O}, \mathcal{E}\}$ we denote by $X_1 X_2 \dots X_k(n)$ the number of all $z = (z_1, \dots, z_k) \in U_1$ such that $z_i \in [0, n]$ and the parity of z_i is X_i . Here \mathcal{A} stands for “any”, \mathcal{E} stands for “even” and \mathcal{O} stands for “odd”.

We will use the convention $\underbrace{X \dots X}_{k \text{ times}} = X^k$ for any $X \in \{\mathcal{A}, \mathcal{O}, \mathcal{E}\}$ and $k \geq 1$. Note that $X_1 X_2 \dots X_k(n) = X_{s(1)} X_{s(2)} \dots X_{s(k)}(n)$, for any permutation s of $\{1, \dots, k\}$, and that $\mathcal{E}^k(n) = 0$ for any $k, n \geq 1$.

The total number of elements in U_1 in $[0, n]^k$ is

$$\mathcal{A}^k(n) = \sum_{i=1}^k \binom{k}{i} \mathcal{E}^{k-i} \mathcal{O}^i(n). \tag{3.3}$$

Let $U_1^{\text{ev}}(n)$ be the set $U_1^{\text{ev}} \cap [0, n]^k$. Then

$$|U_1^{\text{ev}}(n)| = \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2i} \mathcal{E}^{k-2i} \mathcal{O}^{2i}(n). \tag{3.4}$$

We claim that

$$\mathcal{E}^{k-i} \mathcal{O}^i(n) = \mathcal{O}^k(n) + o(n^k) \quad \text{for all } 1 \leq i \leq k. \tag{3.5}$$

Assume first that (3.5) holds. From (3.4) and (3.5) we get

$$|U_1^{\text{ev}}(n)| = \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2i} \mathcal{O}^k(n) + o(n^k),$$

and since $\sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2i} = 2^{k-1} - 1$, we get

$$|U_1^{\text{ev}}(n)| = (2^{k-1} - 1)\mathcal{O}^k(n) + o(n^k).$$

Since $\sum_{i=1}^k \binom{k}{i} = 2^k - 1$, from (3.3) and (3.5) we obtain

$$\mathcal{O}^k(n)(2^k - 1) = \mathcal{A}^k(n) + o(n^k),$$

and hence

$$|U_1^{\text{ev}}(n)| = \frac{2^{k-1} - 1}{2^k - 1} \mathcal{A}^k(n) + o(n^k).$$

Since $\rho_\infty(U_1) = \lim_{n \rightarrow \infty} \frac{\mathcal{A}^k(n)}{n^k} = \frac{1}{\zeta(k)}$, we get

$$\begin{aligned} \rho_\infty(U_1^{\text{ev}}) &= \limsup_{n \rightarrow \infty} \frac{|U_1^{\text{ev}}(n)|}{n^k} \\ &= \lim_{n \rightarrow \infty} \frac{\frac{2^{k-1} - 1}{2^k - 1} \mathcal{A}^k(n) + o(n^k)}{n^k} \\ &= \frac{2^{k-1} - 1}{2^k - 1} \rho_\infty(U_1) = \frac{2^{k-1} - 1}{(2^k - 1)\zeta(k)}. \end{aligned}$$

This completes the proof of the proposition. We now show (3.5). Notice first that

$$\mathcal{O}^i \mathcal{E}^{k-i-1} \mathcal{A}(n) = \mathcal{O}^i \mathcal{E}^{k-i}(n) + \mathcal{O}^{i+1} \mathcal{E}^{k-i-1}(n).$$

Hence it is enough to show

$$\mathcal{O}^i \mathcal{E}^{k-i-1} \mathcal{A}(n) = 2\mathcal{O}^i \mathcal{E}^{k-i}(n) + o(n^k) \quad \text{for all } 1 \leq i \leq k.$$

Let $\mu: \mathbb{N} \rightarrow \{-1, 0, 1\}$ denote the Möbius function and recall that $\sum_{d|n} \mu(d)$ is equal to 1, if $n = 1$ and 0 otherwise. Hence

$$\mathcal{O}^i \mathcal{E}^{k-i-1} \mathcal{A}(n) = \sum_{\substack{0 \leq x_j \leq n, 2 \nmid x_j \\ j=1, \dots, i}} \sum_{\substack{0 \leq x_j \leq n, 2|x_j \\ j=i+1, \dots, k-1}} \sum_{0 \leq x_k \leq n} \sum_{d|\gcd(x_1, \dots, x_k)} \mu(d)$$

and

$$\mathcal{O}^i \mathcal{E}^{k-i}(n) = \sum_{\substack{0 \leq x_j \leq n, 2 \nmid x_j \\ j=1, \dots, i}} \sum_{\substack{0 \leq x_j \leq n, 2|x_j \\ j=i+1, \dots, k}} \sum_{d|\gcd(x_1, \dots, x_k)} \mu(d).$$

Now we switch the order in the summation. We rearrange the terms depending on $d | \gcd(x_1, \dots, x_k)$, writing $x_i = y_i d$. Since there is an odd coordinate, $2 \nmid d$. We obtain

$$\mathcal{O}^i \mathcal{E}^{k-i-1} \mathcal{A}(n) = \sum_{2 \nmid d} \mu(d) \sum_{\substack{0 \leq y_j \leq n/d, 2 \nmid y_j \\ j=1, \dots, i}} \sum_{\substack{0 \leq y_j \leq n/d, 2|y_j \\ j=i+1, \dots, k-1}} \sum_{y_k \leq n/d} 1$$

and

$$\mathcal{O}^i \mathcal{E}^{k-i}(n) = \sum_{2 \nmid d} \mu(d) \sum_{\substack{0 \leq y_j \leq n/d, 2 \nmid y_j \\ j=1, \dots, i}} \sum_{\substack{0 \leq y_j \leq n/d, 2|y_j \\ j=i+1, \dots, k}} 1.$$

Hence $\mathcal{O}^i \mathcal{E}^{k-i-1} \mathcal{A}(n) - 2\mathcal{O}^i \mathcal{E}^{k-i}(n)$ is equal to

$$\sum_{2 \nmid d} \mu(d) \sum_{\substack{0 \leq y_j \leq n/d, 2 \nmid y_j \\ j=1, \dots, i}} \sum_{\substack{0 \leq y_j \leq n/d, 2 \nmid y_j \\ j=i+1, \dots, k-1}} \left(\left[\frac{n}{d} \right] - 2 \left[\frac{n}{2d} \right] \right). \tag{3.6}$$

The term in parenthesis is either 0 or 1, and it is always 0 for $d > n$. Thus the asymptotic behavior of (3.6) is of type

$$\begin{aligned} O\left(\sum_{d \leq n} \sum_{\substack{0 \leq y_j \leq n/d \\ j=1, \dots, k-1}} 1\right) &\subseteq O\left(\sum_{d=1}^n (n/d)^{k-1}\right) \\ &= O\left(n^{k-1} \left(\frac{1}{k-2} - \frac{1}{(k-2)n^{k-2}}\right)\right) \\ &= O(n^{k-1}) \subset o(n^k). \end{aligned} \quad \square$$

4. Densities of visible elements in surface groups

The main result of this section is an extension of [9], Theorem A, that allows us to compute densities of visible elements in free and surface groups. We need to fix some notation.

Notation 4.1. For $k \geq 2$, we denote by F_k the free group of rank k and by S_k the surface group of genus k .

We will work with the standard presentation for F_k ,

$$\langle a_1, \dots, a_k \mid \rangle,$$

and let $A = \{a_1, \dots, a_k\}^{\pm 1}$.

A presentation for S_k has the form

$$\langle a_1, b_1, \dots, a_k, b_k \mid [a_1, b_1] \dots [a_k, b_k] \rangle.$$

In this case we let $A = \{a_1, b_1, \dots, a_k, b_k\}^{\pm 1}$.

Let r denote the rank of the abelianization, that is, $r = k$ for F_k , and $r = 2k$ for S_k .

Our main result is based on the following local limit theorem of Sharp in [13].

Theorem 4.2 (see Theorems 1, 3, 4 in [13]). *Let F be F_k or S_k , and A and r be the corresponding generating set and rank of the abelianization of F , as in Notation 4.1.*

Let $\text{ab}: F \rightarrow \mathbb{Z}^r$ be the abelianization map. Then there exists a symmetric positive definite real matrix D such that, uniformly in $\alpha \in \mathbb{Z}^r$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| (\det D)^{1/2} n^{r/2} \left(\frac{\gamma_A(n, \text{ab}^{-1}(\alpha))}{\gamma_A(n, F)} + \frac{\gamma_A(n+1, \text{ab}^{-1}(\alpha))}{\gamma_A(n+1, F)} \right) \right. \\ \left. - \frac{2}{(2\pi)^{r/2}} e^{-\langle \alpha, D^{-1}\alpha \rangle / 2n} \right| = 0. \end{aligned} \tag{4.1}$$

Proof. For $F = S_k$ this is exactly [13], Theorem 4, with $\mathfrak{g} = r/2$. For $F = F_k$ and D the diagonal matrix with all entries equal to σ^2 , one obtains exactly [13], Theorem 1. \square

Since the proof of the main theorem of this section does not use the fact that F is a free or surface group, but only the conclusions of Theorem 4.2, we will fix the following hypothesis.

Hypothesis 4.3. Let F be a group generated by a finite set A such that $F_{\text{ab}} \cong \mathbb{Z}^r$ and D be a symmetric positive definite real matrix such that the limit (4.1) goes to zero uniformly in $\alpha \in \mathbb{Z}^r$.

By Theorem 4.2, the free group F_k and the surface group S_k of Notation 4.1 satisfy Hypothesis 4.3.

Definition 4.4. Let G_r be the set of all $M \in \text{SL}(r, \mathbb{Z})$ such that $M = I_r$ in $\text{SL}(r, \mathbb{Z}/2\mathbb{Z})$. Then G_r is a finite-index subgroup of $\text{SL}(r, \mathbb{Z})$.

Definition 4.5. We say that a bounded open subset of \mathbb{R}^r is *nice* if its boundary is piecewise smooth.

Proposition 4.6 ([9], Proposition 3.3). *Let $S \subseteq \mathbb{Z}^r$ be a G_r -invariant subset such that $\delta = \rho_\infty(S)$ exists. Let $\Omega \subseteq \mathbb{R}^r$ be a nice bounded open set and for $t \in \mathbb{R}, t > 0$, let*

$$\mu_{t,S}(\Omega) := \frac{\#(S \cap t\Omega)}{t^r}.$$

Then we have

$$\lim_{t \rightarrow \infty} \mu_{t,S}(\Omega) = \delta\lambda(\Omega), \tag{4.2}$$

where λ is the Lebesgue measure.

Although [9] indicates that the proof is similar to that of [9], Proposition 2.3, we include here a proof for Proposition 4.6 for the sake of completeness.

Proof. Each $\mu_{t,S}$ can be regarded as a measure on \mathbb{R}^r . We prove the result by showing that $\mu_{t,S}$ weakly converges to $\delta\lambda$ as $t \rightarrow \infty$.

By Helly’s theorem (see, for instance, [2], Theorem 25.9), there exists a sequence $\{t_i\}$ with $\lim_{i \rightarrow \infty} t_i = \infty$ such that the sequence $\mu_{t_1,S}, \mu_{t_2,S}, \dots$ is weakly convergent to some limiting measure. We now identify this measure by showing that for every convergent subsequence of $\mu_{t_i,S}$ the limiting measure is equal to $\delta\lambda$.

Indeed, we assume that $\eta = \{t_i\}$ is a sequence with $\lim_{i \rightarrow \infty} t_i = \infty$ such that the sequence $\mu_{t_i,S}$ converges to the limiting measure $\mu_\eta = \lim_{i \rightarrow \infty} \mu_{t_i,S}$. Every $\mu_{t_i,S}$ is invariant with respect to the G_r -action on \mathbb{R}^r . Therefore the limiting measure μ_η is also G_r -invariant. Moreover, the measures $\mu_{t,S}$ are dominated by the measures λ_t defined as $\lambda_t(\Omega) = \frac{\#(\mathbb{Z}^r \cap t\Omega)}{t^r}$.

It is well known that if $\Omega \subseteq \mathbb{R}^r$ is a nice bounded open set, then the measures λ_t converge to the Lebesgue measure λ . It follows that μ_η is absolutely continuous with respect to λ . It is also known that the natural action of G_r on \mathbb{R}^r is ergodic with respect to λ (see [15] for the proof of ergodicity). Therefore μ_η is a constant multiple $c\lambda$ of λ . The constant c can be computed for a set such as the open unit ball B in the $\|\cdot\|_\infty$ norm on \mathbb{R}^r defining the length function l_∞ on \mathbb{Z}^r . By assumption we know that

$$\rho_\infty(S) = \lim_{t \rightarrow \infty} \frac{\#\{z \in \mathbb{Z}^r : z \in S \cap tB\}}{\#\{z \in \mathbb{Z}^r : z \in tB\}} = \delta.$$

We also have

$$\lim_{t \rightarrow \infty} \frac{\#\{z \in \mathbb{Z}^r : z \in tB\}}{t^r} = \lambda(B)$$

and hence

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{\#\{z \in \mathbb{Z}^r : z \in tB\}}{t^r} \frac{\#\{z \in \mathbb{Z}^r : z \in S \cap tB\}}{\#\{z \in \mathbb{Z}^r : z \in tB\}} &= \lim_{t \rightarrow \infty} \frac{\#\{z \in \mathbb{Z}^r : z \in S \cap tB\}}{t^r} \\ &= \delta\lambda(B). \end{aligned}$$

Therefore $c = \delta$ and $\mu_\eta = \delta\lambda$. The above argument shows in fact that every convergent subsequence of $\mu_{t,S}$ converges to $\delta\lambda$ and $\lim_{t \rightarrow \infty} \mu_{t,S} = \delta\lambda$. \square

Definition 4.7. Let F be a group generated by the finite set A such that $F_{\text{ab}} \cong \mathbb{Z}^r$.

For an integer $n \geq 1$ and a point $x \in \mathbb{R}^r$, let p_n be given by

$$\begin{aligned} p_n(x) &= \frac{1}{2} \left(\frac{\gamma_A(n-1, \{g \in F : \text{ab}(g) = x\sqrt{n}\})}{\gamma_A(n-1, F)} + \frac{\gamma_A(n, \{g \in F : \text{ab}(g) = x\sqrt{n}\})}{\gamma_A(n, F)} \right). \end{aligned}$$

This is a distribution supported on finitely many points of $\frac{1}{\sqrt{n}} \mathbb{Z}^r$.

We need the following results from [12], [13] about the sequence of distributions p_n .

In our context, we need to restate our Hypothesis 4.3

Proposition 4.8 ([12], [13], [9]). *Let F, A, r satisfy Hypothesis 4.3. Then there exists a normal distribution \mathfrak{n} with density \mathfrak{n} such that:*

(a) *The sequence of distributions p_n converges weakly to \mathfrak{n} and we have*

$$\sup_{x \in \mathbb{Z}^r / \sqrt{n}} |n^{r/2} p_n(x) - \mathfrak{n}(x)| \longrightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{4.3}$$

(b) *For $c > 0$, let $\overline{\Omega}_c := \{x \in \mathbb{R}^r : \|x\| \geq c\}$. Then*

$$\lim_{c \rightarrow \infty} \left(\lim_{n \rightarrow \infty} \sum_{x \in \overline{\Omega}_c} p_n(x) \right) = 0.$$

Proof. Let D be the matrix of Hypothesis 4.3, and let $\mathfrak{n}(x) = \frac{e^{-\langle x, D^{-1}x \rangle/2}}{(2\pi)^r (\det D)^{1/2}}$, the density of a normal distribution \mathfrak{N} . Firstly, we prove the limit in (4.3).

After performing some easy computations,

$$\begin{aligned} & |n^{r/2} p_n(x) - \mathfrak{n}(x)| \\ &= \frac{1}{2(\det D)^{1/2}} \left| (\det D)^{1/2} n^{r/2} \right. \\ &\quad \cdot \left(\frac{\gamma_A(n-1, \text{ab}^{-1}(x\sqrt{n}))}{\gamma_A(n-1, F)} + \frac{\gamma_A(n, \text{ab}^{-1}(x\sqrt{n}))}{\gamma_A(n, F)} \right) - \frac{2}{(2\pi)^r} e^{\langle x, D^{-1}x \rangle/2} \left. \right| \\ &= \frac{1}{2(\det D)^{1/2}} \left| (\det D)^{1/2} n^{r/2} \right. \\ &\quad \cdot \left(\frac{\gamma_A(n-1, \text{ab}^{-1}(\alpha))}{\gamma_A(n-1, F)} + \frac{\gamma_A(n, \text{ab}^{-1}(\alpha))}{\gamma_A(n, F)} \right) - \frac{2}{(2\pi)^r} e^{-\langle \frac{\alpha}{\sqrt{n}}, \frac{D^{-1}\alpha}{\sqrt{n}} \rangle/2} \left. \right| \\ &= \frac{1}{2(\det D)^{1/2}} \left| (\det D)^{1/2} n^{r/2} \right. \\ &\quad \cdot \left(\frac{\gamma_A(n-1, \text{ab}^{-1}(\alpha))}{\gamma_A(n-1, F)} + \frac{\gamma_A(n, \text{ab}^{-1}(\alpha))}{\gamma_A(n, F)} \right) - \frac{2}{(2\pi)^r} e^{-\langle \alpha, D^{-1}\alpha \rangle/2n} \left. \right|, \end{aligned}$$

using the limit (4.1) of the Hypothesis 4.3 and the fact that this limit is uniform in $\alpha = x\sqrt{n}$, we obtain the desired result.

In order to show that the sequence of probability distributions $\{p_n\}$ converges weakly to \mathfrak{n} , we use [2], Theorem 25.8, that is, it is necessary and sufficient that for every bounded continuous function $f(x)$ on \mathbb{R}^r the following holds:

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^r} f(x) p_n(x) d\lambda(x) = \int_{\mathbb{R}^r} f(x) \mathfrak{n}(x) d\lambda(x).$$

We can write

$$\begin{aligned} & \left| \int_{\mathbb{R}^r} f(x) p_n(x) d\lambda(x) - \int_{\mathbb{R}^r} f(x) \mathfrak{n}(x) d\lambda(x) \right| \\ &= \left| \int_{\mathbb{R}^r} f(x) (p_n(x) - \mathfrak{n}(x)) d\lambda(x) \right| \leq \int_{\mathbb{R}^r} |f(x)| |n^{r/2} p_n(x) - \mathfrak{n}(x)| d\lambda(x). \end{aligned}$$

Given that f is a bounded continuous function and by the limit (4.3) proved above, the hypothesis of the Dominated Convergence Theorem is satisfied. Applying this last result, we obtain

$$\left| \int_{\mathbb{R}^r} f(x) p_n(x) d\lambda(x) - \int_{\mathbb{R}^r} f(x) \mathfrak{n}(x) d\lambda(x) \right| \xrightarrow{n \rightarrow \infty} 0,$$

and the weak convergence of the sequence $\{p_n\}$ is proved.

We now prove (b). For $c > 0$, let $\Omega_c = \{x \in \mathbb{R}^r : \|x\| < c\}$, and denote by $\overline{\Omega}_c$ the complement of Ω_c . Then, by the weak convergence of the p_n to \mathfrak{n} , we have that

$$\begin{aligned} \lim_{c \rightarrow \infty} \left(\lim_{n \rightarrow \infty} \sum_{x \in \overline{\Omega}_c} p_n(x) \right) &= \lim_{c \rightarrow \infty} \left(1 - \lim_{n \rightarrow \infty} \sum_{x \in \Omega_c} p_n(x) \right) \\ &= 1 - \lim_{c \rightarrow \infty} \int_{x \in \Omega_c} \mathfrak{n}(x) d\lambda(x) = 0. \quad \square \end{aligned}$$

Theorem 4.9. *Let $\Omega \subseteq \mathbb{R}^r$ be a nice bounded open set. Let $S \subseteq \mathbb{Z}^r$ be a G_r -invariant subset such that $\delta = \rho_\infty(S)$ exists. Then there exists a normal distribution \mathfrak{N} such that*

$$\lim_{n \rightarrow \infty} \sum_{x \in S \cap \sqrt{n}\Omega} p_n(x/\sqrt{n}) = \delta \mathfrak{N}(\Omega).$$

Proof. Note that the proof is the same as that of Theorem 3.4 in [9]. The only difference lies in the use of Proposition 4.6.

There exists a normal \mathfrak{N} distribution with density \mathfrak{n} satisfying the conclusions of Proposition 4.8.

We have

$$\begin{aligned} \sum_{x \in S \cap \sqrt{n}\Omega} p_n(x/\sqrt{n}) &= \sum_{y \in \frac{1}{\sqrt{n}}S \cap \Omega} p_n(y) \\ &= n^{-r/2} \sum_{y \in \frac{1}{\sqrt{n}}S \cap \Omega} \mathfrak{n}(y) + n^{-r/2} \sum_{y \in \frac{1}{\sqrt{n}}S \cap \Omega} (n^{r/2} p_n(y) - \mathfrak{n}(y)). \end{aligned}$$

The local limit theorem of Proposition 4.8 (a) tells us that, as $n \rightarrow \infty$, each summand $n^{-r/2} p_n(y) - \mathfrak{n}(y)$ of the sum in the last line above converges to zero, and hence so does their Cesaro mean.

Using the following convergence of the measures defined in Proposition 4.6,

$$\lim_{n \rightarrow \infty} \mu_{\sqrt{n}, S}(\Omega) = \delta \lambda(\Omega)$$

(recall that $\mu_{\sqrt{n}, S}(\Omega) := \frac{\#(S \cap \sqrt{n}\Omega)}{\sqrt{n}^r}$), we have

$$\lim_{n \rightarrow \infty} \sum_{x \in S \cap \sqrt{n}\Omega} \frac{1}{(\sqrt{n})^r} \mathfrak{n}(x/\sqrt{n}) = \int_{\Omega} \mathfrak{n}(y) \delta d\lambda(y) = \delta \mathfrak{N}(\Omega). \quad \square$$

We obtain the main result of this section by basically following [9], Theorem A. Our theorem provides the formula for the ‘spherical densities’ of visible elements in groups that satisfy Hypothesis 4.3, which include free groups of all finite ranks and surface groups.

Theorem 4.10 (see also [9], Theorem A). *Let F , A , r satisfy Hypothesis 4.3, let $S \subseteq \mathbb{Z}^r$ be a G_r -invariant subset and $\tilde{S} = ab^{-1}(S)$.*

(i) *The strict annular density $\sigma_A(\tilde{S})$ exists and, moreover, $\sigma_A(\tilde{S}) = \rho_\infty(S)$.*

(ii) *Let U_1 denote the set of visible elements in \mathbb{Z}^r and $V_1 = ab^{-1}(U_1)$ denote the visible elements in F . Let $U_1^{\text{ev}} = \{z \in U_1 : l_1(z) \text{ is even}\}$ denote the visible elements of even length. If $ab^{-1}(U_1^{\text{ev}}) = \{v \in V_1 : |v|_A \text{ is even}\}$, then*

$$\lim_{m \rightarrow \infty} \frac{\gamma_A(2m, V_1)}{\gamma_A(2m, F)} = 2\rho_\infty(U_1^{\text{ev}}) = \frac{2^r - 2}{(2^r - 1)\zeta(r)},$$

$$\lim_{m \rightarrow \infty} \frac{\gamma_A(2m - 1, V_1)}{\gamma_A(2m - 1, F)} = 2\rho_\infty(U_1) - 2\rho_\infty(U_1^{\text{ev}}) = \frac{2^r}{(2^r - 1)\zeta(r)}.$$

Proof. For $c > 0$ let $\Omega_c := \{x \in \mathbb{R}^r : \|x\| < c\}$ and let $\overline{\Omega}_c$ be the complement of Ω_c . Then

$$\lim_{c \rightarrow \infty} \mathfrak{N}(\Omega_c) = 1. \tag{4.4}$$

Let $\varepsilon > 0$ be arbitrary. By (4.4) and Proposition 4.8 (b) we can choose $c > 0$ such that

$$|\mathfrak{N}(\Omega_c) - 1| \leq \varepsilon/3 \quad \text{and} \quad \lim_{n \rightarrow \infty} \sum_{x \in \overline{\Omega}_c} p_n(x) \leq \varepsilon/6.$$

Let S be a G_r -invariant subset of \mathbb{Z}^r . By Theorem 4.9 and the above formula there is some $n_0 \geq 1$ such that for all $n \geq n_0$ we have

$$\left| \sum_{x \in S \cap \sqrt{n}\Omega_c} p_n(x/\sqrt{n}) - \rho_\infty(S)\mathfrak{N}(\Omega_c) \right| \leq \varepsilon/3 \tag{4.5}$$

and

$$\sum_{x \in \overline{\Omega}_c} p_n(x) \leq \varepsilon/3. \tag{4.6}$$

Let

$$Q(n) := \frac{\gamma_A(n - 1, ab^{-1}(S))}{2\gamma_A(n - 1, F)} + \frac{\gamma_A(n, ab^{-1}(S))}{2\gamma_A(n, F)}.$$

For $n \geq n_0$ we let

$$\begin{aligned} Q(n) = & \left(\frac{\#\{g \in F : ab(g) \in S, |g|_A = n - 1 \text{ and } \|ab(g)\| < c\sqrt{n}\}}{2\gamma_A(n - 1, F)} \right. \\ & + \left. \frac{\#\{g \in F : ab(g) \in S, |g|_A = n \text{ and } \|ab(g)\| < c\sqrt{n}\}}{2\gamma_A(n - 1, F)} \right) \\ & + \left(\frac{\#\{g \in F : ab(g) \in S, |g|_A = n - 1 \text{ and } \|ab(g)\| \geq c\sqrt{n}\}}{2\gamma_A(n - 1, F)} \right. \\ & + \left. \frac{\#\{g \in F : ab(g) \in S, |g|_A = n \text{ and } \|ab(g)\| \geq c\sqrt{n}\}}{2\gamma_A(n - 1, F)} \right) \end{aligned}$$

$$= \sum_{x \in S \cap \sqrt{n}\Omega_c} p_n(x/\sqrt{n}) + \sum_{x \in S \cap (\mathbb{R}^r \setminus \sqrt{n}\Omega_c)} p_n(x/\sqrt{n}).$$

In the last line of the above equation, by (4.5), the first sum differs from $\rho_\infty(S)\mathfrak{M}(\Omega_c)$ by at most $\varepsilon/3$ since $n \geq n_0$, and by (4.6) the second sum is $\leq \varepsilon/3$ given the choice of c and n_0 . Therefore, again by the choice of c , we have $|Q(n) - \rho_\infty(S)| \leq \varepsilon$. Since ε is arbitrary, this proves (i).

We now prove (ii). First notice that since U_1 is $SL(r, \mathbb{Z})$ -invariant, it is also G_r -invariant. We check that U_1^{ev} is G_r -invariant as well. Let $u \in \mathbb{Z}$. Then $u \in U_1^{\text{ev}}$ if and only if $\sum_{1 \leq i \leq r} (u)_i \pmod 2 = 0$ and $\gcd(u) = 1$. Let $M \in G_r$. As $M \in SL(r, \mathbb{Z})$, $\gcd(Mu) = \gcd(u) = 1$. Also, as $M = I_r$ in $SL(r, \mathbb{Z}/2\mathbb{Z})$,

$$\sum_{1 \leq i \leq r} (Mu)_i \pmod 2 = \sum_{1 \leq i \leq r} (u)_i \pmod 2 = 0.$$

Hence, U_1^{ev} is G_r -invariant.

We now take $S = U_1^{\text{ev}}$, for $n \geq 2$ even. Then

$$Q(n) = \frac{\gamma_A(n-1, \text{ab}^{-1}(U_1^{\text{ev}}))}{2\gamma_A(n-1, F)} + \frac{\gamma_A(n, \text{ab}^{-1}(U_1^{\text{ev}}))}{2\gamma_A(n, F)} = \frac{\gamma_A(n, \text{ab}^{-1}(U_1^{\text{ev}}))}{2\gamma_A(n, F)}.$$

The latter equality follows from the fact that $\text{ab}^{-1}(U_1^{\text{ev}}) = \{v \in V_1 : |v|_A \text{ is even}\}$.

By (i),

$$\lim_{m \rightarrow \infty} \frac{\gamma_A(2m, V_1)}{\gamma_A(2m, F)} = 2 \lim_{m \rightarrow \infty} Q(2m) = 2\rho_\infty(U_1^{\text{ev}}).$$

Thus $\lim_{m \rightarrow \infty} \frac{\gamma_A(2m-1, V_1)}{\gamma_A(2m-1, F)} = 2\rho_\infty(U_1) - 2\rho_\infty(U_1^{\text{ev}})$. By Proposition 3.1 and Proposition 3.5, we obtain the desired results. \square

We now focus on surface and free groups.

Corollary 4.11. *Let $k \geq 2$ and let F be a free group of rank k or a surface group of genus k . Let A and r be as in Notation 4.1. Then*

- (i) $\lim_{m \rightarrow \infty} \frac{\gamma_A(2m, V_1)}{\gamma_A(2m, F)} = \frac{2^r - 2}{(2^r - 1)\zeta(r)},$
- (ii) $\lim_{m \rightarrow \infty} \frac{\gamma_A(2m-1, V_1)}{\gamma_A(2m-1, F)} = \frac{2^r}{(2^r - 1)\zeta(r)}.$

Proof. By Theorem 4.2, F , A and r satisfy the hypotheses of Theorem 4.10. It only remains to show that $\text{ab}^{-1}(U_1^{\text{ev}}) = \{v \in V_1 : |v|_A \text{ is even}\}$. Let f be an element of F such that $\text{ab}(f) = 0 \in \mathbb{Z}^r$. Then any word representing w has the same number of a and a^{-1} and thus it has even length.

Since ab maps elements of A to unit vectors, for $u \in U_1^{\text{ev}}$ there exists $v \in \text{ab}^{-1}(U_1^{\text{ev}})$ of even length. If $\text{ab}(v) = \text{ab}(v')$, then $\text{ab}(v'v^{-1}) = 0$. Hence $v'v^{-1}$ has even length, and so does v' . Thus Theorem 4.10 (ii) applies. \square

5. Asymptotic behavior of homogeneous equations in surface groups

We now study the asymptotic behavior of $e_\gamma(G_n, G_k, s, t)$ when G_n and G_k are surface or free groups, or more generally, satisfy the hypothesis of Theorem 4.10(ii).

Theorem 5.1. *Let G_k and G_n be free or surface groups and let A, B be their respective generating sets, as in Notation 4.1. Let $r(k)$ and $r(n)$ denote the ranks of the abelianization of G_k and G_n , respectively. Let $\varepsilon, \delta \in \{0, 1\}$. Then the following inequalities hold:*

$$\begin{aligned} \frac{2^{r(n)} - 2(1 - \varepsilon)}{(2^{r(n)} - 1)\zeta(r(n))} &\leq \liminf_{s \rightarrow \infty, t \rightarrow \infty} e_\gamma(G_n, G_k, 2s + \varepsilon, 2t + \delta), \\ \limsup_{s \rightarrow \infty, t \rightarrow \infty} e_\gamma(G_n, G_k, 2s + \varepsilon, 2t + \delta) &\leq 1 - \frac{2^{r(k)} - 2(1 - \delta)}{(2^{r(k)} - 1)\zeta(r(k))} \left(1 - \frac{2^{r(n)} - 2(1 - \varepsilon)}{(2^{r(n)} - 1)\zeta(r(n))} \right). \end{aligned}$$

Proof. Let V_t and W_t denote the sets of t -visible elements in G_n and G_k , respectively. Let

$$E(s, t) = \{(u, v) \in G_n \times G_k : |u|_A = s, |v|_B = t, \phi(u) = v \text{ for some } \phi \in \text{Hom}(G_n, G_k)\}.$$

Then $e_\gamma(G_n, G_k, s, t) = \frac{|E(s, t)|}{\gamma_B(s, G_n)\gamma_A(t, G_k)}$.

By Lemma 3.3 we have the following inequalities:

$$\gamma_B(s, W_1)\gamma_A(t, G_k) \leq |E(s, t)| \leq \gamma_B(s, G_n)\gamma_A(t, G_k) - \sum_{r \neq 1} \gamma_B(s, W_r)\gamma_A(t, V_1).$$

The left inequality holds because every element v in G_k is the homomorphic image of a visible element in G_n . The right inequality holds because no visible element in G_k is the homomorphic image of an r -visible element in G_n if $r \neq 1$.

By dividing both sides by $\gamma_B(s, G_n)\gamma_A(t, G_k)$, we get

$$\frac{\gamma_B(s, W_1)}{\gamma_B(s, G_n)} \leq e_\gamma(G_n, G_k, s, t) \leq 1 - \frac{\sum_{r \neq 1} \gamma_B(s, W_r)\gamma_A(t, V_1)}{\gamma_B(s, G_n)\gamma_A(t, G_k)} = f(s, t),$$

where

$$f(s, t) = 1 - \frac{\gamma_A(t, V_1)}{\gamma_A(t, G_k)} \frac{\gamma_B(s, G_n) - \gamma_B(s, W_1)}{\gamma_B(s, G_n)}.$$

Let us use $\beta_{m,k}$ to denote the limits, which depend on the parity of m and the rank of the abelianization of G_n and G_k , found in Corollary 4.11. That is, $\beta_{m,k} = \frac{2^{r(k)} - 2}{(2^{r(k)} - 1)\zeta(r(k))}$ if m is even, and $\beta_{m,k} = \frac{2^{r(k)}}{(2^{r(k)} - 1)\zeta(r(k))}$ if m is odd. In order to simplify the exposition

we will abuse the fact that $\beta_{m,k}$ depends on the parity of m and for the next paragraph ignore the parities of s and t .

Then

$$\lim_{s \rightarrow \infty, t \rightarrow \infty} f(s, t) = 1 - \beta_{t,k}(1 - \beta_{s,n}),$$

and we get the inequalities

$$\beta_{s,n} \leq \liminf_{s \rightarrow \infty, t \rightarrow \infty} e_\gamma(G_n, G_k, s, t) \leq \limsup_{s \rightarrow \infty, t \rightarrow \infty} e_\gamma(G_n, G_k, s, t) \leq 1 - \beta_{t,k}(1 - \beta_{s,n}).$$

Now taking into account the parities of s and t we get the inequalities in the statement of the theorem. \square

Thus the probability of an (s, t) -homogeneous equation to be satisfiable is neither 0 nor 1 as s, t go to infinity. One sees this by choosing G_n to be the free group on n generators and G_k a surface group of genus $g \geq 2$ or a free group of rank ≥ 2 in Theorem 5.1.

Corollary 5.2. *Let G be a surface group of genus $g \geq 2$ or a free group of rank ≥ 2 . Let*

$$A(s, t) = \frac{\#\{\text{satisfiable } (s, t)\text{-homogeneous equations in } G \text{ in } n \text{ variables}\}}{\#\{(s, t)\text{-homogeneous equations in } G \text{ in } n \text{ variables}\}}.$$

Then

$$0 < \liminf_{s \rightarrow \infty, t \rightarrow \infty} A(s, t) \leq \limsup_{s \rightarrow \infty, t \rightarrow \infty} A(s, t) < 1.$$

Similarly, by choosing both G_n and G_k in Theorem 5.1 to be surface groups one obtains the following.

Corollary 5.3. *Let Σ be an orientable closed surface of genus $k \geq 2$. We fix a presentation for $\pi_1(\Sigma)$, $\langle a_1, b_1, \dots, a_k, b_k \mid [a_1, b_1] \dots [a_k, b_k] \rangle$. For a closed curve γ in Σ we denote by $[\gamma]$ the image of γ in $\pi_1(S)$ and by $\|[\gamma]\|$ the length of $[\gamma]$ with respect to $\{a_1, b_1, \dots, a_k, b_k\}$.*

We say that γ_2 is the image of γ_1 if it is the image of γ_1 under a continuous map $S \rightarrow S$.

Let

$$B(s, t) = \frac{\#\{([\gamma_1], [\gamma_2]) \in \pi_1(S)^2, (\|[\gamma_1]\|, \|[\gamma_2]\|) = (s, t) \text{ with } \gamma_2 \text{ the image of } \gamma_1\}}{\#\{([\gamma_1], [\gamma_2]) \in \pi_1(S)^2, (\|[\gamma_1]\|, \|[\gamma_2]\|) = (s, t)\}}.$$

Then

$$0 < \liminf_{s \rightarrow \infty, t \rightarrow \infty} B(s, t) \leq \limsup_{s \rightarrow \infty, t \rightarrow \infty} B(s, t) < 1.$$

Thus for a fixed orientable surface Σ , the probability of a closed curve in Σ to be the image of another closed curve in Σ by a continuous map is neither 0 nor 1 as the curves get more and more “complicated.”

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