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Limits of Baumslag–Solitar groups and dimension estimates in the space of marked groups

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Abstract. We prove that the limits of Baumslag–Solitar groups studied by the authors are non-linear hopfian C*-simple groups with infinitely many twisted conjugacy classes. We exhibit infinite presentations for these groups, classify them up to group isomorphism, describe their automorphisms and discuss the word and conjugacy problems. Finally, we prove that the set of these groups has non-zero Hausdorff dimension in the space of marked groups on two generators.

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Contents

Introduction

The Baumslag–Solitar groups

$$\mathrm{BS}(p,q) = \langle a, b \mid ab^p a^{-1} = b^q \rangle \quad \text{with } p, q \in \mathbb{Z} \setminus \{0\}$$

are defined in [BS62] and give rise to the first examples of non-hopfian one-relator groups for p and q coprime (a group is called *hopfian* if it has no isomorphic proper quotient). The groups $\mathrm{BS}(p,q)$ are ubiquitous in group theory and topology [Mol69], [Gil79], [FM98], [Why01], [JS79] and offer a remarkable test bed for group-theoretic properties. Considering their limits, we obtain in the present paper a Cantor set of pairwise non-isomorphic two-generated groups with a number of combinatorial and geometrical non-closed properties. By a closed property, we mean a property that defines a closed subset of the space of marked groups. We also give the first estimates of non-vanishing Hausdorff dimension in the space of marked groups.

Let $m \in \mathbb{Z} \setminus \{0\}$. For every sequence (ξ_n) of integers in $\mathbb{Z}$ such that $|\xi_n|$ tends to infinity and ξ_n tends to some m-adic integer ξ, define

$$\overline{\mathrm{BS}}(m,\xi) = \lim_{n\to\infty} \mathrm{BS}(m,\xi_n)$$

where the limit is taken with respect to the topology of the space marked groups; the above sequence being proved to be convergent in [Sta06a], Theorem 6. Note that $\overline{\mathrm{BS}}(m,n) \neq \mathrm{BS}(m,n)$ for any $n \in \mathbb{Z} \setminus \{0\} \subset \mathbb{Z}_m$, for stationary sequences are prohibited. As a consequence of its definition, $\overline{\mathrm{BS}}(m,\xi)$ enjoys the following closed properties: it is torsion-free, it contains a non-abelian free group generated by b and bab^{-1} if $|m| > 1$, and satisfies the relation "$[ab^m a^{-1}, b] = 1$", which is actually the shortest relation with respect to a and b by [Sta06a], Proposition 2. We present here results of a different nature, relying on the existence of an HNN decomposition.

The limits $\overline{\mathrm{BS}}(m,\xi)$ are first studied for their own right in [GS08], where it is shown that $\overline{\mathrm{BS}}(m,\xi)$ acts transitively on a tree and maps homomorphically onto the special limit $\overline{\mathrm{BS}}(1,0) = \mathbb{Z} \wr \mathbb{Z}$. These two results are extensively used in this article.

Results. We first prove that $\overline{\mathrm{BS}}(m,\xi)$ is a non-degenerate HNN extension of a free abelian group of infinite countable rank. More precisely, consider the free abelian groups

$$\begin{aligned} E &= \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots, \\ E_{m,\xi} &= \mathbb{Z}me_0 \oplus \mathbb{Z}(e_1 - r_1(\xi)e_0) \oplus \mathbb{Z}(e_2 - r_2(\xi)e_0) \oplus \cdots, \\ E_1 &= \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots, \end{aligned}$$

where $(r_i(\xi))$ is an integer sequence defined through iterated Euclidean divisions by m. Note that the Euclidean division of $x \in \mathbb{Z}_m$ by m is well defined since one can write $x = ms + r$ for a unique *quotient* $s \in \mathbb{Z}_m$ and a unique *remainder* $r \in \{0, \dots, |m|-1\}$. Set $s_0(\xi) = 1$, $r_0(\xi) = 0$ and apply then Euclidean division of $\xi s_i(\xi)$ by m to obtain a new quotient $s_{i+1}(\xi)$ and a new remainder $r_{i+1}(\xi)$. The sequence $(r_i(\xi))$ is thus defined inductively through the formula $\xi s_i(\xi) = ms_{i+1}(\xi) + r_{i+1}(\xi)$. Let $\widetilde{\mathrm{BS}}(m,\xi)$ be the HNN extension of basis E with conjugated subgroups $E_{m,\xi}$, E_1 and stable letter a, where conjugacy from $E_{m,\xi}$ to E_1 is defined by $a(me_0)a^{-1} = e_1$ and $a(e_i - r_i(\xi)e_0)a^{-1} = e_{i+1}$.

Theorem A (Corollary 2.10 and Theorem 3.1). *Let (a,b) be the canonical generating pair of $\overline{\mathrm{BS}}(m,\xi)$. Then the map defined by $f(a) = a$ and $f(b) = e_0$ induces an* *isomorphism from* $\overline{\mathrm{BS}}(m,\xi)$ *to* $\widetilde{\mathrm{BS}}(m,\xi)$. *Moreover, the group* $\overline{\mathrm{BS}}(m,\xi)$ *admits the infinite presentation*

$$\langle a,b \mid [b,b_i]=1,\ i\geq 1\rangle,$$

with $b_1 = ab^m a^{-1}$ *and* $b_i = ab_{i-1}b^{-r_{i-1}(\xi)}a^{-1}$ *for every* $i \geq 2$.

In particular, we have $\overline{\mathrm{BS}}(m,0) = \langle a,b \mid [a^i b^m a^{-i}, b] = 1,\ i\geq 1\rangle$. *In addition, the latter presentation is minimal.*

Thus Theorem A rules out the only possible exception in [GS08], Theorem 4.1, namely $\xi \in m\mathbb{Z}_m$, so that no group $\overline{\mathrm{BS}}(m,\xi)$ can be finitely presented. Note also that $\overline{\mathrm{BS}}(m,\xi)$ enjoys any property shared by all non-degenerate HNN extensions, for instance it is primitive [GG08] and has uniform exponential growth [HB00]. Using the latter HNN decomposition, we show:

Theorem B. *Assume that* $|m| > 1$. *Then we have:*

- $\overline{\mathrm{BS}}(m,\xi)$ *is hopfian but not co-hopfian* (*Theorem* 5.9). *If* m *is prime and* ξ *is algebraic over* $\mathbb{Q}$, *then* $\overline{\mathrm{BS}}(m,\xi)$ *is not residually finite* (*Proposition* 5.10).
- $\overline{\mathrm{BS}}(m,\xi)$ *is C*-simple and inner-amenable* (*Proposition* 4.1 *and Proposition* 4.6).
- $\overline{\mathrm{BS}}(m,\xi)$ *has infinitely many twisted conjugacy classes* (*Proposition* 5.5).
- $\overline{\mathrm{BS}}(m,\xi)$ *is not equationally noetherian and hence not linear* (*Proposition* 5.17).
- *The automorphism group of* $\overline{\mathrm{BS}}(m,\xi)$ *is a split extension of* $\overline{\mathrm{BS}}(m,\xi)$ *by an infinite dihedral group* (*Proposition* 5.13).

From our study of homomorphisms carried out in Section 5, we deduce the following classification result.

Theorem C (Theorem 5.11 and Proposition 2.6). *Let* $m, m' \in \mathbb{Z}\setminus\{0\}$ *and let* $\xi \in \mathbb{Z}_m$, $\xi' \in \mathbb{Z}_{m'}$. *Then* $\overline{\mathrm{BS}}(m,\xi)$ *is abstractly isomorphic to* $\overline{\mathrm{BS}}(m',\xi')$ *if and only if there is* $\epsilon \in \{\pm 1\}$ *and* $d \in \mathbb{N}$ *such that* $m = \epsilon m'$, $d = \gcd(m,\xi) = \gcd(m',\xi')$ *and the* m*-adic numbers* ξ/d, $\epsilon\xi'/d$ *project onto the same element of* $\mathbb{Z}_{m/d}$ *via the canonical map* $\mathbb{Z}_m \to \mathbb{Z}_{m/d}$.

Thus two given limits are isomorphic if and only if they are isomorphic as marked groups [GS08], Theorem 2.1, i.e., if and only if they represent the same point in the space of marked groups on two generators.

The first-named author has shown that the box-counting dimension of $\mathcal{G}_2$, the space of marked groups on two generators, is infinite [Guy07]. Estimating the Hausdorff dimension of $Z_m^{\times}$, the set of marked groups $\overline{\mathrm{BS}}(m,\xi)$ such that ξ is invertible in $\mathbb{Z}_m$ (Theorem 6.5), we deduce the following:

Theorem D (Corollary 6.6). *The Hausdorff dimension of* $\mathcal{G}_2$ *satisfies* $\dim_H(\mathcal{G}_2) \geqslant \log(2)/6$. *In particular, the Hausdorff dimension of* $\mathcal{G}_2$ *does not vanish.*

Our last results pertain to the algorithmic complexity of the word and conjugacy problem. For every problem that can be suitably represented by a language and every m-adic number there is a well-defined degree of complexity, called the *Turing degree* (Section 7).

Theorem E (Corollary 7.7). *Assume that* ξ *is invertible in* $\mathbb{Z}_m$. *Then the following Turing degrees coincide:*

- *the Turing degree of the word problem for* $\overline{\mathrm{BS}}(m,\xi)$;
- *the Turing degree of the conjugacy problem for* $\overline{\mathrm{BS}}(m,\xi)$;
- *the Turing degree of* ξ.

In particular, the word problem is solvable for $\overline{\mathrm{BS}}(m,\xi)$ *if and only if* ξ *is a computable number.*

The resolution degree $r_\Gamma(n)$ of the word problem for a finitely generated group Γ [Gri85], Definition 1, is a quantitative measure of the undecidability of the word problem based on Kolgomorovś ideas. It is, intuitively, the minimal amount of information necessary to decide if $w = 1$ in G for every word w with $|w| \leq n$, where $|w|$ is the length of w with respect to a chosen generating set. For a word w, we denote by $\mathrm{KR}(w)$ the Kolmogorov complexity resolution of w which is, intuitively, the minimal amount of information necessary to obtain, for every natural number $i \leq |w|$, the i-th symbol of the word w.

Theorem F (Proposition 2.5 and Theorem 6.1). *Let* $\Gamma = \mathrm{BS}(m,\xi)$ *and let* $\omega = (r_i(\xi))$. *Then* $r_\Gamma(n)$ *is equivalent to* $\mathrm{KR}(\omega^{(n)})$.

We denote by $\omega^{(n)}$ be the word made of the first n symbols of ω and say that two functions f and g are equivalent if there is some $C > 0$ such that $f(n) \leq g(Cn)+C$ and $g(n) \leq f(Cn) + C$ for every n. For comparison, the resolution degree of a Grigorchuk group $\Gamma = G_\omega$ defined in [Gri84] is equivalent to $\mathrm{KR}(\omega^{([\log n])})$ [Gri85], Theorem 3, the symbols of ω assuming there only three possible values. Thus, for a typical ω in the measure-theoretic sense, $r_\Gamma(n)$ is linear for the corresponding $\mathrm{BS}(m,\xi)$ and logarithmic for G_ω [ZL70].

1. Background

1.1. The space of marked groups. The commonly used definition of the space of marked groups[1] is due to Grigorchuk who proved by a topological argument that his intermediate growth groups cannot be finitely presented [Gri84]. The space of marked

[1]A very similar topology was first considered in [Gro81], Final remarks. The general idea of topologizing sets of groups goes back to Mahlers and Chabauty [Mah46], [Cha50]. The interested reader should consult [Har08] for a thorough account.

groups has then been used to prove both existence and abundance of groups with exotic properties [Ste96], [Cha00] and has turned to be a remarkably suited framework for the study of Sela's *limit groups* [CG05]. Isolated points were investigated in [CGP07] and an isolated group is used by de Cornulier to answer a question of Gromov concerning the existence of sofic groups which do not arise as limits of amenable groups [Cor09b]. Very little is known on its topological type; nevertheless the topological type of the space of metabelian marked groups is unveiled in [Cor09a]. The box-counting dimension of the space of marked groups on k generators is infinite if $k \geq 2$ [Guy07]. We show that its Hausdorff dimension is non-zero in Section 6. These computations are carried out using the class of Hölder equivalent metrics defined below.

Definitions. The free group on k generators will be denoted by $\mathbb{F}_k$, or $\mathbb{F}(S)$ with $S = (s_1, \dots, s_k)$, if we want to precise the names of canonical generating elements. A *marked group on k generators* is a pair (G, S) where G is a group and $S = (s_1, \dots, s_k) \in G^k$ is an ordered generating set of G, also referred to as a *marking* of G. An *isomorphism of marked groups* is an isomorphism which respects the markings. A marked group (G, S) is endowed with a canonical epimorphism $\phi\colon \mathbb{F}_S \to G$, which induces an isomorphism of marked groups between $\mathbb{F}_S / \ker\phi$ and G. Hence a class of marked groups for the relation of marked isomorphism is represented by a unique quotient of $\mathbb{F}_S$. The non-trivial elements of $\ker\phi$ are called *relations* of (G, S). Given $w \in \mathbb{F}_k$ we will often write "$w = 1$ in G" to say that the image of w in G is trivial.

Let $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ be a reduced word in $\mathbb{F}_S$ (with $x_i \in S$ and $\varepsilon_i \in \{\pm 1\}$). The integer n is called the *length* of w and denoted $|w|$. If (G, S) is a marked group on k generators and $g \in G$, the *length* of g is

$$\begin{aligned} |g|_G &:= \min\{n \mid g = s_1 \dots s_n \text{ with } s_i \in S \sqcup S^{-1}\} \\ &= \min\{|w| \mid w \in \mathbb{F}_S,\ \phi(w) = g\}\,. \end{aligned}$$

Let $\mathcal{G}_k$ be the class of marked groups on k generators up to marked isomorphism. Let us recall that the topology on $\mathcal{G}_k$ comes from the following ultrametric distance: for $(G_1, S_1) \neq (G_2, S_2) \in \mathcal{G}_k$ we set $d\big((G_1, S_1), (G_2, S_2)\big) := e^{-\lambda}$ where λ is the length of a shortest element of $\mathbb{F}_k$ which vanishes in one group and not in the other one. Replacing e by e^{α} with $\alpha > 0$ in the definition of $d(\cdot\,,\cdot)$, we obtain an Hölder equivalent metric $d_\alpha(\cdot\,,\cdot)$ with respect to which Hausdorff dimensions are scaled by a factor α.

We will use the following characterization of convergent sequences [Sta06a], Proposition 1, without further notice.

Lemma 1.1. *Let (G_n) be a sequence of marked groups in $\mathcal{G}_k$. The sequence (G_n) converges if and only if for any $w \in \mathbb{F}_k$, we have either $w = 1$ in G_n for n large enough, or $w \neq 1$ in G_n for n large enough.*

The free group $\mathbb{F}_2 = \mathbb{F}(a,b)$, the Baumslag–Solitar groups $\mathrm{BS}(p,q)$ and their limits $\overline{\mathrm{BS}}(m,\xi)$ are marked by their canonical generating pair (a,b).

1.2. The ring of *m*-adic integers. Let $m \in \mathbb{Z} \setminus \{0\}$. Recall that the ring of m-adic integers $\mathbb{Z}_m$ is the projective limit in the category of topological rings of the system

$$\cdots \to \mathbb{Z}/m^h\mathbb{Z} \to \mathbb{Z}/m^{h-1}\mathbb{Z} \to \cdots \to \mathbb{Z}/m^2\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$$

where the arrows are the canonical surjective homomorphisms (Note that $\mathbb{Z}_{-m} = \mathbb{Z}_m$ and $\mathbb{Z}_m$ is the zero ring if $|m| = 1$.). The topological ring $\mathbb{Z}_m$ is compact since all rings in the above system are finite. Its topology is compatible with the ultrametric distance given, for $\xi \neq \eta$, by

$$|\xi - \eta|_m = |m|^{-\max\{k \in \mathbb{N} \mid \xi - \eta \in m^k \mathbb{Z}_m\}},$$

and $\mathbb{Z}$ embeds isomorphically and densely in $\mathbb{Z}_m$. To avoid ambiguity, we call elements of $\mathbb{Z}$ *rational integers*. We only need the following easy facts about m-adic integers. Detailed proofs can be found in the second-named author's Ph.D. thesis [Sta05], Appendix C.

- There is a topological ring isomorphism from $\mathbb{Z}_m$ to $\bigoplus_{p|m} \mathbb{Z}_p$, where p ranges in the set of the prime divisors of m.
- Any non-zero ideal can be uniquely written under the form $d\mathbb{Z}_m$ where d is a positive rational integer whose prime divisors divide m. Moreover we have $\mathbb{Z} \cap d\mathbb{Z}_m = d\mathbb{Z}$. For any $n \in \mathbb{Z}$ and any $\xi \in \mathbb{Z}_m$, we can define the *greatest common divisor* $\gcd(n,\xi)$ of n and ξ as the a unique positive rational integer d such that n and ξ generate $d\mathbb{Z}_m$.
- An m-adic integer ξ is invertible if and only if it does not belong to any of the ideals $p\mathbb{Z}_m$ where p is a prime divisor of m in $\mathbb{Z}$; equivalently $\gcd(m,\xi) = 1$.

Note finally that non-zero rational integers are never zero divisors in $\mathbb{Z}_m$. Hence, we can consider the ring $\mathbb{Z}^{-1}\mathbb{Z}_m$ whose elements are fractions of the form $\frac{a}{b}$ with $a \in \mathbb{Z}_m$ and $b \in \mathbb{Z} \setminus \{0\}$ and whose laws are the classical ones for fractions.

1.3. HNN extensions and tree actions. In this section we fix our notations for HNN extensions and collect several facts concerning their standard tree actions. Let G be a group, let H, K be subgroups of G and let $\tau\colon H \to K$ be an isomorphism. The *HNN extension* of base G whose stable letter t conjugates H to K via τ, is the group

$$\mathrm{HNN}(G,H,K,\tau) = \langle G, t \mid tht^{-1} = \tau(h) \text{ for every } h \in H\rangle.$$

Let $\Gamma = \mathrm{HNN}(G,H,K,\tau)$. We say that Γ is an *ascending* HNN extension if either $G = H$ or $G = K$. We say that Γ is a *degenerate* HNN extension if $G = H = K$, i.e., $\Gamma = G \rtimes \mathbb{Z}$ where $\langle t\rangle$ identifies with $\mathbb{Z}$. Note that a given group (e.g. $\mathbb{Z} \wr \mathbb{Z}$) may have two different HNN decompositions, one being degenerate whereas the other is not.

The Normal Form Theorem. A sequence $g_0, t^{\epsilon_1}, g_1, \dots, t^{\epsilon_n}, g_n$, with $n \geq 0$, is said to be a *reduced sequence* if there is no consecutive subsequence t, g_i, t^{-1} with $g_i \in H$ or t^{-1}, g_i, t with $g_i \in K$. Britton's lemma [LS77], p. 181, asserts that the word $g_0 t^{\epsilon_1} \dots t^{\epsilon_n} g_n$ has a non-trivial image γ in Γ if $g_0, t^{\epsilon_1}, \dots, t^{\epsilon_n}, g_n$ is reduced and $n \geq 1$. Such a word is called a reduced form for γ. Although γ may have different reduced forms, the sequence $t^{\epsilon_1}, \dots, t^{\epsilon_n}$ is uniquely determined by γ and we call its length $n = |\gamma|_t$ the t-length of γ. Fixing a set T_H of representatives of right cosets of H in G and a set T_K of representatives of right cosets of K in G such that $1 \in T_H \cap T_K$, the Normal Form Theorem [LS77], Theorem IV.2.1, asserts that there is a unique sequence $g_0, t^{\epsilon_1}, \dots, t^{\epsilon_n}, g_n$ representing $\gamma \in \Gamma$ with the following properties:

- g_0 is an arbitrary element of G,
- If $\epsilon_i = 1$, then $g_i \in T_H$,
- If $\epsilon_i = -1$, then $g_i \in T_K$,
- there is no consecutive subsequence $t^{\epsilon}, 1, t^{-\epsilon}$.

This sequence is the *normal sequence of* γ *with respect to* T_H *and* T_K and we call the word $g_0 t^{\epsilon_1} \dots t^{\epsilon_n} g_n$ the *normal form* of γ.

Action on the Bass–Serre tree and its boundary. The *Bass–Serre tree* of $\Gamma = \mathrm{HNN}(G, H, K, \tau)$ is the oriented tree T with vertex set $V(T) = \Gamma/G$, with set of oriented edges $E_+(T) = \Gamma/H$ subject to the incidence relations $o(\gamma H) = \gamma G$ and $t(\gamma H) = \gamma t^{-1} G$.

Given a tree T, we define the *boundary* ∂T as the set of cofinal rays. The set ∂T has a natural topology defined as follows. We call a *shadow* the boundary of any connected component of T deprived of one of its edges. The family of shadows generates a topology on ∂T which is Hausdorff and totally discontinuous. If T is a countable non-linear tree, e.g. the standard Bass–Serre tree of a non-degenerate HNN extension, then ∂T is a perfect Baire space [HP11], Proposition 10 and 20.ii′. Note that every automorphism of T induces an homeomorphism of ∂T.

As Γ acts without inversion on its Bass–Serre tree T, every element $\gamma \in \Gamma$, viewed as a tree automorphism, is either elliptic, i.e., γ fixes some vertex of T, or hyperbolic, i.e., γ fixes no vertex of T but exactly two ends of ∂T [Ser77]. The action of Γ on ∂T has no fixed end if Γ is non-ascending, exactly one fixed end if Γ is non-degenerate and ascending, and exactly two fixed ends if Γ is degenerate [HP11], Proposition 20.

2. HNN decomposition of the limits

We fix $m \in \mathbb{Z} \setminus \{0\}$, $\xi \in \mathbb{Z}_m$ and (ξ_n) a sequence of rational integers such that $|\xi_n| \to \infty$ and $\xi_n \to \xi$ in $\mathbb{Z}_m$. A natural HNN decomposition arises from the transitive action of $\overline{\mathrm{BS}}(m, \xi)$ on a tree constructed in [GS08]. We recall this construction.

We denote by H_n (respectively H_n^m) the subgroup of $\mathrm{BS}(m,\xi_n)$ generated by b (respectively b^m) and by T_n the Bass–Serre tree of $\mathrm{BS}(m,\xi_n)$. We set

$$Y = \Big(\prod_{n\in\mathbb{N}} V(T_n)\Big)/\!\sim\; = \Big(\prod_{n\in\mathbb{N}} \mathrm{BS}(m,\xi_n)/H_n\Big)/\!\sim,$$
$$Y^m = \Big(\prod_{n\in\mathbb{N}} E_+(T_n)\Big)/\!\sim\; = \Big(\prod_{n\in\mathbb{N}} \mathrm{BS}(m,\xi_n)/H_n^m\Big)/\!\sim,$$

where $\sim$ is defined by $(x_n) \sim (y_n) \iff \exists n_0\ \forall n \geqslant n_0 \colon x_n = y_n$ in both cases. We now define an oriented graph $X = X_{m,\xi}$ by

$$V(X) = \{x \in Y \mid \exists w \in \mathbb{F}_2 \text{ such that } (x_n) \sim (wH_n)\} = \{(wH_n) \mid w \in \mathbb{F}_2\}/\!\sim,$$
$$E_+(X) = \{y \in Y^m \mid \exists w \in \mathbb{F}_2 \text{ such that } (y_n) \sim (wH_n^m)\} = \{(wH_n^m) \mid w \in \mathbb{F}_2\}/\!\sim,$$
$$o(wH_n^m)) = (wH_n) = (o(wH_n^m)),$$
$$t((wH_n^m)) = (wa^{-1}H_n) = (t(wH_n^m)).$$

The graph $X_{m,\xi}$ is a tree and the obvious action of $\mathbb{F}_2$ on $X_{m,\xi}$ factorizes through the canonical projection $\mathbb{F}_2 \to \overline{\mathrm{BS}}(m,\xi)$ [GS08], Section 3. Let $v_0 = (H_n)$ and $e_0 = (H_n^m)$. We denote by B (respectively B_m) the stabilizer of v_0 (respectively e_0) in $\overline{\mathrm{BS}}(m,\xi)$. We set $B_\xi = aB_m a^{-1}$. As the action of $\overline{\mathrm{BS}}(m,\xi)$ is clearly transitive on vertices and geometrical edges, the group $\overline{\mathrm{BS}}(m,\xi)$ has a HNN decomposition:

Proposition 2.1. *The group $\overline{\mathrm{BS}}(m,\xi)$ is the HNN extension of base group B, stable letter a and conjugated subgroups B_m and $B_\xi = aB_m a^{-1}$.*

We define the inner automorphism $\tau_a \colon \gamma \mapsto a\gamma a^{-1}$ of $\overline{\mathrm{BS}}(m,\xi)$ and we denote by $\mathrm{HNN}(B, B_m, B_\xi, \tau)$ the previous HNN decomposition where τ is the isomorphism from B_m to B_ξ induced by τ_a.

Proof. This follows from results in [Ser77]. See in particular Section I.5 and Remark 1 after Theorem 7 in Section I.4. □

The following lemma is an immediate consequence of the definition of the action.

Lemma 2.2. *Let $w \in \mathbb{F}_2$ and let γ be its image in $\overline{\mathrm{BS}}(m,\xi)$. Then $\gamma \in B$ (respectively $\gamma \in B_m$) if and only if there is rational integer sequence (λ_n) such that $w = b^{\lambda_n}$ (respectively $b^{m\lambda_n}$) in* $\mathrm{BS}(m,\xi_n)$ *for all n large enough.*

As a result, B is abelian. Let $\mathbb{Z}[X]$ be the ring of polynomials in the variable X with integer coefficients. A thorough study of the sequence (λ_n) of Lemma 2.2 enables us to embed B isomorphically into $\mathbb{Z}[X]$ seen as an abelian group:

Proposition 2.3. *Let $w \in \mathbb{F}_2$ with image $\gamma \in B$. Then there is a unique polynomial $P_\gamma(X) \in \mathbb{Z}[X]$ such that $w = b^{P_\gamma(\xi_n/m)}$ in* $\mathrm{BS}(m,\xi_n)$ *for all n large enough.*

The map $q\colon \gamma \mapsto P_\gamma(X)$ defines an injective homomorphism from B into $\mathbb{Z}[X]$.

The construction of the homomorphism q relies on a sequence of polynomials built up from m and ξ in a non obvious way. Its definition is the subject of the next section. The proof of Proposition 2.3 is therefore postponed to Section 2.2 (see Proposition 2.9).

2.1. The functions r_i and s_i. In this section we define functions r_i and s_i on $\mathbb{Z}$ which describe how the b exponents of a given word $w \in \mathbb{F}_2$ behave when we reduce it in $\mathrm{BS}(m,n)$ for n ranging in a given class modulo m^h. In [Sta06a], [GS08], these functions were decisive to describe converging sequences of Baumslag–Solitar groups. It turns out that they extend continuously to $\mathbb{Z}_m$ and that ξ, and hence $\overline{\mathrm{BS}}(m,\xi)$, is uniquely determined by m and the sequence $(r_i(\xi))$.

Definition 2.4. We define the functions $r_0, r_1, \dots, s_0, s_1, \dots$ on $\mathbb{Z}$ depending on a parameter $m \in \mathbb{Z} \setminus \{0\}$ by the inductive formulas

$$r_0(n) = 0, \quad s_0(n) = 1;$$
$$ns_{i-1}(n) = ms_i(n) + r_i(n) \text{ with } r_i(n) \in \{0, \dots, |m|-1\} \text{ for every } i \geq 1.$$

Proposition 2.5. *Let $n, n' \in \mathbb{Z}$ such that* $\gcd(n,m) = \gcd(n',m) = d$. *Let $h \in \mathbb{N} \setminus \{0\}$ and $\widehat{m} = m/d$.*

(1) *Assume that $n \equiv n' \pmod{\widehat{m}^h d\mathbb{Z}}$. Then the following holds:*
   (i) *$r_i(n) = r_i(n')$ for $i = 1, \dots, h$;*
   (ii) *$s_i(n) \equiv s_i(n') \pmod{\widehat{m}^{h-i}\mathbb{Z}}$ for $i = 1, \dots, h$.*

(2) *If $r_i(n) = r_i(n')$ for $i = 1, \dots, h$, then $n \equiv n' \pmod{\widehat{m}^h d\mathbb{Z}}$.*

Proof of Proposition 2.5. Proof of (1). We show by induction on i that $r_i(n) = r_i(n')$, $s_i(n) \equiv s_i(n') \pmod{\widehat{m}^{h-i}\mathbb{Z}}$ and $s_i(n) \cdot n \equiv s_i(n') \cdot n' \pmod{\widehat{m}^{h-i} d\mathbb{Z}}$ for $i = 0, 1, \dots, h$. The case $i = 0$ is obvious. Assume that $r_{i-1}(n) = r_{i-1}(n')$, $s_{i-1}(n) \equiv s_{i-1}(n') \pmod{\widehat{m}^{h-i+1}\mathbb{Z}}$ and $s_{i-1}(n) \cdot n \equiv s_{i-1}(n') \cdot n' \pmod{\widehat{m}^{h-i+1} d\mathbb{Z}}$ for some $1 \leq i \leq h$. By construction

$$s_{i-1}(n) \cdot n = s_i(n) \cdot m + r_i(n)$$

and

$$s_{i-1}(n') \cdot n' = s_i(n') \cdot m + r_i(n').$$

We deduce from the induction hypothesis that $r_i(n) \equiv r_i'(n') \pmod{m\mathbb{Z}}$ and hence $r_i(n) = r_i(n')$. As a result

$$s_i(n) - s_i(n') = \frac{(s_{i-1}(n) \cdot n - s_{i-1}(n') \cdot n')}{m},$$

from which we deduce that

$$s_i(n) \equiv s_i(n') \pmod{\widehat{m}^{h-i}\mathbb{Z}} \tag{1}$$

still by using the induction hypothesis. Look at the right member of the relation

$$s_i(n)\cdot n - s_i(n')\cdot n' = (s_i(n)-s_i(n'))\cdot n + s_i(n')\cdot(n-n').$$

The first term is a multiple of $\widehat{m}^{h-i}d$ because of equation (1) and the fact that d divides n. The second one is a multiple of $\widehat{m}^{h-i}d$ because of hypothesis (1). Therefore, we get $s_i(n)\cdot n \equiv s_i(n')\cdot n' \pmod{\widehat{m}^{h-i}d\mathbb{Z}}$, as desired.

Proof of (2). By construction, we have

$$s_{i-1}(n)\cdot n = s_i(n)\cdot m + r_i(n)$$

and

$$s_{i-1}(n')\cdot n' = s_i(n')\cdot m + r_i(n')$$

for all $i \in \{1,\dots,h\}$. By hypothesis (2), we get $s_i(n)-s_i(n') = (s_{i-1}(n)\cdot n - s_{i-1}(n')\cdot n')/m$, which we write in the form

$$s_i(n)-s_i(n') = \frac{(s_{i-1}(n)-s_{i-1}(n'))\cdot\hat{n}}{\widehat{m}} + \frac{s_{i-1}(\hat{n}')(\hat{n}-\hat{n}')}{\widehat{m}}, \tag{2}$$

where $\hat{n} = n/d$ and $\hat{n}' = n'/d$. We show by induction on k that $\hat{n} \equiv \hat{n}' \pmod{\widehat{m}^k\mathbb{Z}}$ for $k = 1,\dots,h$. The case $k = 1$ follows directly from equation (2) for $i = 1$ (recall that $s_0(n) = 1 = s_0(n')$). For the inductive step, we assume that $2 \le k \le h$ et $\hat{n} \equiv \hat{n}' \pmod{\widehat{m}^{k-1}\mathbb{Z}}$, which implies that the second term on the right-hand side of equation (2) is a multiple of $\widehat{m}^{k-2}$ (it is an integer in particular). We then proceed in k steps using equation (2) and the fact that $\widehat{m}$ and $\hat{n}$ are coprime:

- for $i = k$ we get $s_{k-1}(n) \equiv s_{k-1}(n') \pmod{\widehat{m}\mathbb{Z}}$;
- for $i = k-1$ (if $k \ge 3$) we get $s_{k-2}(n) \equiv s_{k-2}(n') \pmod{\widehat{m}^2\mathbb{Z}}$;
- for $i = k-2,\dots,2$ we get the sequence of congruences

$$s_{k-3}(n) \equiv s_{k-3}(n') \pmod{\widehat{m}^3\mathbb{Z}},\dots, s_1(n) \equiv s_1(n') \pmod{\widehat{m}^{k-1}\mathbb{Z}};$$

- for $i = 1$ we get $\hat{n} \equiv \hat{n}' \pmod{\widehat{m}^k\mathbb{Z}}$.

Finally, we obtain $n \equiv n' \pmod{\widehat{m}^k d\mathbb{Z}}$, which completes the proof. □

Proposition 2.6. (i) *The functions r_i and s_i admit unique continuous extensions*

$$r_i : \mathbb{Z}_m \to \{0,\dots,|m|-1\}, \quad s_i : \mathbb{Z}_m \to \mathbb{Z}_m$$

such that

$$r_0(\xi) = 0, \quad s_0(\xi) = 1; \tag{1_ξ}$$

$$\xi s_{i-1}(\xi) = m s_i(\xi) + r_i(\xi) \text{ with } r_i(\xi) \in \{0,\dots,|m|-1\} \text{ for every } i \ge 1. \tag{2_ξ}$$

Let $\xi,\xi' \in \mathbb{Z}_m$ such that $\gcd(\xi,m) = \gcd(\xi',m) = d$ *and let $h \in \mathbb{N}\setminus\{0\}$. Setting $\widehat{m} = m/d$, the following statements are equivalent:*

(1) $\xi \equiv \xi' \pmod{\widehat{m}^h d\mathbb{Z}_m}$ *holds;*

(2) $r_i(\xi) = r_i(\xi')$ *for* $i = 1, \dots, h$.

(ii) *Let* $\xi, \xi' \in \mathbb{Z}_m$. *The following statements are equivalent:*

(1) *There is some* $d \in \mathbb{N}$ *such that* $\gcd(\xi, m) = \gcd(\xi', m) = d$ *and* $\pi(\xi/d) = \pi(\xi'/d)$ *where* $\pi\colon \mathbb{Z}_m \to \mathbb{Z}_{m/d}$ *is the canonical ring homomorphism;*

(2) $r_i(\xi) = r_i(\xi')$ *for every* i.

(iii) *The mapping* $\xi \mapsto (r_i(\xi))_{i\geq 1}$ *defines a homeomorphism from* $\mathbb{Z}_m^\times$ *onto* $(\mathbb{Z}/m\mathbb{Z})^\times \times (\mathbb{Z}/m\mathbb{Z})^{\mathbb{N}}$ *endowed with its product topology. (Here we identify* $\mathbb{Z}/m\mathbb{Z}$ *with the set* $\{0, \dots, |m|-1\}$ *and* $(\mathbb{Z}/m\mathbb{Z})^\times$ *with the set of integers* $k \in \{0, \dots, |m|-1\}$ *coprime with* m.)

Proof. (i) Proposition 2.5 shows that the functions r_i, s_i are uniformly continuous with respect to the m-adic topology on $\mathbb{Z}$ and that r_i is moreover constant on m-adic balls of radius $|m|^{-i}$. The existence and uniqueness of the extensions to $\mathbb{Z}_m$ follow. The inductive formulas of Definition 2.4 extend continuously on $\mathbb{Z}_m$, which gives (1_ξ) and (2_ξ). Let us now pick $n, n' \in \mathbb{Z}$ such that $n \equiv \xi$ and $n' \equiv \xi' \pmod{m^h\mathbb{Z}_m}$. Then, one has $n \equiv n' \pmod{\widehat{m}^h d)\mathbb{Z}$ if and only if $\xi \equiv \xi' \pmod{\widehat{m}^h d\mathbb{Z}_m}$. As r_i is constant on m-adic balls of radius $|m|^{-i}$ we have $r_i(n) = r_i(\xi)$ and $r_i(n') = r_i(\xi')$ for $i = 1, \dots, h$. The equivalence between (1) and (2) now follows readily from Proposition 2.5.

(ii) As $\gcd(m, \xi) = \gcd(m, r_1(\xi))$ for every $\xi \in \mathbb{Z}_m$ by (2_ξ), the result immediately follows from (i).

(iii) Let $\xi \in \mathbb{Z}_m^\times$. By ($2_\xi$), we have $r_1(\xi) \in (\mathbb{Z}/m\mathbb{Z})^\times$. For any $\xi, \xi' \in \mathbb{Z}_m^\times$, we have $d = \gcd(\xi, m) = \gcd(\xi', m) = 1$. We deduce from (2) that the map $R\colon \xi \mapsto (r_i(\xi))_{i\geq 1}$ defines a continuous embedding from $\mathbb{Z}_m^\times$ into $P := (\mathbb{Z}/m\mathbb{Z})^\times \times (\mathbb{Z}/m\mathbb{Z})^{\mathbb{N}}$. As $\mathbb{Z}_m^\times$ and P are compact, it suffices to show that R has a dense image. Let E_h be the set of integers $k \in \{0, \dots, |m|^h - 1\}$ coprime with m. Consider the map $R_h\colon \xi \mapsto (r_i(\xi))_{1\leq i\leq h}$. This is an injective map from E_h into $P_h := (\mathbb{Z}/m\mathbb{Z})^\times \times (\mathbb{Z}/m\mathbb{Z})^{h-1}$ by the equivalence of (1) and (2). As E_h and P_h have both cardinality $\varphi(m)|m|^{h-1}$, where φ denotes the Euler function, the map R_h is a bijection. Hence R maps the set of integers coprime with $|m|$ onto a dense subset of P. □

The following proposition shows that the restriction of s_i to a suitable congruence class is a polynomial in n with coefficients in $\mathbb{Q}$.

Proposition 2.7. *Let* $\xi \in \mathbb{Z}_m$. *We define recursively* $P_h(X) = P_{h,\xi}(X)$ *by*

$$P_0(X) = m \quad \text{and} \quad P_h(X) = XP_{h-1}(X) - r_h(\xi) \quad \text{for } h \geq 1.$$

Then we have:

(i) $P_h(X) = mX^h - r_1(\xi)X^{h-1} - \cdots - r_h(\xi) \in \mathbb{Z}[X]$.

(ii) $s_h(n) = \frac{1}{m} P_h(\frac{n}{m})$ *for all* $h \geq 0$ *and all* $n \in \mathbb{Z}$ *such that* $n \equiv \xi \pmod{\widehat{m}^h d\mathbb{Z}_m}$.

Proof. (i) The proof is an obvious induction on h.

(ii) First, observe that $r_i(n) = r_i(\xi)$ for all $i \leq h$ and all $n \in \mathbb{Z}$ such that $n \equiv \xi \pmod{\widehat{m}^h d\mathbb{Z}_m}$ by Proposition 2.6. An easy induction on h using the definitions of $s_i(n)$ and $P_i(X)$ gives the conclusion. □

2.2. Stabilizers. This section is devoted to the study of the stabilizers B, B_m and B_ξ.

Let $b_0 = b$, $b_1 = ab^m a^{-1}$, $b_i = ab_{i-1}b_0^{-r_{i-1}(\xi)}a^{-1}$ for $i \geq 2$. The following lemma shows that b_i defines an element of B for every i.

Lemma 2.8. *Let* $i \geq 1$. *We have* $b_i = b^{\frac{\xi_n}{m} P_{i-1}(\frac{\xi_n}{m})}$ *in* $\mathrm{BS}(m, \xi_n)$ *for all* n *large enough.*

Proof. We show by induction on i that:

$$\text{for every } i \geq 1 \text{ we have } b_i = b^{\xi_n s_{i-1}(\xi_n)} \text{ in } \mathrm{BS}(m, \xi_n) \text{ for all } n \text{ large enough.} \quad (3)$$

Since $b_1 = ab^m a^{-1} = b^{\xi_n}$ in $\mathrm{BS}(m, \xi_n)$ and $s_0(\xi_n) = 1$ for all n, (3) holds if $i = 1$. Assume (3) holds for some $i \geq 1$. By the induction hypothesis we have $b_{i+1} = ab_i b^{-r_i(\xi)} a^{-1} = ab^{\xi_n s_{i-1}(\xi_n) - r_i(\xi)} a^{-1}$ for all n large enough. Recall that ξ_n tends to ξ in $\mathbb{Z}_m$ and hence $r_i(\xi_n) = r_i(\xi)$ for all n large enough by Proposition 2.6. By Definition 2.4, we obtain $b_{i+1} = b^{\xi_n s_i(\xi_n)}$ in $\mathrm{BS}(m, \xi_n)$ for all n large enough. Since $\xi_n s_i(\xi_n) = \frac{\xi_n}{m} P_i(\frac{\xi_n}{m})$ for all n large enough by Proposition 2.7, the proof is then complete. □

We can generalize the previous lemma by assigning to every $\gamma \in B$ a polynomial with integer coefficients.

Proposition 2.9. (i) *Let* $w \in \mathbb{F}_2$ *with image* $\gamma \in B$. *Then there is a unique polynomial* $P_\gamma(X) \in \mathbb{Z}[X]$ *independent of* w *such that* $w = b^{P_\gamma(\xi_n/m)}$ *in* $\mathrm{BS}(m, \xi_n)$ *for all* n *large enough.*

(ii) *The map* $q\colon \gamma \mapsto P_\gamma(X)$ *defines an injective homomorphism from* B *into* $\mathbb{Z}[X]$. *The abelian group* $\mathfrak{B} = q(B)$ *is freely generated by* $\{1, XP_0(X), XP_1(X), XP_2(X), \ldots\}$. *Hence* B *is freely generated by the set* $\{b_0, b_1, b_2, b_3, \ldots\}$.

(iii) *Let* $w \in \mathbb{F}_2$ *with image* $\gamma \in B$. *Then* γ *belongs to* B_m *if and only if*

$$P_\gamma(X) = k_0 + k_1 mX + k_2 XP_1(X) + \cdots + k_t XP_{t-1}(X)$$

with $k_0 + k_1 r_1(\xi) + k_2 r_2(\xi) + \cdots + k_t r_t(\xi) \equiv 0 \pmod{m\mathbb{Z}}$. *Moreover the abelian group* $\mathfrak{B}_m = q(B_m)$ *is freely generated by* $P_0(X) = m$, $P_1(X) = XP_0(X) - r_1(\xi)$, $P_2(X) = XP_1(X) - r_2(\xi)$, ... *Hence* B_m *is freely generated by* $\{b_0^m, b_1 b_0^{-r_1(\xi)}, b_2 b_0^{-r_2(\xi)}, \ldots\}$.

(iv) *The abelian group* $\mathfrak{B}_\xi = q(B_\xi)$ *is freely generated by* $\{XP_0(X), XP_1(X), XP_2(X), \dots\}$. *The abelian group* B_ξ *is freely generated by* $\{b_1, b_2, b_3, \dots\}$.

(v) *For any* $\gamma \in B_m$, *we have* $q(a\gamma a^{-1}) = XP_\gamma(X)$. *Moreover the map* $a \mapsto (0,1)$, $b \mapsto (1,0)$ *induces a surjective homomorphism* $q_{m,\xi} \colon \overline{\mathrm{BS}}(m,\xi) \to \mathbb{Z} \wr \mathbb{Z} = \mathbb{Z}[X^{\pm 1}] \rtimes \mathbb{Z}$ *whose restriction to* B *coincides with* q.

(vi) *Let* $d = \gcd(m,\xi)$, $\hat{m} = m/d$, *and let* $\pi \colon \mathbb{Z} \to \mathbb{Z}_{\hat{m}}$ *be the canonical map. The maps* $\chi \colon \gamma \mapsto P_\gamma(\xi/m)$ *and* $\hat{\chi} = \pi \circ \chi$ *define homomorphisms from* B *to* $\mathbb{Z}_m$ *and* $\mathbb{Z}_{\hat{m}}$, *respectively. Their kernels satisfy:* $\ker \chi \subset \bigcap_{i \geq 0} a^{-i} B a^i \subset \ker \hat{\chi}$. *Moreover, if* $\bigcap_{i\geq 0} a^{-i} B a^i$ *is non-trivial, then it contains some* $\gamma \in B_m \setminus B_\xi$.

(vii) *Let* $\mathfrak{C}$ *be the image of* $\mathfrak{B}_\xi$ *by the map* $\iota \colon P(X) \mapsto \frac{X-1}{X} P(X)$. *Then* $\mathfrak{C}$ *is a subgroup of* $\mathfrak{B}$ *and* $\mathfrak{B}/\mathfrak{C}$ *is an infinite cyclic group generated by the image of* 1.

Proof. (i) The uniqueness of $P_\gamma(X)$ follows from the fact that a non-zero polynomial with coefficients in $\mathbb{Q}$ has only finitely many zeros in $\mathbb{Q}$. Using the fact that $\mathrm{BS}(m,\xi_n)$ converges to $\overline{\mathrm{BS}}(m,\xi)$, we also deduce that $P_\gamma(X)$ is independent of w. To show the existence of $P_\gamma(X)$ we write $w = b^{e_0} a^{\varepsilon_1} b^{e_1} \dots a^{\varepsilon_h} b^{e_h}$ with $\varepsilon_j = \pm 1$ for $j = 1, \dots, h$. By Lemma 2.2, w reduces to a power of b in $\mathrm{BS}(m,\xi_n)$ for all n large enough. Since the powers of a must eventually cancel out, we have $h = 2t$ for some $t \in \mathbb{N}$. By [GS08], Lemma 2.6 with $d = 1$, there exist $k_0, \dots, k_t \in \mathbb{Z}$ depending only on w and ξ such that $w = b^{\alpha(n)}$ in $\mathrm{BS}(m,\xi_n)$ for all n such that $\xi_n \equiv \xi \pmod{m^t \mathbb{Z}_m}$ with $|\xi_n|$ large enough and

$$\alpha(n) = k_0 + k_1 \xi_n + k_2 s_1(\xi_n)\xi_n + \dots + k_t s_{t-1}(\xi_n)\xi_n, \tag{4}$$

the latter equation being Formula (*) in the proof of [GS08], Lemma 2.6. Since ξ_n tends to ξ in $\mathbb{Z}_m$ and $|\xi_n|$ tends to infinity as n goes to infinity, the equality (4) holds for all n large enough. Using Proposition 2.7, we can write $\alpha(n) = P_\gamma(\frac{\xi_n}{m})$ with

$$P_\gamma(X) = k_0 + k_1 X P_0(X) + k_2 X P_1(X) + \dots + k_t X P_{t-1}(X). \tag{5}$$

(ii) The map $q \colon \gamma \mapsto P_\gamma(X)$ trivially induces a homomorphism from B to $\mathbb{Z}[X]$ such that $q(b) = 1$. Let $w \in \mathbb{F}_2$ with image $\gamma \in B$. If $P_\gamma(X)$ is the zero polynomial, then w is trivial in $\mathrm{BS}(m,\xi_n)$ for all n large enough and hence it is trivial in $\overline{\mathrm{BS}}(m,\xi)$. Thus q is injective. It is immediate from (5) that $q(B)$ is a subgroup of the free abelian group with basis $\{1, XP_0(X), XP_1(X), \dots\}$. It follows from Lemma 2.8 that $q(b_i) = XP_{i-1}(X)$ for all $i \geq 1$, so that $q(B)$ coincides with the latter group.

(iii) Let $w \in \mathbb{F}_2$ with image $\gamma \in B$. By (i) we can write $w = b^{P_\gamma(\xi_n/m)}$ in $\mathrm{BS}(m,\xi_n)$ for all n large enough. Since $\xi_n s_{i-1}(\xi_n) \equiv r_i(\xi) \pmod{m\mathbb{Z}}$ for every $i \geq 1$ and for all n large enough, we deduce from (5) that $P_\gamma(\xi_n/m) \equiv k_0 + k_1 r_1(\xi) + k_2 r_2(\xi) + \dots + k_t r_t(\xi) \pmod{m\mathbb{Z}}$ for all n large enough. By Lemma 2.2, we have: $\gamma \in B_m$ if and only if $\alpha(n) \equiv 0 \pmod{m\mathbb{Z}}$. This proves the first claim. We easily deduce that $\{m, XP_0(X) - r_1(\xi), XP_1(X) - r_2(\xi), \dots\}$ freely generates $\mathfrak{B}_m$. Since $q(b_0^m) = m$ and $q(b_i b_0^{-r_i(\xi)}) = XP_{i-1}(X) - r_i(\xi)$ the set $\{b_0^m, b_1 b_0^{-r_1(\xi)}, b_2 b_0^{-r_2(\xi)}, \dots\}$ freely generates B_m.

(iv) Since $B_\xi = aB_m a^{-1}$, we deduce from (iii) and the definition of b_i that $\{b_1, b_2, \dots\}$ generates B_ξ. As the elements $q(b_i) = XP_{i-1}(X)$ ($i \geq 1$) freely generate $\mathfrak{B}_\xi$, we deduce that $\{b_1, b_2, \dots\}$ freely generates B_ξ.

(v) Let $w \in \mathbb{F}_2$ with image $\gamma \in B_m$. We can write $w = b^{P_\gamma(\xi_n/m)}$ in $\mathrm{BS}(m, \xi_n)$ for all n large enough with $P_\gamma(\xi_n/m) \equiv 0 \pmod{m\mathbb{Z}}$. Thus $a\gamma a^{-1} = b^{(\xi_n/m)P_\gamma(\xi_n/m)}$ in $\mathrm{BS}(m, \xi_n)$ for all n large enough. Hence $q(a\gamma a^{-1}) = XP_\gamma(X)$ by definition of q.

The map $q_{m,\xi}$ is a well-defined homomorphism by [GS08], Theorem 3.12. Using the first part of (v), we easily deduce by induction that $q(b_i) = q_{m,\xi}(b_i)$ for every $i \geq 0$, which completes the proof.

(vi) The map $\gamma \mapsto P_\gamma(\xi/m)$ is a well-defined homomorphism from B to $\mathbb{Z}^{-1}\mathbb{Z}_m$. By Proposition 2.6 we have $s_i(\xi) \in \mathbb{Z}_m$ for every i. By Proposition 2.7, we have $\frac{1}{m}P_i(\frac{\xi_n}{m}) = s_i(\xi_n) \in \mathbb{Z}$ for every n large enough and hence $\frac{1}{m}P_i(\frac{\xi}{m}) = s_i(\xi) \in \mathbb{Z}_m$ by Proposition 2.6. Now it follows from (ii) that $P_\gamma(\xi/m)$ belongs to $\mathbb{Z}_m$ for every $\gamma \in B$.

Let $\gamma \in \ker \chi$ and let $i \geq 0$. For all n large enough m^i divides $P_\gamma(\xi_n/m)$. We deduce from (i) and Lemma 2.2 that $a^i\gamma a^{-i} \in B$. Thus $\gamma \in \bigcap_{i\geq 0} a^{-i}Ba^i$.

Let $\eta = \pi(\xi/d)$. By (v) we have $\hat{\chi}(a\gamma a^{-1}) = \frac{\eta}{\widehat{m}}\hat{\chi}(\gamma) \in \mathbb{Z}_{\widehat{m}}$ for every $\gamma \in B \cap a^{-1}Ba$. Consider now $\gamma \in B$ such that $a^i\gamma a^{-i} \in B$ for every i. As $\gcd(\widehat{m}, \eta) = 1$, $\widehat{m}^i$ divides $\hat{\chi}(\gamma)$ for every i. Therefore we have $\gamma \in \ker \hat{\chi}$.

Assume that $B' = \bigcap_{i\geq 0} a^{-i}Ba^i$ is non-trivial. Clearly $B' \subset B_m$ and any $\gamma \in B'$ such that $P_\gamma(X)$ has minimal degree does not belong to B_ξ by (v).

(vii) It follows from the definition of $P_i(X)$ that multiplication by $\frac{X-1}{X}$ maps $XP_0(X)$ to $-m + XP_0(X)$ and $XP_i(X)$ to $r_i(\xi) - XP_{i-1}(X) + XP_i(X)$ for every $i \geq 1$. Therefore $\mathfrak{C}$ is a subgroup of $\mathfrak{B}$. A straightforward induction on i shows that the image of $XP_i(X)$ in $\mathfrak{B}/\mathfrak{C}$ lies inside the cyclic subgroup generated by the image of 1. As $\mathfrak{B}_\xi$ does not contain any constant polynomial, neither does $\mathfrak{C}$. Therefore $\mathfrak{B}/\mathfrak{C}$ is infinite. □

Now we can give a simple description of the HNN structure of $\overline{\mathrm{BS}}(m, \xi)$. Let $\widetilde{\mathrm{BS}}(m, \xi)$ be the HNN extension of basis E with conjugated subgroups $E_{m,\xi}$ and E_1 stable letter a, where E, E_m, E_ξ are free abelian groups of countable rank, namely

$$\begin{aligned} E &= \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots, \\ E_{m,\xi} &= \mathbb{Z}me_0 \oplus \mathbb{Z}(e_1 - r_1(\xi)e_0) \oplus \mathbb{Z}(e_2 - r_2(\xi)e_0) \oplus \cdots, \\ E_1 &= \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots, \end{aligned}$$

and where conjugacy from $E_{m,\xi}$ to E_1 is defined by $a(me_0)a^{-1} = e_1$ and $a(e_i - r_i(\xi)e_0)a^{-1} = e_{i+1}$. We denote by ϕ the isomorphism from $E_{m,\xi}$ to E_1 induced by τ_a.

Corollary 2.10. *The map defined by $f(a) = a$ and $f(b) = e_0$ induces an isomorphism from $\overline{\mathrm{BS}}(m, \xi)$ to $\widetilde{\mathrm{BS}}(m, \xi)$. The inverse map is determined by $f^{-1}(a) = a$ and $f^{-1}(e_i) = b_i$ for every $i \geq 0$.*

Proof. This is a straightforward corollary of Propositions 2.1 and 2.9. □

We then have three different notations for the same group, namely

- $\overline{\mathrm{BS}}(m,\xi) = \mathrm{HNN}(B, B_m, B_\xi, \tau)$ where B is generated by $b = b_0, b_1, b_2, \dots$,
- $\widetilde{\mathrm{BS}}(m,\xi) = \mathrm{HNN}(E, E_{m,\xi}, E_1, \phi)$ where E is generated by $e_0, e_1, e_2, \dots$,
- $\mathfrak{BS}(m,\xi) = \mathrm{HNN}(\mathfrak{B}, \mathfrak{B}_m, \mathfrak{B}_\xi, \theta)$ where $\mathfrak{B}$ is generated by 1, $XP_0(X)$, $XP_1(X)$, ... and θ is defined in the obvious way.

Our favoured HNN extension is $\widetilde{\mathrm{BS}}(m,\xi)$ since it considerably simplifies notations when different parameters m and ξ are considered (Section 5 and Section 6). Nevertheless $\overline{\mathrm{BS}}(m,\xi)$ is useful when we still need to see this group as limit in the space of marked groups (Section 5) and the HNN extension $\mathfrak{BS}(m,\xi)$ involving polynomials is advantageous to study relations in the group (Section 5).

3. An infinite presentation built up from ξ

In this section we use of the HNN decomposition of $\overline{\mathrm{BS}}(m,\xi)$ to give an infinite group presentation which depends explicitly on m and the sequence $(r_i(\xi))$. A group presentation $\langle X \mid R\rangle$ is said to be *minimal* if the kernel of the natural homomorphism $\langle X \mid R'\rangle \to \langle X \mid R\rangle$ is non-trivial for all $R' \subsetneq R$.

Theorem 3.1. *The group $\overline{\mathrm{BS}}(m,\xi)$ admits the following infinite presentation:*

$$\langle a, b \mid [b, b_i] = 1,\ i \geq 1\rangle$$

with $b_1 = ab^m a^{-1}$ and $b_i = ab_{i-1}b^{-r_{i-1}(\xi)}a^{-1}$ for every $i \geq 2$.

In particular, we have $\overline{\mathrm{BS}}(m,\xi)(m,0) = \langle a, b \mid [b, a^i b^m a^{-i}] = 1,\ i \geq 1\rangle$. The latter presentation is moreover minimal.

In the case $\xi \neq 0$, we previously showed ([GS08], Theorem 4.1) that $\overline{\mathrm{BS}}(m,\xi)$ cannot be finitely presented. The following consequence is therefore immediate.

Corollary 3.2. *No group $\overline{\mathrm{BS}}(m,\xi)$ is finitely presented.*

Proof of Theorem 3.1. We divide the proof into two parts:

(1) We show that $\overline{\mathrm{BS}}(m,\xi) = \langle a, b \mid [b, b_i] = 1,\ i \geq 1\rangle$ using Propositions 2.1 and 2.9.

(2) We show that this presentation is minimal in the case $\xi = 0$ using basic facts on graph groups.

Proof of (1). By Proposition 2.9 we can define an isomorphism $\phi\colon B_m \to B_\xi$ by $\phi(b_0^m) = b_1$, $\phi(b_i b_0^{-dr_i(\xi)}) = b_{i+1}$ for $i \geq 1$. Making use of the HNN decomposition of Proposition 2.1, we deduce that $\overline{\mathrm{BS}}(m,\xi)$ admits the presentation

$\langle a, B \mid a\gamma a^{-1} = \phi(\gamma),\ \gamma \in B_m\rangle$. Since a and b generate $\overline{\mathrm{BS}}(m,\xi)$ and B is a free abelian group (Proposition 2.9), we deduce that

$$\overline{\mathrm{BS}}(m,\xi) = \langle a, b \mid [b_i, b_j] = 1,\ 0 \leq i < j\rangle. \tag{6}$$

(Note that the relations $ab_0^m a^{-1} = \phi(b_0^m)$ and $ab_i b_0^{-r_i(\xi)} a^{-1} = \phi(b_i b_0^{-r_i(\xi)})$ are satisfied by definition of b_i.) We show by induction on $i \geq 0$ the following claim: any relation $[b_i, b_j] = 1$ with $0 \leq i < j$ is a consequence of relations $[b_0, b_k] = 1$ with $1 \leq k \leq j$. The case $i = 0$ is trivial. Assume now that $i \geq 1$. Using definitions, we can write $[b_i, b_j] = a[b_{i-1}b_0^{-r_{i-1}(\xi)}, b_{j-1}b_0^{-r_{j-1}(\xi)}]a^{-1}$. By induction hypothesis the relation $[b_{i-1}, b_{j-1}] = 1$ is a consequence of relations $[b_0, b_k] = 1$ with $1 \leq k \leq j-1$. Hence $[b_i, b_j] = 1$ is a consequence of relations $[b_0, b_k] = 1$ with $1 \leq k \leq j$. The induction is then complete. Thus all relations $[b_i, b_j] = 1$ with $1 \leq i < j$ can be deleted in presentation (6).

Proof of (2). It remains to show that $\overline{\mathrm{BS}}(m,\xi) = \langle a, b \mid [b_0, b_i] = 1,\ i \geq 1\rangle$ is a minimal presentation when $\xi = \pi(\xi) = 0$. In this case we have $r_i(\xi) = 0$, and hence $b_i = a^i b^m a^{-i}$ for all $i \geq 1$. We fix some $k \geq 1$ and we show that $[b_0, b_k] = 1$ does not hold in the group $G_k := \langle a, b \mid [b_0, b_i] = 1,\ k \neq i \geq 1\rangle$.

To prove this we consider the group $\widetilde{B} = \langle g_0, g_1, \ldots \mid [g_i, g_j] = 1,\ |i - j| \neq k\rangle$ and its subgroups $\widetilde{B}_m = \langle g_0^m, g_1, g_2, \ldots\rangle$ and $\widetilde{B}_\xi = \langle g_1, g_2, \ldots\rangle$. We will use basic facts on *graph groups*, i.e., groups defined by a presentation whose relators are commutators of some pairs of the generators. (Such presentations are often encoded by a graph whose vertices correspond to the generators and whose edges tell which ones commute; this explains the terminology.) Graph groups are also referred to as right-angled Artin groups or partially commutative groups. Let $G = \langle X \mid R\rangle$ be a graph group. Any element of G can be written as a word $c_1 c_2 \ldots c_l$ where each *syllable* c_i belongs to some cyclic group generated by some element of X. We consider three types of moves that we can perform on such words.

1. Remove a syllable $c_i = 1$.
2. Replace consecutive syllables c_i and c_{i+1} in the same cyclic subgroup by the single syllable $(c_i c_{i+1})$.
3. For consecutive syllables $c_i \in \langle x\rangle$ and $c_{i+1} \in \langle x'\rangle$ with $x, x' \in X$, $[x, x'] \in R$, exchange c_i and c_{i+1}.

If $g \in G$ is represented by a word $w = c_1 \ldots c_l$ which cannot be changed to a shorter word using any sequence of the above moves, then w is said to be a *normal form* for g. We will use the following results:

Theorem 3.3 ([Bau81]). (Normal Form Theorem) *A normal form in a graph group represents the trivial element if and only if it is the trivial word.*

(Abelian Subgroups) *Any abelian subgroup of a graph group is a free abelian group.*

We define the partial map $\psi\colon \widetilde{B}_\xi \to \widetilde{B}_m$ by $\psi(g_1) = g_0^m$ and $\psi(g_i) = g_{i-1}$ for every $i \geq 2$. By using the Normal Form Theorem, we can readily show that $\widetilde{B}_\xi = \langle g_1, g_2, \ldots \mid [g_i, g_j] = 1, |i-j| \neq k\rangle$. We clearly have $[\psi(g_i), \psi(g_j)] = 1$ in $\widetilde{B}_m$ for any $i, j \geq 1$ such that $|i-j| \neq k$. Hence ψ induces a surjective homomorphism from $\widetilde{B}_\xi$ onto $\widetilde{B}_m$.

Let $g \in \widetilde{B}_\xi$. Replacing every g_i by $\psi(g_i)$ in any non-trivial normal form for g in $\widetilde{B}$ clearly leads to a non-trivial normal form for $\psi(g)$ in $\widetilde{B}$. Hence ψ is injective, which proves that ψ is an isomorphism. Let $\widetilde{\phi}$ be its inverse homomorphism. We set $\widetilde{G}_k := \langle \widetilde{B}, a \mid aga^{-1} = \widetilde{\phi}(g), g \in \widetilde{B}_m\rangle$. We trivially check that the map $a \mapsto a$, $b \mapsto g_0$ induces a surjective homomorphism from G_k onto $\widetilde{G}_k$ that maps $[b_0, b_k]$ to $[g_0, g_k]$. Since the graph group $\widetilde{B}$ embeds isomorphically into the HNN extension $\widetilde{G}_k$, we have $[g_0, g_k] \neq 1$ and hence $[b_0, b_k] \neq 1$. ☐

4. C*-simplicity

We first recall some definitions. Let $\Gamma = \overline{\mathrm{BS}}(m,\xi)$ and let $\sigma\colon \overline{\mathrm{BS}}(m,\xi) \to \mathbb{Z}$ be the surjective homomorphism defined by $\sigma(a) = 1$ and $\sigma(b) = 0$. Let $\mathbb{Z} \wr \mathbb{Z} = \mathbb{Z}[X^{\pm 1}] \rtimes_X \mathbb{Z}$ where the action of $\mathbb{Z}$ on the additive group of $\mathbb{Z}[X^{\pm 1}]$ is the multiplication by X. This group is generated by $\{(1,0),(0,1)\}$ and the map $a \mapsto (0,1)$, $b \mapsto (1,0)$ induces a surjective homomorphism $q_{m,\xi}$ from Γ onto $\mathbb{Z} \wr \mathbb{Z}$ [GS08], Theorem 3.12. By (v) and (vi) of Proposition 2.9, the restriction of $q_{m,\xi}$ to B (which identifies with a subgroup of $\mathbb{Z}[X]$) is the identity and the map $\gamma \mapsto P_\gamma(\xi/m)$ is a well-defined homomorphism from B to the additive group of $\mathbb{Z}_m$.

Given a group Γ, recall that its *reduced C*-algebra* $C_r^*(\Gamma)$ is the closure for the operator norm of the group algebra $\mathbb{C}[\Gamma]$ acting by the left-regular representation on the Hilbert space $\ell^2(\Gamma)$. For an introduction to group C*-algebras, see for example [Dav96], Chapter VII. A group is C*-simple if it is infinite and if its reduced C*-algebra is a simple topological algebra.

Non-abelian free groups are C*-simple. The first proof of this fact, due to Powers [Pow75], relies on a combinatorial property of free groups shared by many other groups, called for this reason *Powers groups*. Thus Powers groups are C*-simple. The Baumslag–Solitar group $\mathrm{BS}(m,n)$ is C*-simple if and only if $|m| \neq |n|$ [Iva07], Theorem 4.9. In this case, it is actually a strongly Powers group [HP11], Theorem 3.iii with $|m|, |n| \geq 2$. A group G is said to be *strongly Powers* if any of its subnormal subgroups is a Powers group. A subgroup N of G is a *subnormal subgroup* of G if there are subgroups N_i $(1 \leq i \leq n < \infty)$ of G such that $N \subset N_1 \subset \cdots \subset N_{n-1} \subset N_n = G$ and N_i is normal in N_{i+1} for every i.

Theorem 4.1. *Let $|m| > 1$, $\xi \in \mathbb{Z}_m$. Then $\overline{\mathrm{BS}}(m,\xi)$ is a strongly Powers group.*

Amenable groups are not C*-simple, see [Har07], Proposition 3. This is the reason why we have to exclude the groups $\overline{\mathrm{BS}}(m,\xi)$ with $|m| = 1$. Actually, there is only one such marked group [GS08], Theorem 2.1, and it is isomorphic to the solvable group $\mathbb{Z} \wr \mathbb{Z}$ [Sta06a], Theorem 2.

We consider the action of $\Gamma = \overline{\mathrm{BS}}(m, \xi)$ on its Bass–Serre tree $T = X_{m,\xi}$ and use the criterion of de la Harpe and Préaux [HP11], Corollary 15, on tree action to show Theorem 4.1. The latter criterion needs the action of Γ to be faithful in a strong sense: it has to be *slender*. Two other conditions are required, namely minimality and strong hyperbolicity, but both follow immediately from [HP11], Proposition 20, as Γ is a non-ascending HNN extension. Let us define a slender action. A tree automorphism γ of T is *slender* if its fixed point set $(\partial T)^{\gamma}$ has empty interior in ∂T with respect to the shadow topology (Section 1.3). The action of Γ on T is slender if for every $\gamma \in \Gamma \setminus \{1\}$ the automorphism of T induced by γ, also denoted by γ, is slender. A slender action is faithful, it is even strongly faithful in the sense of [HP11], Section 1.

Since hyperbolic elements are obviously slender, we focus now on the fixed point set of elliptic elements. We still need more definitions to describe the fixed point set T^{γ} for any $\gamma \in B$.

Let $l, u \in \mathbb{Z} \cup \{\pm\infty\}$. We denote by $\{l \le \sigma \le u\}$ the subgraph of T whose vertices γB satisfy $l \le \sigma(\gamma) \le u$. We denote by $T[l, u]$ the connected component of $1B$ in $\{l \le \sigma \le u\}$.

Lemma 4.2. *Let $l, u \in \mathbb{Z} \cup \{\pm\infty\}$. Assume either that $l > -\infty$ and $|m| > 1$, or that $u < +\infty$. Then, the set $\partial\{l \le \sigma \le u\} \subset \partial T$ has empty interior in ∂T with respect to the shadow topology.*

Proof. Assume that $l > -\infty$ and $|m| > 1$. Since any vertex γB of T has $|m|$ neighbours $\gamma' B$ such that $\sigma(\gamma') = \sigma(\gamma) - 1$, any shadow contains (the class of) a geodesic ray $(\gamma_1 B, \gamma_2 B, \dots, \gamma_n B, \dots)$ such that $\sigma(\gamma_n)$ tends to $-\infty$. Such a ray does not lie in $\partial\{l \le \sigma \le u\}$. The proof of the second case is analogous (any vertex γB of T has countably many neighbours $\gamma' B$ such that $\sigma(\gamma') = \sigma(\gamma) + 1$). □

Let ν be the natural valuation on $\mathbb{Z}[X]$, i.e., the one defined by $\nu(X^i) = i$ for every i. Let ν be the "valuation" defined on B by $\mu(\gamma) = \sup\{i \ge 1 \mid P_{\gamma}(\xi/m) \in d\widehat{m}^i \mathbb{Z}_m\}$ where $\sup \emptyset = 0$. Let $\gamma \in B$. Observe that

- $\gamma \in B_m$ if and only if $\mu(\gamma) \ge 1$,
- $\gamma \in B_{\xi}$ if and only if $\nu(P_{\gamma}) \ge 1$.

Proposition 4.3. *Let $\gamma \in B$. We have $T^{\gamma} = T[-\mu(\gamma), \nu(P_{\gamma})]$.*

Corollary 4.4. *Let $\Gamma = \overline{\mathrm{BS}}(m, \xi)$.*

- *The action of Γ on T is slender.*
- *The centralizer of b in Γ coincides with B if $|m| > 1$.*
- *The centralizer of a^k in Γ coincides with $\langle a \rangle$ for every $k \neq 0$.*

Proof. Let us show that the action of Γ is slender. Remember that any hyperbolic element of Γ is slender. Since any elliptic element is conjugated to some $\gamma \in B$ by a homeomorphism of ∂T, it suffices to prove that every $\gamma \in B \setminus \{1\}$ is slender. Since $\nu(P_\gamma) < \infty$ for every $\gamma \in B \setminus \{1\}$, the result follows from Proposition 4.3 and Lemma 4.2.

Assume that $|m| > 1$ and let γ be an element of the centralizer of b in Γ. By Proposition 4.3, we have $T^b = 1B$. Since γ commutes with b, we have then $\gamma \cdot 1B = 1B$, i.e., $\gamma \in B$.

Let $k \in \mathbb{Z} \setminus \{0\}$ and let γ be an element of the centralizer of a^k in Γ. Let a_+ (resp. a_-) be the class of rays which are cofinal with $(a^n B)_{n \geq 0}$ (resp. $(a^n B)_{n \leq 0}$). Then γ preserves the fixed point set $\{a_-, a_+\}$ of a^k. Since $\sigma(\gamma a^n) = \sigma(\gamma) + n$ for every $n \in \mathbb{Z}$, γ cannot exchange a_+ and a_- and hence fixes them both. Consequently, there is some $n \in \mathbb{Z}$ such that $\gamma \cdot B = a^n B$, i.e., there is $g \in B$ such that $\gamma = a^n g$. We deduce that g centralizes a^k and hence g fixes a_+ and a_-. As g is elliptic, Proposition 4.3 gives $\nu(P_g) = +\infty$, hence $g = 1$. Thus $\gamma = a^n$. □

Proposition 4.3 is a straightforward consequence of the following lemma.

Lemma 4.5. *Let $g \in \Gamma$ and let $c_0 a^{\epsilon_1} c_1 \dots a^{\epsilon_k} c_k$ with $\epsilon_i \in \{\pm 1\}$, $c_i \in B$, be a reduced form of g. Let $\sigma_i(g) = \sum_{j=1}^{i} \epsilon_j$. Let $\gamma \in B$. The following conditions are equivalent:*

(i) $gB \in T^\gamma$,

(ii) $g^{-1}\gamma g \in B$,

(iii) $-\mu(\gamma) \leq \sigma_i(g) \leq \nu(P_\gamma)$ *for every* $1 \leq i \leq k$.

If the previous conditions hold, then we have $g^{-1}\gamma g = a^{-\sigma(g)}\gamma a^{\sigma(g)}$.

Proof. (i) $\iff$ (ii) is trivial. We show the equivalence (ii) $\iff$ (iii) and the last statement of the lemma by induction on k. If $k = 0$, both are trivial. Assume that $k > 0$ and write $w = c_0 a^{\epsilon_1} g'$.

(ii) $\implies$ (iii): We have $g'^{-1}\gamma' g' \in B$ with $\gamma' = a^{-\epsilon_1}\gamma a^{\epsilon_1}$. By Britton's lemma, we have $g'^{-1}\gamma' g' \in B$ and, either $\gamma \in B_m$ and $\epsilon_1 = -1$, or $\gamma \in B_\xi$ and $\epsilon_1 = 1$. We deduce that $\gamma' \in B$, $\mu(\gamma') = \mu(\gamma) + \epsilon_1$, $\nu(P_{\gamma'}) = \nu(P_\gamma) - \epsilon_1$, $\gamma' \in B$ and $-\mu(\gamma) \leq \epsilon_1 \leq \nu(P_\gamma)$. The result then follows from the induction hypothesis.

(iii) $\implies$ (ii): Since $-\mu(\gamma) \leq \epsilon_1 \leq \nu(P_\gamma)$, we have either $\epsilon_1 = -1$ and hence $\gamma \in B_m$, or $\epsilon_1 = 1$ and hence $\gamma \in B_\xi$. We deduce that $\gamma' \in B$. The result follows from the induction hypothesis.

Last statement: By induction hypothesis, we have

$$g'^{-1}\gamma' g' = a^{-\sigma(g')}\gamma' a^{\sigma(g')} = a^{-\epsilon_1 - \sigma(g')}\gamma a^{\epsilon_1 + \sigma(g')} = a^{-\sigma(g)}\gamma a^{\sigma(g)}. \qquad \square$$

Proof of Theorem 4.1. Since $\Gamma = \overline{\mathrm{BS}}(m, \xi)$ is a non-ascending HNN extension, the action of Γ is strongly hyperbolic on T and minimal on ∂T [HP11], Proposition 20. The action of Γ on T is slender by Corollary 4.4. By [HP11], Corollary 15, Γ is a strongly Powers group. □

Inner amenability. A countable group G is said *inner amenable* if it admits a *mean* (i.e., a non-zero, finite and finitely additive measure) on the set of all the subsets of $G \setminus \{1\}$, which is invariant under inner automorphisms. We say that G has the *icc* property if the conjugacy class of any of its non-trivial element is infinite. These two notions are motivated by the study of the von Neumann algebra $W^*(G)$ of G (see [Eff75], [BH86]). Amenable groups or groups that do not have icc are clearly inner amenable. The second-named author has proved that the Baumslag–Solitar group $\mathrm{BS}(m,n)$ has icc, is inner amenable but not amenable whenever $|m| > |n| > 1$ [Sta06b], Examples 2.4 and 3.2. Note that for every $|m| > 1$, the group $\overline{\mathrm{BS}}(m,\xi)$ also has icc since every Powers group does [Har85], Proposition 1.

Proposition 4.6. *Let $|m| > 1$, $\xi \in \mathbb{Z}_m$. The group $\overline{\mathrm{BS}}(m,\xi)$ is inner amenable and non-amenable.*

The proof relies on:

Theorem 4.7 ([Sta06b], Proposition 0.2). *Let $\Gamma = \mathrm{HNN}(\Lambda, H, K, \phi)$ with $H \neq \Lambda$ or $K \neq \Lambda$. Let $Z(\Lambda)$ be the center of Λ. If for every $n \geq 1$ there exist some non-trivial elements $h_0^{(n)}, h_1^{(n)}, \dots, h_n^{(n)} \in Z(\Lambda) \cap H \cap K$ such that $h_i^{(n)} = \phi(h_{i-1}^{(n)})$ for $i = 1, \dots, n$, then Γ is inner amenable.*

Proof of Proposition 4.6. Let $n \geq 1$ and set $h_i^{(n)} := a^i b^{m^{n+1}} a^{-i}$ for $i = 0, \dots, n$. The hypotheses of Theorem 4.7. are trivially satisfied by these elements, which proves that $\overline{\mathrm{BS}}(m,\xi)$ is inner amenable. Using Britton's lemma, we can readily show that the subgroup generated by a and bab^{-1} is a non-abelian free subgroup of $\overline{\mathrm{BS}}(m,\xi)$. Hence $\overline{\mathrm{BS}}(m,\xi)$ is not amenable by a classical result of von Neumann. □

5. Homomorphisms

This section is devoted to the study of group homomorphisms from a given limit group to another one. We classify the limits of Baumslag–Solitar groups $\overline{\mathrm{BS}}(m,\xi)$ up to abstract group isomorphism (Theorem 5.11), we compute the automorphism group of every limit (Proposition 5.13) and prove that every limit is hopfian (Theorem 5.9). Finally, we show that every limit has infinite twisted conjugacy classes (Proposition 5.5)

Since the map $a \mapsto a$, $b \mapsto b$ induces an isomorphism from $\mathrm{BS}(m,n)$ to $\mathrm{BS}(-m,-n)$, it also induces an isomorphism from $\overline{\mathrm{BS}}(m,\xi)$ to $\overline{\mathrm{BS}}(-m,-\xi)$ for any $\xi \in \mathbb{Z}_m$. For this reason, we will assume that $m > 0$.

The following lemma can be readily deduced from the definition of $\overline{\mathrm{BS}}(m,\xi)$.

Lemma 5.1. *Let $d = \gcd(m,\xi)$ and let $\eta = \pi(\xi/d)$ where $\pi : \mathbb{Z}_m \to \mathbb{Z}_{\widehat{m}}$ is the canonical map. The map $a \mapsto a$, $b \mapsto b^d$ induces an injective homomorphism from $\overline{\mathrm{BS}}(\widehat{m},\eta)$ into $\overline{\mathrm{BS}}(m,\xi)$.*

From now on we will consider $\widetilde{\mathrm{BS}}(m,\xi)$ (see Introduction or paragraph before Corollary 3.2) rather than $\overline{\mathrm{BS}}(m,\xi)$ because it will ease off notations. We fix $m,m' \in \mathbb{N} \setminus \{0\}$ and $\xi \in \mathbb{Z}_m$, $\xi' \in \mathbb{Z}_{m'}$ and we set $\Gamma = \widetilde{\mathrm{BS}}(m,\xi)$ and $\Gamma' = \widetilde{\mathrm{BS}}(m',\xi')$. Recall that $\sigma\colon \Gamma \to \mathbb{Z}$ is the homomorphism defined by $\sigma(a) = 1$ and $\sigma(b) = 0$. We denote also by σ the analogous homomorphism from Γ' onto $\mathbb{Z}$.

The following proposition shows that every surjective homomorphism from Γ onto Γ' is conjugated to a simple one by an element of Γ'.

Proposition 5.2. *Let $p\colon \Gamma \to \Gamma'$ be a surjective homomorphism. Then m' divides m, $\sigma(p(a)) = \pm 1$ and $p(e_0)$ is conjugated to $\pm e_0$. Moreover $\sigma(p(a)) = 1$ if $m' > 1$.*

Before proving Proposition 5.2, we quote the following observation for further reference.

Remark 5.3. For any $Q \in \mathbb{Z}[X^{\pm 1}]$, there exists $\psi_Q \in \mathrm{Aut}(\mathbb{Z}[X^{\pm 1}] \rtimes \mathbb{Z})$ such that $\psi_Q(1,0) = (1,0)$ and $\psi_Q(0,1) = (Q,1)$. This automorphism satisfies $\psi_Q(P,0) = (P,0)$ for all $P \in \mathbb{Z}[X^{\pm 1}]$.

Indeed, this can be readily checked with the well-known presentation

$$\mathbb{Z}[X^{\pm 1}] \rtimes \mathbb{Z} = \mathbb{Z} \wr \mathbb{Z} = \langle a,b \mid [a^i b a^{-i}, b] = 1 \text{ for all } i \geq 1\rangle,$$

where a corresponds to $(0,1)$ and b corresponds to $(1,0)$.

Proof of Proposition 5.2. We set $\alpha = p(a)$, $\beta = p(e_0)$ and $\gamma = \alpha\beta\alpha^{-1}$. First we show that β is an elliptic element of Γ'. Assume by contradiction that β is hyperbolic. Then γ^m is hyperbolic with axis $\alpha(D)$, where D is the axis of β. Since $e_1 = a(me_0)a^{-1}$ commutes with e_0 in Γ, β commutes with γ^m. As a result, γ^m has the same axis as β, namely D. Thus D is invariant under α. There are two cases: either α is hyperbolic (case 1) or α is elliptic (case 2).

Case 1: the bi-infinite ray D is then the axis of α. As α and β generate Γ', the two ends of D are fixed ends of Γ'. This is impossible since a non-degenerate HNN extension has at most one fixed end on the boundary of its Bass–Serre tree.

Case 2: the automorphism $\alpha' = \alpha\beta$ is hyperbolic, since Γ' cannot be generated by two elliptic elements. Indeed, the set of elliptic elements is contained in the kernel of $\sigma\colon \Gamma' \to \mathbb{Z}$. So the argument used in case 1 applies to α' and β.

Therefore, up to conjugacy, we can assume that $\beta \in E$. Now we use the surjective homomorphism $q_{m',\xi'}\colon \Gamma' \to \mathbb{Z} \wr \mathbb{Z}$ (see Proposition 2.9 (v)). Since $\sigma(\beta) = 0$ and $\{\sigma(\alpha), \sigma(\beta)\}$ generates $\mathbb{Z}$, we have $\sigma(\alpha) = \epsilon'$ with $\epsilon' = \pm 1$. We can write $q(\beta) = (P,0)$, $q(\alpha) = (Q,\epsilon')$ with $P, Q \in \mathbb{Z}[X^{\pm 1}]$. If $\epsilon' = 1$, the automorphism ψ_{-Q}, defined as in Remark 5.3, maps $(P,0)$ to $(P,0)$ and $(Q,1)$ to $(0,1)$; if $\epsilon' = 1$, the automorphism ψ_{XQ} maps $(P,0)$ to $(P,0)$ and $(Q,-1)$ to $(0,-1)$. In both cases, the image $\{(P,0),(0,\epsilon')\}$ of $\{q(\alpha), q(\beta)\}$ by the latter automorphism generates $\mathbb{Z} \wr \mathbb{Z}$, i.e., the subgroup $P\mathbb{Z}[X^{\pm 1}] \rtimes \mathbb{Z}$ coincides with $\mathbb{Z} \wr \mathbb{Z}$. This implies that P is invertible in $\mathbb{Z}[X^{\pm 1}]$, which gives $P = \epsilon X^i$ for some $\epsilon = \pm 1$, $i \in \mathbb{Z}$. As $P = q(\beta)$ lies in $\mathbb{Z}[X]$, we have $i \in \mathbb{N}$, whence $q(\beta) = q(a^i(\epsilon e_0)a^{-i})$. Since q is injective on E by Proposition 2.9, we deduce that $\beta = a^i(\epsilon e_0)a^{-i}$. Thus, up to conjugacy, we can assume that $\beta = \epsilon e_0$.

Assume now that $m' > 1$. Let $\alpha = c_1 a^{\epsilon_1} c_2 a^{\epsilon_2} \dots c_l a^{\epsilon_l} c_{l+1} = z a^{\epsilon_l} c_{l+1}$ (with $\epsilon_i = \pm 1$ and $c_i \in E$) be a reduced form in Γ'. As $\gamma^m = \alpha(m\epsilon e_0)\alpha^{-1}$ commutes with $\beta = \epsilon e_0$, we deduce from Corollary 4.4 that $\alpha(me_0)\alpha^{-1} \in E$. It follows that $\epsilon_l = 1$ and m' divides m by Britton's lemma. As $a^{\sigma(\alpha)}(me_0)a^{-\sigma(\alpha)} \in E$ by Lemma 4.5, it follows from Britton's lemma that $\sigma(\alpha) \geq 0$. Since $\sigma(\alpha) = \sigma(z) + 1 \in \{\pm 1\}$, we deduce that $\sigma(z) = 0$, which completes the proof. □

Remark 5.4. For p as in Proposition 5.2, note that we can write $p(a) = zae$ with $e \in E$ and $z \in \Gamma'$ such that $\sigma(z) = 0$ when $m' > 1$; see the proof above.

Let G be a group and let ϕ be an automorphism of G. Two elements $\gamma, \gamma' \in G$ are said to be *ϕ-twisted conjugate* if there is $g \in G$ such that $\gamma' = g\gamma\phi(g^{-1})$. We say that G has infinitely many twisted conjugacy classes if G has infinitely many ϕ-twisted conjugacy classes for every automorphism ϕ. The study of this property is mainly motivated by topological fixed point theory and by the problem of finding a twisted analogue of the classical Burnside–Frobenius theorem (see [FH94] for a introduction to these topics). Baumslag–Solitar groups $\mathrm{BS}(m, n)$ with $(m, n) \neq \pm(1, 1)$ [FG06], [FG08] and the wreath product $\mathbb{Z} \wr \mathbb{Z}$ [GW06], Corollary 4.3, have infinitely many twisted conjugacy classes (the reader may consult [Rom] for an up-to-date list of known examples).

Corollary 5.5. *The group Γ has infinitely many twisted conjugacy classes.*

Proof. Let ϕ be an automorphism of Γ. It follows from Proposition 5.2 that $\sigma \circ \phi = \sigma$. Thus σ is constant on each ϕ-twisted conjugacy class. As σ takes infinitely many values, Γ has infinitely many ϕ-twisted conjugacy classes. □

For every $i \geq 1$, we set

$$w_i = a^i(me_0)a^{-1}(-r_1(\xi)e_0)a^{-1}(-r_2(\xi)e_0)a^{-1}(-r_{i-1}(\xi)e_0)a^{-1}.$$

Then $e_i = w_i$ holds in Γ for every $i \geq 1$. Recall that

$$\begin{aligned} \Gamma &= \langle a, e_0, e_1, \dots \mid [e_i, e_j] = 1 \text{ for all } i, j \geq 0, \\ &\qquad a(me_0)a^{-1} = e_1,\ a(e_i - r_i(\xi)e_0)a^{-1} = e_{i+1} \text{ for all } i \geq 1\rangle \qquad (7) \\ &= \langle a, e_0 \mid [e_0, w_i] = 1 \text{ for all } i \geq 1\rangle \qquad (8) \end{aligned}$$

by Corollary 2.10 and Theorem 3.1.

We now turn to the classification of limits up to group isomorphism. To carry it out, we still need two lemmas. Let J be the map defined by $J(a) = a$ and $J(e_0) = -e_0$. It follows from presentation (7) that J induces an automorphism of Γ such that $J(e_i) = -e_i$ for every i.

Lemma 5.6. *Let $t_1, \dots, t_n \in \{0, \dots, m-1\}$ and let*

$$w(m, t_1, \dots, t_n) = a^{n+1}(me_0)a^{-1}(-t_1 e_0)a^{-1}(-t_2 e_0)a^{-1} \dots (-t_n e_0)a^{-1}.$$

Then the following statements are equivalent:

(i) *The image of $w(m, t_1, \dots, t_n)$ in Γ belongs to E.*

(ii) *The image of $w(m, t_1, \dots, t_n)e_0 w(-m, -t_1, \dots, -t_n)(-e_0)$ in Γ is trivial.*

(iii) *$t_i = r_i(\xi)$ for every $1 \leq i \leq n$.*

Proof. (i) $\implies$ (ii): One has $w(m, t_1, \dots, t_n)e_0 w(-m, -t_1, \dots, -t_n)(-e_0) = 1$ in Γ since $J(w(m, t_1, \dots, t_n)) = w(-m, -t_1, \dots, -t_n)$ and J maps any element of E to its inverse.

(ii) $\implies$ (iii): Assume that $w = w(m, t_1, \dots, t_n)e_0 w(-m, -t_1, \dots, -t_n)(-e_0)$ has a trivial image in Γ. We prove the following claim by induction: we have $w = v_i e_0 J(v_i)(-e_0)$, with $v_i = a^{n+1-i}(e_i - t_i e_0)a^{-1}(-t_{i+1}e_0)a^{-1} \dots (-t_n e_0)a^{-1}$, for every $1 \leq i \leq n$, and $t_j = r_j(\xi)$ for every $1 \leq j \leq i$. For $i = 1$, the claim follows from the relation $a(me_0)a^{-1} = e_1$ in Γ. Assume that the claim holds for some $i \geq 1$. We then have $w = v_i e_0 J(v_i)(-e_0) = 1$ in Γ. By Britton's lemma, the sequence $(a,\, e_i - t_i e_0,\, a^{-1})$ is not reduced and hence $e_i - t_i e_0 \in E_{m,\xi}$. Therefore $t_i = r_i(\xi)$ and $a(e_i - t_i e_0)a^{-1} = e_{i+1}$. We deduce that $w = v_{i+1}e_0 J(v_{i+1})$ with $v_{i+1} = a^{n-i}(e_{i+1} - t_{i+1}e_0)a^{-1}(-t_{i+2}e_0) \dots (-t_n e_0)a^{-1}$, which completes the induction.

(iii) $\implies$ (i): this follows from the fact that $w(m, r_1(\xi), \dots, r_i(\xi)) = w_i$ and $w_i = e_i$ in Γ for $i \geq 1$. ☐

Lemma 5.7. *Assume that $m = m' > 1$. Let $e \in E$, $z \in \Gamma'$ such that $\sigma(z) = 0$ and let θ be the map defined by $\theta(a) = zae$, $\theta(e_0) = e_0$. If θ induces an homomorphism from Γ to Γ' then $r_i(\xi) = r_i(\xi')$ for every $i \geq 1$. In this case, the restriction of θ to E is the identity and we have $|\theta(\gamma)|_a = |\gamma|_a |\theta(a)|_a$ for every $\gamma \in \Gamma$. In particular θ is injective.*

Proof. Assume that θ induces an homomorphism from Γ to Γ'. We show by induction that $\theta(e_i) = w_i$ for every $i \geq 1$. First observe that $\theta(e_i)$ commutes with $\theta(e_0) = e_0$ for every $i \geq 1$. Hence $\theta(e_i) \in E$ for every $i \geq 1$ by Corollary 4.4. In particular, $\theta(e_1) = \theta(w_1) = za(me_0)a^{-1}z^{-1} \in E$. As $\sigma(z) = 0$, we deduce from Lemma 4.5 that $\theta(e_1) = w_1$. Assume now that $\theta(e_i) = w_i$ for some $i \geq 1$. As $e_{i+1} = a(e_i - r_i(\xi)e_0)a^{-1}$ in Γ, we have $\theta(e_{i+1}) = za(w_i - r_i(\xi)e_0)a^{-1}z^{-1} \in E$. We obtain $\theta(e_{i+1}) = w_{i+1}$ by Lemma 4.5, which completes the induction. As $w_i = \theta(e_i) \in E$ for every $i \geq 1$, we deduce from Lemma 5.6 that $r_i(\xi) = r_i(\xi')$ for every $i \geq 1$.

Assume that the latter condition holds and let $\theta(a) = c_1 a^{\epsilon_1} c_2 a^{\epsilon_2} \dots c_l a^{\epsilon_l} c_{l+1}$ (with $\epsilon_j = \pm 1$ and $c_j \in E$ for every j) be a reduced form in Γ'. Because $\theta(a)(me_0)\theta(a)^{-1} = e_1$ and $\theta(a)^{-1}e_1\theta(a) = me_0$, we have $\epsilon_1 = \epsilon_l = 1$ by Britton's lemma. It readily follows that $|\theta(\gamma)|_a = |\gamma|_a |\theta(a)|_a$ for every $\gamma \in \Gamma$. ☐

Hopf property and residual finiteness. An important reason for considering Baumslag–Solitar groups was that originally they were the first examples of non-hopfian one-relator groups [BS62]. Let us recall that a group G is *hopfian* if every surjective endomorphism from G is an isomorphism. It is known that the Hopf property is neither open [ABL+05], [Sta06a] nor closed [CGP07], Proposition 5.10, in the space of marked groups.

A group G is said to be *residually finite* if for every non-trivial element g of G there is a finite quotient F of G such that the image of g in F is non-trivial. For $m = 1$, the only limit is $\widetilde{\mathrm{BS}}(1,0) = \mathbb{Z} \wr \mathbb{Z}$. This group is residually finite, hence hopfian by a well-known theorem of Malcev; see e.g. [LS77], Theorem IV.4.10.

Definition 5.8. A group G is *co-hopfian* if every injective homomorphism from G to itself is an isomorphism.

Theorem 5.9. *The group Γ is hopfian but not co-hopfian.*

We denote by J the automorphism of Γ defined by $J(a) = a$ and $J(e_0) = -e_0$.

Proof. We can assume that $m > 1$. Let p be a surjective endomorphism of Γ. By Proposition 5.2, there is $\epsilon \in \{0,1\}$ and an inner automorphism τ of Γ such that $p' = J^\epsilon \circ \tau \circ p$ satisfies $p'(a) = zae$, $p'(e_0) = e_0$ with $e \in E$, $z \in \Gamma$ such that $\sigma(z) = 0$. By Lemma 5.7, p' is injective and hence so is p. Therefore p is an isomorphism.

Let $k \in \mathbb{N} \setminus \{1\}$ be coprime with m and let θ be the map defined by $\theta(a) = a$, $\theta(e_0) = ke_0$. Considering the group presentation (7), we deduce that θ induces an endomorphism of Γ. We can readily check that $\theta(e) = ke$ for every $e \in E$ and that $|\theta(\gamma)|_a = |\gamma|_a$ for every $\gamma \in \Gamma$. It follows that θ is injective. We also deduce that $\theta(\Gamma) \cap E = \theta(E) = kE \neq E$. Hence θ is not surjective. □

We say that $\xi \in \mathbb{Z}_m = \bigoplus_{p|m} \mathbb{Z}_p$ is *algebraic* if the p-component $\xi_p \in \mathbb{Z}_p$ of ξ is algebraic over $\mathbb{Q}$ for every prime p dividing m. It only means that ξ is a root in $\mathbb{Z}_m$ of some polynomial with coefficients in $\mathbb{Z}$. Let $d = \gcd(m, \xi)$ and $\eta = \pi(\xi/d)$ where $\pi : \mathbb{Z}_m \to \mathbb{Z}_{\widehat{m}}$ is the canonical map. It readily follows from the definition of χ (resp. $\hat{\chi}$) in Proposition 2.9 (vi) that ξ (resp. η) is algebraic if and only if $\ker \chi \neq 1$ (resp. $\ker \hat{\chi} \neq 1$). If ξ is invertible in $\mathbb{Z}_m$, i.e., $\gcd(m,\xi) = 1$, then the two kernels coincide with $\bigcap_{i \geq 0} a^{-i} E a^i$.

Proposition 5.10. *If η is algebraic, then Γ is not residually finite.*

Proof. By Lemma 5.1, $\overline{\mathrm{BS}}(m,\xi)(\widehat{m},\eta)$ embeds isomorphically into $\overline{\mathrm{BS}}(m,\xi)$. Since a subgroup of a residually finite group is residually finite, we can assume that $\gcd(m,\xi) = 1$. By Proposition 2.9 (vi), we can pick $e \in \bigcap_{i \geq 0} a^{-i} E a^i \setminus E_1$. We set $\gamma = [e, ae_0a^{-1}] = (-e)a(-e_0)a^{-1}eae_0a^{-1}$. By Britton's lemma, γ is not trivial in Γ. We show that γ has trivial image in any finite quotient of Γ, which proves that Γ is not residually finite. Let F be a finite quotient of Γ with cardinal n. There is $e' \in E$ such that $e = a^{-n+1}e'a^{n-1}$ in Γ. Since $a^n = 1$ in F, we have $\gamma = [a^{-n+1}e'a^{n-1}, ae_0a^{-1}] = a[e', e_0]a^{-1} = 1$ in F. The proof is then complete. □

Classification of limits up to group isomorphism. First recall that we assume $m, m' > 0$.

Theorem 5.11. *The group Γ is isomorphic to Γ' if and only if $m = m'$ and $r_i(\xi) = r_i(\xi')$ for every $i \geq 1$.*

Proof. Assume that Γ is isomorphic to Γ'. If $m' = 1$, then Γ' is isomorphic to $\mathbb{Z} \wr \mathbb{Z}$ and so is Γ. This forces $m = 1$ for otherwise Γ would contain a non-abelian free subgroup. It follows that $r_i(\xi) = r_i(\xi') = 0$ for every $i \geq 1$. Therefore we can assume that $m' > 1$. By Proposition 5.2, m' divides m, and there is an isomorphism $\theta \colon \Gamma \to \Gamma'$ such that $\theta(a) = zae$, $\theta(e_0) = e_0$ with $e \in E$, $z \in \Gamma$ such that $\sigma(z) = 0$. Considering θ^{-1}, we also deduce that m divides m' and hence $m = m'$. By Lemma 5.7, we have $r_i(\xi) = r_i(\xi')$ for every $i \geq 1$.

The converse follows immediately from the group presentation (7). □

Automorphism group. Let $e \in E$ and let ϕ_e be the map Γ defined by $\phi_e(a) = ae$ and $\phi(e_0) = e_0$. We deduce from the group presentation (8) that ϕ_e induces an automorphism of Γ with inverse map ϕ_{-e}. Moreover, we have $J \circ \phi_e \circ J = \phi_{-e}$. The following lemma is then immediate.

Lemma 5.12. *The automorphisms ϕ_{e_0} and J generate a group isomorphic to an infinite dihedral group, namely the semi-direct product $\mathbb{Z}e_0 \rtimes \mathbb{Z}/2\mathbb{Z}$, where the action of $\mathbb{Z}/2\mathbb{Z}$ on E is multiplication by -1.*

Hence we can consider the semi-direct product $\Gamma \rtimes (\mathbb{Z}e_0 \rtimes \mathbb{Z}/2\mathbb{Z})$ where the action of $\mathbb{Z}e_0 \rtimes \mathbb{Z}/2\mathbb{Z}$ on Γ is the obvious one. We denote by $\mathrm{Inn}(\Gamma)$ the group of inner automorphisms of Γ and by $\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$ the group of outer automorphisms.

Proposition 5.13. *Assume that $m > 1$.*

- $\mathrm{Out}(\Gamma)$ *is isomorphic to* $\mathbb{Z}e_0 \rtimes \mathbb{Z}/2\mathbb{Z}$.
- $\mathrm{Aut}(\Gamma)$ *is isomorphic to* $\Gamma \rtimes (\mathbb{Z}e_0 \rtimes \mathbb{Z}/2\mathbb{Z})$.

We denote by C the subgroup of E generated by $e_1 - me_0$ and the elements $e_i - e_{i-1} + r_{i-1}e_0$ with $i \geq 1$. Proposition 5.13 will follow from:

Lemma 5.14. *Assume that $m > 1$ and let $e \in E$.*

- *The image of $\phi_e \circ J$ in* $\mathrm{Out}(\Gamma)$ *is non-trivial.*

- *The image of ϕ_e is trivial in* $\mathrm{Out}(\Gamma)$ *if and only if* $e \in C$.
- *For every automorphism* ϕ *of* Γ, *there is* $e \in E$, $\epsilon \in \{0,1\}$ *such that* $\phi = \phi_e \circ J^\epsilon$ *holds in* $\mathrm{Out}(\Gamma)$.

Proof. Since Γ is centerless (see e.g. Corollary 4.4) and torsion-free, $\mathrm{Inn}(\Gamma)$ is torsion-free. The first assertion follows from the fact that $\phi_e \circ J$ has order 2.

Let $e \in E$ and assume that there is $z \in \Gamma$ such that $\phi_e = \tau_z$. As z centralizes e_0, we deduce from Corollary 4.4 that $z \in E$. We deduce from the equality $\phi_e(a) = zaz^{-1}$ that $a(e+z)a^{-1} = z$. By Britton's lemma, we have $e + z \in E_{m,\xi}$ and hence $z \in E_1$. Identifying E with B, we deduce from Proposition 2.9.v, that $X(P_e(X) + P_z(X)) = P_z(X)$. Therefore $P_e(X) = -\frac{X-1}{X} P_z(X) = \iota(-P_z(X)) \in \mathfrak{C}$ where ι and $\mathfrak{C}$ are defined in Proposition 2.9 (vii). Since $\mathfrak{C} = q(C)$ and q in injective, we have $e \in C$. Conversely, if $e \in C$, we can readily check that $\phi_e = \tau_z$ where $z \in E_1$ is given by the formula $P_z(X) = -\frac{X}{X-1} P_e(X)$.

Consider now an arbitrary automorphism ϕ and let us show that $\phi = \phi_e \circ J^\epsilon$ holds in $\mathrm{Out}(\Gamma)$ for some $e \in E$ and some $\epsilon \in \{0,1\}$. By Proposition 5.2, we can assume that $\phi(e_0) = \pm e_0$ and $\phi(a) = zae'$ with $e' \in E$ and z such that $\sigma(z) = 0$. Composing possibly by J, we can assume that $\phi(e_0) = e_0$ hence that ϕ and ϕ^{-1} both satisfy the conditions of Lemma 5.7. We deduce from Lemma 5.7 that $1 = |a|_a = |\phi(a)|_a |\phi^{-1}(a)|_a$. Therefore $|z|_a = 0$, i.e., $z \in E$ and hence $\phi = \phi_{e'+z}$ holds in $\mathrm{Out}(\Gamma)$. □

Proof of Proposition 5.13. By Lemma 5.14, $\mathrm{Out}(\Gamma)$ is generated by the images of J, ϕ_e with $e \in E$. By Proposition 2.9 (vii), the quotient E/C is infinite cyclic and generated by the image of e_0. Hence, by Lemma 5.14, the natural map $\mathrm{Aut}(\Gamma) \to \mathrm{Out}(\Gamma)$ induces an isomorphism from the subgroup generated by J and ϕ_{e_0} onto $\mathrm{Out}(\Gamma)$. It follows then from Lemma 5.12 that $\mathrm{Out}(\Gamma)$ is isomorphic to $\mathbb{Z}e_0 \rtimes \mathbb{Z}/2\mathbb{Z}$. As a result, the exact sequence $1 \to \mathrm{Inn}(\Gamma) \to \mathrm{Aut}(\Gamma) \to \mathrm{Out}(\Gamma) \to 1$ splits. Since Γ is centerless, it naturally identifies with $\mathrm{Inn}(\Gamma)$. Thus $\mathrm{Aut}(\Gamma)$ is isomorphic to $\Gamma \rtimes (\mathbb{Z}e_0 \rtimes \mathbb{Z}/2\mathbb{Z})$. □

An immediate consequence of Proposition 5.13 is that every automorphism of Γ is induced by an automorphism of $\mathbb{F}(a, e_0)$. Group presentations with such property are called *almost quasi-free presentations* [LS77], Chapter II.2.

Recall that the map $a \mapsto (0,1)$, $e_0 \mapsto (1,0)$ induces a surjective homomorphism $q_{m,\xi}$ from Γ onto $\mathbb{Z} \wr \mathbb{Z} = \widetilde{\mathrm{BS}}(1,0)$. Another consequence of Proposition 5.13 is:

Corollary 5.15. *If* $|m| > 1$, *then the kernel of* $q_{m,\xi}$ *is a characteristic free subgroup of* Γ *of infinite rank.*

Proof. The normal subgroup $N = \ker q_{m,\xi}$ is a free group by [GS08], Theorem 3.11. Because $N \not\subseteq E_{m,\xi}$ and $NE_{m,\xi}$ has infinite index in Γ, N is not finitely generated by [KS71], Theorem 9.

To conclude, due to Proposition 5.13, it suffices to show that N is invariant under the automorphisms J and ϕ_{e_0}. It is invariant under J since the diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\ J\ } & \Gamma \\ {\scriptstyle q_{m,\xi}}\downarrow & & \downarrow{\scriptstyle q_{m,\xi}} \\ \widetilde{\mathrm{BS}}(1,0) & \xrightarrow{\ J\ } & \widetilde{\mathrm{BS}}(1,0) \end{array}$$

commutes. A similar argument works for the automorphism ϕ_{e_0}. □

Equationally noetherian groups. In this section we determine which groups $\overline{\mathrm{BS}}(m,\xi)$ are equationally noetherian. Equationally noetherian groups play an important role in *algebraic geometry over groups* [BMR99], the state-of-the-art approach to equations over groups. An equationally noetherian group enjoys the following strong form of the Hopf property: any sequence of surjective endomorphisms is stationary (see [MR00], Theorem D1.2, or [OH07], Corollary 2.8). Let us recall the definition. Given $w = w(x_1,\dots,x_n) \in G * \mathbb{F}(x_1,\dots,x_n)$ and an n-tuple $(g_1,\dots g_n) \in G^n$, we denote by $w(g_1,\dots,g_n)$ the element of G obtained by replacing x_i by g_i. For any subset $W \subseteq G * \mathbb{F}(x_1,\dots,x_n)$, we consider the *roots*

$$\mathrm{Root}(W) = \{(g_1,\dots,g_n) \in G^n \mid w(g_1,\dots,g_n) = 1 \text{ for all } w \in W\}.$$

Definition 5.16. A group G is *equationally noetherian* if, for all $n \geqslant 1$ and for all $W \subseteq G * \mathbb{F}(x_1,\dots,x_n)$, there exists a finite subset $W_0 \subseteq W$ such that $\mathrm{Root}(W) = \mathrm{Root}(W_0)$.

Linear groups over a commutative, noetherian, unitary ring (e.g. a field), are equationally noetherian [Bry77], [Gub86] while any wreath product of a non-abelian group by an infinite one is not equationally noetherian [BMR97].

Proposition 5.17. *Let $m \in \mathbb{Z} \setminus \{0\}$ and $\xi \in \mathbb{Z}_m$. The group $\overline{\mathrm{BS}}(m,\xi)$ is equationally noetherian if and only if $|m| = 1$.*

We need the following result on Baumslag–Solitar groups.

Proposition 5.18 ([BMR99], p. 41, Proposition 5). *Let $m, n \in \mathbb{Z} \setminus \{0\}$.*

(1) *If either $|m| = 1$ or $|n| = 1$ or $|m| = |n|$, then the group $\mathrm{BS}(m,n)$ is linear over $\mathbb{R}$ and hence equationally noetherian.*

(2) *In all other cases the group $\mathrm{BS}(m,n)$ is not equationally noetherian.*

Since we need some excerpts of the proof of Proposition 5.18, we provide it in full. Let R be a commutative ring with unity. We will use the following fact without further mention. If a group G has a finite index subgroup which is linear over R, then so is G; see [Weh73], Lemma 2.3.

Proof of Proposition 5.18. (1) Suppose first that $|m| = 1$ or $|n| = 1$. It is well known, and easy to show, that the map $a \mapsto (x \mapsto \frac{m}{n}x)$, $b \mapsto (x \mapsto x+1)$ yields an injective group homomorphism from $\mathrm{BS}(m,n)$ into the affine group over $\mathbb{R}$. The group $\mathrm{BS}(m,n)$ is then linear over $\mathbb{R}$. As $\mathrm{BS}(m,n)$ is soluble in this case, we observe that it is linear over $\mathbb{Z}$ if and only if it is polycyclic[2], i.e., $|n| = |m| = 1$.

If $m = n$, it is easy to check that the normal subgroup $\langle\langle a, b^m\rangle\rangle$ is isomorphic to $\mathbb{F}_{|m|} \times \mathbb{Z}$ and hence linear over $\mathbb{Z}$. Clearly, it has index $|m|$ in $\mathrm{BS}(m,m)$. Thus $\mathrm{BS}(m,n)$ is linear over $\mathbb{Z}$.

Suppose finally that $n = -m$. Let $BS_2(m,n) = \langle\langle a^2, b\rangle\rangle \subset \mathrm{BS}(m,n)$. The subgroups $B_2(m,m)$ and $B_2(m,-m)$ are clearly isomorphic and have index two in $\mathrm{BS}(m,m)$ and $\mathrm{BS}(m,-m)$, respectively. We have shown that $\mathrm{BS}(m,m)$ is linear over $\mathbb{Z}$. We deduce that $B_2(m,m)$ is linear over $\mathbb{Z}$ and hence so is $\mathrm{BS}(m,-m)$.

It follows from the above proof that $\mathrm{BS}(m,n)$ is linear over $\mathbb{Z}$ if and only if $|m| = |n|$.

(2) Since the groups $\mathrm{BS}(m,n)$ and $\mathrm{BS}(n,m)$ are isomorphic, we may assume that $|m| < |n|$. Then there exists $\nu > 0$ and a prime number p such that p^ν divides n but not m. Let us consider the set

$$W = \{w_i := [x^{-i}yx^i, z] \mid i \in \mathbb{N} \setminus \{0\}\} \subseteq \mathbb{F}(x,y,z)$$

and the triples $(x_k = a, y_k = b^{n^k}, z_k = b)$.[3] If n divides an integer β, then we have $a^{-1}b^\beta a = b^{\beta'}$ and the factorization of β' contains (strictly) less factors p than that of β. Consequently, for all k, there exists $N(k) \in \mathbb{N}$ and $\alpha(k) \in \mathbb{Z}$ such that

$$a^{-N(k)}b^{n^k}a^{N(k)} = b^{\alpha(k)} \text{ and } n \text{ does not divide } \alpha(k)\,.$$

Remark 5.19. 1. Since $a^{-k}b^{n^k}a^k = b^{m^k}$, we have $N(k) \geqslant k$.

2. Set $\mu = \mu(m)$ to be the maximal exponent arising in the factorization of m. Then we obtain $N(k) \leqslant (\mu+2)k$. Indeed, we have $a^{-k}b^{n^k}a^k = b^{m^k}$, and the exponent of p in the factorization of m^k is at most $(\mu+1)k$.

The triple $(x_k = a, y_k = b^{n^k}, z_k = b)$ is a root of w_i if and only if $i \leqslant N(k)$. Indeed:

- if $i \leqslant N(k)$, then $a^{-i}b^{n^k}a^i$ is a power of b, so that $[a^{-i}b^{n^k}a^i, b] = 1$;
- if $i > N(k)$, then

$$[a^{-i}b^{n^k}a^i, b] = a^{-(i-N(k))}b^{\alpha(k)}a^{i-N(k)} \cdot b \cdot a^{-(i-N(k))}b^{-\alpha(k)}a^{i-N(k)} \cdot b^{-1}.$$

This is reduced in $\mathrm{BS}(m,n)$, since $|m| > 1$ and n does not divide $\alpha(k)$.

[2] By theorems of Mal'cev and Auslander [Seg83], Chapters 2 and 3, a finitely generated soluble group is linear over $\mathbb{Z}$ if and only if it is polycyclic.

[3] It is possible to use only one variable: replace W by $W' = \{[a^{-i}ya^i, b] : i \in \mathbb{N} \setminus \{0\}\} \subseteq \mathrm{BS}(m,n) * \mathbb{F}(y)$.

If we now consider a finite subset $W_f = \{w_{i_1}, \dots, w_{i_s}\} \subset W$, then, choosing k large enough, we have $N(k) \geqslant i_1, \dots, i_s$. Consequently, the triple (x_k, y_k, z_k) is in $\mathrm{Root}(W_f) \setminus \mathrm{Root}(W)$. This proves that $\mathrm{BS}(m,n)$ is not equationally noetherian. □

Proof of Proposition 5.17. If $|m| = 1$, one has $\overline{\mathrm{BS}}(m,\xi) = \mathbb{Z} \wr \mathbb{Z}$, which is equationally noetherian (e.g. it is linear over the field $\mathbb{Q}(X)$).

Let us now assume that $|m| > 1$. Consider a sequence (ξ_n) of rational integers such that $|\xi_n| \to \infty$ and $\xi_n \to \xi$ in $\mathbb{Z}_m$ for $n \to \infty$. We may assume that $|m| < |\xi_n|$ for all n. Set $W = \{w_i := [x^{-i}yx^i, z] \mid i \in \mathbb{N} \setminus \{0\}\}$ and $(x_k = a, y_k = a^k b^{m^k} a^{-k}, z_k = b)$, as in the proof of Proposition 5.18 (2). We have proved the existence of natural numbers $N_n(k)$ such that (x_k, y_k, z_k) is a root of w_i in $\mathrm{BS}(m,\xi_n)$ if and only if $i \leqslant N_n(k)$. Moreover, Remark 5.19 gives the estimates

$$k \leqslant N_n(k) \leqslant (\mu(m)+2)k$$

for all n. Therefore, if we take a finite subset $W_f = \{w_{i_1}, \dots, w_{i_s}\} \subset W$, then, choosing k large enough, we have $N_n(k) \geqslant k \geqslant i_1, \dots, i_s$ for all n. Therefore, for all j, we have $w_{i_j}(x_k, y_k, z_k) = 1$ in all groups $\mathrm{BS}(m,\xi_n)$, and, passing to the limit, we see that (x_k, y_k, z_k) is a root of W_f in the group $\overline{\mathrm{BS}}(m,\xi)$.

On the other hand, by considering w_i with $i > (\mu(m)+2)k$, we see that $w(x_k, y_k, z_k) \neq 1$ in all $\mathrm{BS}(m,\xi_n)$. Hence, in $\overline{\mathrm{BS}}(m,\xi)$, the triple (x_k, y_k, z_k) is not a root of W. This proves that $\overline{\mathrm{BS}}(m,\xi)$ is not equationally noetherian. □

6. Dimensions

In this section we give the first non-trivial Hausdorff dimension estimates of a subspace of the space of marked groups on two generators. Let us recall that the map $\overline{\mathrm{BS}}_m : \mathbb{Z}_m \to \mathcal{G}_2, \xi \mapsto \overline{\mathrm{BS}}(m,\xi)$, is injective on $\mathbb{Z}_m^\times$, [GS08], Theorem 1. In order to estimate Hausdorff dimensions of the subspaces

$$Z_m^\times = \overline{\mathrm{BS}}_m(\mathbb{Z}_m^\times) = \{\overline{\mathrm{BS}}(m,\xi) \mid \xi \text{ is invertible in } \mathbb{Z}_m\},$$

we will prove that the maps between $Z_m^\times$ and $\mathbb{Z}_m^\times$ satisfy Hölder conditions and then apply classical results about Hausdorff dimension. In this section we always assume that m is a rational integer satisfying $|m| \geqslant 2$.

6.1. Distances between limits.

The first step towards Hausdorff dimension estimates is to estimate the distance between groups $\overline{\mathrm{BS}}(m,\xi)$ and $\overline{\mathrm{BS}}(m,\xi')$ in terms in the m-adic distance between ξ and ξ'.

Theorem 6.1. *Let $h \in \mathbb{N} \setminus \{0\}$ and $\xi, \xi' \in \mathbb{Z}_m$ satisfying $d := \gcd(m,\xi) = \gcd(m,\xi')$. Setting $\hat{m} = m/d$, we have:*

(1) *If* $\overline{\mathrm{BS}}(m,\xi)$ *and* $\overline{\mathrm{BS}}(m,\xi')$ *have the same relations up to length* $2(|m|+1)h+2|m|+6$, *then* $\xi \equiv \xi' \pmod{\hat{m}^h d\mathbb{Z}_m}$*;*

(2) *If* $\xi \equiv \xi' \pmod{\hat{m}^h d\mathbb{Z}_m}$, *then* $\overline{\mathrm{BS}}(m,\xi)$ *and* $\overline{\mathrm{BS}}(m,\xi')$ *have the same relations up to length* $2h$.

Proof. Due to Corollary 2.10 we may work in $\widetilde{\mathrm{BS}}(m,\xi)$ and $\widetilde{\mathrm{BS}}(m,\xi')$ instead of $\overline{\mathrm{BS}}(m,\xi)$ and $\overline{\mathrm{BS}}(m,\xi')$. Recall that, given the free abelian groups of countable rank

$$\begin{aligned} E &= \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots, \\ M &= \mathbb{Z}me_0 \oplus \mathbb{Z}(e_1 - r_1(\xi)e_0) \oplus \mathbb{Z}(e_2 - r_2(\xi)e_0) \oplus \cdots \leqslant E, \\ M' &= \mathbb{Z}me_0 \oplus \mathbb{Z}(e_1 - r_1(\xi')e_0) \oplus \mathbb{Z}(e_2 - r_2(\xi')e_0) \oplus \cdots \leqslant E, \\ E_1 &= \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots \leqslant E, \end{aligned}$$

we have

$$\begin{aligned} \widetilde{\mathrm{BS}}(m,\xi) &= \langle a, E \mid a\psi(x)a^{-1} = x \text{ for all } x \in E_1 \rangle, \\ \widetilde{\mathrm{BS}}(m,\xi') &= \langle a, E \mid a\psi'(x)a^{-1} = x \text{ for all } x \in E_1 \rangle, \end{aligned}$$

where the isomorphism $\psi\colon E_1 \to M$ is defined by $\psi(e_1) = me_0$ and $\psi(e_{i+1}) = e_i - r_i(\xi)e_0$ for $i > 0$, and the isomorphism $\psi'\colon E_1 \to M'$ is defined similarly. Recall also that the element $b \in \overline{\mathrm{BS}}(m,\xi)$ corresponds to $e_0 \in \widetilde{\mathrm{BS}}(m,\xi)$. By Proposition 2.6, the condition $\xi \equiv \xi' \pmod{\hat{m}^h d\mathbb{Z}_m}$ is equivalent to $r_i(\xi) = r_i(\xi')$ for $i = 1,\dots,h$. We will consider the latter condition.

(1) Let $w = w(m, r_1(\xi),\dots,r_h(\xi))e_0 w(-m,-r_1(\xi),\dots,-r_h(\xi))(-e_0)$ be defined as in Lemma 5.6. Since $|w| \le 2(|m|+1)h + 2|m| + 6$ and $w = 1$ in $\widetilde{\mathrm{BS}}(m,\xi)$, we also have $w = 1$ in $\widetilde{\mathrm{BS}}(m,\xi')$. We deduce from Lemma 5.6 that $r_i(\xi) = r_i(\xi')$ for $i = 1,\dots,h$.

(2) Let w be a (freely reduced) word on the alphabet $\{a^{\pm 1}, b^{\pm 1}\}$ satisfying $|w| \leqslant 2h$. By substituting occurrences of b^α by αe_0, we obtain a sequence $s = (x_0, a^{\varepsilon_1}, x_1, \dots, a^{\varepsilon_k}, x_k)$, with $k \ge 0$, of length at most $2h$, where $\varepsilon_i = \pm 1$ and x_i is an element of the subgroup $\mathbb{Z}e_0 \leqslant E$ for all i. What we have to show is that the product of the sequence s vanishes in $\widetilde{\mathrm{BS}}(m,\xi)$ if and only if it vanishes in $\widetilde{\mathrm{BS}}(m,\xi')$.

We *reduce* the sequence s in the HNN-extension $\widetilde{\mathrm{BS}}(m,\xi)$, that is, we perform, as long as possible, substitutions of:

- a subsequence (a, x, a^{-1}), with $x \in M$, by the element $\psi^{-1}(x) \in E_1$;
- a subsequence (a^{-1}, x, a), with $x \in E_1$, by the element $\psi(x) \in M$.

We then obtain a sequence $t = (y_0, a^{\delta_1}, y_1, \dots, a^{\delta_l}, y_l)$, with $l \geqslant 0$, which is reduced in $\widetilde{\mathrm{BS}}(m,\xi)$ and whose product in the latter group is equal to the product of s. The number of substitutions from s to t is trivially at most h. Therefore, it is easy to see that t and the intermediate sequences contain only $a^{\pm 1}$ letters and elements of the subgroup $\mathbb{Z}e_0 \oplus \cdots \oplus \mathbb{Z}e_h$.

Now we use the hypothesis $r_i(\xi) = r_i(\xi')$ for $i = 1,\dots,h$. Therefore, the relation

$$M \cap (\mathbb{Z}e_0 \oplus \cdots \oplus \mathbb{Z}e_h) = M' \cap (\mathbb{Z}e_0 \oplus \cdots \oplus \mathbb{Z}e_h)$$

holds, and ψ and ψ' are equal in restriction to $\mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_{h+1}$. It is thus possible to reduce the sequence s in $\widetilde{\mathrm{BS}}(m,\xi')$ by performing the *same* substitutions as in $\widetilde{\mathrm{BS}}(m,\xi)$. Hence, the sequences s and t have the same product in $\widetilde{\mathrm{BS}}(m,\xi')$. Moreover, the sequence t is also reduced in $\widetilde{\mathrm{BS}}(m,\xi')$ – if not, an argument similar to the above one would show that t is not reduced in $\widetilde{\mathrm{BS}}(m,\xi)$.

Finally, by structure theorems on HNN-extensions, the product of t vanishes in $\widetilde{\mathrm{BS}}(m,\xi)$ (resp. $\widetilde{\mathrm{BS}}(m,\xi')$) if and only if $l = 0$ and $y_0 = 0$ in E. This concludes the proof of part (2). □

We now turn to the case $d = \gcd(m,\xi) = 1$, that is, to the case of invertible m-adic integers. Recall that we chose the metric $d(\cdot,\cdot)$ on $\mathscr{G}_2$ given by $d(N_1,N_2) = e^{-\nu(N_1,N_2)}$ if $N_1 \neq N_2$, where $\nu(N_1,N_2) = \inf\{|w| : w \in N_1 \triangle N_2\}$.

Corollary 6.2. *Let $h \in \mathbb{N}\setminus\{0\}$ and $\xi,\xi' \in \mathbb{Z}_m^\times$ such that $|\xi-\xi'|_m = |m|^{-h}$. Setting $x = \overline{\mathrm{BS}}(m,\xi)$ and $x' = \overline{\mathrm{BS}}(m,\xi')$, we have*

$$e^{-2(|m|+1)(h+1)-2|m|-6} \leqslant d(x,x') \leqslant e^{-2h-1}.$$

Proof. If $d(x,x') < e^{-2(|m|+1)(h+1)-2|m|-6}$, then $\overline{\mathrm{BS}}(m,\xi)$ and $\overline{\mathrm{BS}}(m,\xi')$ have the same relations up to length $2(|m|+1)(h+1)+2|m|+6$. Theorem 6.1 (1) then gives $|\xi-\xi'|_m \leqslant |m|^{-(h+1)}$.

On the other hand, the relation $|\xi-\xi'|_m = |m|^{-h}$ implies that $\xi \equiv \xi' \pmod{m^h\mathbb{Z}_m}$. Theorem 6.1 (2) then gives $d(x,x') \leqslant e^{-2h-1}$. □

6.2. Hausdorff dimension estimates. We set f to be the inverse of the (bijective) map $\overline{\mathrm{BS}}_m\colon \mathbb{Z}_m^\times \to Z_m^\times$. We now show that f and $f^{-1} = \overline{\mathrm{BS}}_m$ both satisfy a Hölder condition.

Proposition 6.3. *For all $x,x' \in Z_m^\times$, we have*

$$|f(x)-f(x')|_m \leqslant C d(x,x')^\alpha,$$

where $\alpha = (2(|m|+1))^{-1}\log|m|$ and C is some positive constant.

Proof. Set $\xi = f(x)$ and $\xi' = f(x')$, so that $x = \overline{\mathrm{BS}}(m,\xi)$ and $x' = \overline{\mathrm{BS}}(m,\xi')$. Write $|\xi-\xi'|_m = |m|^{-h}$ with $h \in \mathbb{N}$. Let us treat the case $h \in \mathbb{N}\setminus\{0\}$ first. Using Corollary 6.2 (at the second line), we get

$$|f(x)-f(x')|_m = |\xi-\xi'|_m = e^{-h\log|m|},$$
$$d(x,x') \geqslant e^{-2(|m|+1)h-4|m|-8} = C_1 e^{-2(|m|+1)h} = C_1\big(e^{-h\log|m|}\big)^{\frac{2(|m|+1)}{\log|m|}},$$

with $C_1 > 0$. Consequently, we have $d(x,x') \geqslant C_1|f(x)-f(x')|_m^{\alpha^{-1}}$, whence $|f(x)-f(x')|_m \leqslant C_2 d(x,x')^\alpha$ for some $C_2 > 0$.

Finally, in case $h = 0$, that is, $\xi \not\equiv \xi' \pmod m$, there is a word

$$a^2 b^m a^{-1} b^{-t} a^{-1} b \cdot a^2 b^{-m} a^{-1} b^t a^{-1} b^{-1} \quad \text{with } 0 \leqslant t \leqslant |m| - 1,$$

which is trivial in one of the marked groups $x = \overline{\mathrm{BS}}(m, \xi)$, $x' = \overline{\mathrm{BS}}(m, \xi')$ but not in the other one. This gives a constant $D > 0$ such that $d(x, x') \geqslant D$, hence a constant $C_3 > 0$ such that $|f(x) - f(x')|_m = 1 \leqslant C_3 d(x, x')^\alpha$. □

Proposition 6.4. *For all* $\xi, \xi' \in \mathbb{Z}_m^\times$, *we have*

$$d(f^{-1}(\xi), f^{-1}(\xi')) \leqslant |\xi - \xi'|_m^\beta,$$

where $\beta = 2(\log|m|)^{-1}$.

Proof. Let us write $|\xi - \xi'|_m = |m|^{-h}$ with $h \in \mathbb{N}$. By Corollary 6.2, we have $d(f^{-1}(\xi), f^{-1}(\xi')) \leqslant e^{-2h-1}$ (note that for $h = 0$ this is trivially true, since $\mathrm{diam}(\mathscr{G}_2) = e^{-1}$). Hence we get

$$d(f^{-1}(\xi), f^{-1}(\xi')) \leqslant e^{-2h} = (e^{-h \log|m|})^{2(\log|m|)^{-1}} = |\xi - \xi'|_m^\beta,$$

which concludes the proof. □

Theorem 6.5. *The Hausdorff dimension of* $Z_m^\times$ *satisfies*

$$\frac{\log|m|}{2(|m|+1)} \leqslant \dim_H(Z_m^\times) \leqslant \frac{\log|m|}{2}$$

(*for all* m *such that* $|m| \geqslant 2$).

Proof. It is well known, and easy to show, that $\dim_H(\mathbb{Z}_m^\times) = 1$ with respect to the metric chosen in Section 1.2. Set $\alpha = (2(|m|+1))^{-1}\log|m|$ and $\beta = 2(\log|m|)^{-1}$, as in Propositions 6.3 and 6.4. Classical theory of Hausdorff dimension (see e.g. [Fal03], Proposition 2.3, or [Rog70], Theorem 29) and these propositions imply that $1 \leqslant \alpha^{-1}\dim_H(Z_m^\times)$ and $\dim_H(Z_m^\times) \leqslant \beta^{-1}$, hence the result. □

Corollary 6.6. *The Hausdorff dimension of* $\mathscr{G}_2$ *with respect to* $d(\cdot\,,\cdot)$ *satisfies* $\dim_H(\mathscr{G}_2) \geqslant \log(2)/6$. *In particular, the Hausdorff dimension of* $\mathscr{G}_2$ *with respect to any Hölder equivalent metric does not vanish.*

We have estimated the Hausdorff dimension of the subspaces $Z_m^\times$, which are homeomorphic to the Cantor set (provided that $|m| \geq 2$). But many interesting subspaces of $\mathscr{G}_2$, or $\mathscr{G}_n$, appeared in the literature, e.g:

- the Cantor set of Grigorchuk groups [Gri84], many such groups have intermediate growth;

- the closure $\mathcal{H}_n \subseteq \mathcal{G}_n$ of non-elementary hyperbolic groups considered by Champetier [Cha00];
- the minimal Cantor subset of $\mathcal{G}_3$ constructed by Nekrashevych [Nek07].

The first-named author has proved that the box-counting dimension (and hence the Hausdorff dimension) of the set of Grigorchuk groups vanishes [Guy07]. So does the box-counting dimension of the set of Nekrashevych groups since they share similar contracting properties with the latter. In the case of hyperbolic groups, we do not know wether or not the Hausdorff dimension vanishes.

7. Complexity of the word and conjugacy problems

In this section we study isomorphism invariants of groups originating from language theory, namely the space complexity and the Turing degree of the word and conjugacy problems. Our results are inspired by the works for Grigorchuk [Gri84] and Garzon and Zalcstein [GZ91] on the word problem of Grigorchuk groups. First, we show that the space complexity of the word problem for $\overline{\mathrm{BS}}(m,\xi)$ is tightly related to the space complexity of the rational integer sequence $(r_i(\xi))$ (Proposition 7.3). Secondly, we show that the conjugacy problem for $\overline{\mathrm{BS}}(m,\xi)$ is Turing reducible to the word problem for $\overline{\mathrm{BS}}(m,\xi)$ (Corollary 7.7). For the sake of simplicity, our emphasis is on the space complexity of the word problem. Analogs of Proposition 7.3 for time complexity and the conjugacy problem could be proved if one is prepared to more technicalities.

Space complexity. Let $\mathcal{A}$ be a set. We denote by $\mathcal{A}^*$ the set all strings (or words) on $\mathcal{A}$. Let $s \in \mathcal{A}^*$. We denote by $|s|_{\mathcal{A}}$ the string length of s, that is, the number of symbols of $\mathcal{A}$ in s. We may simply write $|s|$ when the underlying set is clearly given by the context. A set L is a *language* if it is a subset of $\mathcal{A}^*$ for some finite set $\mathcal{A}$ called *alphabet*.

Let G be a group and let X be a finite generating set of G. We denote by $\mathrm{WP}(G,X)$ the set of strings $s \in (X \cup X^{-1})^*$ such that $s = 1$ in G, i.e., s reduces to the trivial element of G. The decision problem of membership in $\mathrm{WP}(G,X)$ is called the *word problem with respect to* X. The Turing time and space complexity of the language $\mathrm{WP}(G,X)$ are group-theoretic properties independent of X [MO85]; so X will be omitted.

Nota Bene 7.1. A Turing machine M is an *off-line Turing machine* if it has a read-only input tape with endmarkers and finitely many semi-infinite storage tapes. All Turing machines considered in this section are off-line Turing machines that halt on every input. We refer the reader to [HU79] for the complete definitions of terms used in this section.

Let M be an off-line Turing machine and let $f: \mathbb{R}_+ \to \mathbb{N}$ be a function. If for every input word of length n, the machine M scans at most $f(n)$ cells on any storage tape, then M is said to be an $f(n)$ *space-bounded Turing machine*. We denote by DSPACE(f) (resp. NSPACE(f)) the class of languages which are accepted by a deterministic (resp. non-deterministic) $f(n)$ space-bounded Turing machine. A language L is *recursive* if it is accepted by a Turing machine. A function $g: \mathbb{N}^k \to \mathbb{N}^l$ is a *recursive function* if it can be computed by a Turing machine (the k arguments $i_1, \ldots, i_k$ of g are initially placed on the input tape separated by 1's, as $0^{i_1}10^{i_2}1\ldots10^{i_k}$, the l arguments are placed similarly in some output tape). A function $g: \mathbb{N} \to \mathbb{N}^l$ is said to *belong to* DSPACE(f) (resp. NSPACE(f)) if there exists a deterministic (resp. non-deterministic) Turing machine taking as input the binary expansion of j and computing $g(j)$ in space bounded above by $f(n)$ where n is the number of binary digits of j. A language L (resp. a function g) is said to *separate the inclusion* of two space complexity classes

$$\mathrm{DSPACE}(f) \subset \mathrm{NSPACE}(f)$$

if L (resp. g) belongs to NSPACE(f) but not to DSPACE(f). Proofs below use of the Tape Compression Theorem [HU79], Theorem 12.1, without mentioning it: the equality of language classes

$$\mathrm{DSPACE}(f) = \mathrm{DSPACE}(cf)$$

holds for any $c > 0$, with an analogue statement in the non-deterministic case.

Time complexity is analogously defined by counting the number of state transitions of a Turing machine with a read-and-write input tape. Every input word of length n requires at least n state transitions to be entirely read, hence DTIME(n) is the smallest time complexity class. For every function f, we have $\mathrm{DTIME}(f) \subset \mathrm{DSPACE}(f)$. We collect few facts on the word problem of finitely generated groups.

- The language WP(G) is regular if and only if G is a finite group [Ani71]. If G is infinite then WP(G) does not belong to DSPACE(log log) [HS65].
- The language WP(G) is context-free if and only if G is virtually free [MS83], [Dun85].
- The language WP(G) belongs to DSPACE(log) if G is a linear group over a field of characteristic zero [LZ77]. There exists a finitely presented non-linear group G such that $\mathrm{WP}(G) \in \mathrm{DSPACE}(\log)$ [Waa81].
- There is no known example of a "simple" group presentation for which the word problem does not belong to DSPACE(log).
- If G contains a copy of $\mathbb{Z}$ then WP(G) does not belong to DSPACE(g) for any g such that $g(n)/\log(n)$ tends to 0 [AGM92], Theorem 2. In particular, log is a sharp bound for the space complexity of the word problem of any infinite finitely generated linear group.
- The word problem of a word hyperbolic group G is solvable in real time [Hol00]. In particular $\mathrm{WP}(G) \in \mathrm{DTIME}(n)$.

Let $p, q \in \mathbb{Z} \setminus \{0\}$ and let $\mathrm{WP}(p,q)$ (resp. in $\mathrm{WP}(m,\xi)$) be the set of strings $s \in \{a^{\pm 1}, b^{\pm 1}\}^*$ such that $s = 1$ in $\mathrm{BS}(p,q)$ (resp. $\overline{\mathrm{BS}}(m,\xi)$). Given $\overline{\mathrm{BS}}(m,\xi)$, we define the function $r = r_{m,\xi}$ on $\mathbb{N}$ by

$$r(0) = |m|, \quad r(n) := r_n(\epsilon_m \xi), \tag{9}$$

where ϵ_m is the sign of m. This definition is motivated by the fact that $\mathrm{WP}(m,\xi) = \mathrm{WP}(|m|, \epsilon_m \xi)$ since $\overline{\mathrm{BS}}(m,\xi)$ and $\overline{\mathrm{BS}}(-m,-\xi)$ are isomorphic as marked groups. The following proposition can be proved by using arguments similar to those of Lemma 7.4.

Proposition 7.2. $\mathrm{WP}(p,q) \in \mathrm{DSPACE}(n) \cap \mathrm{DTIME}(n^2)$.

As $\mathrm{BS}(p,q)$ is not virtually free, we observe that the language $\mathrm{WP}(p,q)$ is not a context-free language. The complement of $\mathrm{WP}(p,q)$ is not a context-free language either, except for $|p| = |q|$ [HRR+05][4]. Solvable Baumslag–Solitar groups (i.e., groups $\mathrm{BS}(p,q)$ with $|p| = 1$ or $|q| = 1$) have a tidy real-time word problem [HR03], Theorem 2.1. We still ignore wether $\mathrm{WP}(p,q)$ belongs to DSPACE(log) in the case $\mathrm{BS}(p,q)$ is not linear. (Recall that $\mathrm{BS}(p,q)$ is linear if and only if either $|p| = |q|$ or $|p| = 1$ or $|q| = 1$ by Proposition 5.18.) The reader interested in geodesic languages of Baumslag–Solitar groups should consult [Eld05], [DL].

Provided that r belongs to $\mathrm{DSPACE}(n)$, Proposition 7.2 holds for $\mathrm{WP}(m,\xi)$ and it corresponds to the lowest complexity bound we obtain. Our next result relate the space complexity of $\mathrm{WP}(m,\xi)$ to the space complexity of r. Let us stress on the fact that functions $r = r_{m,\xi}$ defined by (9) are "numerous" because of Proposition 2.5 (iii): for any $m \in \mathbb{Z} \setminus \{0\}$, $d \in \mathbb{N} \setminus \{0\}$, and for any $g : \mathbb{N} \to \{0, \dots, |m|-1\}$ there is some $\xi \in \mathbb{Z}_m^\times$ such that $g(n) = r_n(\epsilon_m \xi)$ for all $n \geq 2$. Hence the following proposition can be seen as a result of density in the space hierarchy.

Proposition 7.3. *Let f be a non-decreasing function such that $f(n) \geq n$ and* $\mathrm{DSPACE}(f) \neq \mathrm{NSPACE}(f)$. *Let* $\overline{\mathrm{BS}}(m,\xi)$ *be such that r separates the inclusion* $\mathrm{DSPACE}(f) \subset \mathrm{NSPACE}(f)$. *Then* $\mathrm{WP}(m,\xi)$ *separates the inclusion*

$$\mathrm{DSPACE}(f(n/6|m|)) \subset \mathrm{NSPACE}(f(n)).$$

This result is an immediate consequence of the following two lemmas.

Lemma 7.4. *Assume* $r \in \mathrm{DSPACE}(f(n))$ *for some non-decreasing function f. Then* $\mathrm{WP}(m,\xi) \in \mathrm{DSPACE}(n + f(n))$. *Likewise for* NSPACE.

Lemma 7.5. *Assume* $\mathrm{WP}(m,\xi) \in \mathrm{DSPACE}(f(n/6|m|))$ *for some non-decreasing function f such that $f(n) \geq n$. Then* $r \in \mathrm{DSPACE}(f(n))$. *Likewise for* NSPACE.

[4] It is incorrectly claimed in the proof of [HRR+05], Theorem 13, that $|p| = |q|$ if and only if $\mathrm{BS}(p,q)$ is virtually abelian. The condition $|p| = |q|$ is less restrictive for it means that $\mathrm{BS}(p,q)$ contains a copy of the direct product $\mathbb{F}_{|p|} \times \mathbb{Z}$ as a finite index subgroup, or equally that $\mathrm{BS}(p,q)$ is linear over $\mathbb{Z}$.

Let us summarize the idea of the proof of Lemma 7.4. Applying to a given word $w \in \{a^{\pm 1}, b^{\pm 1}\}^*$ the natural algorithm originating from Britton's lemma, we obtain a reduced sequence for w. This reduction is carried out within at most $|w|_a$ steps and at each step we consider a word whose length is at most $|m|$ times the length of the previous one. As we encode the exponents of a and b by means of their binary expansions, this stretching factor becomes an additive constant which explains the linear part of the space complexity bound. The other part of the bound comes from the fact that we need to compute $r(n)$ to reduce words w such that $|w|_a = n$.

As for the proof of Lemma 7.5, we notice that a Turing machine which can solve the word problem for $\overline{\mathrm{BS}}(m,\xi)$, can decide which of the words defined in Lemma 5.6 are trivial. Hence it can be used to compute $r_i(\xi)$ for every i.

We will work with our favoured HNN extension $\widetilde{\mathrm{BS}}(m,\xi)$ instead of $\overline{\mathrm{BS}}(m,\xi)$. In order to make a careful enough counting of the numbers of scanned cells, we will use following notations. We fix $\mathcal{A} := \{a^{\pm 1}, \pm e_0, \pm e_1, \dots\}$. Let $m \in \mathbb{Z} \setminus \{0\}$ and $\xi \in \mathbb{Z}_m$. Given $w \in \mathcal{A}^*$, we can rewrite w in $\langle a \rangle * E$ under the form

$$w^{(0)} = a^{\alpha_1} c_1 a^{\alpha_2} c_2 \dots a^{\alpha_h} c_h \tag{10}$$

with $c_j = (\beta_{0j} e_0)(\beta_{1j} e_1) \dots (\beta_{k_j j} e_{k_j})$, $\alpha_j, \beta_{lj} \in \mathbb{Z}$ for all l, j.

We denote by ε_j the sign of α_j. We suppose that the following holds: there is some j such that

$$\varepsilon_j = -\varepsilon_{j+1} = -1 \text{ and } c_j \in E_{m,\xi} \quad \text{or} \quad \varepsilon_j = -\varepsilon_{j+1} = 1 \text{ and } c_j \in E_1. \tag{$*$}$$

We denote by $\ell = \ell(w)$ the smallest j such that ($*$) holds. Let w' be the word we get from $w^{(0)}$ by replacing $a^{\alpha_\ell} c_\ell a^{\alpha_{\ell+1}}$ by $a^{\alpha_\ell - \varepsilon_\ell} \phi^{\varepsilon_\ell}(c_\ell) a^{\alpha_{\ell+1} + \varepsilon_\ell}$ in w and reducing this new word as in (10). We write $w' = a^{\alpha'_1} c'_1 a^{\alpha'_2} \dots a^{\alpha'_{h'}} c'_{h'}$. Notice that a given exponent α of a in w either remains unchanged in w', vanishes or is replaced by some α' such that $|\alpha'| = |\alpha| - 1$. The subwords c_j remain unchanged in w' or vanish, except one which is replaced by some subword c' with $|c'| \le (2 + |m|)n$ where $n = |w|$. As long as ($*$) holds for $w^{(i)} = a^{\alpha_1^{(i)}} c_1^{(i)} a^{\alpha_2^{(i)}} \dots a^{\alpha_{h^{(i)}}^{(i)}} c_{h^{(i)}}^{(i)}$ with $i \ge 0$, we can define $w^{(i+1)} = (w^{(i)})'$.

By Britton's lemma, for any $w \in \mathcal{A}^*$, there is some $i = i(w) \ge 0$ such that $w^{(i)} = 1$ is a reduced form for w. We call the previous algorithm the Britton's algorithm.

Proof of Lemma 7.4. By hypothesis, there is an $f(n)$ space-bounded Turing machine M_r computing $r(n)$. We denote by R_1 its input tape and by R_k ($2 \le k \le p$) its storage tapes. We design an off-line Turing machine M that halts on every input $w \in \{a^{\pm 1}, \pm e_0\}^*$: if a non-trivial reduced form for w has been found, it halts without accepting, else w is reduced to 1 and M halts in an accepting state. Tape I is the read-only input tape where w is displayed without accounting for any space. At the beginning, M writes the string $s^{(0)}$ on tape 0 that encodes $w^{(0)}$:

$$s^{(0)} := \cent \varepsilon_1 \bar{\alpha}_1 \bar{c}_1 \varepsilon_2 \bar{\alpha}_2 \bar{c}_2 \dots \varepsilon_h \bar{\alpha}_h \bar{c}_h \$.$$

The strings $\bar{\alpha}_j \in \{0,1\}^*$ are the binary expansions of $|\alpha_j|$; each string $\bar{c}_j \in \{0,1,\pm\}^*$ is the concatenation of the binary expansions of the numbers $|\beta_{lj}|$ separated by sign symbols. If $\alpha_1 = 0$ (respectively $c_h = 0$) then $\varepsilon_1\bar{\alpha}_1$ (respectively $\bar{c}_h$) is replaced by the empty string.

We now describe how M works on its storage tapes R_k $(1 \leq k \leq p)$, T_0, T_1 and D. First, the machine read the input: while the head of tape I scans the first j symbols of w, M stores the number j using a counter situated in tape R_1 and then M computes and stores $r_j(\epsilon_m\xi)$ in some of the tapes R_k by simulating M_r. Meanwhile, the head of tape T_0 writes $s^{(0)}$, following an obvious linearly space-bounded algorithm. Once the input is read, M goes ahead by running Britton's algorithm. During the i-th step of this algorithm, with i even, the head of T_0 writes the string $s^{(i)}$ encoding $w^{(i)}$ over $s^{(i-2)}$ if condition $(*)$ holds for $w^{(i-1)}$. The latter word is encoded by a string $s^{(i-1)}$ stored in tape T_1. In the next step, the head of tape T_1 writes the string $s^{(i+1)}$ encoding $w^{(i+1)}$ over $s^{(i-1)}$ if condition $(*)$ holds for $w^{(i-1)}$. Tape D is a draft tape used to carry out two kind of arithmetical computations on binary expansions: the tests for condition $(*)$ and the computations of $c^{(i)}$. The content of tape D is erased after each step. The machine M halts in a state of acceptation if $s^{(i)}$ is the trivial string. It halts without accepting in case condition $(*)$ does not hold for $w^{(i)}$.

Space bound. The machine M scans at most $f(n)$ cells on the storage tapes R_k while computing $r_1(\epsilon_m\xi), \ldots, r_n(\epsilon_m\xi)$. It also scans at most $C_0 n$ cells while storing each number j and all numbers $r_j(\epsilon_m\xi)$ for $j \leq n$, where $C_0 > 0$ is independent of n.

Since $|s^{(i+1)}| \leq |s^{(i)}| + \log_2(|m|)$ and $|s^{(0)}| \leq 2n + 2$, we deduce that M scans at most $C_1 n$ cells on the storage tapes T_0 and T_1, where $C_1 > 0$ is independent of n. In order to decide if $c_j^{(i)}$ belongs to $E_{m,\xi}$ or E_1, M uses the formula of Proposition 2.9 (iii): according to the signs of $\alpha_j^{(i)}$ and $\alpha_{j+1}^{(i)}$, M carries out the division of $\gamma_m := \beta_{0j}^{(i)} + \beta_{1j}^{(i)} r_1(\epsilon_m\xi) + \beta_{2j}^{(i)} r_2(\epsilon_m\xi) + \cdots + \beta_{k_j j}^{(i)} r_{k_j}(\epsilon_m\xi))$ by $|m|$ or divides $\gamma_\xi := \beta_{1j}^{(i)} + \beta_{2j}^{(i)} r_1(\epsilon_m\xi) + \beta_{2j}^{(i)} r_2(\epsilon_m\xi) + \cdots + \beta_{k_j j}^{(i)} r_{k_j}(\epsilon_m\xi)$ by $|m|$ if moreover $\beta_{0j}^{(i)} = 0$. As $\log_2(1+|\gamma|) \leq |s^{(i+1)}|$, for $\gamma = \gamma_m, \gamma_\xi$, this requires to scan at most $C_2 n$ cells on tape D, where $C_2 > 0$ is independent of n. In order to compute $c^{(i)}$, no more than $|s^{(i+1)}|$ cells need to be scanned on tape D. Hence the number of cells scanned by M on tape D is linearly bounded. All in all, we get $\mathrm{WP}(m,\xi) \in \mathrm{DSPACE}(n + f(n))$. □

The first part of the proof of Lemma 7.5 is based on the following facts.

Lemma 7.6. *Let $m, n \in \mathbb{Z} \setminus \{0\}$ with $|m| > 1$ and let $\xi \in \mathbb{Z}_m$. Let $v_k = [ab^k a^{-1}, b]$ for $k \in \mathbb{Z}$. Then we have: $v_k = 1$ in $\overline{\mathrm{BS}}(m,\xi)$ if and only if $k \equiv 0 \pmod{m\mathbb{Z}}$.*

Proof. Let $|n| > 1$. We deduce from Britton's lemma the following claim: for every $k \in \mathbb{Z}$, we have $v_k = 1$ in $\mathrm{BS}(m,n)$ if and only if $k \equiv 0 \pmod{m\mathbb{Z}}$. As $\mathrm{BS}(m,\xi_n)$

tends to $\overline{\mathrm{BS}}(m, \xi)$ as n goes to infinity, v_k is trivial in $\overline{\mathrm{BS}}(m, \xi)$ if and only if it is trivial in $\mathrm{BS}(m, \xi_n)$ for all n large enough, which completes the proof. □

For $h \geq 1, t_1, \dots, t_h \in \{0, \dots, |m| - 1\}$, we set

$$v(|m|, t_1, \dots, t_h) := w(|m|, t_1, \dots, t_h) b w(-|m|, -t_1, \dots, -t_h) b^{-1},$$

where $w(|m|, t_1, \dots, t_h)$ is defined as in Lemma 5.6.

Proof of Lemma 7.5. By hypothesis, there is a deterministic $f(n/6|m|)$ space-bounded Turing machine M that solves the word problem for $\overline{\mathrm{BS}}(m, \xi)$. We design a Turing machine M′ computing $r(n)$ as follows. The storage tapes of M′ consists of the tapes of M and two other tapes W, and O (output tape). The tape W identifies with the input tape of M and M′ simulates M on every tape of M.

Computation of $|m|$. By Lemma 7.6, we have $|m| = \min\{k \geq 1 \mid v_k = 1$ in $\overline{\mathrm{BS}}(m, \xi)\}$. The machine M′ first writes v_k on tape W for $k = 1$ and runs M. While v_k is not accepted by M, the machine M′ writes v_{k+1} over v_k, adds one to a counter storing k in tape O and clears the storage tapes of M. If v_k is accepted, which means $k = |m|$, then M′ clears tape W.

Computation of $r_n(\epsilon_m \xi)$. Using two counters which store $i \leq n$ and $t \in \{0, \dots, |m| - 1\}$ in tape O, the machine M′ lists recursively the words $w_i(t) = v(|m|, r_1(\epsilon_m \xi), \dots, r_{i-1}(\epsilon_m \xi), t)$ on tape W. Once a word $w_i(t)$ is written on tape W, the machine M′ runs M. If $w_i(t)$ is not accepted by M, then M′ writes $w_i(t + 1)$ over $w_i(t)$, clears the storage tapes of M and runs M again. If the word written on W is accepted by M, which means $t_i = r_i(\epsilon_m \xi)$ by Lemma 5.6, then M′ stores $r_i(\xi)$ in tape O and restarts with $w_{i+1}(0)$ or halts if $i = n$.

Space bound. Obviously the number of cells scanned by M′ to compute $|m|$ is bounded by some constant $C(m)$ independent of n. The number of cells scanned by M′ while writing words $v(|m|, t_1, \dots, t_n)$ on tape W is bounded by $|v(|m|, t_1, \dots, t_n)| \leq 6n|m|$, the number of cells used to store $r_1(\xi), \dots, r_n(\xi)$ is bounded by $n \log_2(|m|+1)$ and the number of cells scanned by M′ while simulating M over its storage tapes is bounded by $f(n)$. Hence $\mathrm{WP}(m, \xi) \in \mathrm{DSPACE}(f)$. □

Turing degree. Let E, F be languages. The language E is said to be *Turing reducible to* F if there is Turing machine M with oracle F whose accepted language is E. The language E is said *Turing equivalent to* F if E is Turing reducible to F and F is Turing reducible to E. The *Turing degree of* E (also called the *degree of unsolvability* of E) is the class all languages that are Turing equivalent to E. Let $f : E \to F$ be a function. We define the *Turing degree of* f as the Turing degree of the graph of f. We denote by $CP(m, \xi)$ the set of pairs $(w, w') \in \{a^{\pm 1}, b^{\pm 1}\}^* \times \{a^{\pm 1}, b^{\pm 1}\}^*$ such that w is conjugated to w' in $\overline{\mathrm{BS}}(m, \xi)$. We call the Turing degrees of $\mathrm{WP}(m, \xi)$, $CP(m, \xi)$ the Turing degrees of the word problem and the conjugacy problem for $\overline{\mathrm{BS}}(m, \xi)$. These Turing degrees does not depend on the choice of a generating set for $\overline{\mathrm{BS}}(m, \xi)$.

Corollary 7.7. *The following Turing degrees coincide:*

- *the Turing degree of the word problem for* $\overline{\mathrm{BS}}(m, \xi)$;
- *the Turing degree of the conjugacy problem for* $\overline{\mathrm{BS}}(m, \xi)$;
- *the Turing degree of* r.

In particular, the word problem is solvable for $\overline{\mathrm{BS}}(m, \xi)$*, i.e.,* $\mathrm{WP}(m, \xi)$ *is a recursive language if and only if* r *is a recursive function.*

In contrast, Britton has proved that the conjugacy problem for any HNN extension with base a finitely generated abelian group is solvable, i.e., both Turing degrees are **0** [Bri79]. The Turing degrees of the word and conjugacy problems need not be equal in general, as shows the optimal result of Miller: for every pair of recursively enumerable Turing degrees **a**, **b** where **a** is Turing reducible to **b**, there is a finitely presented group whose word problem has Turing degree **a** and whose conjugacy problem has Turing degree **b** [Mil71].

Observe that one can define recursive m-adic numbers in the very same way one defines recursive (equivalently computable) real numbers (see [Wei00] for a definition of computable real numbers). The Turing degree of an m-adic number is then defined by means of its Hensel expansion. If $\xi \in \mathbb{Z}_m^\times$, Corollary 7.7 then says that the word problem is solvable in $\overline{\mathrm{BS}}(m, \xi)$ if and only if ξ is a recursive number and that the Turing degree of the word problem coincides with the Turing degree of ξ.

Proof. The last claim directly follows from Lemmas 7.4 and 7.5. From the proofs of these lemmas, we can easily deduce that the Turing degree of $\mathrm{WP}(m, \xi)$ coincides with the Turing degree of r.

To complete the proof we design quite informally a Turing machine with oracle r that solves the conjugacy problem in $\overline{\mathrm{BS}}(m, \xi)$. We fix the set of representatives $T_{m,\xi} = \{0, e_0, \dots, (m-1)e_0\}$ of the cosets of E_1 in E and the set of representatives $T_1 = \{0, e_0, 2e_0, \dots\}$ of the cosets of $E_{m,\xi}$ in E. If r can be computed by means of a Turing machine, Britton's algorithm (see the proof of Lemma 7.4) yields a reduced form in $\widetilde{\mathrm{BS}}(m, \xi)$ of any $w \in \{a^{\pm 1}, \pm e_0\}^*$. The process of working from the right with the relations $a(me_0)a^{-1} = e_1$ and $a(e_i - r_i(\xi)e_0)a^{-1} = e_{i+1}$ yields a normal form for w with respect to the sets of representatives $T_{m,\xi}$ and T_1. Thus we can design a Turing machine with oracle r that computes normal forms $\tilde{v}, \tilde{w}$ of cyclically reduced conjugates of v and w for any $v, w \in \{a^{\pm 1}, \pm e_0\}^*$. If $|\tilde{v}|_a \neq |\tilde{w}|_a$, we deduce from Collin's lemma [LS77], Theorem 2.5, that v is not a conjugate of w in $\widetilde{\mathrm{BS}}(m, \xi)$. The machine can be designed in such a way that it halts in this case in a non-accepting state. Hence we can assume that $|\tilde{v}|_a = |\tilde{w}|_a$. Comparing the normal form $\tilde{v}$ to the normal form of each cyclic permutation of $\tilde{w}$, the machine can decide wether or not there exist, $e \in E$ and some cyclic permutation $(\tilde{w})^*$ of $\tilde{w}$ such that $\tilde{v} = e(\tilde{w})^*(-e)$. By Collin's lemma, it is enough to decide wether v is a conjugate of w, provided either $|\tilde{v}|_a$ or $|\tilde{w}|_a$ is not zero. Hence we can assume that v, w have their images in E. We deduce from Lemma 4.5 that v is a conjugate of w in $\widetilde{\mathrm{BS}}(m, \xi)$ if and only if there is some $n \in \mathbb{Z}$ such that $v = a^n w a^{-n}$ in $\widetilde{\mathrm{BS}}(m,\xi)$. Identifying E with B in Proposition 2.9 (i), we can consider $n(v,w) = \deg P_v(X) - \deg P_w(X)$. By means of a Turing machine with oracle r, we can compare the Laurent polynomials $P_v(X)$ and $X^{n(v,w)} P_w(X)$ and hence decide wether or not v is a conjugate of w. □

Remark 7.8. We can construct a family of public-key cryptosystems based on the word problems of limits of Baumslag–Solitar groups by adapting the construction in [GZ91] based on Grigorchuk groups. The attack conceived in [GHM$^+$04] does not threaten these new cryptosystems since such an attack would require in our case at least $|m|^N$ numbers of computations when the length of the public-key is N, if we follow the cryptanalysis of the authors. However, another attack conceived in [GHM$^+$04], namely the reaction attack against the Magyarik–Wagner protocol, can be proved to be successful.

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