On the separation profile of infinite graphs

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Abstract. Initial steps in the study of inner expansion properties of infinite Cayley graphs and other infinite graphs, such as hyperbolic ones, are taken, in a flavor similar to the well-known Lipton–Tarjan $\sqrt{n}$ separation result for planar graphs. Connections to relaxed versions of quasi-isometries are explored, such as regular and semiregular maps.

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1. Introduction

The best-known separation type result is the Lipton–Tarjan [16] theorem, claiming that there is a way of removing $O(\sqrt{n})$ vertices (and the edges reaching them) from an n-vertex planar graph, so that each of the remaining connected components has at most $n/2$ vertices (see also [1], [2], [17], [18] for other proofs and generalizations).

Rather than considering finite graphs, we will look at infinite graphs, and consider the separation function. The *separation function* at n is the supremum over all subgraphs of size n, of the number of vertices needed to be removed from the subgraph, in order to cut it to connected pieces of size at most $n/2$. We are interested in separation functions up to constant factors. See the exact definitions in Section 1.1 below.

The separation function (or profile) is a natural coarse geometric invariant of infinite graphs and path metric spaces. In this work we would like to view the separation function as such an invariant, in the family of invariants like isoperimetric profiles and volume growth.

One question we are interested in is, which separation functions are possible for transitive graphs or Cayley graphs? What is the separation function of a given Cayley graph? E.g., what is it for groups of intermediate growth? Our results below will give examples of Cayley graphs with finite separation (free groups), logarithmic

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separation (planar hyperbolic groups), $n^{(d-1)/d}$-separation, $d \in \mathbb{N}$ ($\mathbb{Z}^d$, $\mathbb{H}^{d+1}$), $n/\log n$-separation (product of free groups).

We do not know of any example of a vertex transitive graph with separation n^{α}, $0 < \alpha < 1/2$. It is maybe of interest to look at separation function for natural families of groups, e.g., groups generated by finite automata ([19]), lamplighter groups, random groups, Kazhdan groups …. The latter could be natural candidates for groups with linear separation, if there is any.

Question 1.1. Is there linear separation, or $n/\log n$ is the largest growth possible?

In which groups do balls or the minimizers of the isoperimetric problem or convex sets admit the largest separation among all sets of a fixed size?

Understanding separation is useful in figuring out the partial order given by *regular maps* (see Section 1.1 below) on the collection of all spaces, or rather, on the most familiar ones.

In [5] it was shown that bounded degree transient planar graphs admit non-constant bounded Dirichlet harmonic functions. Gil Kalai asked in 1994 whether square root separation can replace planarity? The answer is negative, see [5].

By a *comb* we mean the $\mathbb{Z}^2$ grid when all the edges parallel to x-axis but not on the x-axis removed. A comb $\times \mathbb{Z}$ is an example of a transient graph with square root separation that does not admit non-constant bounded harmonic functions (in particular, non-constant harmonic Dirichlet functions), and thus is not planar [5].

Question 1.2. Does spectral radius < 1 plus an additional separation condition imply the existence of non-constant bounded harmonic functions?

Indeed, finite separation transient graphs admit non-constant bounded harmonic functions: for any such bounded size cut separating the graph to two components if only one component were transient we would get a transient simple random walk escaping to infinity via infinitely many cut sets of bounded size, which is impossible, yet having two transient components connected via a finite set implies the existence of non-constant bounded harmonic functions. What about a weaker separation condition?

In [4] it is shown that if to each level of a binary tree the edges of an expander sequence are added, the result has no non-constant bounded harmonic functions. This explains that some separation condition for Question 1.2 is necessary.

In Section 2 we prove that finite separation is the same as bounded treewidth (see definition there), and we present a theorem about the structure of infinite graphs with finite separation.

Then we study the separation of products of graphs, Section 3. First, a bound on the separation function of a product is given. Later it is shown that for a regular tree T,

$$\mathrm{sep}_{T \times T}(n) \asymp \frac{n}{\log n},$$

where $\asymp$ means up to constant factor. For functions f and g, we write $f = O(g)$ or $g = \Omega(f)$ to denote that there exists a $c > 0$ such that $f(x) \leq cg(x)$ for every x. In Section 4, on Gromov-hyperbolic graphs (or hyperbolic graphs for short), we open with establishing the separation function for $\mathbb{H}^d$ followed by a gap theorem, showing that for hyperbolic graphs the separation function is either a constant, or is growing at least logarithmically.

A regular map between two graphs is a map that increases distances at most by some linear function, and such that there is a uniform bound on the number of preimages of a vertex (see Section 1.1 for the precise definition). Given a regular map, the separators for a constant neighborhood of the image can be pulled back, thus the separation function is monotone non decreasing under regular maps (see Lemma 1.3). Section 5 discusses regular maps.

In the last section it is proved that there is no regular map from $\mathbb{Z}^2$ to the $\mathbb{Z}_2$ lamplighter over $\mathbb{Z}$. The notion of a semi-regular map is introduced and discussed. Asymptotic dimension is monotone under semi-regular maps, hence this seems to be the right type of function for its study (the way that regular maps are suitable for separation).

Many open problems are scattered along the paper.

1.1. Some definitions

Definition. Let G be a graph. Suppose that $S \subset G$ is given. Let $\mathrm{cut}(S)$ denote the infimum of the sizes of subsets C_S of S, so that the largest connected component of $S - C_S$ has size smaller than $|S|/2$. For such a C_S we will say (with a slight sloppiness in terminology) that C_S *separates* S.

The *separation function* $\mathrm{sep}_G(x) \colon \mathbb{R} \to \mathbb{R}$ is

$$\mathrm{sep}_G(x) = \sup_{S \subset G, \ |S| = x} \mathrm{cut}(S).$$

Remark. The separation function can be defined in a wider context of path metric spaces, Riemannian manifolds, in particular. For simplicity of the exposition we stick to graphs. When the separation of a manifold is discussed, the reader can replace the manifold by a rough isometric graph, defined as follows.

Definition. Let $(X, d_X), (Y, d_Y)$ be metric spaces, and let $\kappa < \infty$. A κ-rough isometry (or κ-quasi-isometry) f from X to Y is a (not necessarily continuous) map $f \colon X \to Y$ such that

$$\kappa^{-1} d_X(x_1, x_2) - \kappa \leq d_Y(f(x_1), f(x_2)) \leq \kappa d_X(x_1, x_2) + \kappa$$

holds for all $x_1, x_2 \in X$, and for every $y_1 \in Y$ there is some $x_1 \in X$ such that

$$d_Y(y_1, f(x_1)) \leq \kappa.$$

If such an f exists, we say that X and Y are *roughly isometric*.

It is straightforward to check that roughly isometric graphs have the same separation. In particular, the separation function is a group-invariant for finitely generated groups (it does not depend on which Cayley graph is chosen). However, there is a coarser equivalence relation that still preserves the separation function. This equivalence relation can be defined by the existence of *regular maps* between the two graphs in both directions (and graphs will turn out to have monotone increasing separation under regular maps).

Definition. Let (X, d_X), (Y, d_Y) be metric spaces, and let $\kappa < \infty$. A (not necessarily continuous) map $f : X \to Y$ is κ-regular if the following two conditions are satisfied.

(1) $d_Y(f(x_0), f(x_1)) \leq \kappa(1 + d_X(x_0, x_1))$ holds for every $x_0, x_1 \in X$, and
(2) for every open ball $B = B(y_0, 1)$ with radius 1 in Y, the inverse image $f^{-1}(B)$ can be covered by κ open balls of radius 1 in X.

A *regular* map is a map which is κ-regular for some finite κ. Write $X \to_{\text{reg}} Y$ if there is a regular map from X to Y.

It is easy to check that if there is a rough isometry between bounded degree graphs X and Y, then there is a regular map from X to Y (and also to the other direction, by the symmetry of being roughly isometric – which is not apparent from the definition). Hence being roughly isometric implies the existence of regular maps from X to Y and from Y to X. However, the existence of a regular map from X to Y, and one from Y to X, does not imply that X and Y would be rough isometric. For example, consider two copies of $\mathbb{Z}^2$ "glued" along the axis $\{(0, n)\}$ for X, and $\mathbb{Z}^2$ for Y; see [5].

Regular maps were studied in a somewhat different context by David and Semmes [10]. One use of the separation function is as an obstruction for the existence of a regular map from one graph to another. This is so because if there is a regular map from X into Y, then $\text{cut}_X(x) \leq C \, \text{sep}_Y(x)$ for some $C < \infty$.

Let $\text{cut}^c(G)$ denote the minimal number of vertices that are necessary to separate the finite graph G into c times smaller pieces. Let $\text{sep}_G^c(x) := \sup_{S \subset G, |S| \leq x} \text{cut}^c(S)$ (in particular, $\text{sep}_G^{1/2}(x) = \text{sep}_G(x)$). Then

$$\text{sep}_G^c(x) \asymp \text{sep}_G(x). \tag{1.1}$$

To see this, suppose $c < 1/2$, and consider some graph of size x. After we cut it into pieces of size at most half of the original one, we can repeat the procedure for the smaller pieces, and iterating if necessary, we can get to graphs of size at most c times that of the original one. Now, in the constantly many steps of the procedure, each separating set used along the way is bounded by $\text{sep}_G^{1/2}(x)$, hence their total size is at most a constant times $\text{sep}_G^{1/2}(x)$. (When $c > 1/2$, do the same thing, but starting from a c-separating set.)

Lemma 1.3. *Let X and Y be graphs with a uniform upper bound d on their degrees, and suppose that there is a κ-regular map f from X to Y. Then $\operatorname{sep}_X(x) = O(\operatorname{sep}_Y(x))$.*

Proof. Let $A \subset X$ be an arbitrary set of vertices that induces a connected graph. Define $A' \subset Y$ as the 2κ-neighborhood of $f(A)$. By κ-regularity, A' is connected, and it has size at most $|A|d^{2\kappa}$. Let S' be a minimal subset of A' that separates it into pieces $C'_1, \ldots, C'_m$, each of size at most $A'/(2\kappa d^{2\kappa})$.

Let S_0 be the 2κ-neighborhood of S' in Y. Then $S := f^{-1}(S_0)$ has size $\leq \kappa d^{2\kappa}|S'| = O(\operatorname{cut}_Y(|A'|)) = O(\operatorname{cut}_Y(|A|))$, using (1.1). Denote by C_i the preimage of C'_i by f.

We claim that S is a separating set in A between $C_1 \setminus S, C_2 \setminus S, \ldots, C_m \setminus S$. Suppose not: then there is a path P in $A \setminus S$ between some $C_i \setminus S$ and $C_j \setminus S$ ($i \neq j$). Then $f(P)$ is disjoint from $f(S) = S_0$, thus the 2κ-neighborhood P' of $f(P)$ in A' does not intersect S'. Since P' is connected, this shows that some vertex of $f(C_i \setminus S) \subset C'_i$ and some vertex of $f(C_j \setminus S_0) \subset C'_j$ is connected by a path inside $A' \setminus S'$. This contradicts the fact that C_i and C_j are different components of $A' \setminus S'$.

We have seen that $|S| = O(\operatorname{cut}_Y(|A|))$, and just seen that S is a separating set in A between $C_1 \setminus S, C_2 \setminus S, \ldots, C_m \setminus S$. On the other hand, $|C_i \setminus S| \leq |C_i| \leq \kappa|C'_i| \leq |A|/2$. $\qquad\square$

We know and used three ways to rule out existence of regular maps between spaces: separation, Dirichlet harmonic functions, and growth. See [5] (where κ-regular maps are called κ-quasimonomorphisms). A fourth way is to use that asymptotic dimension is monotone under regular (and more generally, semi-regular) maps; see Section 6.

Separation and growth are monotone with respect to regular maps. Flows with finite energy can be pushed back and forth with a regular map. One should look for additional invariants. A partial motivation to study separation is to try to figure out the partial order given by regular maps on the collection of all spaces, or rather, on the familiar ones.

2. Finite separation and regular maps

A graph admits the finite separation property (FS) iff $\operatorname{sep}_G(n)$ is a bounded function. Trees, and graphs that are roughly isometric to trees, are examples of graphs with finite separation. Yet these are not the only graphs with finite separation. Consider an infinite Sierpiński graph, which we define as some reasonable limit (say, local convergence, with root chosen to be always one of the three extremal vertices) of the sequence of finite Sierpiński graphs. This is an example of a graph with $\operatorname{sep}_G(n) \leq 3$ which is not roughly isometric to a tree, as it contains arbitrary large cycles. In this section we will try to understand the structure of graphs with finite separation.

Theorem 2.1. *If a bounded degree G has finite separation then G admits a regular map to the 3-regular tree.*

Remark. It is easy to check that every locally finite graph has a regular map to a tree if the tree can have arbitrarily large degrees (because the uniform bound on the number of preimages in the definition of a regular map can be ignored by "blowing up" vertices to large enough stars).

The proof for Theorem 2.1 will proceed by showing that finite separation implies bounded treewidth.

Definition. Let G be a finite graph, T be a tree, and consider a family $\mathcal{V} = (V_t)_{t \in T}$ such that $V_t \subset V(G)$ for every t. We say that $(T, \mathcal{V})$ is a *tree-decomposition* of G if the following hold:

(1) $\bigcup_{t \in V(T)} V_t = V(G)$;
(2) for every $e \in E(G)$ there is a t such that both endpoints of e are in V_t;
(3) for every $x \in V(G)$, the set $\{t \in V(T) \mid x \in V_t\}$ induces a connected subgraph of T.

The *width* of the tree decomposition is $\max_{t \in T} |V_t| - 1$.

The *treewidth* of G, denoted by $\mathrm{tw}(G)$, is the minimum of the width of all tree decompositions of G.

An important property of tree decompositions is that for every edge $e = \{x, y\}$ of T, the set $V_x \cap V_y$ is a separating set between $\bigcup_{t \in C_1} V_t \setminus (V_x \cap V_y)$ and $\bigcup_{t \in C_2} V_t \setminus (V_x \cap V_y)$, where C_1 and C_2 are the two components of $T \setminus e$.

The following theorem was proved by Robertson and Seymour [20]. See Theorem 12.4.4 in [11] for a proof and for more details on treewidth.

Theorem 2.2. *For every m there is a k such that a graph of treewidth at least k contains an m-by-m square grid as a minor.*

Lemma 2.3. *For any finite graph G, $\mathrm{cut}(G) \le \mathrm{tw}(G) + 1$. Conversely, for every c there exists a k such that if $\mathrm{tw}(G) \ge k$, then $\mathrm{cut}(G) \ge c$.*

Proof. To see the first claim, let $(T, \mathcal{V})$ be a tree decomposition of G of width $\mathrm{tw}(G)$. Choose $x \in V(T)$ so that the maximum of $|\bigcup_{t \in C} V_t \setminus V_x|$ over all components C of $T \setminus x$ is minimal. It is easy to check that this maximum is at most $|V(G)|/2$; on the other hand, this is the maximum size of components in $G \setminus V_x$. Hence V_x separates, and the first part is proved.

For the other assertion, let c be any positive integer. Choose $m \ge (4c^4 + 1)^{1/2}(c^2 + 1)$ to be an integer, and k as in Theorem 2.2 for this m. We will show that if G has treewidth at least k, then it cannot be separated by c vertices.

By assumption and Theorem 2.2, such a G contains an m by m grid M as a minor. Let G' be a minimal subgraph of G that can be contracted to a graph isomorphic to M. For each vertex x of M, let W_x be the set of vertices of G' that got contracted into x. Suppose that the vertices $x_1, x_2, \ldots, x_{m^2}$ of M are ordered so that

$$|W_{x_1}| \geq |W_{x_2}| \geq \cdots \geq |W_{x_{m^2}}|.$$

Consider the set $X := \{x_1, x_2, \ldots, x_{4c^4}\}$ in M. Subdivide M into (at least $4c^4 + 1$ many) $c^2 + 1$ times $c^2 + 1$ pairwise disjoint subgrids (that is, subgraphs that are grids). At least one of these subgrids does not intersect X; let M_0 be a such subgrid. Let the vertices on the boundary of M_0 be $y_1, y_2, \ldots, y_{4c^2}$. Our plan is to choose pairwise disjoint paths $P_1, \ldots, P_{2c^2}$ from X to $\{y_1, \ldots, y_{4c^2}\}$ in M. Using these, for each y_i that is the endpoint of some P_j, we will replace W_{y_i} by a $\widetilde{W}_{y_i}$ that is still connected, and has "many" vertices (and the same number for each such y_i). Then we will show that $(\bigcup \widetilde{W}_{y_i}) \cup (\bigcup_{y \in M_0} W_y)$ is a subgraph of G with no separation of size at most c. We will explain this construction in more detail later.

It is easy to check, using the isoperimetry of the square grid, that there is no separating set of size $\leq 2c^2$ between X and $\{y_1, \ldots, y_{4c^2}\}$ in M_0. Hence, by the max-flow-min-cut theorem, $P_1, \ldots, P_{2c^2}$ can be chosen. There is no restriction in assuming that their endpoints in $\{y_1, \ldots, y_{4c^2}\}$ are respectively $y_1, \ldots, y_{2c^2}$. For every $i \in \{1, \ldots, 2c^2\}$, define G_i to be the subgraph of G' induced by $\bigcup_{v \in P_i} V(W_v)$. The G_i are connected (by the fact that G_i can be contracted to P_i), and pairwise disjoint. For each $i \in \{1, \ldots, 2c^2\}$, let W'_{y_i} be a connected subgraph of G_i that contains W_{y_i} and has size $\mu := \min\{|W_{x_1}|, \ldots, |W_{x_{4c^4}}|\}$. The choice of such W'_{y_i}s is possible since $|W_{y_i}| \leq \mu$ and $|W_{x_i}| \geq \mu$ (note that $|W_{y_i}| \leq |W_{x_j}|$ for every $i, j \in \{1, \ldots, 4c^4\}$, by the choice of M_0), and by the fact that the endpoint of each P_i is some element of $\{x_1, \ldots, x_{4c^4}\}$. For every $y \in M_0 \setminus \{y_1, \ldots, y_{2c^2}\}$, let $W'_y := W_y$. Finally, let H be the subgraph of G' induced by $\bigcup_{y \in M_0} W'_y$. We claim that H is not separable by $\leq c$ vertices.

To see this, consider some separating set S of size at most c in H. Define S_{M_0} as $\{y \in M_0 \mid S \cap W'_y \neq \emptyset\}$. Then $|S_{M_0}| \leq c$, and hence $M_0 \setminus S_{M_0}$ has a component C of size at least $|M_0| - c^2 = (c^2 + 1)^2 - c^2$. In particular, at least c^2 of the $y_1, \ldots, y_{2c^2}$ are in C. The subgraph of H induced by $\bigcup_{y \in C} W'_y$ is clearly connected, and by the last sentence, it has at least $c^2 \mu$ vertices. On the other hand, $\bigcup_{y \in M_0 \setminus C} W'_y \leq \mu |M_0 \setminus C| \leq c^2 \mu$. This shows that the complement of $\bigcup_{y \in C} W'_y$ in $H \setminus S$ has size at least $c^2 \mu$, while the union of all other components and S has size at most $c^2 \mu$. Thus S does not cut H to pieces of proportion $< 1/2$. Since S was arbitrary, this finishes the proof. □

Proof of Theorem 2.1. The proof is based on the following lemma.

Lemma 2.4. *Let T be a bounded degree tree. Then there is a regular map $T \to_{\mathrm{reg}} T_2$.*

We leave the proof of the lemma as an (easy) exercise to the reader.

The *strong treewidth* of a finite graph H, denoted by $\mathrm{stw}(H)$, is defined as follows. Take a so-called *strong tree decomposition*, which is defined to be a pair $(T, \mathcal{V})$, $\mathcal{V} = (V_i)_{i \in T}$, where T is a tree, and $(V_i)_{i \in T}$ is a *partition* of $V(H)$, and for every edge $\{x, y\} \in E(H)$, $x \in V_i$, $y \in V_j$, then either $i = j$ or $\{i, j\} \in E(T)$. The strong treewidth, $\mathrm{stw}(H)$, is defined as the minimum of $\max_{i \in T} |V_i|$ over all strong tree decompositions.

It is shown in [8] that for any family of bounded degree graphs, having bounded treewidth is equivalent to having bounded strong treewidth (see Theorem 95 in [7]). Take now the family Γ of all finite connected subgraphs of G. From Lemma 2.3 we know that finite separation of G implies that there exists a c such that $\mathrm{tw}(H) \leq c$ for every $H \in \Gamma$. Hence there exists a c' such that $\mathrm{stw}(H) \leq c'$ for every $H \in \Gamma$. Given an H, take a strong tree decomposition $(T, \mathcal{V})$. (We may assume that every $V_i \in \mathcal{V}$ is nonempty.) Then the map $\phi \colon H \to T$, $x \mapsto i$, where $x \in V_i$, is 1-regular. Observe that the maximal degree of T is at most $c'd$, otherwise there were adjacent vertices s, t in T such that there is no edge between $\bigcup_{i \in C_1} V_i$ and $\bigcup_{i \in C_2} V_i$, where C_1 and C_2 are the components one gets after deleting the edge $\{s, t\}$ from T. This would contradict the connectedness of H (since every V_i is nonempty). So, for every $H \in \Gamma$, there is a ϕ 1-regular map into a tree of maximal degree at most $c'd$. A standard compactness argument gives then that there is a 1-regular map from G to a $c'd$-regular tree. Since the composition of regular maps is regular, Lemma 2.4 finishes the proof. $\qquad \square$

3. Products

How does separation behave under products? Below is a lower bound that might be also the right upper bound. At least it is tight for the products $\mathbb{Z}^{d_1 + d_2} = \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}$.

Theorem 3.1. *For $c = (7/8)^{1/2}$ and any finite graphs G, H,*

$$\mathrm{cut}(G \times H) \asymp \min(|H| \, \mathrm{cut}^c(G), |G| \, \mathrm{cut}^c(H)),$$

where $\asymp$ means up to constant factors which are independent of G and H.

Proof. Take a finite graph G, and let K be a subset of $V(G)$. How can we tell if K separates G? Here is how. Pick two vertices v, u in G at random independently and uniformly. Let p be the probability that they are not in the same component of $G \setminus K$ (including the case when at least one of them is in K). Then K separates only if $p > 1/2$.

Now consider some set of vertices K in $G \times H$. Let (g_1, h_1) and (g_2, h_2) be two randomly independently, uniformly chosen points in $G \times H$. Let p_1 be the probability that (g_1, h_1) and (g_1, h_2) are in different components of $g_1 \times H - K$, let p_2 be the probability that (g_1, h_2) and (g_2, h_2) are in different components of $G \times h_2 - K$, and let p be the probability that (g_1, h_1) and (g_2, h_2) are in different

components of $G \times H - K$. Clearly, $p \leq p_1 + p_2$. If K separates, then $p > 1/2$, so at least one of p_1, p_2 is greater than $1/4$. Suppose that it is p_1. That implies that for at least $(1/8) \times |G|$ choices of g_1 in G the probability that h_1 and h_2 are in different components of $g_1 \times H - K \cap (g_1 \times H)$ is at least $1/8$. So for those g_1 we have $K \cap (g_1 \times H)$ separating $g_1 \times H$ (for $c = (7/8)^{1/2}$, that is, the components after deletion have sizes at most c times the original size). Hence, for such g_1 the cardinality of $K \cap (g_1 \times H)$ is at least $\text{cut}^c(H)$. Hence the cardinality of K is at least $C|G|\text{cut}^c(H)$ with some C. This proves the direction

$$\text{cut}(G \times H) > C \min(|H|\text{cut}^c(G), |G|\text{cut}^c(H)).$$

The other direction is obvious. □

Recall (1.1). Using that, Theorem 3.1 provides us with a lower bound for the separation function of product graphs. However, it does not settle the question regarding the separation of a product of infinite graphs, because it does not provide us with an upper bound for the separation of a finite graph $F \subset G \times H$ that is not a product.

Theorem 3.2. *Let G_1 and G_2 be two (possibly infinite) graphs. Then*

$$\text{sep}_{G \times H}(N) = \Omega(\max_k \min(N/k \, \text{sep}_G(k), k \, \text{sep}_H(N/k))).$$

In particular,

$$\text{sep}_{G \times \mathbb{Z}}(m^2/\text{sep}_G(m)) = \Omega(m).$$

Proof. The first half is straightforward from Theorem 3.1 (using also (1.1)). For the second assertion, let $N = m^2/\text{sep}_G(m)$. The numbers whose minimum we take on the right side are equal (up to constant factor) if $k = m/\text{sep}_G(m)$, hence the maximum on the right, given by this case, is m. □

Corollary 3.3. *We have*

$$\text{sep}_{\mathbb{Z}^d}(n) \asymp n^{(d-1)/d},$$
$$\text{sep}_{\mathbb{H}^2 \times \mathbb{H}^2}(n) = \Omega(n^{1/2}/\log n),$$
$$\text{sep}_{\mathbb{R} \times \mathbb{H}^2} = \Omega(n \log n)^{1/2}.$$

Proof. The lower bound for $\text{sep}_{\mathbb{Z}^d}(n)$ follows from the finite separation of $\mathbb{Z}$ using induction and Theorem 3.2. The upper bound is given in Proposition 4.1 (see the details there), and by a similar argument one can get the upper bound for $\mathbb{R} \times \mathbb{H}^2$ (also using Theorem 3.5).

For $\mathbb{H}^2 \times \mathbb{H}^2$, we use that the separation of $\mathbb{H}^2$ is $\log n$ by Theorem 3.2. By symmetry we have $\max_k \min(n/k \, \text{sep}_{\mathbb{H}^2}(k), k \, \text{sep}_{\mathbb{H}^2}(n/k)) = n^{1/2} \log n^{1/2} \asymp n^{1/2} \log n$.

Finally, the lower bound for $\text{sep}_{\mathbb{R} \times \mathbb{H}^2}$ is straightforward from the $G \times \mathbb{Z}$ part of Theorem 3.2. □

The previous lower bound on $\mathrm{sep}_{\mathbb{H}^2 \times \mathbb{H}^2}$ will be improved in Theorem 5.1.

Conjecture 3.4. $\mathrm{sep}_{G \times \mathbb{Z}}(n)$ *is always equal to the lower bound in Theorem* 3.2. *This means that up to constants* $\mathrm{sep}_{G \times \mathbb{Z}}(m^2/\mathrm{sep}_G(m)) \asymp m$.

The value of $\mathrm{sep}_{\mathbb{H}^2 \times \mathbb{R}}(n)$ is a particular case of that, the lower bound was given in Corollary 3.3.

We are bugged by not being able to settle the question of the separation of a graph of the form $G \times \mathbb{Z}$, where the separation function of G is known. If the worst case is a product subgraph of $G \times \mathbb{Z}$ then $\mathrm{sep}_{G \times \mathbb{Z}}(n^2/\mathrm{sep}_G(n)) \asymp n$ would be the answer. We can prove this for the case when G has finite separation. To obtain $\mathrm{sep}_{G \times \mathbb{Z}}(n) \asymp \sqrt{n}$, take a regular map from G to a regular tree T (using Theorem 2.1). This defines a regular map from $\mathbb{Z} \times G$ to $\mathbb{Z} \times T$. The separation of the latter is $\sqrt{n}$; see Lemma 7.2 in [5], hence we have the $\sqrt{n}$ upper bound. For the lower bound, apply Theorem 3.1.

Denote by T the binary tree, we have:

Theorem 3.5.

$$\mathrm{sep}_{T \times T}(n) \asymp \frac{n}{\log n}.$$

Remark. This implies an upper bound of $n/\log n$ for the product of any two bounded valence trees (using Theorem 2.4), and a lower bound for $\mathbb{H}^2 \times \mathbb{H}^2$.

The proof is similar to that of Lemma 7.2 in [5].

Proof. Lower bound first. A useful representation of $T \times T$ is as follows. Consider sequences that are finite, consist of 0's and 1's, and have a decimal point. Two of these are neighbors if one is obtained from the other by adding a digit either on the right or on the left. We will use this model for $T \times T$. Let B_k be the set of points in $T \times T$ that are at distance k from the root (which is "·"). These are just the length $\leq k$ binary sequences with a dot at some place inside. Suppose that W is a separating set for B_k. Pick two length $2k$ binary sequences at random, with the dot in the center of each. Let these be called a and b. At random, pick a k-length subword of a, and denote it by a', and pick a k-length subword of b, denoted by b'. Then a' and b' are just independent uniformly selected random vertices in $B_k \setminus B_{k-1}$. Let a_1 be the left half of a, that is, the part of a to the left of the dot. This is a point in B_k. Let a_2 be the right half of a. Same for $b = b_1 \cdot b_2$, all have length k. Consider the following path from a' to b' through a_2 and b_1. Delete the leftmost digit in a'. Then add to the right the next digit of a_2. Do so repeatedly until you reach a_2. Then delete the rightmost digit, and add on the left the digit of b_1. Continue that until you reach b_1. Then delete digits on the left and add on the right the digits of b_2 until you reach b'. In this path you see length k and $k-1$ subwords of the words a, $b_1 \cdot a_2$, and b. With probability at least 1/4, say, you must pass through the separating set W in this process. Since a, $b_1 \cdot a_2$ and b are equally distributed, the probability that b contains a subword in W (of length k or $k-1$) is at least 1/12. There are 4^k different possible b's. Each word

in W with length k or $k-1$ is contained in 2^k or 2×2^k different b's. Hence the size of W is at least $2^k/24$. But $|B_k| \asymp k \times 2^k$. So $|W| > C|B_k|/\log(|B_k|)$.

Now the upper bound. Let S be a finite set of points in $T \times T$. We shall find a separating set for S of size $Cn/\log n$, where $n = |S|$. Let p_1 and p_2 be the projections into the factors. Let $D(p_1(v)), D(p_2(v))$ be the distance from the root in the respective factors. Let m_1 be the median of $D(p_1(v))$ on S, and let m_2 be the median of $D(p_2(v))$ on S. Let $A_1(m)$ be the set of vertices in S such that $D(p_1(v)) = m_1$, and similarly for $A_2(m)$. If the size of $A_1(m_1)$ is less than $100n/\log n$, then we can take $A_1(m_1)$ as the separating set. Otherwise, let k_1^+ be the least $k \geq m_1$ such that $|A_1(k_1^+)| < 100n/\log n$, let k_1^- be the greatest $k \leq m_1$ such that $|A_1(k_1^-)| < 100n/\log n$. Similarly, define k_2^- and k_2^+. The separating set is $A = A_1(k_1^-) \cup A_1(k_1^+) \cup A_2(k_2^-) \cup A_2(k_2^+)$. The size of A is obviously less than $400n/\log n$. We just need to check that A separates well. Let K be a component of $S \setminus A$, and let v be some vertex in K. If $D(p_1(v)) < k_1^-$, then every vertex w in K satisfies this inequality. Because $m_1 \geq k_1^-$ is the median for $D(p_1(z))$, this clearly implies that $|K| \leq n/2$. So we may assume that $D(p_1(v)) \geq k_1^-$ holds for every vertex v in K. Similarly, one gets $D(p_1(v)) \leq k_1^+, k_2^- \leq D(p_2(v)) \leq k_2^+$. A connected component of $t \in T : k^- \leq D(t) \leq k^+$ has cardinality exactly $2^{k^+ - k^- - 1}$. Hence it follows that $|K| \leq 2^{k_1^+ - k_1^-} \times 2^{k_2^+ - k_2^-}$. But note that $k_j^+ - k_j^- \leq \log(n)/100 + 1$ for $j = 1, 2$, because for each k in the range $k_j^- < k < k_j^+$ we have $|A_j(k)| \geq 100n/\log n$. This gives $|K| \leq 4 \times 2^{\log n/50} < n/2$. Hence A is a separating set. $\qquad \square$

The proof shows also that $\mathrm{sep}_{T \times T \times T}(n) \asymp n/\log n$, and the same for any $T \times T \times \cdots \times T$.

4. Hyperbolic graphs

Before proceeding to the study of more general hyperbolic graphs, we determine the separation function of the hyperbolic space $\mathbb{H}^d$. This follows from the method in [17], [18], which can also be used to give a proof for the separation of $\mathbb{R}^d$.

Proposition 4.1. *For $d = 2$, $\mathrm{sep}_{\mathbb{H}^d} \asymp \log n$, and for $d \geq 3$, $\mathrm{sep}_{\mathbb{H}^d} \asymp n^{(d-2)/(d-1)}$. For $d \geq 1$, $\mathrm{sep}_{\mathbb{Z}^d} \asymp n^{(d-1)/d}$.*

Proof. Suppose that $d \geq 3$. The lower bound follows from the fact that $\mathbb{R}^{d-1}$ embeds isometrically into $\mathbb{H}^d$.

For the upper bound, we adapt the proof of [17] to our setting.

Take a connected fundamental domain Q of $\mathbb{H}^d$ by some group of isometries acting co-compactly and properly discontinuously on $\mathbb{H}^d$. The translates of Q by this group give a tiling $\mathcal{T}$ of $\mathbb{H}^d$; denote the corresponding dual graph by G. For a vertex $x \in G$, let $\tau(x)$ be the corresponding tile. Clearly G is transitive and roughly

isometric to $\mathbb{H}^d$. We want to show that G has separation $n^{(d-2)/(d-1)}$ if $d \geq 3$, and $\log n$ if $d = 2$.

So let H be a finite subgraph of G, and $\mathcal{T}|_H$ be the union of the corresponding set of tiles in $\mathbb{H}^d$. Denote by o a *centerpoint* of $\mathcal{T}|_H$, that is, a point with the property that any hyperplane P through o cuts $\mathcal{T}|_H$ into two pieces whose volumes are the same up to a factor d. (See [17] for references showing the existence of centerpoints.) In particular, $\{x \in H \mid \tau(x) \text{ intersects } P\} =: S$ separates H. So we want to bound $|S|$.

Choose P randomly and uniformly of all hyperspaces through o. Look at $\mathcal{T}$ in the corresponding Poincaré ball model; we may assume that o is the origin (and then S is the intersection of a Euclidean hyperspace with the open unit ball). Let the tile containing the origin be t. If $d \geq 3$, the expected number of tiles in $\mathcal{T}|_H$ intersected by P can be estimated as follows. The set of all tiles in $\mathcal{T}$ at graph-distance k from t is of order $c^{(d-1)k}$ with some $c > 1$; the set of those intersecting P is of order $c^{(d-2)k}$ (here we are using the fact that the tiles all have the same volume, hence the same c works for both cases). Hence, if m_k is the number of tiles in $\mathcal{T}|_H$ intersecting P and at graph distance k from t, we obtain $\mathbf{E}(|S|) = O(\sum_{k=1}^{\infty} m_k c^{(d-2)k}/c^{(d-1)k}) = O(\sum m_k c^{-k})$. This is maximized when H is a ball, in which case we have $\mathbf{E}(|S|) = O(c^{(d-1)\log n} c^{-\log n})$ (where the base of the logarithm is c^{d-1}). Hence, there exists a P such that the corresponding S has $O(c^{(d-2)\log n}) = O(n^{(d-2)/(d-1)})$ elements, which is what we wanted to prove.

The proofs for $\mathbb{H}^2$ and for $\mathbb{Z}^d$ proceed similarly, so we omit the details. $\qquad\square$

From this proof technique of [17], one also gets that a graph which can be sphere packed in $\mathbb{R}^d$ such that the spheres in the packing have bounded ratios, have separation at most $O(n^{(d-1)/d})$. This implies, e.g., that $\mathbb{Z}^{d+1}$ cannot be sphere packed in $\mathbb{R}^d$. For more on this, see [6].

Remark. In [15] the following is proved. Say that a graph G has growth rate d if every ball of radius r in G contains at most r^d vertices. Then there is an injective graph homomorphism from G to $\mathbb{Z}_{\infty}^{d \log d}$, where $\mathbb{Z}_{\infty}^{d \log d}$ is the graph on $\mathbb{Z}^{d \log d}$ where two vertices are adjacent if each of their coordinates differ by at most one. Together with the method of the previous proof, we can conclude that $\text{sep}_G(n) = O(n^{(d \log d - 1)/(d \log d)})$ for such a G.

In the standard hyperbolic plane $\mathbb{H}^2$, triangles do not have the same properties as in the Euclidean plane. For instance, in the Euclidean plane, in any large isosceles triangle, the midpoint of the hypotenuse is far away from the other two sides. This cannot happen in hyperbolic space. That observation led E. Rips to the following definition.

Definition. Let $G = (V, E)$ be a graph. Given three vertices $u, v, w \in V$, pick geodesics between any two to get a geodesic triangle. Denote the geodesics by

$[u, v], [v, w], [w, u]$. Say the triangle is δ-thin if for any $v' \in [u, v]$,

$$\min(d(v', [w, u]), d(v', [v, w])) \leq \delta.$$

G is said to be δ-hyperbolic if there is some $\delta \geq 0$ such that all geodesic triangles in G are δ-thin.

Hyperbolic groups were introduced by Gromov [13]. They are among the central objects in geometric group theory.

We do not have any general upper bounds on the separation of hyperbolic graphs. Still, we have the following gap theorem.

Theorem 4.2. *Let $G = (V, E)$ be a graph which is Δ-hyperbolic and has maximal degree $M < \infty$. Let N be some integer. There is a $c > 0$, which depends only on M, N and Δ, such that if*

$$\mathrm{sep}_G(n) < c \log n \quad \text{for all } n > N, \tag{4.1}$$

then G is roughly-isometric to a tree.

Lemma 4.3. *Let $G = (V, E)$ be a graph which is Δ-hyperbolic and has maximal degree $M < \infty$, and let v_0 be some vertex of G.*

Let us write $u \sim v$ whenever u, v are two vertices of G with $d(u, v_0) = d(v, v_0)$ and there is a sequence of vertices $u = u_1, u_2, \ldots, u_k = v$ in G with $d(u_j, v_0) = d(u, v_0)$ and $d(u_j, u_{j+1}) \leq 2\Delta$ for each $j = 1, \ldots, k - 1$.

Let N be some integer. There are $D, c > 0$, which depend only on M, N and Δ such that if (4.1) holds, then $d(u, v) < D$ whenever $u \sim v$.

Proof. For each vertex $v \neq v_0$, let $\sigma_1 v$ be one of the neighbors of v that is closer to v_0, and set $\sigma_1 v_0 = v_0$. Inductively, define $\sigma_{n+1} v = \sigma_1 \sigma_n v$, and set $\sigma(v) = \{v, \sigma_1 v, \sigma_2 v, \ldots, v_0\}$. Note that this is a geodesic from v to v_0, and $\sigma(u) \subset \sigma(v)$ holds whenever $u \in \sigma(v)$.

Now assume that $u_1, u_2, \ldots, u_k$ is a sequence of vertices such that $d(u_j, v_0) = d(u_1, v_0)$ and $d(u_j, u_{j+1}) \leq 2\Delta$ for each $j = 1, \ldots, k - 1$. Also assume that (4.1) holds for some c satisfying

$$c^{-1} > (2\Delta + 3) \log M, \tag{4.2}$$

and for some N. We need to show that $d(u_1, u_k) < D$ for some $D = D(\Delta, M, N)$.

It follows from (4.2) that

$$c \log 11 + c \log Q + c Q \log M < \frac{Q - \Delta}{2\Delta + 3} \tag{4.3}$$

holds for all sufficiently large Q. So let Q be an integer large enough to satisfy (4.3) and

$$10 M^Q > \max\{N, 2\Delta\}.$$

Also set

$$L = 10M^Q, \quad R = d(u_1, v_0).$$

It is clear that we may assume $R > L + Q$, for otherwise $D = 2(L + Q)$ suffices. Let t be the largest integer in $\{1, \ldots, k\}$ such that the cardinality of $A = \{\sigma_{L+Q}u_j \mid j = 1, \ldots, t\}$ is at most Q. See figure below. Our plan is to prove that $|A| < Q$. The definition of t then shows that $t = k$. As we shall see, the diameter of A is bounded by $2\Delta|A|$. Hence it follows from $t = k$ and $|A| < Q$ that $d(u_1, u_k) \leq D$ with $D = 2\Delta Q + 2L + 2Q$ (consider the path from u_1 to u_k constructed by following $\sigma(u_1)$ to A, taking the shortest path to $A \cap \sigma(u_k)$, and then following $\sigma(u_k)$ to u_k), and the lemma follows. So the main task is to show that $|A| < Q$.

For every $w \in A$ let x_w be one of the u_j's in $\{u_1, \ldots, u_t\}$ so that $w = \sigma_{L+Q}x_w$, and let s_w be the segment of $\sigma(x_w)$ between x_w and w; that is, $s_w = \{\sigma_j x_w \mid j = 1, 2, \ldots, L + Q\}$.

Let U be the set of vertices of G that are at distance at most Q from A or that are in one of the segments s_w, $w \in A$, and let H be the graph obtained by restricting G to U. By construction, the segments s_w, $w \in A$ are disjoint, and consequently,

$$|U| \geq |A|(L + Q) > |A|L = 10|A|M^Q.$$

On the other hand, a ball of radius Q in G has clearly no more than M^Q vertices, and it follows that

$$|U| \leq |A|(L + M^Q) = 11|A|M^Q.$$

From (4.1) and $10M^Q > N$ it follows that there is a set of vertices $U_0 \subset U$ with

$$|U_0| \leq c \log |U| \leq c \log(11|A|M^Q) \tag{4.4}$$

such that every component of $H - U_0$ has less than $6|A|M^Q$ vertices.

We are trying to prove that $|A| < Q$. By construction $|A| \leq Q$, so assume that $|A| = Q$. Therefore, from (4.4) and (4.3) it follows that

$$|U_0| \leq c \log 11 + c \log Q + c Q \log M < \frac{Q - \Delta}{2\Delta + 3} \leq Q/3. \tag{4.5}$$

Let A' be the set of $w \in A$ such that the segment s_w does not meet U_0. Because these segments are disjoint, and $|A| = Q$, it follows from (4.5) that

$$|A'| > 2Q/3.$$

Let r be an integer in the range $[R - L - Q, R - L - \Delta]$, and let

$$V_r = \{v \in V \mid |d(v, v_0) - r| \leq \Delta\}.$$

We claim that there is a path whose vertices are in $V_r \cap U$ that intersects each s_w, $w \in A$. For each $j = 1, \ldots, t - 1$, let y_j be the vertex on $\sigma(u_j)$ that has distance r from v_0. Consider the triangle in G formed by taking $\sigma(u_j)$, $\sigma(u_{j+1})$ and a shortest

curve α_j joining u_j and u_{j+1}. Since G is Δ-hyperbolic, there is a vertex z_{j+1} in $\alpha_j \cup \sigma(u_{j+1})$ whose distance to y_j is at most Δ. Because the length of α_j is at most 2Δ, the distance from α_j to v_0 is at least $R - \Delta$. On the other hand,

$$d(z_{j+1}, v_0) \le d(z_{j+1}, y_j) + d(y_j, v_0) \le \Delta + r < R - \Delta,$$

and it follows that $z_{j+1} \notin \alpha_j$. Hence $z_{j+1} \in \sigma(u_{j+1})$, and there is a path β_j in U from y_j to z_{j+1} with all vertices in V_r (since $d(z_{j+1}, v_0) < R - L$). Let γ_{j+1} be the arc of $\sigma(u_{j+1})$ connecting z_{j+1} with y_{j+1}. It is now clear that the union of all the arcs β_j and γ_j is a curve whose vertices are in $V_r \cap U$ that intersects each s_w, $w \in A$. This construction with $r = R - Q - L$ also shows that the diameter of A is bounded by $2\Delta|A|$, as we have promised.

Let m be the largest integer smaller than $(Q - \Delta)/(2\Delta + 2)$. Set $r_j = R - Q - L + 2j(\Delta + 1)$ for $j = 0, 1, \ldots, m$, and note that the sets V_{r_j} are disjoint. Since

$$|U_0| < m + 1$$

follows from (4.5), there must be some such r_j so that U_0 does not meet V_{r_j}. From the above it then follows that there is a connected component of $H - U_0$ that intersects each s_w, $w \in A$. Consequently, there is a connected component X of $H - U_0$ that contains each s_w, $w \in A'$. The number of vertices of X is at least

$$|A'|L \ge 2QL/3 = 20|A|M^Q/3 > 6|A|M^Q.$$

This contradicts the definition of U_0, and the contradiction establishes $|A| < Q$ and the lemma. $\qquad\square$

Lemma 4.4. *Let G be a Δ-hyperbolic graph, let v_0 be some vertex in G, and let $\sim$ be the equivalence relation defined in Lemma 4.3. Let V_1 be the set of vertices of G whose distance from v_0 is divisible by $3\Delta + 1$, and let $\widetilde{V} = V_1/\sim$ be the set of equivalence classes of $\sim$ in V_1. Let $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ be defined by letting $[\tilde{v}, \tilde{u}] \in \widetilde{E}$ whenever for some $v \in \tilde{v}$, $u \in \tilde{u}$ there is a geodesic passing through v_0, v, u and $d(v, u) = 3\Delta + 1$. Then $\widetilde{G}$ is a tree.*

Proof. Let $\tilde{d}$ denote the distance in $\widetilde{G}$, and set $\tilde{v}_0 = \{v_0\} \in \widetilde{V}$, the equivalence class of v_0. It is clear that if $v \in \tilde{v} \in \widetilde{V}$, then $\tilde{d}(\tilde{v}, \tilde{v}_0) = (3\Delta + 1)d(v, v_0)$.

Let $\tilde{v} \in \widetilde{V} - \{\tilde{v}_0\}$. By construction, $\tilde{v}$ has no neighbor $\tilde{u}$ with $\tilde{d}(\tilde{v}, \tilde{v}_0) = \tilde{d}(\tilde{u}, \tilde{v}_0)$. We claim that there is exactly one neighbor $\tilde{u}$ of $\tilde{v}$ such that $\tilde{d}(\tilde{v}, \tilde{v}_0) = \tilde{d}(\tilde{u}, \tilde{v}_0) + 1$. Indeed, let $v_1, v_2 \in \tilde{v}$, and suppose that $d(v_1, v_2) \le 2\Delta$. Let s_1 be a shortest path from v_1 to v_0, and let s_2 be a shortest path from v_2 to v_0. Let u_1 be the vertex on s_1 whose distance from v_1 is equal to $3\Delta + 1$, and let u_2 be the vertex on s_2 whose distance from v_2 is $3\Delta + 1$. Let s_3 be a shortest segment joining v_1 and v_2. Because G is Δ-hyperbolic, there is a vertex w in $s_3 \cup s_2$ whose distance to u_1 is at most Δ. But the length of s_3 is at most 2Δ, and hence the distance from s_3 to u_1 is at least $\Delta + 1$. Thus,

$w \notin s_3$ and $w \in s_2$. Since $|d(w, v_0) - d(u_2, v_0)| = |d(w, v_0) - d(u_1, v_0)| \leq \Delta$, and $w \in s_2$, we get $d(w, u_2) \leq \Delta$. So $d(u_1, u_2) \leq d(u_1, w) + d(w, u_2) \leq 2\Delta$, which implies that u_1 and u_2 belong to the same equivalence class in $\widetilde{V}$. It now follows that $\tilde{v}$ has exactly one neighbor $\tilde{u} \in \widetilde{V}$ with $\tilde{d}(\tilde{v}, \tilde{v}_0) = \tilde{d}(\tilde{u}, \tilde{v}_0) + 1$.

Suppose that $\widetilde{G}$ is not a tree. Then there is a simple closed loop $\widetilde{C}$ in $\widetilde{G}$. Consider a vertex $\tilde{v}$ in $\widetilde{C}$, where $\tilde{d}(\tilde{v}, \tilde{v}_0)$ is maximal. The two edges adjacent to $\tilde{v}$ in $\widetilde{C}$ must connect $\tilde{v}$ with neighbors $\tilde{u}$ with $\tilde{d}(\tilde{v}, \tilde{v}_0) \geq \tilde{d}(\tilde{u}, \tilde{v}_0)$, but $\tilde{v}$ has at most one such neighbor. This contradicts the existence of $\widetilde{C}$, and hence $\widetilde{G}$ is a tree. $\square$

Proof of Theorem 4.2. Let v_0 be some vertex in G, and let $V_1, \widetilde{G}$ be as in Lemma 4.4. For each $v \in V$ choose some geodesic $\sigma(v)$ from v to v_0, and let v' be the vertex closest to v in $\sigma(v) \cap V_1$. Let πv denote the equivalence class of v' in $\widetilde{V} = V/\sim$. It is an easy exercise to check that the Δ-hyperbolicity of G implies that π is a contraction. On the other hand, it follows from Lemma 4.3 that $\tilde{d}(\pi v, \pi u) \geq C^{-1} d(v, u) - C$ holds for some $C > 0$ and every $v, u \in V$. We conclude that π is a rough-isometry. Now Lemma 4.4 shows that $\widetilde{G}$ is a tree, which completes the proof of the theorem. $\square$

Question 4.5. Show that for any planar hyperbolic graph G,

$$\mathrm{sep}_G(n) < C \log n$$

for some $C(G) = C > 0$.

Note that, by Proposition 4.1, this cannot be true in general if we omit planarity.

5. More on regular maps

Let T denote the binary tree. The next result is shown to be true in [9], even with a quasi-isometric embedding.

Theorem 5.1. $\mathbb{H}^2 \to_{\mathrm{reg}} T \times T$.

Using Theorem 3.5 and the remark after its proof, the previous theorem implies that $\mathrm{sep}_{\mathbb{H}^2 \times \mathbb{H}^2}(n) \asymp n/\log n$ because $T \times T \to_{\mathrm{reg}} \mathbb{H}^2 \times \mathbb{H}^2 \to_{\mathrm{reg}} T \times T \times T \times T$.

Question 5.2. Is $\mathbb{H}^2 \to_{\mathrm{reg}} T \times \mathbb{Z}$?

There may be something special about the $n/\log(n)$ separation function. The following problems are weaker than Question 1.1.

Question 5.3. (1) Suppose that G, H have separation functions $\leq n/\log n$. Is it true that $G \times H$ has separation function $\leq n/\log n$?

(2) Is it true that balls in infinite transitive graphs have

$$\mathrm{cut}(B(r)) \leq |B(r)|/\log |B(r)|?$$

The last question, if true, would answer negatively an old question of the first author, namely, if there is any Cayley graph where the balls form an expander family (see [4]). Without assuming transitivity, the answer to this question is clearly no: take some infinite tree with expanders put on each set of points at the same distance from the origin (as in [4]).

We have found that $\text{sep}_{\mathbb{H}^2 \times \mathbb{R}}(n)$ is at least $\sqrt{n \log n}$ (Corollary 3.3). This implies that $\mathbb{H}^2 \times \mathbb{R}$ does not have a regular map to $\mathbb{H}^3$ which has separation only of order $\sqrt{n}$ (Proposition 4.1).

Question 5.4. Is $\mathbb{H}^3 \to_{\text{reg}} \mathbb{H}^2 \times \mathbb{R}$?

We doubt that the answer would be positive, but as we mention in Section 1.1, we know only four ways to rule out regular maps: separation, Dirichlet harmonic functions, asymptotic dimension and growth. Neither does the job here.

6. Semi-regular maps

In a private communication, Yehuda Shalom asked: Is it true that a Cayley graph of an amenable group which is not virtually cyclic admits a bilipschitz embedding of $\mathbb{Z}^2$? Yehuda suggested that the lamplighter group $(\mathbb{Z}/2\mathbb{Z})_2 \wr \mathbb{Z}$ is a counterexample (see e.g. [19] for a detailed definition).

Here is a proof not using any of the arguments applied in the sections before but rather using the notion of semi-regular maps defined below. This will allow us to prove that there is no regular map $\mathbb{Z}^2 \to_{\text{reg}} \text{Lamplighter}(\mathbb{Z})$.

Definition. Call a map $f : X \to Y$ *semi-regular*, denoted by $X \to_{\text{s-reg}} Y$, if f is Lipschitz and for every r there is a $c(r) < \infty$ such that for every $y \in Y$ every connected component of $f^{-1}(B(y, r))$ has diameter at most $c(r)$.

It is an easy exercise to verify that a composition of semi-regular maps is semi-regular.

Also, a regular map is semi-regular (we assume bounded degree; there are analogous definitions appropriate for general metric spaces).

Note that the canonical map $\text{Lamplighter}(\mathbb{Z}) \to \mathbb{Z}$ (location of the lamplighter) is semi-regular, with $c(r)$ of order $r2^r$.

Proposition 6.1. *There is no semi-regular map $\mathbb{Z}^2 \to_{\text{s-reg}} \mathbb{Z}$.*

By combining with the above, it follows that there is no semi-regular map $\mathbb{Z}^2 \to_{\text{s-reg}}$ Lamplighter($\mathbb{Z}$). In particular, there is no regular map $\mathbb{Z}^2 \to_{\text{reg}}$ Lamplighter($\mathbb{Z}$).

Proof. Let $g \colon \mathbb{Z}^2 \to \mathbb{Z}$ be Lipschitz. We look at "quasi-level sets" of g, and show that there is some quasi-connected component of a quasi-level set that is large (at this point, this is a vague statement; the precise formulation will become clear shortly). Take k large, and consider the map $a(z) := \lfloor (z/k) \rfloor$, $a \colon \mathbb{Z} - > \mathbb{Z}$. Fix k large enough so that $|a(g(z)) - a(g(z'))| < 2$ if z and z' are within distance 2 in $\mathbb{Z}^2$. Let X be the graph obtained from $\mathbb{Z}^2$ by adding diagonals. Given $r \in \mathbb{Z}$, let $S(r)$ denote the set of connected components in X of $\{x \in X \mid a(g(z)) = r\}$. Let $S* := \bigcup_{r \in \mathbb{Z}} S(r)$. This is a partition of X. Note that if $S \in S(r)$ is finite, then there is a unique $S' \in S(r + 1) \cup S(r - 1)$ that is adjacent to S and "surrounds" it. Let $r \in \mathbb{Z}$ be such that $S(r)$ is nonempty. Let $S_0 \in S(r)$. Inductively, define S_{n+1} to be S_n if S_n is infinite, and if not, let S_{n+1} be some $S \in S*$ which surrounds S_n. Clearly, $|S_n| \geq n$. Hence g is not semi-regular. $\square$

Note: This also shows that there is no regular map

$$\mathbb{Z}^2 \to_{\mathrm{reg}} \mathrm{Lamplighter}(\mathrm{Lamplighter}(\mathbb{Z})).$$

There is a more general way to see the proposition, which we will sketch in a later remark.

Before Question 5.2, maybe consider

Question 6.2. Does $\mathbb{H}^2 \to_{\text{s-reg}} T \times \mathbb{Z}$?

Gromov [14] defined the notion of asymptotic dimension of a metric space, which is defined as follows. Say that X has asymptotic dimension at most D if for every $s > 0$ there is a partition (i.e., coloring) $X = Y_1 \cup Y_2 \cup \cdots \cup Y_{D+1}$ such that each Y_j can be partitioned into a collection of pieces of bounded diameter (depending only on s), with the distance between any two of the pieces at least s. See [3] for a nice survey on the asymptotic dimension.

Now, if X maps semi-regularly into $\mathbb{Z}^d$, then the asymptotic dimension of X is bounded from above by the asymptotic dimension of $\mathbb{Z}^d$ (which is d, [3]). To see this, just pull back the colorings in the definition of asymptotic dimension using the semi-regular map. By a similar argument one can show that there is no (semi-)regular map from $T \times T \times T$ to $T \times T$ (because the asymptotic dimension of the former is 3, and that of the latter is 2), and that there is no (semi-)regular map from $\mathbb{R}^d$ $(d \geq 3)$ to $T \times T$ (even though the separation of $T \times T$ is larger).

The question is whether this observation captures the partial order among "good" classes of spaces. To better understand the partial order given by semi-regular embeddings, the following is a relevant question.

Question 6.3. Suppose that G is a transitive (homogeneous) locally finite graph with asymptotic dimension d. Does it follow that $\mathbb{Z}^d \to_{\text{s-reg}} G$ and $G \to_{\text{s-reg}} \mathbb{Z}^d$?

Remark. We have just mentioned that the asymptotic dimension is monotone increasing under semi-regular maps. Hence there is no semi-regular map from $\mathbb{Z}^n$ to $\mathbb{Z}^m$ when $n > m$, giving an alternative proof for the above proposition.

Remark. Later Oded showed that $\mathbb{H}^2$ and $\mathbb{R}^2$ are semi-regularly equivalent. The proof is unfortunately lost.

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