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Interval exchanges that do not occur in free groups

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Abstract. A disjoint rotation map is an interval exchange transformation (IET) on the unit interval that acts by rotation on a finite number of invariant subintervals. It is currently unknown whether the group $\mathcal{E}$ of all IETs possesses any non-abelian free subgroups. It is shown that it is not possible for a disjoint rotation map to occur in a subgroup of $\mathcal{E}$ that is isomorphic to a non-abelian free group.

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1. Introduction

An interval exchange transformation (IET) is an invertible map $[0,1) \to [0,1)$ defined by a finite partition of $[0,1)$ into half-open subintervals and a reordering of these intervals by translation. The dynamics of single IETs have been actively studied since the late 1970s. See the recent survey of Viana [16] for a presentation of many early results in this area. The study of IET dynamics is currently quite active, due in part to the close connection between IETs and the moduli space of translation surfaces (see the survey of Zorich [17]), and also due to the recent resolution of some long-outstanding problems in the area (e.g., [2], [3], and [6]).

More recently, group actions by interval exchanges have begun to be studied; see, for instance, [1], [12], and [13]. The set $\mathcal{E}$ of all interval exchanges forms a group under composition, and an *interval exchange action* of a group G is a homomorphism $G \to \mathcal{E}$. The study of such actions is motivated by the analogous study of group actions on manifolds, particularly 1-dimensional ones, by means of homeomorphisms or diffeomorphisms. The subject of group actions on 1-manifolds is well-developed and quite active; see, for instance, [7] or [10]–[11].

In contrast, many fundamental questions which are well understood for groups acting on 1-manifolds are currently open for the group of IETs. Perhaps foremost among these is the following question, due to A. Katok:

Question 1.1. Does $\mathcal{E}$ contain a subgroup isomorphic to F_2, the non-abelian free group on two generators?

It is easy to construct examples of non-abelian free subgroups in $\mathrm{Diff}(S^1)$ and $\mathrm{Diff}(\mathbb{R})$ by means of the ping-pong construction. More detailed results, analogous to the Tits' alternative, are known for $\mathrm{Homeo}_+(S^1)$ and $\mathrm{Diff}_+^{\omega}(S^1)$; see [9] and [5], respectively. It is also shown in [7] that for a residual set of pairs (f, g) in $\mathrm{Homeo}_+(S^1)$, the group $\langle f, g\rangle$ is isomorphic to F_2. However, there are examples of groups of homeomorphisms of 1-manifolds that do not contain non-abelian free subgroups. For instance, it is known from work of Brin and Squier [4] that this is the case for the group $\mathrm{PL}_+([0,1])$ of piecewise-linear homeomorphisms of the interval.

Remark 1.2. The paper [4] also shows that the mechanism by which $\mathrm{PL}_+([0,1])$ fails to contain non-abelian free subgroups is not an obvious one. In particular, this work proves that $\mathrm{PL}_+([0,1])$ does not satisfy a law; i.e., there does *not* exist $\omega \in F_2 \setminus \{e\}$ such that $\phi(\omega) = \mathrm{id}$ for every homomorphism $\phi\colon F_2 \to \mathrm{PL}_+([0,1])$. It is also the case that $\mathcal{E}$ does not satisfy a law. If such a law were to exist, then it would have to be satisfied by every finite group, since every finite group occurs as a subgroup of $\mathcal{E}$. The existence of such a universal law for finite groups is impossible; in particular, it would imply that F_2 is not residually finite, which is false (see Section III.18 of [8]).

The current work shows that a particular class of IETs, the *disjoint rotation maps*, cannot occur in a non-abelian free subgroup of $\mathcal{E}$, if such subgroups actually do exist. Briefly, a disjoint rotation map r is an IET for which there is a finite partition of $[0,1)$ into r-invariant subintervals I_j, such that r restricted to each I_j is an exchange of two further subintervals; a graphical depiction is given in Figure 1, and a formal definition is provided below.

Theorem 1.3. *Let r be conjugate in $\mathcal{E}$ to a disjoint rotation map, and let $g \in \mathcal{E}$ be an arbitrary interval exchange. Then the subgroup $\langle r, g\rangle$ is not isomorphic to the free group on two generators.*

The author has learned that a proof of this result has also been obtained by F.Dahmani, K.Fujiwara, and V.Guirardel. The proof given here is essentially constructive, in that it describes a nontrivial word in the generators r and g that forms the identity map. The construction strongly relies on two features of the disjoint rotation map r; such maps have iterates that are arbitrarily close to the identity in an L^1 sense, and the iterates r^n have essentially the same number of discontinuities as r itself. It is known ([15], Theorems 1.3 and 1.4) that almost every irreducible IET possesses iterates that are L^1 close to the identity, which raises the question of whether the construction described below can be adapted to such maps. The immediate obstruction is that for most such IETs f, the number of discontinuities of f^n grows linearly with n; this prevents one from showing that a particular word formed from f^n and g has support contained in a neighborhood of a fixed finite set for infinitely many n.

It is interesting to note that conjugates of disjoint rotation maps are precisely those IETs that can occur in the image of a continuous homomorphism $\mathbb{R} \to \mathcal{E}$; see [13]. Thus, Theorem 1.3 and the Tits' Alternative [14] imply that any linear Lie group that continuously embeds in $\mathcal{E}$ must be virtually solvable.

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2. Proof of Theorem 1.3

2.1. Notation. We now give a precise definition and notation for interval exchanges. Let $\pi \in \Sigma_n$ be a permutation of $\{1, 2, \dots, n\}$, and let λ be a vector in the simplex

$$\Lambda_n = \{\lambda = (\lambda_1, \dots, \lambda_n) \mid \lambda_i > 0,\ \sum \lambda_i = 1\} \subseteq \mathbb{R}^n.$$

The vector λ induces a partition of $[0, 1)$ into intervals

$$I_j = \big[\beta_{j-1} := \textstyle\sum_{i=1}^{i=j-1} \lambda_i, \beta_j := \sum_{i=1}^{i=j} \lambda_i\big), \quad 1 \le j \le n. \tag{1}$$

Let $f_{(\pi,\lambda)}$ be the IET that translates each I_j so that the ordering of these intervals within $[0, 1)$ is permuted according to π. More precisely,

$$f_{(\pi,\lambda)}(x) = x + \omega_j \quad \text{if } x \in I_j,$$

where

$$\omega_j = \Omega_\pi(\lambda)_j = \sum_{i\,:\,\pi(i)<\pi(j)} \lambda_i - \sum_{i\,:\,i<j} \lambda_i.$$

Note that $\Omega_\pi \colon \Lambda_n \to \mathbb{R}^n$ is a linear map depending only on π.

The above notation is adapted in the following way to represent a disjoint rotation map. Given $\lambda \in \Lambda_n$ for some n, define the points β_j and the intervals I_j by equation (1). Let $\alpha \in \mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$ be given, where $\mathbb{T}^n$ is to be identified with $[0, 1)^n$.

Define the *disjoint rotation map* $r_{[\alpha,\lambda]}$ by

$$r_{[\alpha,\lambda]}(x) = \begin{cases} x + \lambda_j\alpha_j, & x \in [\beta_{j-1}, \beta_j - \lambda_j\alpha_j), \\ x + \lambda_j\alpha_j - \lambda_j, & x \in [\beta_j - \lambda_j\alpha_j, \beta_j). \end{cases} \tag{2}$$

See Figure 1 for a graphical representation of a disjoint rotation map.

2.2. Construction of a relation in $\langle r, g\rangle$. To begin the proof of Theorem 1.3, it can be assumed after a conjugacy in $\mathcal{E}$ that the map r has the form in equation (2) for some $\alpha \in \mathbb{T}^n$ and $\lambda \in \Lambda_n$. If all points are periodic under r, then r has finite order, and Theorem 1.3 holds trivially. Thus, assume that r has infinite order; equivalently, assume $\alpha \notin (\mathbb{Q}/\mathbb{Z})^n$. If the set of periodic points of r is nonempty, then after

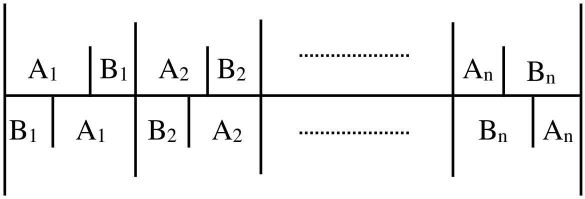

Figure 1. A disjoint rotation map.

replacing r by an iterate it can be assumed that all periodic points of r are fixed. After a further conjugacy and possibly redefining n, it can be assumed that $\mathrm{Fix}(r) = I_n$; in this case, $\alpha = (\alpha_1, \ldots, \alpha_{n-1}, 0)$ where $\alpha_i \in [0, 1)$ is irrational for $1 \le i \le n-1$. Define the *support* of r, denoted by $\mathrm{supp}(r)$, to be the complement of its fixed point set.

A relation in $\langle r, g\rangle$ is constructed using the map $h = [g^{-1}, s^{-1}] \circ [g^{-1}, s]$, where $[x, y]$ denotes the commutator $xyx^{-1}y^{-1}$ and $s = r^M$ for some integer M to be chosen later. It is to be shown that for suitably chosen $s = r^M$, the support of h is contained in a small neighborhood of a finite set.

In particular, define the finite set P by

$$P = \{\beta_i \mid 0 \le i < n\} \cup \{g^{-1}(\beta_i) \mid 0 \le i < n\} \cup \{x \mid g \text{ is discontinuous at } x\}.$$

Let $P' = P \cap \mathrm{supp}(r)$. Since by assumption all non-fixed orbits of r are infinite, it is possible to choose an integer $d > 0$ such that $r^d(P') \cap P' = \emptyset$. For $\varepsilon > 0$ and $p \in [0, 1) \cong \mathbb{R}/\mathbb{Z}$, let $N_\varepsilon(p)$ denote the open ε-ball centered at p in $\mathbb{R}/\mathbb{Z}$; define the sets $X = X_\varepsilon = \bigcup_{p\in P} N_\varepsilon(p)$ and $X' = (X \cap \mathrm{supp}(r))$. Next, fix $\varepsilon > 0$ sufficiently small so that

(i) the collection of sets $\{N_\varepsilon(p) \colon p \in P\}$ are pairwise disjoint, and

(ii) the sets X' and $r^d(X')$ are disjoint.

Finally, choose an integer $M > 0$ such that the rotation rates $M\alpha_i \in \mathbb{R}/\mathbb{Z}$ of $s = r^M$ are in $N_{\varepsilon/10}(0)$ for $i = 1, \ldots, n$. To see that such an M exists, associate to r the translation $\hat{r} \colon \mathbb{T}^n \to \mathbb{T}^n$ defined by $x \mapsto x + \alpha$, and note that the $\hat{r}$-orbit $\{n\alpha \mid n \in \mathbb{Z}\}$ is a dense subset of a nontrivial subgroup of $\mathbb{T}^n$.

Recall that $h = [g^{-1}, s^{-1}] \circ [g^{-1}, s]$, with $s = r^M$ for M as chosen above.

Lemma 2.1. *With notation as defined above, the support of h is contained in X.*

Proof. Let $y \in [0, 1) \setminus X$ be given; it will be shown that h fixes y. Let $j, k \in \{1, \ldots, n\}$ be such that $y \in I_j$ and $g(y) \in I_k$. By the definition of X, y is located a distance of at least ε away from each of the endpoints β_{j-1} and β_j of I_j. Also, y is at least ε away from any discontinuity of g; thus, the entire neighborhood $N_\varepsilon(y)$

is translated under g by $\omega = g(y) - y$. Since P also contains the points $g^{-1}(\beta_i)$ for $0 \leq i < n$, it follows that $g(y)$ is located a distance of at least ε away from the endpoints of I_k.

With these conditions in mind, we trace the orbit of y through the composition

$$h = (g^{-1}s^{-1}gs)(g^{-1}sgs^{-1}).$$

Let γ_j denote the rotation (mod $|I_j|$) induced by s on I_j; define γ_k similarly. By the construction of $s = r^M$, we have $|\gamma_\iota| < \varepsilon/10$ for $\iota \in \{j, k\}$. Thus, $s^{-1}(y) = y - \gamma_j$. As $s^{-1}(y)$ is still located in $N_\varepsilon(y)$, we have $g(s^{-1}(y)) = y - \gamma_j + \omega = g(y) - \gamma_j$. Since $g(y) - \gamma_j$ is still at least a distance of $\varepsilon/2$ away from the endpoints of I_k, we have $s(gs^{-1}(y)) = y - \gamma_j + \omega + \gamma_k = g(y) - \gamma_j + \gamma_k$. The map g^{-1} translates $N_\varepsilon(g(y))$ by $-\omega$, and $(g(y) - \gamma_j + \gamma_k) \in N_\varepsilon(g(y))$. Thus $g^{-1}(sgs^{-1}(y)) = y - \gamma_j + \gamma_k$.

Let $z = g^{-1}sgs^{-1}(y)$, and note that $|y - z| < \varepsilon/5$. Consequently, upon tracing the action of $g^{-1}s^{-1}gs$ on z, reasoning similar to the previous paragraph shows that $g^{-1}s^{-1}gs(z) = z + \gamma_j - \gamma_k = y$. Thus, $h(y) = y$, as desired. □

If it happens that the map h is the identity, then this suffices to show that $\langle r, g\rangle$ is not isomorphic to F_2. Suppose, however, that $h \neq \mathrm{id}$. Recall that d is chosen so that $r^d(X')$ and X' are disjoint. Consider the interval exchange k defined by

$$k = r^d h r^{-d}.$$

By Lemma 2.1, h is supported in X, and consequently, k is supported in $r^d(X)$. If the map r has no periodic points, then $\mathrm{supp}(r) = [0, 1)$ and $X' = X$. Thus, in this situation the maps h and k have disjoint supports. It follows that h and k are commuting, nontrivial elements of $\langle r, g\rangle$ such that $k \notin \langle h\rangle$, which proves that the group $\langle r, g\rangle$ is not isomorphic to F_2 when r has no periodic points.

To handle the general situation where $\mathrm{Fix}(r) \neq \emptyset$, it is shown below that the commutator $T = kh^{-1}k^{-1}h$ has finite order that divides six. This again implies the existence of a nontrivial relation in $\langle r, g\rangle$, completing the proof of Theorem 1.3. The proof that $T^6 = \mathrm{id}$ does not rely on the maps involved being IETs; it follows from the existence of a conjugacy and the relation between the supports of h and k. In the argument below, the *support* of a bijective map again refers to the complement of its set of fixed points.

Proposition 2.2. *Let h and ϕ be bijections of a set Ω. Write* $\mathrm{supp}(h) = A \sqcup B$*, where* $A = \mathrm{supp}(h) \cap \mathrm{supp}(\phi)$ *and* $B = \mathrm{supp}(h) \cap \mathrm{Fix}(\phi)$*. Let $k = \phi h \phi^{-1}$, and let $T = kh^{-1}k^{-1}h$. If $A \cap \phi(A) = \emptyset$, then $T^6 = \mathrm{id}$.*

Remark 2.3. By Lemma 2.1 and the condition that $X' \cap r^d(X') = \emptyset$, it follows that the previously defined IETs h and $\phi = r^d$ satisfy the hypotheses of Proposition 2.2.

Remark 2.4. We give an informal description of the proof of Proposition 2.2 before presenting a detailed argument below. To show that $T = kh^{-1}k^{-1}h$ has finite order, we first show that the dynamics of T can be fully described in a finite, combinatorial manner. In particular, under the given hypotheses the support of T is a disjoint union of three sets, denoted A, B, and C. Using the structure of T and the assumed behavior of the maps h and k, it is shown that for each pair of sets $(\Delta, \Gamma) \in \{A, B, C\}^2$, the action of T is uniform with respect to h and k in mapping a point from Δ to Γ. For example, it is shown that if $p \in A$ and $T(p) \in B$, then it must follow that $T(p) = h(p)$; it is similarly seen that if $p \in B$ and $T(p) \in A$, then $T(p) = h^{-1}(p)$.

Using this combinatorial description, we trace through the various possibilities for the orbit of a point under the map T. It is seen that every point is either fixed or has a periodic orbit of length two or three. For example, since T acts as the map h in sending points from A to B, and since T acts as h^{-1} in sending points back from B to A, any point of A following such an itinerary must be fixed by T^2.

Proof. For notation, let $C = \phi(A)$. Note that the sets A, B and C are pairwise disjoint; $A \cap C = \emptyset$ by assumption, while both $A \cap B = \emptyset$ and $C \cap B = \emptyset$ since $B \subseteq \mathrm{Fix}(\phi)$ and $A \sqcup C \subseteq \mathrm{supp}(\phi)$. Since $A \sqcup B = \mathrm{supp}(h)$, it follows that h fixes all points in C. Similarly, since $k = \phi h \phi^{-1}$, we have $\mathrm{supp}(k) = \phi(\mathrm{supp}(h)) = C \sqcup B$. Thus, k fixes all points in A. We will assume that B is nonempty since otherwise h and k have disjoint supports, in which case $T = [k, h^{-1}] = \mathrm{id}$.

Since $k = \phi h \phi^{-1}$ and ϕ fixes all points in B, the following are true:

(a) If $q \in B$ and $h^{\pm 1}(q) \in B$, then $k^{\pm 1}(q) = h^{\pm 1}(q)$.

(b) If $q \in B$ and $h^{\pm 1}(q) \in A$, then $k^{\pm 1}(q) \in C$.

In addition, since $\mathrm{supp}(h) = A \sqcup B$, one of (a) or (b) must hold for every $q \in B$.

Note that if $p \in \Omega \setminus (A \sqcup B \sqcup C)$, then $T(p) = p$. All other $p \in \Omega$ satisfy the hypotheses of exactly one of the following assertions:

(I) If $p \in A$ and $h(p) \in A$, then $T(p) = p$. **(∗)**

(II) If $p \in A$ and $h(p) \in B$, then $T(p) = h(p)$. **(A → B)**

(III) If $p \in C$ and $k^{-1}(p) \in C$, then $T(p) = p$. **(∗)**

(IV) If $p \in C$ and $k^{-1}(p) \in B$, then one of the following holds:

- (IVa) $h^{-1}k^{-1}(p) \in A$, in which case $T(p) = h^{-1}k^{-1}(p)$. **(C → A)**
- (IVb) $h^{-1}k^{-1}(p) \in B$, in which case $T(p) = k^{-1}(p)$. **(C → B)**

(V) If $p \in B$ and $h(p) \in B$, then one of the following holds:

- (Va) $h^{-1}(p) \in A$, in which case $T(p) = h^{-1}(p)$. **(B → A)**
- (Vb) $h^{-1}(p) \in B$, in which case $T(p) = p$. **(∗)**

(VI) If $p \in B$ and $h(p) \in A$, then $T(p) = k(p)$. **(B → C)**

The assertions in each of the above situations can be verified by tracing the location (either A, B, or C) of the point p through the commutator $T = kh^{-1}k^{-1}h$; in doing this, one uses the conditions (a) and (b) listed above, as well as the fact that $h^{\pm 1} = \mathrm{id}$ on C and $k^{\pm 1} = \mathrm{id}$ on A.

For instance, situation (II) is checked as follows. By assumption, $h(p) \in B$. Then, by (b) applied to $q = h(p)$, we have $k^{-1}h(p) \in C$. Since h^{-1} acts trivially on C, we have $h^{-1}k^{-1}h(p) = k^{-1}h(p)$, and thus $T(p) = k(h^{-1}k^{-1}h(p)) = h(p)$, as claimed. This trace can be summarized in the following way:

$$\text{(II):}\quad \begin{bmatrix} p \\ A \end{bmatrix} \overset{h}{\longmapsto} \begin{bmatrix} h(p) \\ B \end{bmatrix} \overset{k^{-1}}{\longmapsto} \begin{bmatrix} k^{-1}h(p) \\ C \end{bmatrix} \overset{h^{-1}}{\longmapsto} \begin{bmatrix} k^{-1}h(p) \\ C \end{bmatrix} \overset{k}{\longmapsto} \begin{bmatrix} h(p) \\ B \end{bmatrix}.$$

To verify situation (IVa) as a further example, the trace is illustrated by:

$$\text{(IVa):}\quad \begin{bmatrix} p \\ C \end{bmatrix} \overset{h}{\longmapsto} \begin{bmatrix} p \\ C \end{bmatrix} \overset{k^{-1}}{\longmapsto} \begin{bmatrix} k^{-1}(p) \\ B \end{bmatrix} \overset{h^{-1}}{\longmapsto} \begin{bmatrix} h^{-1}k^{-1}(p) \\ A \end{bmatrix} \overset{k}{\longmapsto} \begin{bmatrix} h^{-1}k^{-1}(p) \\ A \end{bmatrix}.$$

In situation (IVa), the application of h is trivial since h is the identity on C, the application of k is trivial since k is the identity on A, and all other labels and locations follow by the assumptions of (IVa). The remaining situations labelled by roman numerals can be verified through similar reasoning.

Several useful observations about T are drawn from the above list of statements. First, note that for each ordered pair $(\Delta, \Gamma) \in \{A, B, C\}^2$, there is at most one situation in the above list such that $p \in \Delta$ and $T(p) \in \Gamma$; all such pairs are represented, with the exception of (A, C). Consequently, having knowledge about how T acts on a point p with respect to the sets $\{A, B, C\}$ implies information about how T acts on p in terms of the maps h and k. As an example, using statement (II), we see that if $p \in A$ and $T(p) \in B$, then it must follow that $T(p) = h(p)$. As a further important example, we see that if p and $T(p)$ are both in the same $\Delta \in \{A, B, C\}$, then p is fixed by T; this observation corresponds to the situations (I), (III), and (Vb) that are marked by **(∗)** in the above list.

These properties are now used to show that $T^6|_B = \mathrm{id}_B$. Let $x \in B$; if $T(x) \in B$, then $T(x) = x$ by statement (Vb). Otherwise, there are two cases to consider:

Case I: Suppose that $T(x) \in A$. Then x must satisfy situation (Va), which implies that $h(x) \in B$ and $h^{-1}(x) \in A$. Thus $y = T(x) = h^{-1}(x)$. It is now the case that $y \in A$ and $h(y) = x \in B$, so y satisfies situation (II). It follows that $T(y) = h(y) = x$, which shows that $T^2(x) = x$ in this case.

Case II: Suppose that $T(x) \in C$. Then x satisfies situation (VI), so $h(x) \in A$ and $y = T(x) = k(x)$. Thus $y \in C$ and $k^{-1}(y) = x \in B$, so y must satisfy either situation (IVa) or (IVb). If y satisfies (IVb), then $T(y) = k^{-1}(y) = x$, in which case $T^2(x) = x$. Otherwise, assume that y satisfies (IVa). Then $h^{-1}k^{-1}(y) = h^{-1}(x) \in A$ and $z = T(y) = h^{-1}k^{-1}(y) = h^{-1}(x) \in A$. Since $z \in A$ and $T(z) \neq z$, it must be the case that z satisfies situation (II). Thus, $T(z) = h(z) = x$, and it follows that $T^3(x) = x$.

As all points $x \in B$ have a T-orbit consisting of one, two, or three points, it follows that $T^6|_B = \mathrm{id}_B$. Considering $T^6|_A$, note that points in $A \cap T^{-1}(A)$ are fixed by T, and note that points in $A \cap T^{-1}(B)$ are fixed by T^6 from the above argument. Since $T(A) \subset A \cup B$, it follows that $T^6|_A = \mathrm{id}_A$. Similarly, since points in $C \cap T^{-1}(C)$ are fixed by T and all other points in C map to $A \cup B$, we also have that $T^6|_C = \mathrm{id}_C$, which completes the proof that $T^6 = \mathrm{id}$. □

References

[1] P. Arnoux, Échanges d'intervalles et flots sur les surfaces. In *Théorie ergodique* (Séminaire des Plans-sur-Bex, 1980), Monograph. Enseign. Math. 29, Univ. Genève, Geneva 1981, 5–38. Zbl 0471.28014 MR 609891

[2] A. Avila and G. Forni, Weak mixing for interval exchange transformations and translation flows. *Ann. of Math.* (2) **165** (2007), 637–664. Zbl 1136.37003 MR 2299743

[3] A. Avila and M. Viana, Simplicity of Lyapunov spectra: proof of the Zorich-Kontsevich conjecture. *Acta Math.* **198** (2007), 1–56. Zbl 1143.37001 MR 2316268

[4] M. G. Brin and C. C. Squier, Groups of piecewise linear homeomorphisms of the real line. *Invent. Math.* **79** (1985), 485–498. Zbl 0563.57022 MR 782231

[5] B. Farb and P. Shalen, Groups of real-analytic diffeomorphisms of the circle. *Ergodic Theory Dynam. Systems* **22** (2002), 835–844. Zbl 1007.37013 MR 1908556

[6] S. Ferenczi, C. Holton, and L. Q. Zamboni, Structure of three-interval exchange transformations III: ergodic and spectral properties. *J. Anal. Math.* **93** (2004), 103–138. Zbl 1094.37005 MR 2110326

[7] É. Ghys, Groups acting on the circle. *Enseign. Math.* (2) **47** (2001), 329–407. Zbl 1044.37033 MR 1876932

[8] P. de la Harpe, *Topics in geometric group theory*. Chicago Lectures in Math., The University of Chicago Press, Chicago 2000. Zbl 0965.20025 MR 1786869

[9] G. Margulis, Free subgroups of the homeomorphism group of the circle. *C. R. Acad. Sci. Paris Sér. I Math.* **331** (2000), 669–674. Zbl 0983.37029 MR 1797749

[10] A. Navas, *Grupos de difeomorfismos del círculo*. Ensaios Matemáticos 13, Sociedade Brasileira de Matemática, Rio de Janeiro 2007. Zbl 1163.37002 MR 2394157

[11] A. Navas, Groups of circle diffeomorphisms. English transl. of [10]. arXiv:math/0607481

[12] C. F. Novak, Discontinuity-growth of interval-exchange maps. *J. Mod. Dyn.* **3** (2009), 379–405. Zbl 1183.37077 MR 2538474

[13] C. F. Novak, Continuous interval exchange actions. *Algebr. Geom. Topol.* **10** (2010), 1609–1625. Zbl 1203.37071 MR 2683746

[14] J. Tits, Free subgroups in linear groups. *J. Algebra* **20** (1972), 250–270. Zbl 0236.20032 MR 0286898

[15] W. A. Veech, The metric theory of interval exchange transformations I. Generic spectral properties. *Amer. J. Math.* **106** (1984), 1331–1359. Zbl 0631.28006 MR 765582

[16] M. Viana, Ergodic theory of interval exchange maps. *Rev. Mat. Complut.* **19** (2006), 7–100. Zbl 1112.37003 MR 2219821

[17] A. Zorich, Flat surfaces. In *Frontiers in number theory, physics, and geometry I*, Springer, Berlin 2006, 437–583. Zbl 1129.32012 MR 2261104

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