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Chain recurrence in β-compactifications of topological groups

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Abstract. Let G be a topological group. In this paper limit behavior in the Stone–Čech compactification βG is studied. It depends on a family of translates of a reversible subsemigroup S. The notion of semitotal subsemigroup is introduced. It is shown that the semitotality property is equivalent to the existence of only two maximal chain transitive sets in βG whenever S is centric. This result links an algebraic property to a dynamical property. The concept of a chain recurrent function is also introduced and characterized via the compactification βG. Applications of chain recurrent function to linear differential systems and transformation groups are done.

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1. Introduction

The Stone–Čech compactification βG of a topological group G is an interesting tool in topological dynamics since each minimal left ideal of βG is a universal minimal set. Many studies in topological dynamics go into the investigation of minimal sets of transformation groups (as reference sources we mention J. Auslander et al. [1], [2], R. Ellis et al. [8], [10], and J. de Vries [17]). Minimal sets have usually been the basic objects of studying dynamics of semigroup actions (see, for instance, [10]). Recently, other dynamical objects were defined for semigroup actions on topological spaces ([3], [4]). Abstract concepts of attractor and chain recurrence are defined in terms of a family of subsets of the semigroup that establishes a direction for limit behavior. In the present paper we use the topological methodology of [3] to study the action of G on βG and determine its chain transitive sets.

Regarding the group of the real numbers $\mathbb{R}$, we could imagine the chain recurrence of the dynamical system $(\mathbb{R}, \beta\mathbb{R})$ occurs in two distinct places of $\beta\mathbb{R}$. In Section 3, we

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go into a general demonstration of this conjecture. We consider a transformation group $(G, \beta G)$ with the assumption that G admits a generating reversible subsemigroup S, that is, $S \subset G$ is a reversible subsemigroup such that $G = S^{-1}S$. The limit behavior in βG is directed by S in the sense that it depends on the family of translates$\mathcal{F} = \{St : t \in S\}$ (see Section 2.1). For example, if $G = \mathbb{R}^n$ and $u_1, \ldots, u_n$ is an ordered basis, the limit behavior in $\beta\mathbb{R}^n$ directed by the cone $S_i = \{(x_1, \ldots, x_n) : x_i \geq 0\}$ means the limit behavior directed by the vector u_i. If S is weakly centric, we verify that the chain recurrence of $(G, \beta G)$ occurs in the limit sets of the identity in G whenever S is semitotal, which means the existence of an element $s \in S$ such that $s^{-1}S \cup sS^{-1} = G$. Moreover, if S is just centric, we show in Theorem 3.9 that the limit sets of the identity are the maximal chain transitive sets in βG if and only if S is semitotal.

In Section 4 we go into the analysis of chain recurrence in hulls of functions. We introduce the concept of a chain recurrent function, which is a generalization of a recurrent (or minimal) function studied by Ellis et al. [8], [9], and R. Johnson [11]. For a group G, a reversible subsemigroup $S \subset G$, and a compact space M, we say that a uniformly continuous function $f : G \to M$ is chain recurrent if its hull $H(f)$ is chain recurrent. Our approach is to investigate the chain recurrence in $H(f)$ from the chain recurrence in βG, since $H(f)$ is a quotient space of βG. If S is centric and semitotal, we show in Theorem 4.4 that a function $f : G \to M$ is chain recurrent if and only if the projections of the maximal chain transitive sets from βG intersect each other in $H(f)$. It follows that either $H(f)$ is chain recurrent or it has exactly two distinct maximal chain transitive sets.

The concept of a chain recurrent function is implicit in the theories of differential systems and topological dynamics. The initial idea involved a linear differential equation

$$\dot{x} = A(t)x \quad (t \in \mathbb{R},\ x \in V),$$

where $A : \mathbb{R} \to L(V)$ is uniformly bounded and uniformly continuous, $V = \mathbb{R}^n$ or $\mathbb{C}^n$, $L(V)$ is the space of linear self-maps of V, with the intention of studying chain recurrence in the hull $H(A)$. Ellis and Johnson [9] found out (1) could be profitably studied by using the theory of cocycles and cocycle flows whenever A is recurrent, that is, whenever $H(A)$ is minimal. Furthermore, Johnson [12] has utilized chain recurrence to study the spectrum of the linear skew-product flow in $V \times H(A)$, and R. J. Sacker and G. R. Sell [15] investigated the gradient-like structure of the flow in $H(A)$. We give some applications to those theories. In Section 5, we present a different way to determine precisely the possibilities of gradient-like structure in $H(A)$. We also show that the chain recurrence for a certain class of transformation groups can be characterized from chain recurrent functions, since each chain recurrent point is associated to a chain recurrent function.

2. Preliminaries

In this section the basic definitions and results are given which enable the study of limit behavior for transformation groups.

Definition 2.1. Let M be a topological space and let G be a topological group with identity e. A *left transformation group* (G, M) is defined by a jointly continuous in each variable separately map $\sigma: G \times M \to M: (g, x) \mapsto gx$, satisfying (i) $ex = x$, and (ii) $(gh)x = g(hx)$, for all $g, h \in G$ and $x \in M$. A *right transformation group* (M, G) is defined by a separately continuous (that is, continuous in each variable separately) map $\sigma: M \times G \to M: (x, g) \mapsto xg$, satisfying (i) $xe = x$, and (ii) $x(gh) = (xg)h$, for all $x \in M$ and $g, h \in G$.

Let (G, M, σ) be a left transformation group. For each $g \in G$ and $x \in M$, we denote by $\sigma_g: M \to M$ the map defined as $\sigma_g(x) = gx$, and we denote by $\sigma_x: G \to M$ the map defined as $\sigma_x(g) = gx$. The map σ_g is a homeomorphism of M and σ_x is continuous, since σ is continuous in each variable separately. For a point $x \in M$, the *orbit of* x is the set $Gx = \{gx : g \in G\}$. A set $X \subset M$ is said to be *invariant* if $Gx \subset X$ whenever $x \in X$. The set X is said to be *minimal* if X is nonempty, closed and invariant, and X has no proper subset with these properties. A closed invariant set X is said to be *isolated* if there is a neighborhood N of X with the following property: if $Y \subset N$ is any invariant set, then $Y \subset X$. One also defines restricted invariance, as follows. A set $X \subset M$ is said to be invariant by a subsemigroup $S \subset G$ (or simply S-invariant) if $Sx \subset X$ whenever $x \in X$.

A subsemigroup $S \subset G$ is *right reversible* if $Ss \cap St \neq \emptyset$ for all $s, t \in S$; and it is *left reversible* if $sS \cap tS \neq \emptyset$ for all $s, t \in S$. The semigroup is *reversible* if it is right and left reversible. If S is a subsemigroup that generates G, then $G = S^{-1}S$ if and only if S is right reversible. This is well known as *Ore's conditions* (see, for instance, [5]). In this case, a subset $X \subset M$ is invariant if and only if it is S-invariant and S^{-1}-invariant. The semigroup S is *weakly left centric* if given any principal left ideal Ss, there is $t \in S$ such that $tS \subseteq Ss$; it is *weakly right centric* if given any principal right ideal sS, there is $t \in S$ such that $St \subseteq sS$. The semigroup is *weakly centric* if it is weakly left centric and weakly right centric. The subsemigroup is *centric* if $sS = Ss$ for all $s \in S$. Note that S is right reversible if it is weakly left centric, and it is left reversible if it is weakly right centric. Summarily, one has the following diagram:

$$\text{centric} \to \text{weakly centric} \to \text{reversible}.$$

Example 2.2. Let G be the group of affine transformations of the real line of the form $t \mapsto at + b$, where $a > 0$, and let S be all those such that $0 < a \leq 1$ and $0 \leq b \leq 1 - a$. Then S is a weakly left centric semigroup.

Definition 2.3. A *homomorphism* of the transformation group (G, M) into the transformation group (G, N) is a continuous map $\pi: M \to N$ such that $\pi(gx) = g\pi(x)$

for all $x \in M$ and $g \in G$. If a homomorphism π is also a homeomorphism of M onto N, it is an *isomorphism* of (G, M) onto (G, N).

Definition 2.4. A *pointed transformation group* (G, X, x_0) is a transformation group, with X compact Hausdorff, together with a distinguished point $x_0 \in X$ such that the orbit closure $\mathrm{cls}(Gx_0) = X$.

Note that a transformation group (G, X) is minimal if and only if (G, X, x) is a pointed transformation group, for all $x \in X$.

2.1. Limit sets and chain recurrence. Now we introduce the notions of limit sets, attractors, and chain recurrence for transformation groups. We refer to papers [3], [4] for unexplained dynamical concepts for semigroup actions. Throughout, there is a fixed transformation group (G, M), with M compact, and a fixed generating reversible subsemigroup $S \subset G$.

Definition 2.5. The following relation in S is defined:

$$\text{for } t, s \in S \text{ let } t \geqslant s \text{ if and only if } t = s \text{ or } t \in Ss.$$

The relation $\geqslant$ is the reverse of the well-known Green's $\mathcal{L}$-preorder of semigroup theory: $t \leq_{\mathcal{L}} s$ if and only if $t = s$ or $t \in Ss$ ([5]). Since S is right reversible, the preorder $\geqslant$ is directed. We consider the limit behavior of (G, M) in this direction. It means the family of translates $\mathcal{F} = \{St : t \in S\}$ may be used to define dynamical concepts. We start by defining limit sets. For all $X \subset M$ and $A \subset G$ we set $AX = \{ax : x \in X, a \in A\}$.

Definition 2.6. The *ω-limit set* of $X \subset M$ is defined as

$$\omega(X) = \bigcap_{t \in S} \mathrm{cls}(StX),$$

and the *ω^*-limit set* of X as

$$\omega^*(X) = \bigcap_{t \in S} \mathrm{cls}(t^{-1}S^{-1}X).$$

The ω-limit set and the ω^*-limit set of X are called *limit sets* of X.

Definition 2.6 generalizes limit sets for classical dynamical systems. In fact, for a dynamical system $\sigma \colon \mathbb{T} \times M \to M$, where $\mathbb{T}$ denotes the real numbers $\mathbb{R}$ or the integers $\mathbb{Z}$, the limit sets of $X \subset M$ are defined by

$$\omega(X) = \bigcap_{t \geq 0} \mathrm{cls}(\sigma(\{\mathbb{T}^+ + t\} \times X)) \quad \text{and} \quad \omega^*(X) = \bigcap_{t \geq 0} \mathrm{cls}(\sigma(\{\mathbb{T}^- - t\} \times X)).$$

By the right reversibility property, the family of translates $\mathcal{F} = \{St : t \in S\}$ is a filter basis on the subsets of S (that is, $\emptyset \notin \mathcal{F}$ and given $A, B \in \mathcal{F}$ there is $C \in \mathcal{F}$ with $C \subset A \cap B$), and it satisfies the following hypotheses:

(1) For all $s \in S$ and $A \in \mathcal{F}$ there exists $B \in \mathcal{F}$ such that $sB \subset A$.
(2) For all $s \in S$ and $A \in \mathcal{F}$ there exists $B \in \mathcal{F}$ such that $Bs \subset A$.
(3) For all $s \in S$ and $A \in \mathcal{F}$ there exists $B \in \mathcal{F}$ such that $B \subset As$.

Hypothesis (1) derives from left reversibility, while (2) and (3) from right reversibility. These hypotheses have been considered in [3] and [4]. Thus, we can refer back to their results. The next one is proved in [3], Propositions 2.10 and 2.12.

Proposition 2.7. *Let $X \subset M$ be a nonempty subset. The limit sets $\omega(X)$ and $\omega^*(X)$ are nonempty and compact; $\omega(X)$ is S-invariant and $\omega^*(X)$ is invariant.*

An immediate consequence of Proposition 2.7 is that a subset $X \subset M$ is minimal if and only if $X = \omega^*(x)$ for all $x \in X$, since $\omega^*(x)$ is compact and invariant.

The next result is proved in [4], Theorem 3.1. It presents a preservation property of limit sets under equivariant maps.

Proposition 2.8. *Let (G, M) and (G, N) be two transformation groups. Suppose that $\pi : M \to N$ is a homomorphism. For a subset X in M, one has*

$$\pi(\omega(X)) = \omega(\pi(X)).$$

Now we define an attractor and a repeller.

Definition 2.9. An *attractor* for a transformation group (G, M) is a set $\mathcal{A}$ which admits a neighborhood V such that $\omega(V) = \mathcal{A}$. A *repeller* is a set $\mathcal{R}$ that has a neighborhood U with $\omega^*(U) = \mathcal{R}$. The neighborhoods V and U are called *attractor neighborhood* of $\mathcal{A}$ and *repeller neighborhood* of $\mathcal{R}$, respectively. We consider the empty set and M as attractors and repellers.

Note that attractors are compact S-invariant sets and repellers are compact invariant sets. Attractors are invariant if S is a centric subsemigroup of G.

For an attractor $\mathcal{A}$, we define

$$\mathcal{A}^* = M \backslash \{x \in M : \omega(x) \subset \mathcal{A}\}.$$

The set $\mathcal{A}^*$ is a repeller ([3], Section 3) called the *complementary repeller* of $\mathcal{A}$, and $(\mathcal{A}, \mathcal{A}^*)$ is called an *attractor-repeller pair*. Each repeller is a complementary repeller of some attractor (see [3], Proposition 3.5). The next result is proved in [3], Proposition 3.6, which is the main property of an attractor-repeller pair.

Proposition 2.10. *Let $\mathcal{A}$ be an attractor and assume that $x \notin \mathcal{A} \cup \mathcal{A}^*$. Then $\omega^*(x) \subset \mathcal{A}^*$ and $\omega(x) \subset \mathcal{A}$.*

Now we define the concept of chain recurrence.

Definition 2.11. For $x, y \in M$, an open covering $\mathcal{U}$ of M and $t \in S$, we define a $(\mathcal{U}, t)$-*chain* from x to y as a sequence $x_0 = x, x_1, \ldots, x_n = y$ in M, elements $t_0, \ldots, t_{n-1} \geqslant t$ and open sets $U_0, \ldots, U_{n-1} \in \mathcal{U}$, such that $t_i x_i, x_{i+1} \in U_i$, for $i = 0, \ldots, n-1$.

Definition 2.12. Let $\mathcal{O}$ be the family of all open coverings of M. The Ω-*chain limit set* of X is defined as

$$\Omega(X) = \bigcap_{\mathcal{U} \in \mathcal{O}, t \in S} \Omega(X, \mathcal{U}, t),$$

where $\Omega(X, \mathcal{U}, t) = \{y \in M :$ there is a point $x \in X$ and a $(\mathcal{U}, t)$-chain from x to $y\}$, and the Ω^*-*chain limit set* of X is defined as

$$\Omega^*(X) = \bigcap_{\mathcal{U} \in \mathcal{O}, t \in S} \Omega^*(X, \mathcal{U}, t),$$

where $\Omega^*(X, \mathcal{U}, t) = \{y \in M :$ there is a point $x \in X$ and a $(\mathcal{U}, t)$-chain from y to $x\}$. The Ω-chain limit set and the Ω^*-chain limit set of X are called *chain limit sets* of X. A point $x \in M$ is *chain recurrent* if $x \in \Omega(x)$. A subset $Y \subset M$ is *chain recurrent* if all the points in Y are chain recurrent. A subset $Y \subset M$ is *chain transitive* if $Y \subset \Omega(x)$ for all $x \in Y$. We denote by $\mathfrak{R}$ the *chain recurrence set*, that is, the set of all chain recurrent points of (G, M).

Remark 2.13. The maximal chain transitive sets (with respect to set inclusion) are the sets

$$E_x = \Omega(x) \cap \Omega^*(x),$$

with $x \in \mathfrak{R}$. Indeed, let E be a maximal chain transitive set and take $x, y \in E$. Then $y \in \Omega(x)$ and $x \in \Omega(y)$, that is, $y \in \Omega(x)$ and $y \in \Omega^*(x)$. Hence, $E \subset \Omega(x) \cap \Omega^*(x)$. On the other hand, for $y, z \in \Omega(x) \cap \Omega^*(x)$, we have $y \in \Omega(x)$ and $x \in \Omega(z)$, hence $y \in \Omega(z)$. This means that $\Omega(x) \cap \Omega^*(x)$ is chain transitive. Since E is maximal satisfying chain transitivity, it follows that $E = \Omega(x) \cap \Omega^*(x)$.

Proposition 2.14. *The minimal subsets of* (G, M) *are chain transitive.*

Proof. Let $X \subset M$ be a minimal subset. For $x, y \in X$, $\mathcal{U} \in \mathcal{O}$ and $t \in S$, choose $t_0 \geqslant t$ and $U_0, U_1 \in \mathcal{U}$ such that $t_0 x \in U_0$ and $y \in U_1$. Since $X = \omega^*(x)$, we have $s^{-1}x \in U_1$ for some $s \in S$. Since $s^{-1}x, t_0 x \in X$, we have $t_0 x \in \omega^*(s^{-1}x)$, hence, $t_1^{-1}s^{-1}x \in U_0$ for some $t_1 \geqslant t$. Then $t_1(t_1^{-1}s^{-1}x) = s^{-1}x \in U_1$. Thus, the points $x, t_1^{-1}s^{-1}x, y \in X$, the elements $t_0, t_1 \geqslant t$, and the open sets $U_0, U_1 \in \mathcal{U}$, define a $(\mathcal{U}, t)$-chain from x to y. $\square$

Note that a set is chain recurrent if it is chain transitive. On the other hand, a chain recurrent set is chain transitive if it is compact and connected. This result is proved in [4], Proposition 4.5.

Proposition 2.15. *Suppose that $N \subset M$ is connected, compact and chain recurrent. Then N is chain transitive.*

The next theorem, which is proved in [3], Propositions 4.3 and 4.7, and [4], Proposition 4.2, relates the limit sets to the chain limit sets.

Proposition 2.16. *For every $x \in M$, the limit set $\omega(x)$ is chain transitive. The chain limit set $\Omega(x)$ is the intersection of all attractors containing $\omega(x)$, and the chain limit set $\Omega^*(x)$ is the intersection of all repellers $\mathcal{A}^*$ such that $x \notin \mathcal{A}$.*

In particular, both chain limit sets $\Omega(x)$ and $\Omega^*(x)$ are compact sets; $\Omega(x)$ is S-invariant and $\Omega^*(x)$ is invariant, for all $x \in M$. We note that $\Omega(x)$ is invariant if S is a centric subsemigroup of G.

Proposition 2.17. *Assume that S is centric. The limit set $\omega^*(x)$ is chain transitive.*

Proof. For $y, z \in \omega^*(x)$, $t \in S$, and $\mathcal{U} \in \mathcal{O}$, take $t_0 \geqslant t$ and $U_0, U_1 \in \mathcal{U}$ such that $t_0 y \in U_0$ and $z \in U_1$. Then $s^{-1}x \in U_1$ for some $s \in S$. Since $t_0 y \in \omega^*(x)$, we have $t^{-1}s^{-1}S^{-1}x \cap U_0 \neq \emptyset$. Since $s^{-1}S^{-1} = S^{-1}s^{-1}$, there is $t_1 \geqslant t$ such that $t_1^{-1}s^{-1}x \in U_0$ and $t_1(t_1^{-1}s^{-1}x) = s^{-1}x \in U_1$. Thus, the points $y, t_1^{-1}s^{-1}x, z \in M$, the elements $t_0, t_1 \geqslant t$, and the open sets $U_0, U_1 \in \mathcal{U}$ define a $(\mathcal{U}, t)$-chain from y to z. □

The following theorem is proved in [3], Theorem 4.1. It extends the well-known Conley theorem in dynamical systems that characterizes the chain recurrence set in terms of attractors

Proposition 2.18. *The chain recurrent set $\mathfrak{R}$ is the set*

$$\bigcap \{\mathcal{A} \cup \mathcal{A}^* : \mathcal{A} \textit{ is an attractor}\}.$$

3. Stone–Čech compactification

This section contains the main results of the paper. We study the action of a topological group G on its Stone–Čech compactification βG. In order to determine the chain transitive sets, we investigate a specific attractor-repeller pair of $(G, \beta G)$. We refer to [18] for the ultrafilter version of the Stone–Čech compactification.

Let G be a noncompact T_4 topological group. The Stone–Čech compactification βG can be described as the set of closed ultrafilters on G provided with the hull-kernel topology. For a closed subset $A \subset G$, the set $\mathrm{h_c}(A) = \{u \in \beta G : A \in u\}$ is a basic closed subset of βG, and for an open subset $U \subset G$, the set $\mathrm{h_o}(U) = \{u \in \beta G :$ there is $A \in u$ with $A \subset U\}$ is a basic open subset of βG.

For each $g \in G$, we have the ultrafilter $u_g = \{A \subset G : g \in A\}$. The mapping $g \to u_g$ is an embedding of G into βG. Thus, we might consider $G \subset \beta G$. Given a subset $B \subset G$, we have $\mathrm{cls}_{\beta G}(B) = \mathrm{h_c}(\mathrm{cls}_G(B))$. The group G acts on the left on βG, as follows. For $(g, u) \in G \times \beta G$, we denote $gu = \{gA : A \in u\}$. Since $\mathrm{cls}(Gu_e) = \beta G$, the transformation group $(G, \beta G, u_e)$ is pointed. If (G, X, x_0) is another pointed transformation group, there is an epimorphism π of βG onto X such that $\pi(u_e) = x_0$. Similarly, G acts on the right on βG by $ug = \{Ag : A \in u\}$.

Remark 3.1. Let $A \subset G$ be a closed set and $U \subset G$ an open set. It is easily seen that $g\mathrm{h_c}(A) = \mathrm{h_c}(gA)$ and $g\mathrm{h_o}(U) = \mathrm{h_o}(gU)$ for all $g \in G$.

Let $S \subset G$ be a proper generating weakly centric subsemigroup, and assume that it is closed and has nonempty interior. The next proposition presents interesting attractors and repellers of $(G, \beta G)$.

Proposition 3.2. *For each $g \in G$, the set $\omega(\mathrm{h_c}(Sg))$ is an attractor and the set $\omega^*(\mathrm{h_c}(S^{-1}g))$ is a repeller in βG.*

Proof. For $s \in \mathrm{int}(S)$, we have $Ssg \subset \mathrm{int}(Sg)$ and $\mathrm{h_c}(Ssg) \subset \mathrm{int}(\mathrm{h_c}(Sg))$. Since S is weakly (left) centric, there is $\tau \in S$ such that $\tau S \subset Ss$. From Remark 3.1, we have

$$S\tau\mathrm{h_c}(Sg) = \bigcup_{t \in S} t\tau\mathrm{h_c}(Sg) = \bigcup_{t \in S} \mathrm{h_c}(t\tau Sg) \subset \mathrm{h_c}(Ssg) \subset \mathrm{int}(\mathrm{h_c}(Sg)).$$

Hence,

$$\omega(\mathrm{h_c}(Sg)) = \bigcap_{t \in S} \mathrm{cls}(St\mathrm{h_c}(Sg)) \subset \mathrm{cls}(S\tau\mathrm{h_c}(Sg)) \subset \mathrm{int}(\mathrm{h_c}(Sg)).$$

Thus, $\omega(\mathrm{h_c}(Sg))$ is an attractor with attractor neighborhood $\mathrm{h_c}(Sg)$. Similarly, we have $\mathrm{h_c}(s^{-1}S^{-1}g) \subset \mathrm{int}(\mathrm{h_c}(S^{-1}g))$. Hence,

$$\begin{aligned}\omega^*(\mathrm{h_c}(S^{-1}g)) &= \bigcap_{t \in S} \mathrm{cls}(t^{-1}S^{-1}\mathrm{h_c}(S^{-1}g)) \subset \mathrm{cls}(s^{-1}S^{-1}\mathrm{h_c}(Sg)) \\ &\subset \mathrm{h_c}(s^{-1}S^{-1}g) \subset \mathrm{int}(\mathrm{h_c}(S^{-1}g)).\end{aligned}$$

Thus, $\omega^*(\mathrm{h_c}(S^{-1}g))$ is a repeller with repeller neighborhood $\mathrm{h_c}(S^{-1}g)$. □

Let $e \in G$ be the identity. The following consequence of Proposition 3.2 presents specific properties of the limit sets of u_e.

Corollary 3.3. *The limit set $\omega^*(u_e)$ is a repeller, and the limit set $\omega(u_e)$ is an intersection of attractors.*

Proof. Note that $\omega^*(u_e) = \omega^*(\mathrm{h_c}(S^{-1}))$, hence, $\omega^*(u_e)$ is a repeller. For all $s, t \in S$, we have $\omega(u_e) \subset \mathrm{h_c}(Stss) \subset St\mathrm{h_c}(Ss)$. Hence, $\omega(u_e) \subset \bigcap_{s\in S}\omega(\mathrm{h_c}(Ss))$. On the other hand, for all $t \in S$, we have

$$\bigcap_{s\in S}\omega(\mathrm{h_c}(Ss)) \subset \omega(\mathrm{h_c}(St)) \subset \mathrm{h_c}(St).$$

Thus, $\bigcap\limits_{s\in S}\omega(\mathrm{h_c}(Ss)) \subset \omega(u_e)$ and $\omega(u_e) = \bigcap\limits_{s\in S}\omega(\mathrm{h_c}(Ss))$. □

We can relate the repeller $\omega^*(\mathrm{h_c}(S^{-1}))$ to the complementary repeller $\omega(\mathrm{h_c}(S))^*$ of $\omega(\mathrm{h_c}(S))$. In general, we have $\omega^*(\mathrm{h_c}(S^{-1})) \subset \omega(\mathrm{h_c}(S))^*$. Indeed, since S is a proper subsemigroup of G, there is an element $t \in S$ such that $t \notin S^{-1}$. Hence, $\mathrm{h_c}(S) \cap \mathrm{h_c}(t^{-1}S^{-1}) = \emptyset$, and $\omega(\mathrm{h_c}(S)) \cap \omega^*(\mathrm{h_c}(S^{-1})) = \emptyset$. Since $\omega^*(\mathrm{h_c}(S^{-1}))$ is compact and invariant, it follows that $\omega^*(\mathrm{h_c}(S^{-1})) \subset \omega(\mathrm{h_c}(S))^*$. Nevertheless, the equality $\omega^*(\mathrm{h_c}(S^{-1})) = \omega(\mathrm{h_c}(S))^*$ does not hold unless S is a semitotal subsemigroup, as follows.

Definition 3.4. A subsemigroup H of G is called *semitotal* if there is an element $h \in H$ such that $h^{-1}H \cup hH^{-1} = G$.

A subsemigroup H of G is called *total* if $H \cup H^{-1} = G$. It is well known that a maximal subsemigroup of a nilpotent group is total (and centric), and a maximal subsemigroup with nonempty interior of a finite dimensional connected solvable Lie group is total (see [13], Theorem 8.3 and Corollary 11.2). Definition 3.4 introduces a semigroup property that is more general than the totality property. Note that total subsemigroups contain the identity of G. Thus, a total subsemigroup is semitotal. On the other hand, there are semitotal subsemigroups which are not total. For instance, let $G = \mathrm{GL}(n, \mathbb{R})^+$ be the group of the real matrices with positive determinant and take a real number $b \geq 1$. The subset $S_b = \{g \in G : \det g \geq b\}$ is a centric semitotal subsemigroup of G, and S_b is not total if $b > 1$.

Theorem 3.5. *The subsemigroup S is semitotal if and only if $\omega^*(\mathrm{h_c}(S^{-1})) = \omega(\mathrm{h_c}(S))^*$.*

Proof. Suppose that S is semitotal and let $u \in \beta G \backslash \omega^*(\mathrm{h_c}(S^{-1}))$. There is an element $\tau \in S$ such that $u \notin \mathrm{h_c}(\tau^{-1}S^{-1})$. Hence, there is $A \in u$ with $A \subset G \setminus \tau^{-1}S^{-1}$. Take $s \in S$ such that $s^{-1}S \cup sS^{-1} = G$. We have $\tau^{-1}s^{-2}S \cup \tau^{-1}S^{-1} = G$, hence, $G \setminus \tau^{-1}S^{-1} \subset \tau^{-1}s^{-2}S$. Thus, $u \in \mathrm{h_c}(\tau^{-1}s^{-2}S)$ and $Ss^2\tau u \subset \mathrm{h_c}(S)$. Since $\omega(\mathrm{h_c}(S))^*$ is invariant and $\omega(\mathrm{h_c}(S))^* \cap \mathrm{h_c}(S) = \emptyset$, we have $u \in \beta G \setminus \omega(\mathrm{h_c}(S))^*$. Hence, $\omega(\mathrm{h_c}(S))^* \subset \omega^*(\mathrm{h_c}(S^{-1}))$, and since $\omega^*(\mathrm{h_c}(S^{-1})) \subset \omega(\mathrm{h_c}(S))^*$, we have $\omega(\mathrm{h_c}(S))^* = \omega^*(\mathrm{h_c}(S^{-1}))$. Conversely, suppose that $\omega^*(\mathrm{h_c}(S^{-1})) = \omega(\mathrm{h_c}(S))^*$. If $u \notin \omega(\mathrm{h_c}(S)) \cup \omega^*(\mathrm{h_c}(S^{-1}))$, Proposition 2.10 says that $\omega(u) \subset \omega(\mathrm{h_c}(S))$ and $\omega^*(u) \subset \omega^*(\mathrm{h_c}(S^{-1}))$. Suppose by contradiction that S is not semitotal, that is,

$t^{-1}S \cup tS^{-1} \neq G$ for every $t \in S$. This implies the set

$$F = \bigcap_{t\in S} \mathrm{h_c}(G \setminus (\mathrm{int}(t^{-1}S) \cup \mathrm{int}(tS^{-1})))$$

is nonempty and invariant. In fact, take the family of compact subsets in βG

$$\mathcal{F} = \{\mathrm{h_c}(G \setminus (\mathrm{int}(t^{-1}S) \cup \mathrm{int}(tS^{-1}))) : t \in S\}.$$

Since S is reversible, for a finite sequence of elements $t_1, \ldots, t_n \in S$, we can take $t \in t_1St_1 \cap \cdots \cap t_nSt_n$. Then $t_i^{-1}S \subset t^{-1}S$ and $t_iS^{-1} \subset tS^{-1}$, for $i = 1, \ldots, n$. It follows that

$$\bigcup_{i=1}^{n} \mathrm{int}(t_i^{-1}S) \cup \mathrm{int}(t_iS^{-1}) \subset t^{-1}S \cup tS^{-1}.$$

Since $t^{-1}S \cup tS^{-1} \neq G$, the set $G \setminus \bigcup_{i=1}^{n} \mathrm{int}(t_i^{-1}S) \cup \mathrm{int}(t_iS^{-1})$ is not empty, that is,

$$\bigcap_{i=1}^{n} G \setminus \mathrm{int}(t_i^{-1}S) \cup \mathrm{int}(t_iS^{-1}) \neq \emptyset.$$

Hence, $\bigcap_{i=1}^{n} \mathrm{h_c}(G \setminus \mathrm{int}(t_i^{-1}S) \cup \mathrm{int}(t_iS^{-1}))$ is a nonempty subset of βG. Since βG is compact Hausdorff, it follows that F is nonempty and compact. For showing the invariance of F, let $u \in F$ and $s \in S$. For $t \in S$, take $s_1, s_2 \in S$ such that $ss_1 = ts_2$. Then

$$G \setminus (\mathrm{int}(s^{-1}t^{-1}s_1^{-1}S) \cup \mathrm{int}(s_1tsS^{-1})) \in u,$$

and

$$G \setminus (\mathrm{int}(t^{-1}s_1^{-1}S) \cup \mathrm{int}(ts_2tsS^{-1})) \in su.$$

Since $t^{-1}S \subset t^{-1}s_1^{-1}S$ and $tS^{-1} \subset ts_2tsS^{-1}$, we have

$$G \setminus (\mathrm{int}(t^{-1}s_1^{-1}S) \cup \mathrm{int}(ts_2tsS^{-1})) \subset G \setminus (\mathrm{int}(t^{-1}S) \cup \mathrm{int}(tS^{-1})).$$

Thus, $G \setminus (\mathrm{int}(t^{-1}S) \cup \mathrm{int}(tS^{-1})) \in su$. Since t is arbitrary, it follows that $su \in F$. Now, take $s_1, s_2 \in S$ such that $s_1s = s_2t$. Then

$$G \setminus (\mathrm{int}(s_1^{-1}t^{-1}s^{-1}S) \cup \mathrm{int}(sts_1S^{-1})) \in u,$$

and

$$G \setminus (\mathrm{int}(t^{-1}s_2^{-1}t^{-1}s^{-1}S) \cup \mathrm{int}(ts_1S^{-1})) \in s^{-1}u.$$

Hence, $G \setminus (\mathrm{int}(t^{-1}S) \cup \mathrm{int}(tS^{-1})) \in s^{-1}u$, and $s^{-1}u \in F$. Thus, F is invariant by S and S^{-1}, whence the invariance of F. Finally, for $t \in \mathrm{int}(S)$, we have $tS \subset \mathrm{int}(S)$ and $t^{-1}S^{-1} \subset \mathrm{int}(S^{-1})$. Since

$$\begin{aligned} F &\subset \mathrm{h_c}(G \setminus (\mathrm{int}(S) \cup \mathrm{int}(S^{-1}))) \\ &= \mathrm{h_c}((G \setminus \mathrm{int}(S)) \cap (G \setminus \mathrm{int}(S^{-1}))) \\ &= \mathrm{h_c}(G \setminus \mathrm{int}(S)) \cap \mathrm{h_c}(G \setminus \mathrm{int}(S^{-1})), \end{aligned}$$

we have $F \cap (\omega(\mathrm{h_c}(S)) \cup \omega^*(\mathrm{h_c}(S^{-1}))) = \emptyset$, which is a contradiction since an attractor-repeller pair intersects all invariant closed sets (in contrast with Proposition 2.10). Therefore, S is semitotal. □

3.1. Chain recurrence. Now we go into the investigation of the chain transitive sets in βG. Let $\mathcal{O}$ be the family of all open coverings of βG.

Proposition 3.6. *The limit set $\omega(u_e)$ is a maximal chain transitive set in βG. If S is centric, the limit set $\omega^*(u_e)$ is a maximal chain transitive set in βG.*

Proof. For all $u \in \omega(u_e)$, we have $\omega(u) \subset \omega(u_e)$. Since $\omega(u_e)$ is an intersection of attractors and it is chain transitive, Proposition 2.16 says that $\Omega(u) \subset \omega(u_e)$ and $\Omega(u) \cap \Omega^*(u) = \omega(u_e)$, whence $\omega(u_e)$ is a maximal chain transitive set. By assuming that S is centric, Proposition 2.17 says that $\omega^*(u_e)$ is chain transitive. Since $\omega^*(u_e)$ is a repeller, Proposition 2.18 guarantees that it is a maximal chain transitive set. □

Since $\omega(u_e) \neq G$, we have the following consequence.

Corollary 3.7. *The transformation group $(G, \beta G)$ is not chain transitive.*

The next result shows that every point in the maximal chain transitive set $\omega(u_e)$ is chain attainable from any point in βG . If S is centric, every point of βG is chain attainable from any point of the limit set $\omega^*(u_e)$.

Proposition 3.8. *For all $u \in \omega(u_e)$, one has $\Omega^*(u) = \beta G$ and $\Omega(u) = \omega(u_e)$. If S is centric, then $\Omega(v) = \beta G$ and $\Omega^*(v) = \omega^*(u_e)$ for all $v \in \omega^*(u_e)$.*

Proof. Let $v \in \beta G$, $\mathcal{U} \in \mathcal{O}$ and $t \in S$. Let $\{\mathrm{h_o}(U_1), \ldots, \mathrm{h_o}(U_n)\}$ be a refinement covering of $\mathcal{U}$ given by open sets of the topology basis. For a point $u \in \omega(u_e)$, take $\mathrm{h_o}(U_i), \mathrm{h_o}(U_j) \in \mathcal{U}$ and $t_0 \geqslant t$ such that $u \in \mathrm{h_o}(U_i)$ and $t_0 v \in \mathrm{h_o}(U_j)$. Choose $g \in U_j$. We have

$$t_0 v, u_g \in \mathrm{h_o}(U_j).$$

By writing $g = s_1^{-1} s_2$, with $s_1, s_2 \in S$, take $s \in St \cap Ss_1$. Then $sg \in S$. Since $\omega(u_e) = \bigcap_{t \in S} \mathrm{h_c}(St)$, it follows that $Ssg \in u$, and $Ssg \cap U_i \neq \emptyset$. Hence, there is $t_1 \geqslant t$ such that $t_1 g \in U_i$. Thus, we have

$$t_1 u_g, u \in \mathrm{h_o}(U_i).$$

Take $V_0, V_1 \in \mathcal{U}$ such that $\mathrm{h_o}(U_j) \subset V_0$ and $\mathrm{h_o}(U_i) \subset V_1$. The points $v, u_g, u \in \beta G$, the elements $t_0, t_1 \geqslant t$, and the open sets $V_0, V_1 \in \mathcal{U}$, define a $(\mathcal{U}, t)$-chain from v to u. Therefore, $\beta G = \Omega^*(u)$. The equality $\Omega(u) = \omega(u_e)$ follows from Proposition 3.6. The proof of the second part of the proposition is similar. Suppose that S is centric. Let $z \in \beta G$, $\mathcal{U} \in \mathcal{O}$, and $t \in S$. Let $\{\mathrm{h_o}(U_1), \ldots, \mathrm{h_o}(U_n)\}$ be a refinement covering of $\mathcal{U}$ given by open sets of the topology basis. For $v \in \omega^*(u_e)$, take $\mathrm{h_o}(U_i), \mathrm{h_o}(U_j) \in \mathcal{U}$ and $t_0 \geqslant t$ such that $t_0 v \in \mathrm{h_o}(U_i)$ and $z \in \mathrm{h_o}(U_j)$. Choose $g \in U_j$. By the invariance of S, there is $s \geqslant t$ such that $s^{-1}g \in S^{-1}t^{-1}$. Since $\omega^*(u_e) = \bigcap_{t \in S} \mathrm{h_c}(S^{-1}t^{-1})$, we have $S^{-1}t_0^{-1}s^{-1}g \in v$. Hence, $S^{-1}s^{-1}g \in t_0 v$ and $S^{-1}s^{-1}g \cap U_i \neq \emptyset$. It implies that there are $t_1 \geqslant t$ and $g_1 \in U_i$ such that $t_1 g_1 = g$. Thus, we have

$$t_0 v, u_{g_1} \in \mathrm{h_o}(U_i) \quad \text{and} \quad t_1 u_{g_1}, z \in \mathrm{h_o}(U_j).$$

Take $V_0, V_1 \in \mathcal{U}$ such that $\mathrm{h_o}(U_i) \subset V_0$ and $\mathrm{h_o}(U_j) \subset V_1$. The points $v, u_{g_1}, z \in \beta G$, the elements $t_0, t_1 \geqslant t$, and the open sets $V_0, V_1 \in \mathcal{U}$, define a $(\mathcal{U}, t)$-chain from v to z. Therefore, $\beta G = \Omega(v)$. Since $\omega^*(u_e)$ is a maximal chain transitive set, we have the equality $\Omega^*(u) = \omega^*(u_e)$. □

The main result can now be proved. Note that $u \in \omega(\mathrm{h_f}(S))^*$ whenever $u \in \mathfrak{R} \setminus \omega(u_e)$. Hence, we have $\mathfrak{R} \subset \omega(u_e) \cup \omega(\mathrm{h_f}(S))^*$. Since $\omega^*(u_e) = \omega^*(\mathrm{h_f}(S^{-1}))$, Theorem 3.5 means the subsemigroup S is semitotal if and only if $\mathfrak{R} \subset \omega(u_e) \cup \omega^*(u_e)$. If $\omega^*(u_e)$ is chain transitive, it follows that S is semitotal if and only if $\mathfrak{R} = \omega(u_e) \cup \omega^*(u_e)$. The main theorem is completed from Proposition 2.17, as follows.

Theorem 3.9. *Assume that S is centric. Then S is semitotal if and only if $\omega(u_e)$ and $\omega^*(u_e)$ are the maximal chain transitive sets in βG. Equivalently, S is semitotal if and only if $(\omega(u_e), \omega^*(u_e))$ is the only nontrivial attractor-repeller pair in βG.*

3.2. Nonuniversality. The universality is the main property of a minimal left ideal $M \subset \beta G$. For each minimal transformation group (G, X), there is an epimorphism $M \to X$. Naturally, we have thought about an aspect of universality of a maximal chain transitive set $E \subset \beta G$. We have conjectured that, for each chain transitive transformation group (G, X), there is an epimorphism $E \to X$. Nevertheless, from simple argument, we show that E does not have such a property. Indeed, let X be a connected compact space with $\mathrm{card}(X) > \mathrm{card}(\beta G)$. As an example of such a space, take the family $Y = \mathcal{P}(\beta G)$ of all subsets of βG provided with a topology. We have $\mathrm{card}(Y) > \mathrm{card}(\beta G)$. The pathwise connectification X of βY is a connected compact space and $\mathrm{card}(X) > \mathrm{card}(\beta G)$. Consider the trivial action of G on X, that is, G fixes the points in X. Each point in X is a minimal subset, and X is chain recurrent. By Proposition 2.15, it follows that X is chain transitive. But, there is not any surjective function of βG onto X, since $\mathrm{card}(X) > \mathrm{card}(\beta G)$. In particular, an epimorphism $E \to X$ does not exist.

4. Chain recurrent functions

In this section we start the second part of the paper. We introduce the notion of a chain recurrent function. Let G be a T_4 compactly generated group provided with the left uniformity. Fix a proper generating weakly centric subsemigroup $S \subset G$, and assume that it is closed and it has nonempty interior. Let M be a compact Hausdorff space. We denote by $C(G, M)$ the space of all continuous functions of G into M provided with the compact-open topology.

Definition 4.1. Let $f: G \to M$ be a uniformly continuous function. The *hull* of f, $H(f)$, is the subspace $\mathrm{cls}\{f \cdot g : g \in G\} \subset C(G, M)$, where $f \cdot g(h) = f(gh)$ for all $g, h \in G$.

The triple $(H(f), G, f)$ is a pointed transformation group. By Ascoli's theorem, it follows that $H(f)$ is compact. It assures $H(f)$ is isomorphic with a quotient space of βG.

Definition 4.2. Let $f: G \to M$ be a uniformly continuous function and let $\tilde{f}: \beta G \to M$ be the extension of f. Then f defines an equivalence relation (f) in βG, as follows. Let $u \equiv v$ if and only if $\tilde{f}(ug) = \tilde{f}(vg)$ for all $g \in G$. The quotient space $\beta G / (f)$ is denoted by $\mathrm{sp}(f)$, and called *space of* f.

The action of G on βG induces an action on $\mathrm{sp}(f)$ via the quotient map $\pi: \beta G \to \mathrm{sp}(f)$. The triple $(\mathrm{sp}(f), G, \pi(u_e))$ is a pointed transformation group. Then the map $\Phi\colon f \cdot g \to \pi(u_g)$ extends to an isomorphism of $(H(f), G)$ onto $(\mathrm{sp}(f), G)$.

Definition 4.3. A uniformly continuous function $f: G \to M$ is called *chain recurrent* if $H(f)$ is chain recurrent.

A uniformly continuous function $f: G \to M$ is chain recurrent if and only if $H(f)$ is chain transitive. Indeed, we have $f \cdot g \in \Omega^*(f \cdot g)$ for all $g \in G$ if f is chain recurrent. Since $\Omega^*(f \cdot g)$ is compact and invariant, it follows that $H(f) = \Omega^*(f \cdot g)$ for all $g \in G$. Thus, $H(f)$ is chain transitive.

The following theorem characterizes the chain recurrent functions.

Theorem 4.4. *Assume that S is centric and semitotal. A uniformly continuous function $f: G \to M$ is chain recurrent if and only if $\omega(f) \cap \omega^*(f) \neq \emptyset$ in $H(f)$. Equivalently, the function f is not chain recurrent if and only if $\omega(f)$ and $\omega^*(f)$ are the maximal chain transitive sets in $H(f)$.*

Proof. Suppose that f is chain recurrent and let $\pi: \beta G \to \mathrm{sp}(f)$ be the quotient map. Suppose by contradiction that $\omega(f) \cap \omega^*(f) = \emptyset$ in $H(f)$. Then $\omega(\pi(u_e)) \cap \omega^*(\pi(u_e)) = \emptyset$ in $\mathrm{sp}(f)$. Hence,

$$\bigcap_{t,s \in S} \mathrm{cls}(\pi(u_e)St) \cap \mathrm{cls}(\pi(u_e)S^{-1}s^{-1}) = \emptyset.$$

Since $\mathrm{h_c}(St) = \mathrm{cls}(St)$, we have $\pi(\mathrm{h_c}(St)) \subset \mathrm{cls}(\pi(St))$. On the other hand, since $St \subset \mathrm{h_c}(St)$, we have $\pi(St) \subset \pi(\mathrm{h_c}(St))$. The compactness of $\pi(\mathrm{h_c}(St))$ implies that $\mathrm{cls}(\pi(St)) \subset \pi(\mathrm{h_c}(St))$. Hence, $\pi(\mathrm{h_c}(St)) = \mathrm{cls}(\pi(St)) = \mathrm{cls}(\pi(u_e)St)$. Analogously, $\pi(\mathrm{h_c}(S^{-1}t^{-1})) = \mathrm{cls}(\pi(u_e)S^{-1}t^{-1})$. Thus, we have

$$\bigcap_{t,s\in S} \pi(\mathrm{h_c}(St)) \cap \pi(\mathrm{h_c}(S^{-1}s^{-1})) = \emptyset.$$

Since $\mathrm{sp}(f)$ is compact and Hausdorff, it follows that the family of closed sets

$$\mathcal{F} = \{\pi(\mathrm{h_c}(St)) \cap \pi(\mathrm{h_c}(S^{-1}s^{-1})) : t, s \in S\}$$

does not satisfy the property of finite intersection. It means there are elements $t_1, \ldots, t_n, s_1, \ldots, s_n \in S$ such that

$$\pi(\mathrm{h_c}(St_1)) \cap \pi(\mathrm{h_c}(S^{-1}s_1^{-1})) \cap \cdots \cap \pi(\mathrm{h_c}(St_n)) \cap \pi(\mathrm{h_c}(S^{-1}s_n^{-1})) = \emptyset.$$

Take $t_0 \in St_1 \cap \cdots \cap St_n$ and $s_0 \in Ss_1 \cap \cdots \cap Ss_n$. We have

$$\pi(\mathrm{h_c}(St_0)) \subset \bigcap_{i=1}^{n} \pi(\mathrm{h_c}(St_i)) \quad \text{and} \quad \pi(\mathrm{h_c}(S^{-1}s_0^{-1})) \subset \bigcap_{i=1}^{n} \pi(\mathrm{h_c}(S^{-1}s_i^{-1})).$$

It follows that $\pi(\mathrm{h_c}(St_0)) \cap \pi(\mathrm{h_c}(S^{-1}s_0^{-1})) = \emptyset$. By the semitotality property, there is $s \in S$ such that $G = Ss^{-2} \cup S^{-1}$. Then $G = Ss^{-2}s_0^{-1} \cup S^{-1}s_0^{-1}$ and $\beta G = \mathrm{h_c}(Ss^{-2}s_0^{-1}) \cup \mathrm{h_c}(S^{-1}s_0^{-1})$. Hence, $\mathrm{sp}(f) = \pi(\mathrm{h_c}(Ss^{-2}s_0^{-1})) \cup \pi(\mathrm{h_c}(S^{-1}s_0^{-1}))$. Since $\pi(\mathrm{h_c}(S^{-1}s_0^{-1}))$ is compact, the set $\mathrm{sp}(f) \setminus \pi(\mathrm{h_c}(S^{-1}s_0^{-1}))$ is open and it is contained in $\pi(\mathrm{h_c}(Ss^{-2}s_0^{-1}))$. Moreover, $\pi(\mathrm{h_c}(St_0)) \subset \mathrm{sp}(f) \setminus \pi(\mathrm{h_c}(S^{-1}s_0^{-1}))$. Hence, $\pi(\mathrm{h_c}(St_0)) \subset \mathrm{int}(\pi(\mathrm{h_c}(Ss^{-2}s_0^{-1})))$, and $\pi(\mathrm{h_c}(St_0s_0s^2)) \subset \mathrm{int}(\pi(\mathrm{h_c}(S)))$. By Proposition 2.8, we have $\omega(\pi(\mathrm{h_c}(S))) = \pi(\omega(\mathrm{h_c}(S)))$. Hence,

$$\omega\big(\pi(\mathrm{h_c}(S))\big) = \pi\Big(\bigcap_{t,s\in S} \mathrm{cls}(\mathrm{h_c}(S)St)\Big) \subset \pi(\mathrm{h_c}(St_0s_0s^2)) \subset \mathrm{int}(\pi(\mathrm{h_c}(S))),$$

which means $\omega(\pi(\mathrm{h_c}(S)))$ is an attractor with attractor neighborhood $\pi(\mathrm{h_c}(S))$. Similarly, we show that $\omega(\pi(\mathrm{h_c}(Ss)))$ is an attractor for all $s \in S$. By Proposition 2.18, the chain transitivity of $\mathrm{sp}(f)$ implies that $\omega(\pi(\mathrm{h_c}(Ss))) = \mathrm{sp}(f)$ for all $s \in S$. Since $\omega(u_e) = \bigcap_{s\in S}\omega(\mathrm{h_c}(Ss))$, we have $\omega(\pi(u_e)) = \bigcap_{s\in S}\omega(\pi(\mathrm{h_c}(Ss))) = \mathrm{sp}(f)$, and $\omega(f) = H(f)$. Thus, $\omega^*(f) = \emptyset$, which contradicts the Proposition 2.7. Therefore, $\omega(f) \cap \omega^*(f) \neq \emptyset$. For showing the converse, it is enough to show that, if f is not chain recurrence, then $\omega(f)$ and $\omega^*(f)$ are the maximal chain transitive sets in $H(f)$. Suppose that f is not chain recurrent. Let $z \in \mathrm{sp}(f) \setminus (\omega(\pi(u_e)) \cup \omega^*(\pi(u_e)))$. Then $z \notin \pi(\omega(u_e)) \cup \pi(\omega^*(u_e))$ since $\pi(\omega(u_e)) \subset \omega(\pi(u_e))$ and $\pi(\omega^*(u_e)) \subset \omega^*(\pi(u_e))$. Hence, $z = \pi(u)$ for some $u \in \beta G \setminus (\omega(u_e) \cup \omega^*(u_e))$. Since $\omega(u) \subset \omega(u_e)$ and $\omega^*(u) \subset \omega^*(u_e)$, we have

$$\omega(z) \cap \omega(\pi(u_e)) \neq \emptyset \quad \text{and} \quad \omega^*(z) \cap \omega^*(\pi(u_e)) \neq \emptyset.$$

Hence, z is chain recurrent if the union $\omega(\pi(u_e)) \cup \omega^*(\pi(u_e))$ is chain transitive. Since $\mathrm{sp}(f)$ is not chain recurrent, it follows that the union $\omega(\pi(u_e)) \cup \omega^*(\pi(u_e))$

is not chain transitive. Thus, $\omega(\pi(u_e)) \subset E$ and $\omega^*(\pi(u_e)) \subset E^*$, where E and E^* are distinct maximal chain transitive sets in $\mathrm{sp}(f)$. Now, let $z \in \mathrm{sp}(f) \setminus (E \cup E^*)$. Then $z \notin \pi(\omega(u_e)) \cup \pi(\omega^*(u_e))$, hence, $\omega(z) \subset E$ and $\omega^*(z) \subset E^*$ since $\omega(z) \cap \omega(\pi(u_e)) \neq \emptyset$ and $\omega^*(z) \cap \omega^*(\pi(u_e)) \neq \emptyset$. Thus, E and E^* are the only maximal chain transitive sets in $\mathrm{sp}(f)$. Finally, for $w \in E$, we have $\omega(w), \omega^*(w) \subset E$, which implies $w \in \pi(\omega(u_e))$. Therefore, $E = \omega(\pi(u_e))$. Similarly, for $w' \in E^*$, we have $w' \in \pi(\omega^*(u_e))$, and $E^* = \omega^*(\pi(u_e))$. The theorem is proved. □

Note that if S is centric and semitotal, a function $f : G \to M$ is not chain recurrent if and only if the transformation group $(\mathrm{sp}(f), G)$ has the same dynamical behavior as $(\beta G, G)$. In other words, the function f is not chain recurrent if and only if the structure of the transformation group $(\beta G, G)$ does not change under the quotient map on $\mathrm{sp}(f)$. Thus, we have only two alternatives for chain recurrence in $H(f)$, as follows.

Corollary 4.5. *Assume that the subsemigroup S is centric and semitotal. Either $H(f)$ is chain transitive or $\omega(f)$ and $\omega^*(f)$ are the only maximal chain transitive sets in $H(f)$.*

Let us see a consequence of Theorem 4.4 that justifies our intention of generalizing recurrent function. A uniformly continuous function $f : G \to M$ is recurrent if and only if $H(f)$ is minimal (contrast with [1], [8], [11]). Since a minimal set is chain transitive, the function f is chain recurrent if it is recurrent. Nevertheless, there are chain recurrent functions which are not recurrent functions, as shown by the next result.

Corollary 4.6. *Let $f : G \to M$ be a uniformly continuous function and consider the two nets $(f(t))_{t\in S}$ and $(f(t^{-1}))_{t\in S}$ directed by $\geqslant$. Assume that the two limits*

$$L_1 = \lim_{t\in S} f(t^{-1}) \quad \textit{and} \quad L_2 = \lim_{t\in S} f(t)$$

exist. Then f is chain recurrent if and only if $L_1 = L_2$.

Proof. It is enough to observe that L_1 and L_2 define constant functions in $H(f)$, where $\omega^*(f) = \{L_1\}$, and $\omega(f) = \{L_2\}$. The proof follows from Theorem 4.4. □

An asymptotic chain recurrent function f like in Corollary 4.6 is not recurrent, except if it is constant. Indeed, if $L = \lim_{t\in S} f(t)$, then $\mathrm{cls}(L \cdot G) = \{L\}$. If f is not constant, it follows that $\mathrm{cls}(L \cdot G) \neq H(f)$, and $H(f)$ is not minimal. Thus, the class of chain recurrent functions is really larger than the class of recurrent functions.

5. Applications

In this last section we apply the results of the present paper to linear differential systems and to topological dynamics.

5.1. Linear differential systems. For an $n \times n$ matrix-valued function $A(t) \subset L(V)$, we consider its linear skew-product flow on $V \times H(A)$, and formulate an alternative theorem for the main result of Sacker–Seel [15].

Let V be the space $\mathbb{R}^n$ or $\mathbb{C}^n$ and let $L(V)$ be the space of linear self maps of V. Assume that $A\colon \mathbb{R} \to L(V)$ is uniformly bounded and uniformly continuous. Consider the linear differential equation

$$\dot{x} = A(t)x \quad (t \in \mathbb{R},\ x \in V).$$

We define $\widetilde{A}\colon H(A) \to L(V)$ by $\widetilde{A}(\xi) = \xi(0)$, and consider the collection of ODEs

$$\dot{x} = \widetilde{A}(\xi \cdot t)x \quad (\xi \in H(A)).$$

The solution of the initial value problem $\dot{x} = \widetilde{A}(\xi \cdot t)x$, $x(0) = x$, is denoted by $\varphi(x, \xi, t)$. The linear skew-product flow in $V \times H(A)$ is defined by

$$\sigma(x, \xi, t) = (\varphi(x, \xi, t), \xi \cdot t).$$

The main result of [15] discusses the gradient-like structure of the flow in $H(A)$ from the skew-product flow in $V \times H(A)$. To explain it we need some definitions. The bounded set $\mathfrak{B}$, the stable set $\mathfrak{S}$, and the unstable set $\mathfrak{U}$, are defined as

$$\begin{aligned}
\mathfrak{B} &= \{(x, \xi) \in V \times H(A) : \|\varphi(x, \xi, t)\| \text{ is uniformly bounded in } t\},\\
\mathfrak{S} &= \{(x, \xi) \in V \times H(A) : \|\varphi(x, \xi, t)\| \to 0 \text{ as } t \to +\infty\},\\
\mathfrak{U} &= \{(x, \xi) \in V \times H(A) : \|\varphi(x, \xi, t)\| \to 0 \text{ as } t \to -\infty\},
\end{aligned}$$

where $\|\cdot\|$ denotes a norm on V. For every $\xi \in H(A)$, the sections $\mathfrak{S}(\xi)$, $\mathfrak{U}(\xi)$ are defined by

$$\begin{aligned}
\mathfrak{S}(\xi) &= \{x \in V : (x, \xi) \in \mathfrak{S}\},\\
\mathfrak{U}(\xi) &= \{x \in V : (x, \xi) \in \mathfrak{U}\},
\end{aligned}$$

which are linear subspaces of V. For $n = \dim V$ and $k = 0, 1, \ldots, n$, we define

$$Y_k = \{\xi \in H(A) : \dim \mathfrak{S}(\xi) = k \text{ and } \dim \mathfrak{U}(\xi) = n - k\}.$$

The sets Y_1, …, Y_n are isolated and pairwise disjoin (see [15], Lemmas 10 and 13). The main results of [15] are done under the *Standing Hypotheses*, which means here the hypothesis $\mathfrak{B} = \{0\} \times H(A)$. Two cases are distinguished:

(1) There is precisely one nonempty Y_k. In this case, $H(A) = Y_k$ for some k (see [15], Theorem 2).

(2) There are at least two nonempty Y_k. In this case, define

$$Q = \max\{k : Y_k \text{ is nonempty}\} \quad \text{and} \quad q = \min\{k : Y_k \text{ is nonempty}\}.$$

Then Y_q is an attractor and Y_Q is a repeller in $H(A)$, with $H(A) \neq Y_q$. Moreover, every point $\xi \in H(A)$ has its limit sets in some Y_k (see [15], Theorem 3).

Note that if there are at least two nonempty Y_k, the flow in $H(A)$ has a nontrivial Morse decomposition. However, there are only two possibilities for dynamical behavior in $H(A)$: either $H(A)$ is chain transitive or there is only one nontrivial attractor-repeller pair in $H(A)$, which is $(\omega(A), \omega^*(A))$ (Corollary 4.5). Hence, the attractor-repeller pair $(\omega(A), \omega^*(A))$ is the only nontrivial Morse decomposition in $H(A)$. Therefore, we can establish an alternative theorem for the result of [15] mentioned above.

Theorem 5.1. *Assume that $\mathfrak{B} = \{0\} \times H(A)$. There are only two alternatives on $(H(A), \mathbb{R})$:*

(1) *There is precisely one nonempty Y_k, which is $H(A)$. It occurs if and only if A is a chain recurrent function.*

(2) *There are precisely two nonempty Y_k, which are $Y_q = \omega(A)$ and $Y_Q = \omega^*(A)$. It occurs if and only if A is not a chain recurrent function.*

We have another comment. Assume that there are constant matrices A^- and A^+ such that

$$\lim_{t\to-\infty} A(t) = A^- \quad \text{and} \quad \lim_{t\to\infty} A(t) = A^+.$$

Then $\omega(A) = A^+$ and $\omega^*(A) = A^-$. Furthermore, the number of eigenvalues having negative real parts of A^+ and A^- are $\dim \mathfrak{S}(A^+)$ and $\dim \mathfrak{S}(A^-)$, respectively. We assume that none of the eigenvalues of A^- and A^+ have zero real parts. The following assertions are proved in [15], Theorem 4, and [14], Theorem 6.2, respectively:

(1) The equation $\dot{x} = A(t)x$ has at least k linearly independent bounded solutions where

$$k = \dim \mathfrak{S}(A^+) - \dim \mathfrak{S}(A^-).$$

(2) If $\dim \mathfrak{S}(A^+) = \dim \mathfrak{S}(A^-) = d$ and the equation $\dot{x} = A(t)x$ has no bounded solutions (except $x \equiv 0$), then $\dim \mathfrak{S}(\xi) = d$, $\dim \mathfrak{U}(\xi) = n - d$ and $\mathfrak{S}(\xi) \oplus \mathfrak{U}(\xi) = V$ at every $\xi \in H(A)$.

Statement (1) above follows because the Standing Hypotheses must fail. If A is chain recurrent, we have $A^- = A^+$, hence statement (1) does not guarantee the existence of linearly independent bounded solution. On the other hand, if A is a chain recurrent function, we do not know if the equation $\dot{x} = A(t)x$ has no bounded solution. However, by assuming that A is chain recurrent and the equation $\dot{x} = A(t)x$ has no bounded solutions, we apply the statement (2) above to conclude that the Standing Hypotheses hold and $\mathfrak{S}(\xi) \oplus \mathfrak{U}(\xi) = V$ at every $\xi \in H(A)$.

5.2. General transformation groups. Now we present a characterization of the chain recurrence in other transformation groups. Let (M, G, σ) be a transformation group, where M is a compact Hausdorff space. Assume that $\sigma_x \colon G \to M$ is uniformly continuous, for all $x \in M$. We can characterize the hull $H(\sigma_x)$, as follows.

Proposition 5.2. *For each $x \in M$, the hull $H(\sigma_x)$ is isomorphic with* $\mathrm{cls}(xG)$*, and*

$$H(\sigma_x) = \{\sigma_y : y \in \mathrm{cls}(xG)\}.$$

Proof. Let $\xi \in H(\sigma_x)$ and take a net $(\sigma_x \cdot g_i)_{i \in I}$ such that $\xi = \lim_i (\sigma_x \cdot g_i)$. By taking a subnet if necessary, we can assume that the net (xg_i) converges in M. Hence,

$$\xi(e) = \lim_i \sigma_x(g_i) = \lim_i xg_i \in \mathrm{cls}(xG).$$

For all $g \in G$, we have

$$\xi(g) = \lim_i \sigma_g(xg_i) = \sigma_g(\lim_i xg_i) = \sigma_g(\xi(e)) = \sigma_{\xi(e)}(g).$$

Hence, $\xi = \sigma_{\xi(e)}$, with $\xi(e) \in \mathrm{cls}(xG)$. On the other hand, if $y \in \mathrm{cls}(xG)$, we use the inverse process for obtaining $\sigma_y \in H(\sigma_x)$. Thus, $H(\sigma_x) = \{\sigma_y : y \in \mathrm{cls}(xG)\}$. An isomorphism of $H(\sigma_x)$ onto $\mathrm{cls}(xG)$ is given by $\xi \in H(\sigma_x) \to \xi(e) \in \mathrm{cls}(xG)$. □

We obtain the following relationship between the chain recurrent points in M and the chain recurrent functions.

Corollary 5.3. *A point $x \in M$ is chain recurrent if and only if the function $\sigma_x \colon G \to M$ is chain recurrent.*

Proof. A point $x \in M$ is chain recurrent if and only if $\mathrm{cls}(xG)$ is a chain transitive set. The proof follows from the previous proposition. □

References

[1] J. Auslander, *Minimal flows and their extensions*. North-Holland Math. Stud. 153, North-Holland, Amsterdam 1988. Zbl 0654.54027 MR 956049

[2] J. Auslander and H. Furstenberg, Product recurrence and distal points. *Trans. Amer. Math. Soc.* **343** (1994), 221–232. Zbl 0801.54031 MR 1170562

[3] C. J. B. Barros and J. A. Souza, Attractors and chain recurrence for semigroup actions. *J. Dynam. Differential Equations* **22** (2010), 723–740. Zbl 1209.37013 MR 2734477

[4] C. J. Braga Barros and J. A. Souza, Finest Morse decompositions for semigroup actions on fiber bundles. *J. Dynam. Differential Equations* **22** (2010), 741–760. Zbl 1243.37014 MR 2734478

[5] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*. Vol. II, Amer. Math. Soc., Providence, R.I., 1967. Zbl 0178.01203 MR 0218472

[6] F. Colonius and W. Kliemann, *The dynamics of control*. Birkhäuser, Boston 2000. Zbl 1020.93500 MR 1752730

[7] C. Conley, *Isolated invariant sets and the Morse index*. CBMS Regional Conf. Ser. in Math. 38, Amer. Math. Soc., Providence, R.I., 1978. Zbl 0397.34056 MR 511133

[8] R. Ellis, *Lectures on topological dynamics*. W. A. Benjamin, Inc., New York 1969. Zbl 0193.51502 MR 0267561

[9] R. Ellis and R. A. Johnson, Topological dynamics and linear differential systems. *J. Differential Equations* **44** (1982), 21–39. Zbl 0501.58033 MR 651685

[10] D. B. Ellis, R. Ellis, and M. Nerurkar, The topological dynamics of semigroup actions. *Trans. Amer. Math. Soc.* **353** (2001), 1279–1320. Zbl 0976.54039 MR 1806740

[11] R. A. Johnson, Minimal functions with unbounded integral. *Israel J. Math.* **31** (1978), 133–141. Zbl 0406.58030 MR 516249

[12] R. A. Johnson, On a Floquet theory for almost-periodic, two-dimensional linear systems. *J. Differential Equations* **37** (1980), 184–205. Zbl 0508.34031 MR 587221

[13] J. D. Lawson, Maximal subsemigroups of Lie groups that are total. *Proc. Edinburgh Math. Soc.* (2) **30** (1987), 479–501. Zbl 0649.22004 MR 908455

[14] R. J. Sacker, Existence of dichotomies and invariant splittings for linear differential systems. IV. *J. Differential Equations* **27** (1978), 106–137. Zbl 0359.34044 MR 0477315

[15] R. J. Sacker and G. R. Sell, Existence of dichotomies and invariant splittings for linear differential systems. II. *J. Differential Equations* **22** (1976), 478–496. Zbl 0339.58013 MR 0440620

[16] G. R. Sell, Linear Differential Systems. Lecture Notes, University of Minnesota, Minneapolis 1975.

[17] J. de Vries, *Elements of topological dynamics*. Math. Appl. 257, Kluwer Academic Publishers, Dordrecht 1993. Zbl 0783.54035 MR 1249063

[18] S. Willard, *General topology*. Dover Publications, Mineola, NY, 2004. Zbl 1052.54001 MR 2048350

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