

Conjugacy p -separability of right-angled Artin groups and applications

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Abstract. We prove that every subnormal subgroup of p -power index in a right-angled Artin group is conjugacy p -separable. As an application, we prove that every right-angled Artin group is conjugacy separable in the class of torsion-free nilpotent groups. As another application, we prove that the outer automorphism group of a right-angled Artin group is virtually residually p -finite. We also prove that the Torelli group of a right-angled Artin group is residually torsion-free nilpotent, hence residually p -finite and bi-orderable.

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1. Introduction

Let $\Gamma = (V, E)$ be a finite simplicial graph. The *right-angled Artin group* associated to Γ is the group G_Γ defined by the presentation:

$$G_\Gamma = \langle V \mid vw = wv \text{ for all } \{v, w\} \in E \rangle.$$

Note that if Γ is a discrete graph, then G_Γ is the free group F_r on r generators (where $r = |V|$), and if Γ is a complete graph, then G_Γ is the free abelian group \mathbb{Z}^r . Thus, right-angled Artin groups can be seen as interpolating between free groups and free abelian groups. The *rank* of G_Γ is by definition the number of vertices of Γ . A *special subgroup* of G_Γ is a subgroup generated by a subset W of the set of vertices V of Γ – it is naturally isomorphic to the right-angled Artin group $G_{\Gamma(W)}$, where $\Gamma(W)$ denotes the full subgraph of Γ spanned by W . Let v be a vertex of Γ . The *link* of v , denoted by $\text{link}(v)$, is the subset of V consisting of all vertices that are adjacent to v . The *star* of v , denoted by $\text{star}(v)$, is $\text{link}(v) \cup \{v\}$. We refer to [C] for a general survey of right-angled Artin groups.

Little is known about the automorphism groups of right-angled Artin groups. In 1989, Servatius conjectured a generating set for $\text{Aut}(G_\Gamma)$ (see [Ser]). He proved his conjecture in certain special cases, for example when the graph is a tree. Thereafter Laurence proved the conjecture in the general case (see [L]). Charney and Vogtmann

showed that $\text{Out}(G_\Gamma)$ is virtually torsion-free and has finite virtual cohomological dimension (see [CV1]). Day gave a finite presentation for $\text{Aut}(G_\Gamma)$ (see [D1]). More recently, Minasyan proved that $\text{Out}(G_\Gamma)$ is residually finite (see [M]). This result was obtained independently by Charney and Vogtmann in [CV2], where they also proved that, for a large class of graphs, $\text{Out}(G_\Gamma)$ satisfies the Tits alternative.

Let \mathcal{K} be a class of group. A group G is said to be *residually \mathcal{K}* if for all $g \in G \setminus \{1\}$, there exists a homomorphism φ from G to some group of \mathcal{K} such that $\varphi(g) \neq 1$. Note that if \mathcal{K} is the class of all finite groups, this notion reduces to residual finiteness.

For a group G and for $g, h \in G$, we use the notation $g \sim h$ to mean that g and h are conjugate. A group G is said to be *conjugacy \mathcal{K} -separable* (or *conjugacy separable in the class \mathcal{K}*) if for all $g, h \in G$, either $g \sim h$, or there exists a homomorphism φ from G to some group of \mathcal{K} such that $\varphi(g) \not\sim \varphi(h)$. Note that if \mathcal{K} is the class of all finite groups, this notion reduces to conjugacy separability. Clearly, if a group is conjugacy \mathcal{K} -separable, then it is residually \mathcal{K} .

Our focus here is on conjugacy separability in the class of finite p -groups. Let p be a prime number. If \mathcal{K} is the class of all finite p -groups, then, instead of saying “ G is residually \mathcal{K} ”, we shall say that G is *residually p -finite*. Note that this implies residually finite as well as residually nilpotent. Instead of saying “ G is conjugacy \mathcal{K} -separable”, we shall say that G is *conjugacy p -separable*. Following Ivanova (see [I]), we say that a subset S of a group G is *finitely p -separable* if for every $g \in G \setminus S$, there exists a homomorphism φ from G onto a finite p -group P such that $\varphi(g) \notin \varphi(S)$. Note that G is conjugacy p -separable if and only if every conjugacy class of G is finitely p -separable.

Examples of groups which are known to be conjugacy p -separable include free groups (see, e.g. [LS]) and fundamental groups of oriented closed surfaces (see [P]).

There is a connection between these notions and a topology on G , the “pro- p topology” on G . The *pro- p topology* on G is defined by taking the normal subgroups of p -power index in G as a basis of neighbourhoods of 1 (see [RZ]). Equipped with the pro- p topology, G becomes a topological group. Observe that G is Hausdorff if and only if it is residually p -finite. One can show that a subset S of G is closed in the pro- p topology on G if and only if it is finitely p -separable. Thus, G is conjugacy p -separable if and only if every conjugacy class of G is closed in the pro- p topology on G .

In [CZ], Chagas and Zalesskii constructed an example of a conjugacy separable group possessing a non conjugacy separable subgroup of finite index. This led them to introduce the notion of “hereditarily conjugacy separable group”. A group G is said to be *hereditarily conjugacy separable* if every subgroup of finite index in G is conjugacy separable.

Recall that a *subnormal subgroup* of a group G is a subgroup H of G such that there exists a finite sequence of subgroups of G :

$$H = H_0 < H_1 < \dots < H_n = G,$$

such that H_i is normal in H_{i+1} for all $i \in \{0, \dots, n - 1\}$.

A subgroup H of a group G is open in the pro- p topology on G if and only if it is subnormal of p -power index (see Lemma A.1). This leads us to the following definition, which naturally generalizes that of [CZ]:

Definition 1.1. Let G be a group. We say that G is hereditarily conjugacy p -separable if every subnormal subgroup of p -power index in G is conjugacy p -separable.

In [M], Minasyan proved that right-angled Artin groups are hereditarily conjugacy separable. Our main theorem is the following:

Theorem 6.15. *Every right-angled Artin group is hereditarily conjugacy p -separable.*

We will now discuss some applications of Theorem 6.15. The first application that we mention is an application of Theorem 6.15 to separability properties of G_Γ :

Corollary 7.1. *Every right-angled Artin group is conjugacy separable in the class of torsion-free nilpotent groups.*

Let \mathcal{P} be a group property. A group G is said to be *virtually \mathcal{P}* if there exists a finite index subgroup $H < G$ such that H has Property \mathcal{P} . Combining Theorem 6.15 with a result of Paris (see [P]), we obtain the following:

Corollary 7.4. *The outer automorphism group of a right-angled Artin group is virtually residually p -finite.*

On the other hand, combining Theorem 6.15 with a result of Myasnikov (see [My]), we obtain the following:

Corollary 7.6. *The outer automorphism group of a right-angled Artin group is residually \mathcal{K} , where \mathcal{K} is the class of all outer automorphism groups of finite p -groups.*

The next application was suggested to the author by Ruth Charney and Luis Paris.

The natural action $\text{Aut}(G_\Gamma) \rightarrow \text{GL}_r(\mathbb{Z})$ of $\text{Aut}(G_\Gamma)$ on $H_1(G_\Gamma, \mathbb{Z})$ gives rise to a homomorphism $\text{Out}(G_\Gamma) \rightarrow \text{GL}_r(\mathbb{Z})$, whose kernel is called the *Torelli group* of G_Γ – by analogy with the Torelli group of a mapping class group. In Section 7, we combine well-known results of Bass–Lubotzky (see [BL]), and Duchamp–Krob (see [DK1], [DK2]) with Theorem 6.15 to attain the following:

Theorem 7.14. *The Torelli group of a right-angled Artin group is residually torsion-free nilpotent.*

Corollary 7.15. *The Torelli group of a right-angled Artin group is residually p -finite.*

Recall that a group G is said to be *bi-orderable* if it can be endowed with a total order \leq such that if $g \leq h$, then $kg \leq kh$ and $gk \leq hk$ for all $g, h, k \in G$.

Corollary 7.16. *The Torelli group of a right-angled Artin group is bi-order-able.*

Our proof follows closely that of Minasyan (see [M]). Both proofs proceed by induction on the rank of G_Γ . The key observation is that a right-angled Artin group of rank r can be written as an HNN extension of any of its special subgroups of rank $r - 1$. After passing to an HNN extension of a finite group (which is known to be virtually free), Minasyan applies a theorem of Dyer stating that virtually free groups are conjugacy separable (see [Dy1]).

This paper is organized as follows. In Section 3, we introduce the p -centralizer condition which is the analogue of the centralizer condition in [M], and we prove that a group is hereditarily conjugacy p -separable if and only if it is conjugacy p -separable and satisfies the p -centralizer condition. In Section 4, we prove the following analogue of Dyer's theorem for conjugacy p -separability:

Theorem 1.2. *Every extension of a free group by a finite p -group is conjugacy p -separable.*

Section 5 deals with retractions that are key tools in the proof of our main theorem, which is the object of Section 6.

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2. HNN extensions

In this section, we recall the definition and basic properties of HNN extensions (see [LS]).

Let H be a group. Then by the notation

$$\langle H, s, \dots \mid r, \dots \rangle$$

we mean the group defined by the presentation whose generators are the generators of H together with s, \dots and the relators of H together with r, \dots .

Let H be a group, and let K be a subgroup of H . The *HNN extension of H relative to K* is the group defined by the presentation

$$G = \langle H, t \mid t^{-1}kt = k \text{ for all } k \in K \rangle.$$

Every element of G can be written as a word $x_0 t^{a_1} x_1 \dots t^{a_n} x_n$ ($n \geq 0$, $x_0, \dots, x_n \in H$, $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$). Following Minasyan (see [M]), we will say that the word

$x_0 t^{a_1} x_1 \dots t^{a_n} x_n$ is reduced if $x_0 \in H$, $x_1, \dots, x_{n-1} \in H \setminus K$, and $x_n \in H$. Every element of G can be written as a reduced word. Note that our definition of a reduced word is stronger than the definition of a reduced word in [LS].

Lemma 2.1 (Britton’s Lemma). *If a word $x_0 t^{a_1} x_1 \dots t^{a_n} x_n$ is reduced with $n \geq 1$, then $x_0 t^{a_1} x_1 \dots t^{a_n} x_n \neq 1$.*

Proof. Proved in [LS] (see Theorem IV.2.1). □

Lemma 2.2. *If $x_0 t^{a_1} x_1 \dots t^{a_n} x_n$ and $y_0 t^{b_1} y_1 \dots t^{b_m} y_m$ are reduced words such that $x_0 t^{a_1} x_1 \dots t^{a_n} x_n = y_0 t^{b_1} y_1 \dots t^{b_m} y_m$, then $m = n$ and $a_i = b_i$ for all $i \in \{1, \dots, n\}$.*

Proof. Proved in [LS] (see Lemma IV.2.3). □

A cyclic permutation of the word $t^{a_1} x_1 \dots t^{a_n} x_n$ is a word

$$t^{a_k} x_k \dots t^{a_n} x_n t^{a_1} x_1 \dots t^{a_{k-1}} x_{k-1}$$

with $k \in \{1, \dots, n\}$. A word $t^{a_1} x_1 \dots t^{a_n} x_n$ is said to be *cyclically reduced* if any cyclic permutation of $t^{a_1} x_1 \dots t^{a_n} x_n$ is reduced. Note that if $t^{a_1} x_1 \dots t^{a_n} x_n$ is reduced and $n \geq 2$, then $t^{a_1} x_1 \dots t^{a_n} x_n$ is cyclically reduced if and only if $x_n \in H \setminus K$. Every element of G is conjugate to a cyclically reduced word.

Lemma 2.3 (Collins’ Lemma). *If the words $g = t^{a_1} x_1 \dots t^{a_n} x_n$ ($n \geq 1$) and $h = t^{b_1} y_1 \dots t^{b_m} y_m$ ($m \geq 1$) are cyclically reduced and conjugate, then there exists a cyclic permutation h^* of h and an element $\alpha \in K$ such that $g = \alpha h^* \alpha^{-1}$.*

Proof. Proved in [LS] (see Theorem IV.2.5). □

Remark 2.4. There exists a natural homomorphism $f : G \rightarrow H$, defined by $f(h) = h$ for all $h \in H$, and $f(t) = 1$.

Remark 2.5. Let P be a group and let $\varphi : H \rightarrow P$ be a homomorphism. Let Q be the HNN extension of P relative to $\varphi(K)$:

$$Q = \langle P, \bar{t} \mid \bar{t}^{-1} \varphi(k) \bar{t} = \varphi(k) \text{ for all } k \in K \rangle.$$

Then φ induces a homomorphism $\bar{\varphi} : G \rightarrow Q$, defined by $\bar{\varphi}(h) = \varphi(h)$ for all $h \in H$, and $\bar{\varphi}(t) = \bar{t}$.

Lemma 2.6. *With the notations of Remark 2.5, $\ker(\bar{\varphi})$ is the normal closure of $\ker(\varphi)$ in G .*

Proof. Proved in [M] (see Lemma 7.5). □

The following observation is the key in the proof of our main theorem.

Remark 2.7. Let G be a right-angled Artin group of rank r ($r \geq 1$). Let H be a special subgroup of G of rank $r - 1$. In other words, there is a partition of V , $V = W \cup \{t\}$ such that $H = \langle W \rangle$. Then G can be written as the HNN extension of H relative to the special subgroup $K = \langle \text{link}(t) \rangle$ of H :

$$G = \langle H, t \mid t^{-1}kt = k \text{ for all } k \in K \rangle.$$

3. Hereditary conjugacy p -separability and p -centralizer condition

We start with an observation that the reader has to keep in mind, because it will be used repeatedly in the rest of the paper: if H and K are two normal subgroups of p -power index in a group G , then $H \cap K$ is a normal subgroup of p -power index in G .

The centralizer condition was first introduced by Chagas and Zalesskii as a sufficient condition for a conjugacy separable group to be hereditarily conjugacy separable (see [CZ]). Thereafter Minasyan showed that this condition is also necessary; that is, a group is hereditarily conjugacy separable if and only if it is conjugacy separable and satisfies the centralizer condition (see [M]). We make the following definition, which naturally generalizes that of [M]:

Definition 3.1. We say that G satisfies the p -centralizer condition (pCC) if, for every normal subgroup H of p -power index in G , and for all $g \in G$, there exists a normal subgroup K of p -power index in G such that $K < H$ and

$$C_{G/K}(\varphi(g)) \subset \varphi(C_G(g)H),$$

where $\varphi: G \rightarrow G/K$ denotes the canonical projection.

We shall show that a group G is hereditarily conjugacy p -separable if and only if it is conjugacy p -separable and satisfies the p -centralizer condition (see Proposition 3.6). If H is a subgroup of G , and $g \in G$, we set $C_H(g) = \{h \in H \mid gh = hg\}$. For technical reasons, we have to introduce the following definitions.

Definition 3.2. Let G be a group, H be a subgroup of G , and $g \in G$. We say that the pair (H, g) satisfies the p -centralizer condition in G (pCC $_G$) if, for every normal subgroup K of p -power index in G , there exists a normal subgroup L of p -power index in G such that $L < K$, and

$$C_{\varphi(H)}(\varphi(g)) \subset \varphi(C_H(g)K),$$

where $\varphi: G \rightarrow G/L$ denotes the canonical projection. We say that H satisfies the p -centralizer condition in G (pCC $_G$) if the pair (H, g) satisfies the p -centralizer condition in G for all $g \in G$.

If G is a group, H is a subgroup of G , and $g \in G$, we set $g^H = \{\alpha g \alpha^{-1} \mid \alpha \in H\}$. In order to prove Proposition 3.6, we need the following statements, which are the analogues of some statements obtained in [M] (Lemma 3.4, Corollary 3.5, and Lemma 3.7, respectively):

Lemma 3.3. *Let G be a group, H be a subgroup of G , and $g \in G$. Suppose that the pair (G, g) satisfies pCC_G , and that g^G is finitely p -separable in G . If $C_G(g)H$ is finitely p -separable in G , then g^H is also finitely p -separable in G .*

Proof. Let $h \in G$ such that $h \notin g^H$. If $h \notin g^G$, then, since g^G is finitely p -separable in G , there exists a homomorphism φ from G onto a finite p -group P such that $\varphi(h) \notin \varphi(g^G)$. In particular, $\varphi(h) \notin \varphi(g^H)$. Thus we can assume that $h \in g^G$. Let $\alpha \in G$ be such that $h = \alpha g \alpha^{-1}$. Suppose that $C_G(g) \cap \alpha^{-1}H \neq \emptyset$. Let $k \in C_G(g) \cap \alpha^{-1}H$. Then $\alpha k \in H$, and $h = \alpha g \alpha^{-1} = \alpha k g (\alpha k)^{-1} \in g^H$, which is a contradiction. Thus $C_G(g) \cap \alpha^{-1}H = \emptyset$, i.e., $\alpha^{-1} \notin C_G(g)H$. As $C_G(g)H$ is finitely p -separable in G , there exists a normal subgroup K of p -power index in G such that $\alpha^{-1} \notin C_G(g)HK$. Now the condition pCC_G implies that there exists a normal subgroup L of p -power index in G such that $L < K$ and

$$C_{G/L}(\varphi(g)) \subset \varphi(C_G(g)K),$$

where $\varphi: G \rightarrow G/L$ denotes the canonical projection. We claim that $\varphi(h) \notin \varphi(g^H)$. Indeed, if there is $\beta \in H$ such that $\varphi(h) = \varphi(\beta g \beta^{-1})$, then

$$\varphi(\alpha^{-1}\beta)\varphi(g) = \varphi(\alpha^{-1}\beta)\varphi(\beta^{-1}h\beta) = \varphi(\alpha^{-1}h\alpha)\varphi(\alpha^{-1}\beta) = \varphi(g)\varphi(\alpha^{-1}\beta),$$

i.e., $\varphi(\alpha^{-1}\beta) \in C_{G/L}(\varphi(g))$. But then $\varphi(\alpha^{-1}) \in C_{G/L}(\varphi(g))\varphi(H) \subset \varphi(C_G(g)KH)$. Hence $\alpha^{-1} \in C_G(g)HKL = C_G(g)HK$ (because $L < K$), which is a contradiction. \square

Corollary 3.4. *Let G be a conjugacy p -separable group satisfying pCC , and H be a subgroup of G such that $C_G(h)H$ is finitely p -separable in G for all $h \in H$. Then H is conjugacy p -separable. Moreover, h^H is finitely p -separable in G for all $h \in H$.*

Proof. Let $h \in H$. Since G satisfies pCC , the pair (G, h) satisfies pCC_G . Since G is conjugacy p -separable, h^G is finitely p -separable in G . Lemma 3.3 now implies that h^H is finitely p -separable in G . Therefore h^H is finitely p -separable in H . \square

Lemma 3.5. *Let G be a group, H be a subgroup of G , and $g \in G$. Let K be a normal subgroup of p -power index in G . If $g^{H \cap K}$ is finitely p -separable in G , then there exists a normal subgroup L of p -power index in G such that $L < K$ and*

$$C_{\varphi(H)}(\varphi(g)) \subset \varphi(C_H(g)K),$$

where $\varphi: G \rightarrow G/L$ denotes the canonical projection.

Proof. Note that $H \cap K$ is of finite index n in H . Actually, $H \cap K$ is of p -power index in H (because $\frac{H}{H \cap K} \simeq \frac{KH}{K} < \frac{G}{K}$), but this is not needed here. There exist $\alpha_1, \dots, \alpha_n \in H$ such that $H = \sqcup_{i=1}^n \alpha_i(H \cap K)$. Up to renumbering, we can assume that there exists $l \in \{0, \dots, n\}$ such that $\alpha_i^{-1}g\alpha_i \in g^{H \cap K}$ for all $i \in \{1, \dots, l\}$ and $\alpha_i^{-1}g\alpha_i \notin g^{H \cap K}$ for all $i \in \{l+1, \dots, n\}$. By the assumptions, there exists a normal subgroup L of p -power index in G such that $\alpha_i^{-1}g\alpha_i \notin g^{H \cap K}L$ for all $i \in \{l+1, \dots, n\}$. Up to replacing L by $L \cap K$, we can assume that $L < K$. Let $\varphi: G \rightarrow G/L$ be the canonical projection. Let $\bar{h} \in C_{\varphi(H)}(\varphi(g))$. There exists $h \in H$ such that $\bar{h} = \varphi(h)$. There exist $i \in \{1, \dots, n\}$ and $k \in H \cap K$ such that $h = \alpha_i k$. We have $\varphi(h^{-1}gh) = \varphi(h)^{-1}\varphi(g)\varphi(h) = \varphi(g)$. Thus $h^{-1}gh \in gL$. But then $\alpha_i^{-1}g\alpha_i = kh^{-1}ghk^{-1} \in kgLk^{-1} = kgk^{-1}L \subset g^{H \cap K}L$. Therefore $i \leq l$. Then there exists $\beta \in H \cap K$ such that $\alpha_i^{-1}g\alpha_i = \beta g \beta^{-1}$. This is to say that $\alpha_i \beta \in C_H(g)$, and then $h = \alpha_i k = (\alpha_i \beta)(\beta^{-1}k) \in C_H(g)(H \cap K) \subset C_H(g)K$. We have shown that $C_{\varphi(H)}(\varphi(g)) \subset \varphi(C_H(g)K)$. \square

We are now ready to prove:

Proposition 3.6. *A group is hereditarily conjugacy p -separable if and only if it is conjugacy p -separable and satisfies pCC.*

Proof. Suppose that G is conjugacy p -separable and satisfies pCC. Let H be a subnormal subgroup of p -power index in G . Thus H is closed in the pro- p topology on G (because $G \setminus H = \bigcup \{gH \mid g \notin H\}$). Let $h \in H$. The set $C_G(h)H$ is a finite union of left cosets modulo H and thus is closed in the pro- p topology on G . Corollary 3.4 now implies that H is conjugacy p -separable. Therefore G is hereditarily conjugacy p -separable. Suppose now that G is hereditarily conjugacy p -separable. In particular, G is conjugacy p -separable. We shall show that G satisfies pCC. Let $g \in G$. Let K be a normal subgroup of p -power index in G . Let $H = K\langle g \rangle$. Since $K < H$, $[G : H]$ is a power of p . As $\frac{G}{K}$ is a finite p -group, every subgroup of it is subnormal. Thus H is subnormal in G . Therefore H is conjugacy p -separable. Note that $g^{G \cap K} = g^K = g^H \subset H$. As g^H is closed in the pro- p topology on H , it is closed in the pro- p topology on G , because the topology induced on H by the pro- p topology on G coincides with the pro- p topology on H (see, for example, [RZ2], Corollary 5.8). The result now follows from Lemma 3.5. \square

4. Extensions of free groups by finite p -groups are conjugacy p -separable

We start with an observation that the reader has to keep in mind because it will be used repeatedly in the proof of Theorem 4.2: if $\varphi: G \rightarrow H$ is a homomorphism from a group G to a group H whose kernel is torsion-free, then the restriction of φ to any finite subgroup of G is injective.

We need the following lemma.

Lemma 4.1. *Let $G = G_1 * \dots * G_n$ be a free product of n conjugacy p -separable groups G_1, \dots, G_n . Let $g, h \in G \setminus \{1\}$ be two non-trivial elements of finite order in G such that $g \sim h$. There exists a homomorphism φ from G onto a finite p -group P such that $\varphi(g) \sim \varphi(h)$.*

Proof. Since g is of finite order in G , there exists $i \in \{1, \dots, n\}$ such that g is conjugate to an element of finite order in G_i . Thus we may assume that g belongs to G_i . Similarly, we may assume that there exists $j \in \{1, \dots, n\}$ such that h belongs to G_j . Suppose that $i \neq j$. Let $\varphi: G_i \rightarrow P$ be a homomorphism from G_i onto a finite p -group P such that $\varphi(g) \neq 1$. Let $\tilde{\varphi}: G \rightarrow P$ be the natural homomorphism extending φ . Then $\tilde{\varphi}(g) \sim \tilde{\varphi}(h)$. Suppose that $i = j$. Then g and h are not conjugate in G_i – otherwise they would be conjugate in G . Since G_i is conjugacy p -separable, there exists a homomorphism $\varphi: G_i \rightarrow P$ from G_i onto a finite p -group P such that $\varphi(g) \sim \varphi(h)$. Let $\tilde{\varphi}: G \rightarrow P$ be defined as above. We have $\tilde{\varphi}(g) \sim \tilde{\varphi}(h)$. \square

In Section 4, by a graph, we mean a unoriented graph, possibly with loops or multiple edges.

Recall that a *graph of groups* is a connected graph $\Gamma = (V, E)$, together with a function \mathcal{G} which assigns

- to each vertex $v \in V$ a group G_v , and
- to each edge $e = \{v, w\} \in E$ a group G_e together with two injective homomorphisms $\alpha_e: G_e \rightarrow G_v$ and $\beta_e: G_e \rightarrow G_w$ – we are not assuming that $v \neq w$.

(See [Se], see also [Dy1].) The groups G_v ($v \in V$) are called the *vertex groups* of Γ , the groups G_e ($e \in E$) are called the *edge groups* of Γ . The monomorphisms α_e and β_e ($e \in E$) are called the *edge monomorphisms*. The images of the edge groups under the edge monomorphisms are called the *edge subgroups*.

Choose disjoint presentations $G_v = \langle X_v \mid R_v \rangle$ for the vertex groups of Γ . Choose a maximal tree T in Γ . Assign a direction to each edge of Γ . Let $\{t_e \mid e \in E\}$ be a set in one-to-one correspondence with the set of edges of Γ , and disjoint from the X_v ($v \in V$). The *fundamental group* of the above graph of groups Γ is the group G_Γ defined by the presentation whose generators are

$$X_v \ (v \in V), \quad t_e \ (e \in E)$$

(called vertex and edge generators, respectively) and whose relations are

$$R_v \ (v \in V), \quad t_e = 1 \ (e \in T), \quad t_e \alpha_e(g_e) t_e^{-1} = \beta_e(g_e) \quad \text{for all } g_e \in G_e \ (e \in E)$$

(called vertex, tree, and edge relations, respectively). One can prove that this is well defined, that is, independent of our choice of T , etc. Note that it suffices to write the edge relations for g_e in a set of generators for G_e .

Convention. The groups G_v ($v \in V$) and G_e ($e \in E$) will be regarded as subgroups of G_Γ .

Let $\{\Gamma_i\}_{i \in I}$ be a collection of connected and pairwise disjoint subgraphs of Γ . We may define a graph of groups Γ^* from Γ by *contracting* Γ_i to a point for all $i \in I$ as follows. The graph Γ^* is obtained from Γ by contracting Γ_i to a point p_i for all $i \in I$. The function \mathcal{G}^* is obtained from \mathcal{G} by using the fundamental group of Γ_i for the vertex group at p_i , and by composing the edge monomorphisms of Γ by the natural inclusions of the vertex groups of Γ_i into the fundamental group of Γ_i , if necessary. Clearly, G_Γ is isomorphic to the fundamental group G_{Γ^*} of Γ^* .

If $\pi : G_\Gamma \rightarrow H$ is a homomorphism from G_Γ to a group H , such that the restriction of π to each edge subgroup of Γ is injective, then we may define a graph of groups Γ' from Γ by *composing with* π as follows. The vertex set of Γ' is V , and the edge set of Γ' is E . The vertex groups of Γ' are the groups $G'_v = \pi(G_v)$ ($v \in V$), and the edge groups of Γ' are the groups $G'_e = G_e$ ($e \in E$). The edge monomorphisms are the monomorphisms $\alpha'_e = \pi \circ \alpha_e$ and $\beta'_e = \pi \circ \beta_e$ ($e \in E$). Present G_Γ and $G_{\Gamma'}$ using the same symbols for edge generators and with the same choice of maximal tree. There exist two homomorphisms, $\pi_V : G_\Gamma \rightarrow G_{\Gamma'}$ and $\pi_E : G_{\Gamma'} \rightarrow H$ such that the diagram

$$\begin{array}{ccc} G_\Gamma & \xrightarrow{\pi} & H \\ \pi_V \downarrow & \nearrow \pi_E & \\ G_{\Gamma'} & & \end{array}$$

commutes, and that the restriction of π_E to each vertex group of $G_{\Gamma'}$ is injective. The homomorphism π_V is given by

$$\begin{aligned} (\pi_V)|_{G_v} &= \pi|_{G_v} && \text{for all } v \in V, \\ \pi_V(t_e) &= t_e && \text{for all } e \in E. \end{aligned}$$

And the homomorphism π_E is given by

$$\begin{aligned} (\pi_E)|_{G'_v} &= (\text{id}_H)|_{G'_v} && \text{for all } v \in V, \\ \pi_E(t_e) &= \pi(t_e) && \text{for all } e \in E. \end{aligned}$$

In [Dy1], Dyer proved that every extension of a free group by a finite group is conjugacy separable. The following theorem is the analogue of Dyer’s theorem for conjugacy p -separability.

Theorem 4.2. *Every extension of a free group by a finite p -group is conjugacy p -separable.*

Proof. Our proof is inspired by that of Dyer (see [Dy1]). Let G be an extension of a free group by a finite p -group. In other words, there exists a short exact sequence

$$1 \rightarrow F \rightarrow G \xrightarrow{\pi} P \rightarrow 1,$$

where F is a free group, and P is a finite p -group. Let $g \in G$. Let $h \in G$ such that $g \sim h$.

Step 1. We show that we may assume that G satisfies a short exact sequence,

$$1 \rightarrow F \rightarrow G \xrightarrow{\pi} C_{p^n} \rightarrow 1$$

where F is a free group, $n \geq 1$, C_{p^n} denotes the cyclic group of order p^n , and $\pi(g) = \pi(h)$.

Since G is an extension of a free group by a finite p -group, G is residually p -finite by [G], Lemma 1.5. Therefore, if $g = 1$, then $g^G = \{1\}$ is finitely p -separable in G . On the other hand, if g is of infinite order in G , then g^G is finitely p -separable in G by [I], Proposition 5. Therefore we may assume that $g \neq 1$ and that g is of finite order in G . Similarly, we may assume that $h \neq 1$ and that h is of finite order in G . If $\pi(g)$ and $\pi(h)$ are not conjugate in P , we are done. Thus, up to replacing h by a conjugate of itself, we may assume that $\pi(g) = \pi(h)$. Since $\ker(\pi) = F$ is torsion-free, g and h have the same order p^n ($n \in \mathbb{N}^*$). Let $H = F\langle g \rangle$. Note that H is a subgroup of p -power index in G , and that g and h belong to H . As $\frac{G}{F} = P$ is a finite p -group, every subgroup of it is subnormal. Thus H is subnormal in G . Then we may replace G by H , by [I], Proposition 4¹, so as to assume that G satisfies the short exact sequence

$$1 \rightarrow F \rightarrow G \xrightarrow{\pi} C_{p^n} \rightarrow 1.$$

Now G is the fundamental group of a graph of groups Γ , whose vertex groups are all finite groups, by [S], Theorem. As $\pi|_{G_v}$ is injective for all $v \in V$, G_v is isomorphic to a subgroup of C_{p^n} for all $v \in V$. From now on, the groups G_v ($v \in V$) will be regarded as subgroups of C_{p^n} .

Step 2. We show that we may assume that all edge groups are non-trivial, that if two different vertices are connected by an edge, then the corresponding edge group is a proper subgroup of C_{p^n} , and that g and h belong to two different vertex groups.

First, we show that we may assume that all edge groups are non-trivial. Indeed, Let Γ_0 be the subgraph of Γ whose vertices are all the vertices of Γ and whose edges are the edges of Γ for which the edge group is non-trivial. Let $\Gamma_1, \dots, \Gamma_r$ be the connected components of Γ_0 . Let Γ^* be the graph of groups obtained from Γ by contracting Γ_i to a point for all $i \in \{1, \dots, r\}$. Let T be a maximal tree of Γ^* . Then G is isomorphic to the fundamental group G^* of Γ^* . Observe that G^* is the free product of the free group on $\{t_e \mid e \in E \setminus T\}$ and the fundamental groups of the Γ_i ($i \in \{1, \dots, r\}$). Thus, it suffices to consider the case where $\Gamma = \Gamma_i$ ($i \in \{1, \dots, r\}$), by Lemma 4.1. Since each Γ_i ($i \in \{1, \dots, r\}$) is a graph of groups whose edge groups are all non-trivial, the first part of the assertion is proved.

¹Strictly speaking, it follows from the proof of [I], Proposition 4, that if there exists a homomorphism $\varphi: H \rightarrow P$ from H onto a finite p -group P such that $\varphi(g) \sim \varphi(h)$, then there exists a homomorphism $\psi: G \rightarrow Q$ from G onto a finite p -group Q such that $\psi(g) \sim \psi(h)$. The exact statement of [I], Proposition 4, is slightly different.

Now we show that we may assume that if two different vertices are connected by an edge, then the corresponding edge group is a proper subgroup of C_{p^n} . Indeed, let Γ_0 be the subgraph of Γ whose vertices are all the vertices of Γ , and whose edges are the edges of Γ for which the edge group is isomorphic to C_{p^n} . Let $\Gamma_1, \dots, \Gamma_r$ be the connected components of Γ_0 . Choose a maximal tree T_i in Γ_i for all $i \in \{1, \dots, r\}$. Let Γ^* be the graph of groups obtained from Γ by contracting T_i to a point for all $i \in \{1, \dots, r\}$. Then G is isomorphic to the fundamental group G^* of Γ^* . Note that a vertex group of Γ^* is either a vertex group of Γ , or the fundamental group of T_i , for some $i \in \{1, \dots, r\}$, in which case it is isomorphic to C_{p^n} (because each T_i ($i \in \{1, \dots, r\}$) is a tree of groups whose vertex and edge groups are all equal to C_{p^n}). Thus, we may replace Γ by Γ^* , so that the second part of the assertion is proved.

Since g is of finite order in G , there exists a vertex v of Γ , an element g_0 of finite order in the vertex group G_v of v , and an element α of G such that $g = \alpha g_0 \alpha^{-1}$. Similarly, there exists a vertex w of Γ , an element h_0 of finite order in the vertex group G_w of w , and an element β of G such that $h = \beta h_0 \beta^{-1}$. As C_{p^n} is abelian, we have $\pi(g_0) = \pi(h_0)$. Thus, up to replacing g by g_0 and h by h_0 , we may assume that g belongs to G_v , and h belongs to G_w . Since $\pi|_{G_v}$ is injective, and $\pi(g) = \pi(h)$, we have $v \neq w$.

Step 3. We show that we may assume that Γ has exactly two vertices, and that all edges join these two vertices.

Indeed, choose a maximal tree T in Γ . There is a path P in T joining v to w . Choose an edge e on this path. Then $T \setminus \{e\}$ is the disjoint union of two trees, T_v and T_w , with $v \in T_v$ and $w \in T_w$. Let Γ_v be the full subgraph of Γ generated by the vertices of T_v , and Γ_w be the full subgraph of Γ generated by the vertices of T_w . Let Γ^* be the graph of groups obtained from Γ by contracting Γ_v to a point v^* and Γ_w to a point w^* . Observe that Γ^* has exactly two vertices and that all edges join these two vertices. The vertex groups of Γ^* are the fundamental groups of Γ_v and Γ_w , respectively. The edge groups of Γ^* are non-trivial proper subgroups of C_{p^n} . And G is isomorphic to the fundamental group G^* of Γ^* . Now, since the restriction of π to each edge subgroup of Γ^* is injective, we may define a graph of groups Γ' from Γ^* by composing with π , as described above. Denote by G' the fundamental group of Γ' . There exist two homomorphisms, $\pi_V : G \rightarrow G'$ and $\pi_E : G' \rightarrow C_{p^n}$, such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi} & C_{p^n} \\ \pi_V \downarrow & \nearrow \pi_E & \\ G' & & \end{array}$$

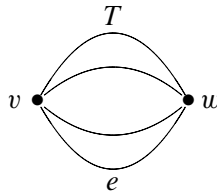
commutes, and that the restriction of π_E to each vertex group of Γ' is injective. Consequently, $\ker(\pi_E)$ is free by [Se], II, 2.6., Lemma 8.

Set $g' = \pi_V(g)$ and $h' = \pi_V(h)$. As g' and h' have order p^n , the vertex groups of Γ' are equal to C_{p^n} . The edge groups of Γ' are non-trivial proper subgroups of C_{p^n} .

Observe that g' and h' belong to two different vertex groups, and that g' (resp. h') is not conjugate to an element of one of the edge groups. Let e be an edge of Γ' . Then g' and h' are not conjugate in $G'_v *_{G'_e} G'_w$, by [MKS], Theorem 4.6. Observe that G' is an HNN extension (in the general sense) of $G'_v *_{G'_e} G'_w$ with stable letters t_a ($a \in E \setminus \{e\}$), and associated subgroups $\alpha'_a(G'_a)$ and $\beta'_a(G'_a)$ ($a \in E \setminus \{e\}$). Therefore g' and h' are not conjugate in G' (see, for example, [Dy2], Theorem 3). Thus, we may replace Γ by Γ' , G by G' , g by g' , and h by h' , so as to assume that Γ has two vertices and that all edges join these two vertices.

Step 4. We show that we may assume that Γ has at most two edges.

Suppose that Γ has more than two edges. Choose a maximal tree T in Γ , that is, an edge of Γ . Present $G_v = \langle g \mid g^{p^n} = 1 \rangle$, $G_w = \langle h \mid h^{p^n} = 1 \rangle$, and G as described above. Choose an edge $e \in E \setminus T$.



The edge relations corresponding to e can be reduced to the following:

$$t_e \alpha_e(g_e) t_e^{-1} = \beta_e(g_e),$$

where g_e is a generator of G_e . Let p^s be the order of G_e ($s \in \{1, \dots, n - 1\}$). Then $\alpha_e(g_e)$ generates a subgroup of order p^s of G_v . But there exists a unique subgroup of order p^s in G_v ; it is cyclic, generated by g^{p^r} , where $r = n - s$. Thus, up to replacing g_e by the preimage of g^{p^r} under α_e , we may assume that $\alpha_e(g_e) = g^{p^r}$. There exists $k \in \mathbb{N}$ such that p and k are coprime, and that $\beta_e(g_e) = h^{kp^r}$. Therefore the edge relation corresponding to e can be written

$$t_e g^{p^r} t_e^{-1} = h^{kp^r},$$

where $r \in \{1, \dots, n - 1\}$, $k \in \mathbb{N}$, and p and k are coprime. Now since $\pi: G \rightarrow C_{p^n}$ satisfies $\pi(g) = \pi(h)$, we have $\pi(g)^{p^r} = \pi(h)^{kp^r} = \pi(g)^{kp^r}$, and then $\pi(g)^{(k-1)p^r} = 1$ (in C_{p^n}). As $\pi(g)$ has order p^n in C_{p^n} , we deduce that p^{n-r} divides $k - 1$. There exists $a \in \mathbb{Z}$ such that $k = ap^{n-r} + 1$. We conclude that the edge relation corresponding to e can be written

$$t_e g^{p^r} t_e^{-1} = h^{p^r},$$

where $r \in \{1, \dots, n - 1\}$.

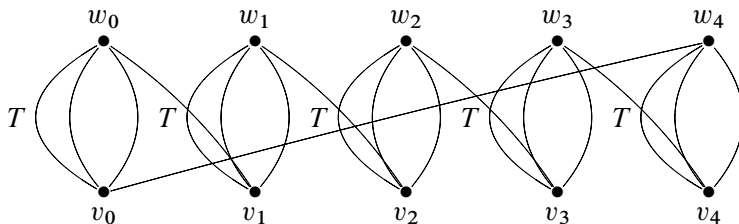
Let H be the normal subgroup of G generated by the elements

$$g, h, t_a \ (a \in E \setminus \{e\}), t_e^p.$$

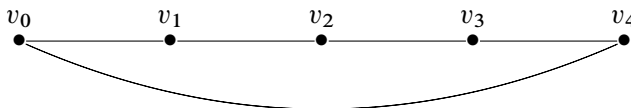
Then H has index p in G , and g and h belong to H . Thus we may replace G by H by [I], Proposition 4. Let G_0 be the fundamental group of the graph of groups $\Gamma \setminus \{e\}$. Set $G_0 = \langle X_0 \mid R_0 \rangle$, where the presentation is as fundamental group of the graph of groups $\Gamma \setminus \{e\}$. Set $G_i = t_e^i G_0 t_e^{-i} = \langle X_i \mid R_i \rangle$ for all $i \in \{1, \dots, p-1\}$. Clearly $\{1, t_e, \dots, t_e^{p-1}\}$ is a Schreier transversal for H in G . The Reidemeister–Schreier method yields the presentation

$$H = \langle X_0, X_1, \dots, X_{p-1}, u \mid R_0, R_1, \dots, R_{p-1}, g_1^{p'} = h_0^{p'}, g_2^{p'} = h_1^{p'}, \dots, \dots, g_{p-1}^{p'} = h_{p-2}^{p'}, u g_0^{p'} u^{-1} = h_{p-1}^{p'} \rangle,$$

where $u = t_e^p$, $g_i = t_e^i g t_e^{-i}$ ($i \in \{0, \dots, p-1\}$), and $h_j = t_e^j h t_e^{-j}$ ($j \in \{0, \dots, p-1\}$). Replace g by g_0 , and h by h_1 . Observe that H is the fundamental group of a graph of groups $\tilde{\Gamma}$ as follows. The graph $\tilde{\Gamma}$ has $2p$ vertices, say $v_0, w_0, v_1, w_1, \dots, v_{p-1}, w_{p-1}$, and $p|E|$ edges. Let $\tilde{\Gamma}_i$ be the full subgraph of $\tilde{\Gamma}$ generated by $\{v_i, w_i\}$ for all $i \in \{0, \dots, p-1\}$. Then $\tilde{\Gamma}_i$ is isomorphic to $\Gamma \setminus \{e\}$. There is one edge joining w_0 to v_1 , one edge joining w_1 to v_2 , ..., one edge joining w_{p-2} to v_{p-1} , and one edge joining v_0 to w_{p-1} , and the edge groups associated to these edges are isomorphic to G_e . Note that g belongs to the vertex group of v_0 and h belongs to the vertex group of w_1 .



Let Γ^* be the graph of groups obtained from $\tilde{\Gamma}$ by contracting $\tilde{\Gamma}_i$ to a point for all $i \in \{1, \dots, p-1\}$. Then G is isomorphic to the fundamental group of Γ^* . The graph Γ^* has p vertices, say v_0, \dots, v_{p-1} . There is one edge joining v_0 to v_1 , one edge joining v_1 to v_2 , ..., one edge joining v_{p-2} to v_{p-1} , and one edge joining v_0 to v_{p-1} , and the edge groups associated to these edges are all isomorphic to G_e . Note that g belongs to the vertex group of v_0 and h belongs to the vertex group of v_1 .



Let T be the maximal tree $T = v_0 v_1 \dots v_{p-2} v_{p-1}$. Then $T \setminus \{v_0 v_1\}$ is the disjoint union of two trees: v_0 and $v_1 v_2 \dots v_{p-2} v_{p-1}$. Set $\Gamma_1^* = v_0$ and $\Gamma_2^* = v_1 v_2 \dots v_{p-2} v_{p-1}$. Let Λ be the graph of groups obtained from Γ^* by contracting Γ_i^* to a point for all $i \in \{1, 2\}$. Let Λ' be the graph of groups obtained from Λ by composing with π . As in Step 3, we may replace Γ by Λ' , so as to assume that Γ has two vertices and two edges joining these two vertices.

End of the proof. Present $G_v = \langle g \mid g^{p^n} = 1 \rangle$, $G_w = \langle h \mid h^{p^n} = 1 \rangle$, and G as described above. There are two cases:

Case 1. Γ has one edge.

In this case, G is an amalgamated product of two finite abelian p -groups. Since G is residually p -finite, G is conjugacy p -separable by [1], Theorem 2. Thus, there exists a homomorphism φ from G onto a finite p -group P such that $\varphi(g) \sim \varphi(h)$.

Case 2. Γ has two edges.

We have

$$G = \langle g, h, t \mid g^{p^n} = 1, h^{p^n} = 1, g^{p^r} = h^{p^r}, t g^{p^s} t^{-1} = h^{p^s} \rangle,$$

where $r \in \{1, \dots, n - 1\}$, $s \in \{1, \dots, n - 1\}$. Let

$$A = C_{p^n} \times \underbrace{C_{p^s} \times \dots \times C_{p^s}}_{p^r - 1} \times C_{p^r}.$$

Set $m = p^r + 1$. Present each factor of this product in the natural way, using generators x_1, \dots, x_m respectively. Let α be the automorphism of A defined by

$$\begin{aligned} \alpha(x_1) &= x_1 x_2 x_m, \\ \alpha(x_i) &= x_{i+1} \text{ for all } i \in \{2, \dots, m - 2\}, \\ \alpha(x_{m-1}) &= (x_2 \dots x_{m-1})^{-1}, \\ \alpha(x_m) &= x_m. \end{aligned}$$

It is easily seen that α has order $m - 1 = p^r$. We have

$$\begin{aligned} \alpha^0(x_1) &= x_1, \\ \alpha^1(x_1) &= x_1 x_2 x_m, \\ \alpha^2(x_1) &= x_1 x_2 x_3 x_m^2, \\ &\vdots \\ \alpha^{m-2}(x_1) &= x_1 x_2 x_3 \dots x_{m-1} x_m^{m-2}. \end{aligned}$$

Let $B = A \rtimes \langle \alpha \rangle$ be the semidirect product of A by $\langle \alpha \rangle$. Note that B is a finite p -group. Let $\varphi: G \rightarrow B$ be the homomorphism defined by

$$\varphi(g) = x_1, \quad \varphi(h) = x_1 x_m, \quad \varphi(t) = \alpha.$$

Observe that the conjugacy class of $\varphi(g)$ in B is $\varphi(g)^B = \{\alpha^k(x_1) \mid k \in \{0, \dots, m - 2\}\}$. Thus, $\varphi(g)$ and $\varphi(h)$ are not conjugate in B . □

Corollary 4.3. *Let P be a finite p -group. Let A be a subgroup of P . Let Q be the HNN extension of P relative to A :*

$$Q = \langle P, t \mid t^{-1} a t = a \text{ for all } a \in A \rangle.$$

Then Q is hereditarily conjugacy p -separable.

Proof. Let R be an arbitrary subgroup of Q . Let $f: Q \rightarrow P$ be the natural homomorphism. We have $\ker(f) \cap P = \{1\}$. Therefore $\ker(f)$ is free by [KS], Theorem 6. That is, Q is an extension of a free group by a finite p -group. Thus R is itself an extension of a free group by a finite p -group. Therefore R is conjugacy p -separable by Theorem 4.2. \square

Remark 4.4. It is known that a fundamental group of a graph of groups, whose vertex groups are all finite p -groups is residually p -finite if and only if it is an extension of a free group by a finite p -group (see, for example, [AF1], Lemma 3.3). Thus, as an immediate consequence of Theorem 4.2, we have that a fundamental group of a graph of groups whose vertex groups are all finite p -groups is conjugacy p -separable if and only if it is residually p -finite.

5. Retractions

In this section, we shall prove several results on retractions that will allow us to control the growth of the intersection of Lemma 6.5 in finite p -group quotients of G_Γ .

Definition 5.1. Let G be a group, and H be a subgroup of G . We say that H is a retract of G if there exists a homomorphism $\rho_H: G \rightarrow G$ such that $\rho_H(G) = H$ and $\rho_H(h) = h$ for all $h \in H$. The homomorphism ρ_H is called a retraction of G onto H .

Remark 5.2. If G is a right-angled Artin group, and $H = \langle W \rangle$ is a special subgroup of G , then H is a retract of G . A retraction of G onto H is given by

$$\rho_H(v) = \begin{cases} v & \text{if } v \in W, \\ 1 & \text{if } v \in V \setminus W. \end{cases}$$

Lemma 5.3. Let G be a group and H be a subgroup of G . Suppose that H is a retract of G . Let ρ_H be a retraction of G onto H . Let N be a normal subgroup of G such that $\rho_H(N) \subset N$. Then ρ_H induces a retraction $\rho_{\bar{H}}: G/N \rightarrow G/N$ of G/N onto the canonical image \bar{H} of H in G/N , defined by $\rho_{\bar{H}}(gN) = \rho_H(g)N$ for all $gN \in G/N$.

Proof. Proved in [M] (see Lemma 4.1). \square

Remark 5.4. Let G be a group and let H, H' be two subgroups of G . Suppose that H and H' are retracts of G and that the corresponding retractions, ρ_H and $\rho_{H'}$, commute. Then $\rho_H(H') = \rho_{H'}(H) = H \cap H'$. Moreover, $H \cap H'$ is a retract of G . A retraction of G onto $H \cap H'$ is given by $\rho_{H \cap H'} = \rho_H \circ \rho_{H'} = \rho_{H'} \circ \rho_H$.

Proposition 5.5. *Let G be a group and H_1, \dots, H_n be n subgroups of G . Suppose that H_1, \dots, H_n are retracts of G and that the corresponding retractions pairwise commute. Then, for every normal subgroup K of p -power index in G , there exists a normal subgroup N of p -power index in G such that $N < K$ and $\rho_{H_i}(N) \subset N$ for all $i \in \{1, \dots, n\}$. Consequently, for every $i \in \{1, \dots, n\}$, the retraction ρ_{H_i} induces a retraction $\rho_{\bar{H}_i}$ of G/N onto the canonical image \bar{H}_i of H_i in G/N .*

Proof. Proved in [M] (see Proposition 4.3 and Remark 4.4). □

Lemma 5.6. *Let G be a group and let H, H' be two subgroups of G . Suppose that H and H' are retracts of G and that the corresponding retractions, ρ_H and $\rho_{H'}$, commute. Let N be a normal subgroup of G such that $\rho_H(N) \subset N$ and $\rho_{H'}(N) \subset N$. Then $\varphi(H \cap H') = \varphi(H) \cap \varphi(H')$, where $\varphi: G \rightarrow G/N$ denotes the canonical projection.*

Proof. Proved in [M] (see Lemma 4.5). □

The next statement is the analogue of Lemma 4.6 in [M]:

Corollary 5.7. *Let G be a group and H_1, \dots, H_n be n subgroups of G . Suppose that H_1, \dots, H_n are retracts of G and that the corresponding retractions $\rho_{H_1}, \dots, \rho_{H_n}$ pairwise commute. Then, for every normal subgroup K of p -power index in G , there exists a normal subgroup N of p -power index in G such that $N < K$ and $\rho_{H_i}(N) \subset N$ for all $i \in \{1, \dots, n\}$. Moreover, if $\varphi: G \rightarrow G/N$ denotes the canonical projection, then $\varphi(\bigcap_{i=1}^n H_i) = \bigcap_{i=1}^n \varphi(H_i)$.*

Proof. By Proposition 5.5, there exists a normal subgroup N of p -power index in G such that $N < K$ and $\rho_{H_i}(N) \subset N$ for all $i \in \{1, \dots, n\}$. We denote by $\varphi: G \rightarrow G/N$ the canonical projection. We argue by induction on $k \in \{1, \dots, n\}$ to prove that $\varphi(\bigcap_{i=1}^k H_i) = \bigcap_{i=1}^k \varphi(H_i)$. If $k = 1$, then the result is trivial. Thus we can assume that $k \geq 2$ and that the result has been proved for $k - 1$. We set $H' = \bigcap_{i=1}^{k-1} H_i$. By Remark 5.4, H' is a retract of G . A retraction of G onto H' is given by $\rho_{H'} = \rho_{H_1} \circ \dots \circ \rho_{H_{k-1}}$. We have

$$\begin{aligned} \rho_{H'}(N) &= \rho_{H_1}(\dots(\rho_{H_{k-2}}(\rho_{H_{k-1}}(N)))) \\ &\subset \rho_{H_1}(\dots(\rho_{H_{k-2}}(N))) \subset \dots \subset \rho_{H_1}(N) \subset N. \end{aligned}$$

The retractions $\rho_{H'}$ and ρ_{H_k} commute, so we can apply Lemma 5.6 to conclude that $\varphi(H' \cap H_k) = \varphi(H') \cap \varphi(H_k)$. By the induction hypothesis, $\varphi(H') = \bigcap_{i=1}^{k-1} \varphi(H_i)$. Finally $\varphi(\bigcap_{i=1}^k H_i) = \bigcap_{i=1}^k \varphi(H_i)$. □

In the following lemmas, G is a group and A, B are two subgroups of G . We assume that A and B are retracts of G and that the corresponding retractions, ρ_A and ρ_B , commute.

Lemma 5.8. *Let $x, y \in G$. We set $\alpha = \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1}) \in AxB$ and $\beta = \rho_A(\rho_B(y)y^{-1})y\rho_B(y^{-1}) \in AyB$. Then the following are equivalent:*

- (1) $y \in AxB$,
- (2) $\beta \in \alpha^{A \cap B}$.

Proof. Proved in [M] (see Lemma 5.1). □

Lemma 5.9. *Let $x \in G$. We set $\alpha = \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1}) \in AxB$ and $\gamma = \rho_A(\rho_B(x)x^{-1}) \in A$. Then we have*

$$A \cap xBx^{-1} = \gamma^{-1}C_{A \cap B}(\alpha)\gamma.$$

Proof. Proved in [M] (see Lemma 5.2). □

The next five statements are the analogues of some statements in [M] (Lemma 5.3, Corollary 5.4, Lemma 5.5, Lemma 5.6, and Lemma 5.7, respectively):

Lemma 5.10. *Let $x \in G$. We set: $\alpha = \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1}) \in AxB$. If $\alpha^{A \cap B}$ is finitely p -separable in G , then AxB is also finitely p -separable in G .*

Proof. Let $y \in G$ such that $y \notin AxB$. We set $\beta = \rho_A(\rho_B(y)y^{-1})y\rho_B(y^{-1})$. By Lemma 5.8, we have $\beta \notin \alpha^{A \cap B}$. Since $\alpha^{A \cap B}$ is finitely p -separable in G , there exists a normal subgroup K of p -power index in G such that if $\psi: G \rightarrow G/K$ denotes the canonical projection, we have $\psi(\beta) \notin \psi(\alpha^{A \cap B}) = \psi(\alpha)^{\psi(A \cap B)}$. By Corollary 5.7, there exists a normal subgroup N of p -power index in G such that $N < K$, $\rho_A(N) \subset N$, $\rho_B(N) \subset N$, and if $\varphi: G \rightarrow G/N$ denotes the canonical projection, then $\varphi(A \cap B) = \varphi(A) \cap \varphi(B)$. Assume that $\varphi(\beta) \in \varphi(\alpha)^{\varphi(A \cap B)}$. Let $g \in A \cap B$ be such that $\varphi(\beta) = \varphi(g)\varphi(\alpha)\varphi(g)^{-1}$. Then $\beta \in g\alpha g^{-1}N$. Since $N < K$, we obtain $\beta \in g\alpha g^{-1}K$. But this contradicts the fact that $\psi(\beta) \notin \psi(\alpha)^{\psi(A \cap B)}$. Therefore we have $\varphi(\beta) \notin \varphi(\alpha)^{\varphi(A \cap B)}$, i.e., $\varphi(\beta) \notin \varphi(\alpha)^{\varphi(A) \cap \varphi(B)}$. We set $\bar{A} = \varphi(A)$ and $\bar{B} = \varphi(B)$. By Lemma 5.3, ρ_A induces a retraction $\rho_{\bar{A}}$ of G/N onto \bar{A} and ρ_B induces a retraction $\rho_{\bar{B}}$ of G/N onto \bar{B} . We set $\bar{x} = \varphi(x)$ and $\bar{y} = \varphi(y)$. We have $\varphi(\alpha) = \rho_{\bar{A}}(\rho_{\bar{B}}(\bar{x})\bar{x}^{-1})\bar{x}\rho_{\bar{B}}(\bar{x}^{-1})$ and $\varphi(\beta) = \rho_{\bar{A}}(\rho_{\bar{B}}(\bar{y})\bar{y}^{-1})\bar{y}\rho_{\bar{B}}(\bar{y}^{-1})$. By Lemma 5.8, we have $\bar{y} \notin \bar{A}\bar{x}\bar{B}$, i.e., $\varphi(y) \notin \varphi(AxB)$. □

Corollary 5.11. *Let G be a group, and A, B be two subgroups of G . Suppose that G is residually p -finite. If A and B are retracts of G such that the corresponding retractions commute, then AB is finitely p -separable in G .*

Proof. We apply Lemma 5.10 to $x = 1$. □

Lemma 5.12. *Let G be a group, and A be a subgroup of G . Suppose that G is residually p -finite and that A is a retract of G . Then if a subset S of A is closed in the pro- p topology on A , it is also closed in the pro- p topology on G .*

Proof. We denote by \bar{S} the closure of S in G , equipped with the pro- p topology. We shall show that $\bar{S} \subset S$. By Corollary 5.11, A is closed in G . Therefore $\bar{S} \subset A$. Let $a \in G \setminus S$. We can assume that $a \in A$. There exists a homomorphism ψ from A onto a finite p -group P such that $\psi(a) \notin \psi(S)$. We set $\varphi = \psi \circ \rho_A$. We have $\varphi(a) = \psi(a) \notin \psi(S) = \varphi(S)$. Then $a \notin \bar{S}$. \square

Lemma 5.13. *Let $x \in G$. We set $\alpha = \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1})$. Suppose that the pair $(A \cap B, \alpha)$ satisfies the p -centralizer condition in G . Then, for every normal subgroup K of p -power index in G , there exists a normal subgroup N of p -power index in G such that $N < K$, $\rho_A(N) \subset N$, $\rho_B(N) \subset N$, and if $\varphi: G \rightarrow G/N$ denotes the canonical projection, then $\varphi(A) \cap \varphi(xBx^{-1}) \subset \varphi(A \cap xBx^{-1})\varphi(K)$.*

Proof. Suppose that K is a normal subgroup of p -power index in G . We set $\gamma = \rho_A(\rho_B(x)x^{-1}) \in A$. By Lemma 5.9, we have $A \cap xBx^{-1} = \gamma^{-1}C_{A \cap B}(\alpha)\gamma$. Since the pair $(A \cap B, \alpha)$ satisfies pCC_G , there exists a normal subgroup L of p -power index in G such that $L < K$ and if $\psi: G \rightarrow G/L$ denotes the canonical projection, $C_{\psi(A \cap B)}(\psi(\alpha)) \subset \psi(C_{A \cap B}(\alpha)K)$. This is equivalent to $\psi^{-1}(C_{\psi(A \cap B)}(\psi(\alpha))) \subset C_{A \cap B}(\alpha)K$. Indeed, assume that $g \in \psi^{-1}(C_{\psi(A \cap B)}(\psi(\alpha)))$. We then have $\psi(g) \in C_{\psi(A \cap B)}(\psi(\alpha)) \subset \psi(C_{A \cap B}(\alpha)K)$, and hence $g \in C_{A \cap B}(\alpha)KL \subset C_{A \cap B}(\alpha)K$ (because $L < K$).

By Corollary 5.7, there exists a normal subgroup N of p -power index in G such that $N < L$, $\rho_A(N) \subset N$, $\rho_B(N) \subset N$, and if $\varphi: G \rightarrow G/N$ denotes the canonical projection, $\varphi(A \cap B) = \varphi(A) \cap \varphi(B)$. We set $\bar{A} = \varphi(A)$, $\bar{B} = \varphi(B)$. By Lemma 5.3, $\rho_{\bar{A}}$ induces a retraction $\rho_{\bar{A}}$ of G/N onto \bar{A} , and $\rho_{\bar{B}}$ induces a retraction $\rho_{\bar{B}}$ of G/N onto \bar{B} . Obviously $\rho_{\bar{A}}$ and $\rho_{\bar{B}}$ commute. We set $\bar{x} = \varphi(x)$, $\bar{\alpha} = \rho_{\bar{A}}(\rho_{\bar{B}}(\bar{x})\bar{x}^{-1})\bar{x}\rho_{\bar{B}}(\bar{x}^{-1})$ ($\in G/N$) and $\bar{\gamma} = \rho_{\bar{A}}(\rho_{\bar{B}}(\bar{x})\bar{x}^{-1})$ ($\in \bar{A}$). Observe that $\bar{\alpha} = \varphi(\alpha)$ and $\bar{\gamma} = \varphi(\gamma)$. Then, by Lemma 5.9, we have $\bar{A} \cap \bar{x}\bar{B}\bar{x}^{-1} = \bar{\gamma}^{-1}C_{\bar{A} \cap \bar{B}}(\bar{\alpha})\bar{\gamma}$. Now, $\bar{A} \cap \bar{B} = \varphi(A \cap B)$. Thus,

$$\varphi^{-1}(\bar{A} \cap \bar{x}\bar{B}\bar{x}^{-1}) = \varphi^{-1}(\bar{\gamma}^{-1}C_{\varphi(A \cap B)}(\bar{\alpha})\bar{\gamma}) = \gamma^{-1}\varphi^{-1}(C_{\varphi(A \cap B)}(\bar{\alpha}))\gamma.$$

We claim that

$$\varphi^{-1}(C_{\varphi(A \cap B)}(\varphi(\alpha))) \subset \psi^{-1}(C_{\psi(A \cap B)}(\psi(\alpha))).$$

Indeed let $g \in \varphi^{-1}(C_{\varphi(A \cap B)}(\varphi(\alpha)))$. We have $\varphi(g) \in \varphi(A \cap B)$, i.e., $g \in (A \cap B)N$, which implies that $g \in (A \cap B)L$, i.e., $\psi(g) \in \psi(A \cap B)$; and $\varphi(g)\varphi(\alpha) = \varphi(\alpha)\varphi(g)$, i.e., $g\alpha g^{-1}\alpha^{-1} \in N$, which implies that $g\alpha g^{-1}\alpha^{-1} \in L$, i.e., $\psi(g)\psi(\alpha) = \psi(\alpha)\psi(g)$. We deduce that

$$\varphi^{-1}(C_{\varphi(A \cap B)}(\varphi(\alpha))) \subset C_{A \cap B}(\alpha),$$

and hence

$$\varphi^{-1}(\bar{A} \cap \bar{x}\bar{B}\bar{x}^{-1}) \subset \gamma^{-1}C_{A \cap B}(\alpha)\gamma K = (A \cap xBx^{-1})K.$$

We conclude that

$$\varphi(A) \cap \varphi(xBx^{-1}) \subset \varphi(A \cap xBx^{-1})\varphi(K). \quad \square$$

Lemma 5.14. *Let $x, y \in G$. We set*

$$C = xBx^{-1} (< G) \quad \text{and} \quad \alpha = \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1}).$$

If $\alpha^{A \cap B}$ and $y^{A \cap C}$ are finitely p -separable in G and if the pair $(A \cap B, \alpha)$ satisfies pCC_G , then $C_A(y)C$ is finitely p -separable in G .

Proof. Let $z \in G$ such that $z \notin C_A(y)C$. Suppose first that $z \notin AC$. Since $\alpha^{A \cap B}$ is finitely p -separable in G , AxB is finitely p -separable in G by Lemma 5.10. Therefore $AC = AxBx^{-1}$ is also finitely p -separable in G . Consequently there exists a normal subgroup N of p -power index in G such that $z \notin ACN$. We obviously have $z \notin C_A(y)CN$. Thus we can assume that $z \in AC$. Let $a \in A, c \in C$ be such that $z = ac$. Since $z \notin C_A(y)C, a^{-1}ya \notin y^{A \cap C}$. Indeed, if there is $g \in A \cap C$ such that $a^{-1}ya = gyg^{-1}$, then $(ag)^{-1}y(ag) = y$ i.e., $ag \in C_A(y)$. We obtain $a \in C_A(y)C$, and then $z \in C_A(y)C$, which is a contradiction. Now $y^{A \cap C}$ is finitely p -separable in G . Then there exists a normal subgroup K of p -power index in G such that $a^{-1}ya \notin y^{A \cap C}K$. By Lemma 5.13, there exists a normal subgroup N of p -power index in G such that $N < K$ and, if $\varphi: G \rightarrow G/N$ denotes the canonical projection, $\varphi(A) \cap \varphi(C) \subset \varphi(A \cap C)\varphi(K)$. For a subset S of G , we set $\bar{S} = \varphi(S)$. For an element g of G , we set $\bar{g} = \varphi(g)$. We have $\bar{y}^{\bar{A} \cap \bar{C}} \subset \bar{y}^{\bar{A} \cap \bar{C}} \bar{K}$. Note that $\bar{K} \triangleleft G/N$. Then $\bar{y}^{\bar{A} \cap \bar{C}} \subset \bar{y}^{\bar{A} \cap \bar{C}} \bar{K}$. Observe that $\bar{a}^{-1} \bar{y} \bar{a} \notin \bar{y}^{\bar{A} \cap \bar{C}} \bar{K}$ – otherwise we would have $a^{-1}ya \in y^{A \cap C}KN$ and then $a^{-1}ya \in y^{A \cap C}K$ (because $N < K$). We deduce that $\bar{a}^{-1} \bar{y} \bar{a} \notin \bar{y}^{\bar{A} \cap \bar{C}}$. Now it suffices to show that $\varphi(z) \notin \varphi(C_A(y)C)$. Suppose the contrary. Let $a' \in C_A(y), c' \in C$ be such that $\varphi(z) = \varphi(a'c')$. Then $\varphi(ac) = \varphi(a'c')$. Thus $\varphi(a^{-1}a) = \varphi(c'c^{-1})$. We set $\bar{g} = \varphi(a^{-1}a) = \varphi(c'c^{-1})$ ($\in \bar{A} \cap \bar{C}$). We have $\varphi(z) = \varphi(a')\bar{g}\varphi(c)$ and $\bar{a} = \varphi(z)\varphi(c)^{-1} = \varphi(a')\bar{g}$. Then $\bar{a}^{-1} \bar{y} \bar{a} = \bar{g}^{-1}\varphi(a'^{-1}ya')\bar{g} = \bar{g}^{-1}\varphi(y)\bar{g} = \bar{g}^{-1} \bar{y} \bar{g} \in \bar{y}^{\bar{A} \cap \bar{C}}$, a contradiction. We have shown that $C_A(y)C$ is finitely p -separable in G . □

6. Proof of the main theorem

We turn now to the proof that right-angled Artin groups are hereditarily conjugacy p -separable. We need the following theorem, which is due to Duchamp and Kroh (see [DK2], Theorem 2.3).

Theorem 6.1. *Right-angled Artin groups are residually p -finite.*

(Note that the exact statement of [DK2], Theorem 2.3, is that right-angled Artin groups are residually torsion-free nilpotent; Theorem 6.1 then follows from [G],

Theorem 2.1.) This theorem can also be proved using HNN extensions (see [Lo], Theorem 2.11).

Basically, Proposition 6.2 establishes the main result. Proposition 6.2(1) and Proposition 6.2(2) will be proved simultaneously by induction on the rank of G .

Proposition 6.2. *Let G be a right-angled Artin group.*

- (1) *Every special subgroup S of G satisfies the p -centralizer condition in G (pCC_G).*
- (2) *For all $g \in G$ and for every special subgroup S of G , g^S is finitely p -separable in G .*

From now on we assume that G is a right-angled Artin group of rank r ($r \geq 1$), and that $H = \langle W \rangle$ is a special subgroup of G of rank $r - 1$. Thus, G can be written as an HNN extension of H relative to the special subgroup $K = \langle \text{link}(t) \rangle$ of H :

$$G = \langle H, t \mid t^{-1}kt = k \text{ for all } k \in K \rangle.$$

Recall that H is a retract of G . A retraction of G onto H is given by

$$\rho_H(v) = \begin{cases} v & \text{if } v \in W, \\ 1 & \text{if } v \in V \setminus W. \end{cases}$$

We also assume that

- every special subgroup S of H satisfies the p -centralizer condition in H (pCC_H),
- for all $h \in H$ and for every special subgroup S of H , h^S is finitely p -separable in H .

The next results (Lemma 6.3 to Lemma 6.14) are preliminaries to the proof of Proposition 6.2.

In general, if A and B are subgroups of a group G , the image of the intersection of A and B under a homomorphism $\varphi: G \rightarrow H$ does not coincide with the intersection of the images of A and B in H . However, the p -centralizer condition and the above results on retractions will allow us to obtain the following lemma, which will be used to apply Minasyan’s criterion for conjugacy in HNN extensions (see Lemma 6.5).

Lemma 6.3. *Let be given A_0 , a conjugate of a special subgroup of H , A_1, \dots, A_n , n special subgroups of H , and $\alpha, x_0, \dots, x_n, y_1, \dots, y_n$, $2(n + 1)$ elements of H . Then, for every normal subgroup L of p -power index in H , there exists a normal subgroup N of p -power index in H such that $N < L$ and if $\varphi: H \rightarrow H/N$ denotes the canonical projection, then*

$$\bar{\alpha}C_{A_0}(\bar{x}_0) \cap \bigcap_{i=1}^n \bar{x}_i \bar{A}_i \bar{y}_i \subset \varphi((\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^n x_i A_i y_i)L),$$

where $\bar{A}_i = \varphi(A_i)$ ($i \in \{0, \dots, n\}$), $\bar{\alpha} = \varphi(\alpha)$, $\bar{x}_j = \varphi(x_j)$ ($j \in \{0, \dots, n\}$), $\bar{y}_k = \varphi(y_k)$ ($k \in \{1, \dots, n\}$).

Proof. Let L be a subgroup of p -power index in H . We argue by induction on n . Strictly speaking, the basis of our induction is $n = 0$ but we will need the case $n = 1$. By the assumptions, there exist a special subgroup A of H and an element β of H such that $A_0 = \beta A \beta^{-1}$.

$n = 0$: We set $x = \beta^{-1} x_0 \beta$. The pair (A, x) satisfies pCC_H by the assumptions. There exists a normal subgroup N of p -power index in H such that $N < L$ and if $\varphi: H \rightarrow H/N$ denotes the canonical projection, then $C_{\varphi(A)}(\varphi(x)) \subset \varphi(C_A(x)L)$. But $C_{A_0}(x_0) = \beta C_A(x) \beta^{-1}$. We deduce that $\varphi(\alpha) C_{\varphi(A_0)}(\varphi(x_0)) \subset \varphi(\alpha C_{A_0}(x_0)L)$.

$n = 1$: There are two cases:

Case 1. $\alpha C_{A_0}(x_0) \cap x_1 A_1 y_1 = \emptyset$. This is equivalent to saying that $x_1 \notin \alpha C_{A_0}(x_0) y_1^{-1} A_1$. Put $B = (y_1 \beta)^{-1} A_1 y_1 \beta$ so that $x_1 \notin \alpha \beta (C_A(x) B) \beta^{-1} y_1^{-1}$. Now the intersection of conjugates of two special subgroups of H is a conjugate of a special subgroup of H (see [M], Lemma 6.5). Then $A \cap A_1$ is a conjugate of a special subgroup C of H . There exists $\gamma \in H$ such that $A \cap A_1 = \gamma C \gamma^{-1}$. Therefore if $h \in H$, $h^{A \cap A_1} = \gamma (\gamma^{-1} h \gamma)^C \gamma^{-1}$. Now $(\gamma^{-1} h \gamma)^C$ is finitely p -separable in H by the assumptions. We deduce that $h^{A \cap A_1}$ is finitely p -separable in H . With the same argument, $x^{A \cap B}$ is finitely p -separable in H . Now the pair $(A \cap A_1, h)$ satisfies pCC_H by the assumptions. We deduce that $C_A(x) B$ is finitely p -separable in H by Lemma 5.14. This implies that $\alpha C_{A_0}(x_0) y_1^{-1} A_1$ is finitely p -separable in H . There exists a normal subgroup M of p -power index in H such that $x_1 \notin \alpha C_{A_0}(x_0) y_1^{-1} A_1 M$. Up to replacing M by $M \cap L$, we can assume that $M < L$. Now the pair (A_0, x_0) satisfies pCC_H by the assumptions. There exists a normal subgroup N of p -power index in H such that $N < M$ and if $\varphi: H \rightarrow H/N$ denotes the canonical projection, then $C_{\varphi(A_0)}(\varphi(x_0)) \subset \varphi(C_{A_0}(x_0) M)$, or, equivalently, $\varphi^{-1}(C_{\varphi(A_0)}(\varphi(x_0))) \subset C_{A_0}(x_0) M$. Then $\varphi^{-1}(\bar{\alpha} C_{\bar{A}_0}(\bar{x}_0) \bar{y}_1^{-1} \bar{A}_1) \subset \alpha \varphi^{-1}(C_{\bar{A}_0}(\bar{x}_0)) y_1^{-1} A_1 \subset \alpha C_{A_0}(x_0) y_1^{-1} A_1 M$ (with the same notations as in the statement of the lemma). Therefore $x_1 \notin \varphi^{-1}(\bar{\alpha} C_{\bar{A}_0}(\bar{x}_0) \bar{y}_1^{-1} \bar{A}_1)$. Finally, we have $\bar{\alpha} C_{\bar{A}_0}(\bar{x}_0) \cap \bar{x}_1 \bar{A}_1 \bar{y}_1 = \emptyset$.

Case 2. $\alpha C_{A_0}(x_0) \cap x_1 A_1 y_1 \neq \emptyset$.

Remark 6.4. If G is a group and H, K are two subgroups of G such that $aH \cap bKc \neq \emptyset$, where $a, b, c \in G$, then we have $aH \cap bKc = g(H \cap c^{-1}Kc)$ for all $g \in aH \cap bKc$.

Choose $g \in \alpha C_{A_0}(x_0) \cap x_1 A_1 y_1$. By Remark 6.4, we have $\alpha C_{A_0}(x_0) \cap x_1 A_1 y_1 = g(C_{A_0}(x_0) \cap y_1^{-1} A_1 y_1)$. We set $D = A_0 \cap y_1^{-1} A_1 y_1$. Then $\alpha C_{A_0}(x_0) \cap x_1 A_1 y_1 = g C_D(x_0)$. Now, D is a conjugate of a special subgroup E of H by [M], Lemma 6.5. There exists $\delta \in H$ such that $D = \delta E \delta^{-1}$. As above, the pair (D, x_0) satisfies pCC_H . There exists a normal subgroup M of p -power index in H such that $M < L$ and if $\psi: H \rightarrow H/M$ denotes the canonical projection, we have $C_{\psi(D)}(\psi(x_0)) \subset \psi(C_D(x_0)L)$. Now by Lemma 5.13, there exists a normal subgroup N of p -power

index in H such that $N < M$ and if $\varphi: H \rightarrow H/N$ denotes the canonical projection, then $\varphi(A) \cap \varphi((y_1\beta)^{-1}A_1y_1\beta) \subset \varphi(A \cap (y_1\beta)^{-1}A_1y_1\beta)\varphi(M)$. Therefore

$$\begin{aligned} \bar{A}_0 \cap \bar{y}_1^{-1}\bar{A}_1\bar{y}_1 &= \varphi(\beta A \beta^{-1}) \cap \varphi(y_1^{-1}A_1y_1) \\ &= \varphi(\beta)(\varphi(A) \cap \varphi((y_1\beta)^{-1}A_1y_1\beta))\varphi(\beta^{-1}) \\ &\subset \varphi(\beta)(\varphi(A \cap (y_1\beta)^{-1}A_1y_1\beta)\varphi(M))\varphi(\beta^{-1}) \quad (*) \\ &= \varphi(A_0 \cap y_1^{-1}A_1y_1)\varphi(M) \\ &= \varphi(D)\varphi(M)(*) \end{aligned}$$

(with the same notations as in the statement of the lemma). We set $\bar{g} = \varphi(g)$. Note that $\bar{g} \in \bar{\alpha}C_{\bar{A}_0}(\bar{x}_0) \cap \bar{x}_1\bar{A}_1\bar{y}_1$. Therefore $\bar{\alpha}C_{\bar{A}_0}(\bar{x}_0) \cap \bar{x}_1\bar{A}_1\bar{y}_1 = \bar{g}(C_{\bar{A}_0}(\bar{x}_0) \cap \bar{y}_1^{-1}\bar{A}_1\bar{y}_1)$. Considering $(*)$, we obtain

$$\bar{\alpha}C_{\bar{A}_0}(\bar{x}_0) \cap \bar{x}_1\bar{A}_1\bar{y}_1 = \bar{g}C_{\bar{A}_0 \cap \bar{y}_1^{-1}\bar{A}_1\bar{y}_1}(\bar{x}_0) \subset \bar{g}C_{\varphi(D)\varphi(M)}(\bar{x}_0).$$

Recall that $N < M$. Then $\psi: H \rightarrow H/M$ induces a homomorphism $\tilde{\psi}: H/N \rightarrow H/M$ such that $\psi = \tilde{\psi} \circ \varphi$. Note that $\tilde{\psi}(\varphi(D)\varphi(M)) = \psi(D)$. Let $z \in C_{\varphi(D)\varphi(M)}(\bar{x}_0)$. Then

$$\tilde{\psi}(z) \in C_{\psi(D)}(\psi(x_0)) \subset \psi(C_D(x_0)L) = \tilde{\psi}(\varphi(C_D(x_0)L)).$$

Therefore $z \in \varphi(C_D(x_0)L) \ker(\tilde{\psi}) = \varphi(C_D(x_0)L)$ (because $\ker(\tilde{\psi}) = \varphi(M) < \varphi(L)$). We deduce that $C_{\varphi(D)\varphi(M)}(\bar{x}_0) \subset \varphi(C_D(x_0)L)$. We conclude that

$$\begin{aligned} \bar{\alpha}C_{\bar{A}_0}(\bar{x}_0) \cap \bar{x}_1\bar{A}_1\bar{y}_1 &\subset \bar{g}\varphi(C_D(x_0)L) = \varphi(gC_D(x_0)L) \\ &= \varphi((\alpha C_{A_0}(x_0) \cap x_1A_1y_1)L). \end{aligned}$$

Inductive step: Suppose that $n \geq 1$ and that the result has been proved for $n - 1$. Note that if $\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_i A_i y_i = \emptyset$, then by the induction hypothesis, there exists a normal subgroup N of p -power index in H such that if $\varphi: H \rightarrow H/N$ denotes the canonical projection, then

$$\bar{\alpha}C_{\bar{A}_0}(\bar{x}_0) \cap \bigcap_{i=1}^{n-1} \bar{x}_i \bar{A}_i \bar{y}_i \subset \varphi((\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_i A_i y_i)L) = \emptyset.$$

Obviously

$$\bar{\alpha}C_{\bar{A}_0}(\bar{x}_0) \cap \bigcap_{i=1}^n \bar{x}_i \bar{A}_i \bar{y}_i = \emptyset \subset \varphi((\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^n x_i A_i y_i)L).$$

Thus we can assume that $\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_i A_i y_i \neq \emptyset$. Therefore $\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_i A_i y_i = g(C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} y_i^{-1}A_i y_i)$ for some $g \in H$. We set $F = A_0 \cap \bigcap_{i=1}^{n-1} y_i^{-1}A_i y_i$. Again, F is a conjugate of a special subgroup of H by [M],

Lemma 6.5. We have: $\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_i A_i y_i = g C_F(x_0)$. Now, by the case $n = 1$, there exists a normal subgroup M of p -power index in H such that $M < L$ and if $\psi: H \rightarrow H/M$ denotes the canonical projection, then

$$\psi(g)C_{\psi(F)}(\psi(x_0)) \cap \psi(x_n A_n y_n) \subset \psi((g C_F(x_0) \cap x_n A_n y_n)L),$$

or, equivalently,

$$\psi^{-1}(\psi(g)C_{\psi(F)}(\psi(x_0)) \cap \psi(x_n A_n y_n)) \subset (g C_F(x_0) \cap x_n A_n y_n)L.$$

On the other hand, by the induction hypothesis, there exists a normal subgroup N of p -power index in H such that $N < M$ and if $\varphi: H \rightarrow H/N$ denotes the canonical projection, then

$$\bar{\alpha} C_{\bar{A}_0}(\bar{x}_0) \cap \bigcap_{i=1}^{n-1} \bar{x}_i \bar{A}_i \bar{y}_i \subset \varphi((\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_i A_i y_i)M)$$

or, equivalently,

$$\varphi^{-1}(\bar{\alpha} C_{\bar{A}_0}(\bar{x}_0) \cap \bigcap_{i=1}^{n-1} \bar{x}_i \bar{A}_i \bar{y}_i) \subset (\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_i A_i y_i)M.$$

Thus we have

$$\begin{aligned} \varphi^{-1}(\bar{\alpha} C_{\bar{A}_0}(\bar{x}_0) \cap \bigcap_{i=1}^n \bar{x}_i \bar{A}_i \bar{y}_i) &= \varphi^{-1}(\bar{\alpha} C_{\bar{A}_0}(\bar{x}_0) \cap \bigcap_{i=1}^{n-1} \bar{x}_i \bar{A}_i \bar{y}_i) \cap \varphi^{-1}(\bar{x}_n \bar{A}_n \bar{y}_n) \\ &\subset (\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_i A_i y_i)M \cap x_n A_n y_n N \\ &= g C_F(x_0)M \cap x_n A_n y_n N. \end{aligned}$$

Recall that $N < M$. Finally we have

$$\begin{aligned} \varphi^{-1}(\bar{\alpha} C_{\bar{A}_0}(\bar{x}_0) \cap \bigcap_{i=1}^n \bar{x}_i \bar{A}_i \bar{y}_i) &\subset g C_F(x_0)M \cap x_n A_n y_n M \\ &\subset \psi^{-1}(\psi(g)C_{\psi(F)}(\psi(x_0))) \cap \psi^{-1}(\psi(x_n A_n y_n)) \\ &= \psi^{-1}(\psi(g)C_{\psi(F)}(\psi(x_0)) \cap \psi(x_n A_n y_n)) \\ &\subset (g C_F(x_0) \cap x_n A_n y_n)L \\ &= (\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^n x_i A_i y_i)L. \quad \square \end{aligned}$$

We need the following criterion for conjugacy in HNN extensions:

Lemma 6.5. *Let $G = \langle H, t \mid t^{-1}kt = k \text{ for all } k \in K \rangle$ be an HNN extension. Let S be a subgroup of H . Let $g = x_0 t^{a_1} x_1 \dots t^{a_n} x_n$ ($n \geq 1$) and $h = y_0 t^{b_1} y_1 \dots t^{b_m} y_m$ be elements of G in reduced form. Then $h \in g^S$ if and only if all of the following conditions hold:*

- (1) $m = n$ and $a_i = b_i$ for all $i \in \{1, \dots, n\}$,
- (2) $y_0 \dots y_n \in (x_0 \dots x_n)^S$,
- (3) if $\alpha \in S$ satisfies $y_0 \dots y_n = \alpha x_0 \dots x_n \alpha^{-1}$, then

$$\alpha C_S(x_0 \dots x_n) \cap y_0 K x_0^{-1} \cap (y_0 y_1) K (x_0 x_1)^{-1} \cap \dots \\ \dots \cap (y_0 \dots y_{n-1}) K (x_0 \dots x_{n-1})^{-1} \neq \emptyset.$$

Proof. Proved in [M] (see Lemma 7.11). □

The following is the analogue of Lemma 6.8 in [M]:

Lemma 6.6. *Let S be a special subgroup of H . Let $g \in G \setminus H$. Let $h \in G \setminus g^S$. There exists a normal subgroup L of p -power index in H such that if $\varphi: H \rightarrow P = H/L$ denotes the canonical projection if Q denotes the HNN extension of P relative to $\varphi(K)$ and if $\bar{\varphi}: G \rightarrow Q$ denotes the homomorphism induced by φ , we have $\bar{\varphi}(h) \notin \bar{\varphi}(g)^{\bar{\varphi}(S)}$.*

Proof. Write $g = x_0 t^{a_1} x_1 \dots t^{a_n} x_n$ and $h = y_0 t^{b_1} y_1 \dots t^{b_m} y_m$ in reduced forms. We have $n \geq 1$ as $g \notin H$.

Step 1. We assume that the first condition in Minasyan’s criterion (see Lemma 6.5) is not satisfied by g and h .

It follows from Lemma 2.10 in [Lo], and Theorem 6.1 (see, alternatively, [Lo], Theorem 2.11) that the special subgroup K is closed in the pro- p topology on H . (Note that this can also be obtained by combining Corollary 5.11 and Theorem 6.1.) Thus, there exists a normal subgroup L of p -power index in H such that

$$x_i \notin KL \quad \text{for all } i \in \{1, \dots, n-1\}, \tag{*}$$

$$y_j \notin KL \quad \text{for all } j \in \{1, \dots, m-1\}. \tag{**}$$

We denote by $\varphi: H \rightarrow P = H/L$ the canonical projection. If Q denotes the HNN extension of P relative to $\varphi(K)$,

$$Q = \langle P, \bar{t} \mid \bar{t}^{-1} \varphi(k) \bar{t} = \varphi(k) \text{ for all } k \in K \rangle,$$

and if $\bar{\varphi}: G \rightarrow Q$ denotes the homomorphism induced by φ , with $\bar{\varphi}|_H = \varphi$ and $\bar{\varphi}(t) = \bar{t}$, then $\bar{\varphi}(g) = \bar{x}_0 \bar{t}^{a_1} \bar{x}_1 \dots \bar{t}^{a_n} \bar{x}_n$ and $\bar{\varphi}(h) = \bar{y}_0 \bar{t}^{b_1} \bar{y}_1 \dots \bar{t}^{b_m} \bar{y}_m$ are reduced forms in Q by (*) and (**), where $\bar{x}_i = \bar{\varphi}(x_i)$ ($i \in \{0, \dots, n\}$) and $\bar{y}_j = \bar{\varphi}(y_j)$ ($j \in \{0, \dots, m\}$). But then the first condition in Minasyan’s criterion will not hold for $\bar{\varphi}(g)$ and $\bar{\varphi}(h)$.

Conclusion of Step 1. We can assume that $m = n$ and $a_i = b_i$ for all $i \in \{1, \dots, n\}$.

Step 2. We assume that the second condition in Minasyan’s criterion is not satisfied by g and h . We set $x = x_0 \dots x_n$ and $y = y_0 \dots y_n$. Thus $y \notin x^S$.

By the assumptions, x^S is finitely p -separable in H . Therefore there exists a homomorphism φ from H onto a finite p -group P such that $\varphi(y) \notin \varphi(x)^{\varphi(S)}$. Denote by Q the HNN extension of P relative to $\varphi(K)$, and by $\bar{\varphi}: G \rightarrow Q$ the homomorphism induced by φ . Now let $f: Q \rightarrow P$ be the natural homomorphism. We have

$$\begin{aligned} f(\bar{\varphi}(g)) &= f(\bar{x}_0 \bar{t}^{a_1} \bar{x}_1 \dots \bar{t}^{a_n} \bar{x}_n) = \bar{x}_0 \dots \bar{x}_n = \varphi(x), \\ f(\bar{\varphi}(h)) &= f(\bar{y}_0 \bar{t}^{a_1} \bar{y}_1 \dots \bar{t}^{a_n} \bar{y}_n) = \bar{y}_0 \dots \bar{y}_n = \varphi(y) \end{aligned}$$

(with the same notations as above). Since $\varphi(y) \notin \varphi(x)^{\varphi(S)}$, we see that $\bar{\varphi}(h) \notin \bar{\varphi}(g)^{\bar{\varphi}(S)}$.

Conclusion of Step 2. We can assume that $y \in x^S$. There exists $\alpha \in S$ such that $y = \alpha x \alpha^{-1}$.

End of the proof. Considering Minasyan’s criterion, since $h \notin g^S$, we must have

$$\begin{aligned} \alpha C_S(x_0 \dots x_n) \cap y_0 K x_0^{-1} \cap (y_0 y_1) K (x_0 x_1)^{-1} \cap \dots \\ \dots \cap (y_0 \dots y_{n-1}) K (x_0 \dots x_{n-1})^{-1} = \emptyset. \end{aligned}$$

As we noted above, K is closed in the pro- p topology on H ; thus, there exists a normal subgroup L of p -power index in H such that:

$$x_i \notin KL \text{ for all } i \in \{1, \dots, n-1\}, \tag{*}$$

$$y_j \notin KL \text{ for all } j \in \{1, \dots, n-1\}. \tag{**}$$

Now by Lemma 6.3, there exists a normal subgroup N of p -power index in H such that $N < L$ and if $\varphi: H \rightarrow P = H/N$ denotes the canonical projection, then

$$\begin{aligned} \bar{\alpha} C_{\bar{S}}(\bar{x}) \cap \bar{y}_0 \bar{K} \bar{x}_0^{-1} \cap \bar{y}_0 \bar{y}_1 \bar{K} (\bar{x}_0 \bar{x}_1)^{-1} \cap \dots \\ \dots \cap \bar{y}_0 \dots \bar{y}_{n-1} \bar{K} (\bar{x}_0 \dots \bar{x}_{n-1})^{-1} \\ \subset \varphi((\alpha C_S(x) \cap y_0 K x_0^{-1} \cap y_0 y_1 K (x_0 x_1)^{-1} \cap \dots \\ \dots \cap y_0 \dots y_{n-1} K (x_0 \dots x_{n-1})^{-1}) L) = \emptyset, \end{aligned} \tag{***}$$

where $\bar{S} = \varphi(S)$, $\bar{\alpha} = \varphi(\alpha)$, $\bar{x} = \varphi(x)$, $\bar{x}_i = \varphi(x_i)$ ($i \in \{0, \dots, n\}$), $\bar{y}_j = \varphi(y_j)$ ($j \in \{0, \dots, n\}$). Let Q be the HNN extension of P relative to $\varphi(K)$ and let $\bar{\varphi}: G \rightarrow Q$ be the homomorphism induced by φ . Then, by (*) and (**), $\bar{\varphi}(g) = \bar{x}_0 \bar{t}^{a_1} \bar{x}_1 \dots \bar{t}^{a_n} \bar{x}_n$ and $\bar{\varphi}(h) = \bar{y}_0 \bar{t}^{a_1} \bar{y}_1 \dots \bar{t}^{a_n} \bar{y}_n$ are reduced forms in Q . So, in view of (***), we have $\bar{\varphi}(h) \notin \bar{\varphi}(g)^{\bar{\varphi}(S)}$. \square

The following is the analogue of Lemma 8.8 in [M]:

Lemma 6.7. *Let $g_0 = t^{a_1} x_1 \dots t^{a_n} x_n$ ($n \geq 1$) and $h_0 = t^{b_1} y_1 \dots t^{b_m} y_m$ be cyclically reduced elements of G . Let h_1, \dots, h_k be elements of G . If $h_i \notin g_0^K$ for all $i \in \{1, \dots, k\}$, then there exists a normal subgroup L of p -power index in H*

such that if $\varphi: H \rightarrow P = H/L$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\varphi(K)$, and if $\bar{\varphi}: G \rightarrow Q$ denotes the homomorphism induced by φ , we have

- (1) $\bar{\varphi}(g_0) = \bar{t}^{a_1} \bar{x}_1 \dots \bar{t}^{a_n} \bar{x}_n$ and $\bar{\varphi}(h_0) = \bar{t}^{b_1} \bar{y}_1 \dots \bar{t}^{b_m} \bar{y}_m$ are cyclically reduced in Q , where $\bar{x}_i = \bar{\varphi}(x_i)$ ($i \in \{1, \dots, n\}$) and $\bar{y}_j = \bar{\varphi}(y_j)$ ($j \in \{1, \dots, n\}$),
- (2) $\bar{\varphi}(h_i) \notin \bar{\varphi}(g_0)^{\bar{\varphi}(K)}$ for all $i \in \{1, \dots, k\}$.

Proof. As we noted above, K is closed in the pro- p topology on H ; thus, there exists a normal subgroup L_0 of p -power index in H such that

$$x_i \notin KL_0 \quad \text{for all } i \in \{1, \dots, n-1\}, \tag{*}$$

$$y_j \notin KL_0 \quad \text{for all } j \in \{1, \dots, n-1\}. \tag{**}$$

Let $i \in \{1, \dots, k\}$. Since $h_i \notin g_0^K$, there exists a normal subgroup L_i of p -power index in H such that if $\varphi_i: H \rightarrow P_i = H/L_i$ denotes the canonical projection, if Q_i denotes the HNN extension of P_i relative to $\varphi_i(K)$ and if $\bar{\varphi}_i: G \rightarrow Q_i$ denotes the homomorphism induced by φ_i , we have $\bar{\varphi}_i(h_i) \notin \bar{\varphi}_i(g_0)^{\bar{\varphi}_i(K)}$ by Lemma 6.6. Set $L = L_0 \cap L_1 \dots \cap L_k$. Let $\varphi: H \rightarrow P = H/L$ be the canonical projection, let Q be the HNN extension of P relative to $\varphi(K)$, and let $\bar{\varphi}: G \rightarrow Q$ be the homomorphism induced by φ . Since $L < L_0$, $\bar{\varphi}(g_0) = \bar{t}^{a_1} \bar{x}_1 \dots \bar{t}^{a_n} \bar{x}_n$ and $\bar{\varphi}(h_0) = \bar{t}^{b_1} \bar{y}_1 \dots \bar{t}^{b_m} \bar{y}_m$ are cyclically reduced in Q by (*) and (**) (with the same notations as in the statement of the lemma). As $L < L_i$ for all $i \in \{1, \dots, k\}$, we have $\bar{\varphi}(h_i) \notin \bar{\varphi}(g_0)^{\bar{\varphi}(K)}$ for all $i \in \{1, \dots, k\}$. □

Lemma 6.8. *Let $G = \langle H, t \mid t^{-1}kt = k \text{ for all } k \in K \rangle$ be an HNN extension. Let S be a subgroup of H . Let $g = x_0 t^{a_1} x_1 \dots t^{a_n} x_n$ be an element of G in reduced form ($n \geq 1$). Then*

$$C_S(g) = C_S(x_0 \dots x_n) \cap x_0 K x_0^{-1} \cap (x_0 x_1) K (x_0 x_1)^{-1} \cap \dots \\ \dots \cap (x_0 \dots x_{n-1}) K (x_0 \dots x_{n-1})^{-1}.$$

Proof. Proved in [M] (see Lemma 7.12). □

The following is the analogue of Lemma 8.9 in [M]:

Lemma 6.9. *Let S be a special subgroup of H . Let L be a normal subgroup of p -power index in G , and let $g = x_0 t^{a_1} x_1 \dots t^{a_n} x_n$ be an element of G in reduced form and not contained in H . Then there exists a normal subgroup N of p -power index of H such that if $\varphi: H \rightarrow P = H/N$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\varphi(K)$, and if $\bar{\varphi}: G \rightarrow Q$ denotes the homomorphism induced by φ , we have*

- (1) $C_{\bar{\varphi}(S)}(\bar{\varphi}(g)) \subset \bar{\varphi}(C_S(g)L)$,
- (2) $\ker(\varphi) = N < H \cap L$,

$$(3) \ker(\bar{\varphi}) < L.$$

Proof. We have $n \geq 1$ as $g \notin H$.

As we noted above, K is closed in the pro- p topology on H . Therefore there exists a normal subgroup M of p -power index in H such that

$$x_i \notin KM \quad \text{for all } i \in \{1, \dots, n - 1\}. \tag{*}$$

We set $L' = H \cap L$. Note that L' is a normal subgroup of p -power index in H . Thus, up to replacing M by $M \cap L'$, we can assume that $M < L'$. We set $x = x_0 \dots x_n$. We have

$$C_S(g) = C_S(x) \cap x_0 K x_0^{-1} \cap (x_0 x_1) K (x_0 x_1)^{-1} \cap \dots \cap (x_0 \dots x_{n-1}) K (x_0 \dots x_{n-1})^{-1}$$

by Lemma 6.8. We denote by I the intersection on the right-hand side. By Lemma 6.3, there exists a normal subgroup N of p -power index in H such that $N < M$ and if $\varphi: H \rightarrow P = H/N$ denotes the canonical projection, we have

$$C_{\bar{S}}(\bar{x}) \cap \bar{x}_0 \bar{K} \bar{x}_0^{-1} \cap \bar{x}_0 \bar{x}_1 \bar{K} (\bar{x}_0 \bar{x}_1)^{-1} \cap \dots \cap \bar{x}_0 \dots \bar{x}_{n-1} \bar{K} (\bar{x}_0 \dots \bar{x}_{n-1})^{-1} \subset \varphi(IM),$$

where $\bar{S} = \varphi(S)$, $\bar{x} = \varphi(x)$, $\bar{x}_i = \varphi(x_i)$ ($i \in \{0, \dots, n - 1\}$). We denote by J the intersection on the left-hand side. Let Q be the HNN extension of P relative to $\varphi(K)$, and let $\bar{\varphi}: G \rightarrow Q$ be the homomorphism induced by φ . Then $\bar{x}_0 \bar{t}^{a_1} \bar{x}_1 \dots \bar{t}^{a_n} \bar{x}_n$ is a reduced form of $\bar{\varphi}(g)$ in Q by (*). But then $C_{\bar{\varphi}(S)}(\bar{\varphi}(g)) = J$ by Lemma 6.8. Now $\varphi(M) < \varphi(L') = \bar{\varphi}(L') < \bar{\varphi}(L)$. Therefore

$$\begin{aligned} C_{\bar{\varphi}(S)}(\bar{\varphi}(g)) &= J \subset \varphi(IM) = \varphi(I)\varphi(M) \\ &\subset \bar{\varphi}(I)\bar{\varphi}(L) = \bar{\varphi}(C_S(g))\bar{\varphi}(L) = \bar{\varphi}(C_S(g)L). \end{aligned}$$

Finally we remark that $\ker(\varphi) = N < M < L' = H \cap L < L$. Since $\ker(\bar{\varphi})$ is the normal closure of $\ker(\varphi)$ in G , we conclude that $\ker(\bar{\varphi}) < L$ (because L is normal in G). □

A prefix of $t^{a_1} x_1 \dots t^{a_n} x_n$ is an element of G of the form $t^{a_1} x_1 \dots t^{a_k} x_k$ for some $k \in \{0, \dots, n\}$. We need the following result:

Proposition 6.10. *Let $G = \langle H, t \mid t^{-1}kt = k \text{ for all } k \in K \rangle$ be an HNN extension. Let $g = t^{a_1} x_1 \dots t^{a_n} x_n$ be a cyclically reduced element of G ($n \geq 1$). Let $\{p_1, \dots, p_{n+1}\}$ be the set of all prefixes of g – we are not assuming that p_1, \dots, p_{n+1} are ordered. There are two cases:*

- (1) *If $x_n \in K$, then $n = 1$ and $C_G(g) = \langle t \rangle C_K(g)$.*
- (2) *If $x_n \in H \setminus K$, let $\{p_1, \dots, p_m\}$ be the set of prefixes of g satisfying $p_i^{-1} g p_i \in g^K$ ($m \in \{0, \dots, n + 1\}$). For each $i \in \{1, \dots, m\}$, we choose $\alpha_i \in K$ such that $p_i^{-1} g p_i = \alpha_i^{-1} g \alpha_i$. We set $\Omega = \{\alpha_i p_i^{-1} \mid i \in \{1, \dots, m\}\}$. Then $C_G(g) = C_K(g)\langle g \rangle \Omega$.*

Proof. Proved in [M] (see Proposition 7.8). □

The following is the analogue of Lemma 8.10 in [M]:

Lemma 6.11. *Let L be a normal subgroup of p -power index in G . Let $g_0 = t^{a_1}x_1 \dots t^{a_n}x_n$ ($n \geq 1$) be a cyclically reduced element of G . There exists a normal subgroup N of p -power index in H such that if $\varphi: H \rightarrow P = H/N$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\varphi(K)$, and if $\bar{\varphi}: G \rightarrow Q$ denotes the homomorphism induced by φ , we have*

- (1) $C_Q(\bar{\varphi}(g_0)) \subset \bar{\varphi}(C_G(g_0)L)$,
- (2) $\ker(\varphi) = N < H \cap L$,
- (3) $\ker(\bar{\varphi}) < L$.

Proof. Let $\{p_1, \dots, p_{n+1}\}$ be the set of all prefixes of g_0 . Renumbering p_1, \dots, p_{n+1} if necessary, we can assume that there exists $m \in \{1, \dots, n+1\}$ such that $p_i^{-1}g_0p_i \in g_0^K$ for all $i \in \{1, \dots, m\}$, and $p_i^{-1}g_0p_i \notin g_0^K$ for all $i \in \{m+1, \dots, n+1\}$. For each $i \in \{1, \dots, m\}$, we choose $\alpha_i \in K$ such that $p_i^{-1}g_0p_i = \alpha_i^{-1}g_0\alpha_i$. We set $\Omega = \{\alpha_i p_i^{-1} \mid i \in \{1, \dots, m\}\}$. We set $h_i = p_i^{-1}g_0p_i$ for all $i \in \{m+1, \dots, n+1\}$. By Lemma 6.7, there exists a normal subgroup N_1 of p -power index in H such that if $\varphi_1: H \rightarrow P_1 = H/N_1$ denotes the canonical projection, if Q_1 denotes the HNN extension of P_1 relative to $\varphi_1(K)$, and if $\bar{\varphi}_1: G \rightarrow Q_1$ denotes the homomorphism induced by φ_1 , then $\varphi_1(g_0)$ is cyclically reduced in Q_1 , and $\bar{\varphi}_1(h_i) \notin \bar{\varphi}_1(g_0)^{\varphi_1(K)}$ for all $i \in \{m+1, \dots, n+1\}$. On the other hand, by Lemma 6.9, there exists a normal subgroup N_2 of p -power index in H such that if $\varphi_2: H \rightarrow P_2 = H/N_2$ denotes the canonical projection, if Q_2 denotes the HNN extension of P_2 relative to $\varphi_2(K)$ and if $\bar{\varphi}_2: G \rightarrow Q_2$ denotes the homomorphism induced by φ_2 , we have $C_{\bar{\varphi}_2(K)}(\bar{\varphi}_2(g_0)) \subset \bar{\varphi}_2(C_K(g_0)L)$, $\ker(\varphi_2) < H \cap L$, and $\ker(\bar{\varphi}_2) < L$. Set $N = N_1 \cap N_2$. Let $\varphi: H \rightarrow P = H/N$ be the canonical projection, let Q be the HNN extension of P relative to $\varphi(K)$, and let $\bar{\varphi}: G \rightarrow Q$ be the homomorphism induced by φ . Since $N < N_1$, $\bar{\varphi}(g_0)$ is cyclically reduced in Q and $\bar{\varphi}(h_i) \notin \bar{\varphi}(g_0)^{\varphi(K)}$ for all $i \in \{m+1, \dots, n+1\}$. On the other hand, since $N < N_2$, we have

$$\bar{\varphi}^{-1}(C_{\bar{\varphi}(K)}(\bar{\varphi}(g_0))) \subset \bar{\varphi}_2^{-1}(C_{\bar{\varphi}_2(K)}(\bar{\varphi}_2(g_0))) \subset C_K(g_0)L. \tag{*}$$

There are two cases:

Case 1. $x_n \in K$. Then $n = 1$, $C_G(g_0) = \langle t \rangle C_K(g_0)$, and $C_Q(\bar{\varphi}(g_0)) = \langle \bar{t} \rangle C_{\varphi(K)}(\bar{\varphi}(g_0))$ by Proposition 6.10. Now (*) implies that

$$C_Q(\bar{\varphi}(g_0)) \subset \langle \bar{\varphi}(t) \rangle \bar{\varphi}(C_K(g_0)L) = \bar{\varphi}(\langle t \rangle C_K(g_0)L) = \bar{\varphi}(C_G(g_0)L).$$

Case 2. $x_n \in H \setminus K$. If $i \in \{1, \dots, m\}$, $\bar{\varphi}(p_i)^{-1}\bar{\varphi}(g_0)\bar{\varphi}(p_i) = \bar{\varphi}(p_i^{-1}g_0p_i) \in \bar{\varphi}(g_0)^{\varphi(K)}$ – because $p_i^{-1}g_0p_i \in g_0^K$ –, whereas if $i \in \{m+1, \dots, n+1\}$, $\bar{\varphi}(p_i)^{-1}\bar{\varphi}(g_0)\bar{\varphi}(p_i) = \bar{\varphi}(h_i) \notin \bar{\varphi}(g_0)^{\varphi(K)}$. Therefore $\{\bar{\varphi}(p_1), \dots, \bar{\varphi}(p_m)\}$ is the

set of all prefixes of $\bar{\varphi}(g_0)$ satisfying $\bar{\varphi}(p_i)^{-1}\bar{\varphi}(g_0)\bar{\varphi}(p_i) \in \bar{\varphi}(g_0)^{\varphi(K)}$. By Proposition 6.10, $C_G(g_0) = C_K(g_0)\langle g_0 \rangle \Omega$, and $C_Q(\bar{\varphi}(g_0)) = C_{\varphi(K)}(\bar{\varphi}(g_0))\langle \bar{\varphi}(g_0) \rangle \bar{\Omega}$, where $\bar{\Omega} = \bar{\varphi}(\Omega) = \{\bar{\varphi}(\alpha_i)\bar{\varphi}(p_i)^{-1} \mid i \in \{1, \dots, m\}\}$. We deduce that

$$C_Q(\bar{\varphi}(g_0)) \subset \bar{\varphi}(C_K(g_0)L)\langle \bar{\varphi}(g_0) \rangle \bar{\varphi}(\Omega) = \bar{\varphi}(C_K(g_0)L\langle g_0 \rangle \Omega) = \bar{\varphi}(C_G(g_0)L). \quad \square$$

Proposition 6.12. *Let G be a right-angled Artin group of rank r ($r \geq 1$). Let $g \in G$. If $g \neq 1$, there exists a special subgroup H of rank $r - 1$ of G such that $g \notin H^G$, where $H^G = \bigcup_{h \in H} h^G$.*

Proof. Proved in [M] (see Lemma 6.8). □

Lemma 6.13. *Every special subgroup S of G satisfies the p -centralizer condition in G (pCC_G).*

Proof. Let $g \in G$. Let L be a normal subgroup of p -power index in G . There are two cases:

Case 1. $S \neq G$. Let H be a special subgroup of rank $r - 1$ of G such that $S < H$. Then G can be written as an HNN extension of H , relative to a special subgroup K of H :

$$G = \langle H, t \mid t^{-1}kt = k \text{ for all } k \in K \rangle.$$

We set $L' = H \cap L$. We note that L' is a normal subgroup of p -power index in H . There are two cases:

Subcase 1. $g \in H$. By the assumptions, the pair (S, g) satisfies the p -centralizer condition in H (pCC_H). There exists a normal subgroup M of p -power index in H such that $M < L'$ and if $\psi : H \rightarrow P = H/M$ denotes the canonical projection, we have

$$C_{\psi(S)}(\psi(g)) \subset \psi(C_S(g)L'). \tag{*}$$

We denote by $f : G \rightarrow H$ the natural homomorphism. We note that $f^{-1}(M)$ is a normal subgroup of p -power index in G (because $f^{-1}(M)$ is the kernel of the homomorphism $\psi \circ f$). Therefore, $N = L \cap f^{-1}(M)$ is a normal subgroup of p -power index in G . Moreover $N < L$ and $f(N) < M$. We denote by $\varphi : G \rightarrow Q = G/N$ the canonical projection. We observe that $\ker(\psi) = M$, $\ker(\varphi) = N$, $M < f^{-1}(M) \cap L \cap H = N \cap H$, and $N \cap H \subset f(N) < M$. Therefore $M = N \cap H$. Thus we can assume that $P < Q$ and $\varphi|_H = \psi$. But then $\psi(L') = \varphi(L') \subset \varphi(L)$. Recall that $g \in H$ and $S < H$. Thus considering (*), we obtain

$$C_{\varphi(S)}(\varphi(g)) = C_{\psi(S)}(\psi(g)) \subset \psi(C_S(g))\psi(L') \subset \varphi(C_S(g))\varphi(L) = \varphi(C_S(g)L).$$

Subcase 2. $g \in G \setminus H$. Write $g = x_0 t^{a_1} x_1 \dots t^{a_n} x_n$ in a reduced form ($n \geq 1$). Then, by Lemma 6.9, there exists a normal subgroup M of p -power index in H such that if $\psi : H \rightarrow P = H/M$ denotes the canonical projection, if Q denotes the

HNN extension of P relative to $\psi(K)$, and if $\bar{\psi}: G \rightarrow Q$ denotes the homomorphism induced by ψ , then $C_{\bar{\psi}(S)}(\bar{\psi}(g)) \subset \bar{\psi}(C_S(g)L)$, $\ker(\psi) < H \cap L$, and $\ker(\bar{\psi}) < L$. We note that $\bar{\psi}(S) \cap \bar{\psi}(L) = \psi(S) \cap \bar{\psi}(L) < P$ is finite. Since Q is residually p -finite, $\bar{\psi}(g)^{\bar{\psi}(S) \cap \bar{\psi}(L)}$ is finitely p -separable in Q . Therefore, by Lemma 3.5, there exists a normal subgroup N of p -power index in Q such that $N < \bar{\psi}(L)$ and if $\chi: Q \rightarrow R = Q/N$ denotes the canonical projection, then

$$C_{\chi(\bar{\psi}(S))}(\chi(\bar{\psi}(g))) \subset \chi(C_{\bar{\psi}(S)}(\bar{\psi}(g))\bar{\psi}(L)).$$

We set $\varphi = \chi \circ \bar{\psi}: G \rightarrow R$. We have $\ker(\varphi) = \bar{\psi}^{-1}(\ker(\chi)) = \bar{\psi}^{-1}(N) \subset \bar{\psi}^{-1}(\bar{\psi}(L)) = L \ker(\bar{\psi})$. Now $\ker(\bar{\psi}) < L$. Then $\ker(\varphi) < L$ and

$$\begin{aligned} C_{\varphi(S)}(\varphi(g)) &= C_{\chi(\bar{\psi}(S))}(\chi(\bar{\psi}(g))) \subset \chi(C_{\bar{\psi}(S)}(\bar{\psi}(g))\bar{\psi}(L)) \\ &\subset \chi(\bar{\psi}(C_S(g)L)\bar{\psi}(L)) = \varphi(C_S(g)L). \end{aligned}$$

Case 2. $S = G$. If $g = 1$, then the result is trivial. Thus we can assume that $g \neq 1$. Then, by Proposition 6.12, there exists a special subgroup H of rank $r - 1$ of G such that $g \notin H^G$. As above, G can be written as an HNN extension of H relative to a special subgroup K of H :

$$G = \langle H, t \mid t^{-1}kt = k \text{ for all } k \in K \rangle.$$

Let $g_0 = t^{a_1}x_1 \dots t^{a_n}x_n$ be a cyclically reduced element in G conjugate to g . Choose $\alpha \in G$ such that $g = \alpha g_0 \alpha^{-1}$. Note that $g \notin H^G$ implies that $n \geq 1$. By Lemma 6.11, there exists a normal subgroup M of p -power index in H such that if $\psi: H \rightarrow P = H/M$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\psi(K)$, and if $\bar{\psi}: G \rightarrow Q$ denotes the homomorphism induced by ψ , then $C_Q(\bar{\psi}(g_0)) \subset \bar{\psi}(C_G(g_0)L)$, $\ker(\psi) < H \cap L$, and $\ker(\bar{\psi}) < L$. Now Q is hereditarily conjugacy p -separable by Corollary 4.3. Then Q satisfies the p -centralizer condition by Proposition 3.6. There exists a normal subgroup N of p -power index in Q such that $N < \bar{\psi}(L)$, and if $\chi: Q \rightarrow R = Q/N$ denotes the canonical projection, we have

$$C_R(\chi(\bar{\psi}(g_0))) \subset \chi(C_Q(\bar{\psi}(g_0))\bar{\psi}(L)).$$

We set $\varphi = \chi \circ \bar{\psi}: G \rightarrow R$. As above, we have $\ker(\varphi) = \bar{\psi}^{-1}(\ker(\chi)) = \bar{\psi}^{-1}(N) \subset \bar{\psi}^{-1}(\bar{\psi}(L)) = L \ker(\bar{\psi})$. Now $\ker(\bar{\psi}) < L$. Then $\ker(\varphi) < L$ and

$$\begin{aligned} C_R(\varphi(g_0)) &= C_{\varphi(G)}(\varphi(g_0)) = C_{\chi(\bar{\psi}(G))}(\chi(\bar{\psi}(g_0))) \subset \chi(C_{\bar{\psi}(G)}(\bar{\psi}(g_0))\bar{\psi}(L)) \\ &\subset \chi(\bar{\psi}(C_G(g_0)L)\bar{\psi}(L)) = \varphi(C_G(g_0)L). \end{aligned}$$

Finally,

$$\varphi(\alpha)C_R(\varphi(g_0))\varphi(\alpha)^{-1} \subset \varphi(\alpha)\varphi(C_G(g_0)L)\varphi(\alpha)^{-1},$$

that is,

$$C_R(\varphi(g)) \subset \varphi(C_G(g)L).$$

□

Lemma 6.14. *For every $g \in G$ and for every special subgroup S of G , g^S is finitely p -separable in G .*

Proof. There are two cases:

Case 1. $S \neq G$. Let H be a special subgroup of rank $r - 1$ of G such that $S < H$. As above, G can be written as an HNN extension of H relative to a special subgroup K of H :

$$G = \langle H, t \mid t^{-1}kt = k \text{ for all } k \in K \rangle.$$

Let $g \in G$. There are two cases:

Subcase 1. $g \in H$. Then g^S is finitely p -separable in H by the assumptions. Since G is residually p -finite by Theorem 6.1, g^S is finitely p -separable in G by Lemma 5.12.

Subcase 2. $g \in G \setminus H$. Let $h \in G \setminus g^S$. By Lemma 6.6, there exists a normal subgroup L of p -power index in H such that if $\psi: H \rightarrow P = H/L$ denotes the canonical projection, if \underline{Q} denotes the HNN extension of P relative to $\psi(K)$, and if $\bar{\psi}: G \rightarrow \underline{Q}$ denotes the homomorphism induced by ψ , we have $\bar{\psi}(h) \notin \bar{\psi}(g)^{\bar{\psi}(S)}$. Now $\bar{\psi}(S) = \psi(S) < P$ is finite and \underline{Q} is residually p -finite. Then there exists a homomorphism $\chi: \underline{Q} \rightarrow R$ from \underline{Q} onto a finite p -group R such that $\chi(\bar{\psi}(h)) \notin \chi(\bar{\psi}(g)^{\bar{\psi}(S)})$. Thus the homomorphism $\varphi = \chi \circ \bar{\psi}: G \rightarrow R$ satisfies the condition $\varphi(h) \notin \varphi(g^S)$, as required.

Case 2. $S = G$. Let $g \in G$. If $g = 1$, then, since G is residually p -finite by Theorem 6.1, $g^G = \{1\}$ is finitely p -separable in G . Thus we can assume that $g \neq 1$. Then, by Proposition 6.12, there exists a special subgroup H of rank $r - 1$ of G such that $g \notin H^G$. As above, G can be written as an HNN extension of H relative to a special subgroup K of H :

$$G = \langle H, t \mid t^{-1}kt = k \text{ for all } k \in K \rangle.$$

Let $h \in G \setminus g^G$. Let $g_0 = t^{a_1}x_1 \dots t^{a_n}x_n$ and $h_0 = t^{b_1}y_1 \dots t^{b_m}y_m$ be cyclically reduced elements of G conjugate to g and h , respectively. Note that $g \notin H^G$ implies that $n \geq 1$. There are two cases:

Subcase 1. $h_0 \in H$. Then, by Lemma 6.7, there exists a normal subgroup L of p -power index in H such that if $\psi: H \rightarrow P = H/L$ denotes the canonical projection, if \underline{Q} denotes the HNN extension of P relative to $\psi(K)$, and if $\bar{\psi}: G \rightarrow \underline{Q}$ denotes the homomorphism induced by ψ , then $\bar{\psi}(g_0) = \bar{t}^{a_1}\bar{x}_1 \dots \bar{t}^{a_n}\bar{x}_n$ is cyclically reduced in \underline{Q} , where $\bar{x}_i = \bar{\psi}(x_i)$ ($i \in \{1, \dots, n\}$). Since $n \geq 1$, we have $\bar{\psi}(g_0) \notin P^Q = \bar{\psi}(H^G)$. Therefore $\bar{\psi}(g_0) \notin \bar{\psi}(h_0)^Q = \bar{\psi}(h_0^G) \subset \bar{\psi}(H^G)$. Now \underline{Q} is conjugacy p -separable by Corollary 4.3. Then there exists a homomorphism χ from \underline{Q} onto a finite p -group R such that $\chi(\bar{\psi}(g_0)) \notin \chi(\bar{\psi}(h_0)^G)^R$. Therefore $\chi(\bar{\psi}(g)) \notin \chi(\bar{\psi}(h))^R$. Thus the homomorphism $\varphi = \chi \circ \bar{\psi}: G \rightarrow R$ satisfies the condition $\varphi(h) \notin \varphi(g^S)$, as desired.

Subcase 2. $h_0 \in G \setminus H$. Let $\{h_1, \dots, h_m\}$ be the set of all cyclic permutations of h_0 . Then, since $h \notin g^G$, we have: $h_i \notin g_0^G$ for all $i \in \{1, \dots, m\}$. Therefore,

by Lemma 6.7, there exists a normal subgroup L of p -power index in H such that, if $\psi: H \rightarrow P = H/L$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\psi(K)$, and if $\bar{\psi}: G \rightarrow Q$ denotes the homomorphism induced by ψ , then $\bar{\psi}(g_0) = \bar{t}^{a_1} \bar{x}_1 \dots \bar{t}^{a_n} \bar{x}_n$ and $\bar{\psi}(h_0) = \bar{t}^{b_1} \bar{y}_1 \dots \bar{t}^{b_m} \bar{y}_m$ are cyclically reduced in Q , where $\bar{x}_i = \bar{\psi}(x_i)$ ($i \in \{1, \dots, n\}$) and $\bar{y}_j = \bar{\psi}(y_j)$ ($j \in \{1, \dots, m\}$) – and $\bar{\psi}(h_i) \notin \bar{\psi}(g_0)^{\bar{\psi}(K)}$ for all $i \in \{1, \dots, m\}$. Consequently, by Lemma 2.3, $\bar{\psi}(g_0) \notin \bar{\psi}(h_0)^Q$. Now Q is conjugacy p -separable by Corollary 4.3. Then there exists a homomorphism χ from Q onto a finite p -group R such that $\chi(\bar{\psi}(g_0)) \notin \chi(\bar{\psi}(h_0))^R$. Hence $\chi(\bar{\psi}(g)) \notin \chi(\bar{\psi}(h))^R$. Thus the homomorphism $\varphi = \chi \circ \bar{\psi}: G \rightarrow R$ satisfies the condition $\varphi(h) \notin \varphi(g^S)$, as required. \square

Proof of Proposition 6.2. We argue by induction on the rank r of G . If $r = 0$, then the result is trivial. Thus we can assume that $r \geq 1$ and that the result has been proved for $1, \dots, r - 1$. Now, Proposition 6.2 (1) follows from Lemma 6.13, and Proposition 6.2 (2) follows from Lemma 6.14. \square

We are now ready to prove:

Theorem 6.15. *Every right-angled Artin group is hereditarily conjugacy p -separable.*

Proof. Let G be a right-angled Artin group. Let $g \in G$. Then g^G is finitely p -separable in G by Proposition 6.2 (1). We deduce that G is conjugacy p -separable. On the other hand, G satisfies the p -centralizer condition by Proposition 6.2 (2). We conclude that G is hereditarily conjugacy p -separable by Proposition 3.6. \square

7. Applications

The first application that we mention is an application of our main theorem to separability properties of G_Γ .

For a group G , we denote by $(C^n(G))_{n \geq 1}$ the lower central series of G . Recall that $(C^n(G))_{n \geq 1}$ is defined inductively by $C^1(G) = G$, and $C^{n+1}(G) = [G, C^n(G)]$ for all $n \geq 1$.

Corollary 7.1. *Every right-angled Artin group is conjugacy separable in the class of torsion-free nilpotent groups.*

Proof. Let G be a right-angled Artin group. Let $g, h \in G$ such that $g \sim h$. Let p be a prime number. Then G is conjugacy p -separable by Theorem 6.15. Thus, there exists a homomorphism φ from G onto a finite p -group P such that $\varphi(g) \sim \varphi(h)$. Now P is nilpotent. Therefore, there exists $n \geq 1$ such that $C^n(P) = \{1\}$. Let $\pi: G \rightarrow \frac{G}{C^n(G)}$ be the canonical projection. It follows from [DK2], Theorem 2.1,

that for all $n \geq 1$, there exists $d_n \in \mathbb{N}$ such that

$$\frac{C^n(G)}{C^{n+1}(G)} \simeq \mathbb{Z}^{d_n}.$$

Thus, an easy induction on n shows that $\frac{G}{C^n(G)}$ is torsion-free for all $n \geq 1$. Hence $\frac{G}{C^n(G)}$ is a torsion-free nilpotent group for all $n \geq 1$. Since $\varphi(C^n(G)) < C^n(P) = \{1\}$, φ induces a homomorphism $\tilde{\varphi}: \frac{G}{C^n(G)} \rightarrow P$ such that $\varphi = \tilde{\varphi} \circ \pi$. As $\varphi(g) \sim \varphi(h)$, we have $\pi(g) \sim \pi(h)$. \square

We now turn to applications of our main theorem to residual properties of $\text{Out}(G_\Gamma)$.

An automorphism φ of a group G is said to be *conjugating* if $\varphi(g) \sim g$ for every $g \in G$. We say that G has *Property A* if every conjugating automorphism of G is inner. The following proposition is due to Minasyan (see [M], Proposition 6.9):

Proposition 7.2. *Right-angled Artin groups have Property A.*

For a group G , we denote by $\mathcal{I}_p(G)$ the kernel of the natural homomorphism $\text{Out}(G) \rightarrow \text{GL}(H_1(G, \mathbb{F}_p))$ (where \mathbb{F}_p denotes the finite field with p elements). The following theorem is due to Paris (see [P], Theorem 2.5):

Theorem 7.3. *Let G be a finitely generated group. If G is conjugacy p -separable and has Property A, then $\mathcal{I}_p(G)$ is residually p -finite.*

Thus, combining Theorem 7.3 and Proposition 7.2 with Theorem 6.15, we obtain:

Corollary 7.4. *The outer automorphism group of a right-angled Artin group is virtually residually p -finite.*

The following theorem is due to Myasnikov (see [My], Theorem 1):

Theorem 7.5. *Let G be a finitely generated group. If G is conjugacy p -separable and has property A, then $\text{Out}(G)$ is residually \mathcal{K} , where \mathcal{K} is the class of all outer automorphism groups of finite p -groups.*

Thus, combining Theorem 7.5 and Proposition 7.2 with Theorem 6.15, we obtain:

Corollary 7.6. *The outer automorphism group of a right-angled Artin group is residually \mathcal{K} , where \mathcal{K} is the class of all outer automorphism groups of finite p -groups.*

In the remainder of this paper, we prove Theorem 7.14. Let $G = G_\Gamma$ be a right-angled Artin group. Let r be the rank of G . We denote by $T(G)$ the kernel of the natural homomorphism $\text{Aut}(G) \rightarrow \text{GL}_r(\mathbb{Z})$, and by $\mathcal{T}(G)$ the kernel of the natural homomorphism $\text{Out}(G) \rightarrow \text{GL}_r(\mathbb{Z})$. Note that $\mathcal{T}(G) = T(G)/\text{Inn}(G)$. Day proved

that $T(G)$ is finitely generated (see [D2], Theorem B). Therefore $\mathcal{T}(G)$ is finitely generated.

In order to prove Theorem 7.14, we have to introduce the notion of separating \mathbb{Z} -linear central filtration.

Recall that a *central filtration* on a group G is a sequence $(G_n)_{n \geq 1}$ of subgroups of G satisfying the conditions:

$$G_1 = G, \quad G_n > G_{n+1}, \quad [G_m, G_n] < G_{m+n} \quad \text{for all } m, n \geq 1.$$

Let $\mathcal{F} = (G_n)_{n \geq 1}$ be a central filtration. Then the mapping $G \times G \rightarrow G$, $(x, y) \mapsto xyx^{-1}y^{-1}$ induces on

$$\mathcal{L}_{\mathcal{F}}(G) = \bigoplus_{n \geq 1} \frac{G_n}{G_{n+1}}$$

a Lie bracket which makes $\mathcal{L}_{\mathcal{F}}(G)$ into a graded Lie \mathbb{Z} -algebra.

We say that $(G_n)_{n \geq 1}$ is a *separating filtration* if $\bigcap_{n \geq 1} G_n = \{1\}$. We say that $(G_n)_{n \geq 1}$ is *\mathbb{Z} -linear* if the \mathbb{Z} -module $\frac{G_n}{G_{n+1}}$ is free of finite rank for all $n \geq 1$.

For a group G , we denote by $(C_{\mathbb{Z}}^n(G))_{n \geq 1}$ the sequence of subgroups of G defined inductively by $C_{\mathbb{Z}}^1(G) = G$, $[G, C_{\mathbb{Z}}^n(G)] < C_{\mathbb{Z}}^{n+1}(G)$, and $\frac{C_{\mathbb{Z}}^{n+1}(G)}{[G, C_{\mathbb{Z}}^n(G)]}$ is the torsion subgroup of $\frac{C_{\mathbb{Z}}^n(G)}{[G, C_{\mathbb{Z}}^n(G)]}$ for all $n \geq 1$.

Proposition 7.7. *For all $m, n \geq 1$, $[C_{\mathbb{Z}}^m(G), C_{\mathbb{Z}}^n(G)] < C_{\mathbb{Z}}^{m+n}(G)$.*

Proof. Proved in [BL] (see Proposition 7.2). □

Thus, $(C_{\mathbb{Z}}^n(G))_{n \geq 1}$ is a central filtration on G . We denote by $\mathcal{L}_{\mathbb{Z}}(G)$ the corresponding graded Lie \mathbb{Z} -algebra.

For a Lie algebra \mathfrak{g} , we denote by $Z(\mathfrak{g})$ the center of \mathfrak{g} . Let G be a group. For $n \geq 1$, we denote by A_n the kernel of the natural homomorphism $\text{Aut}(G) \rightarrow \text{Aut}(\frac{G}{C_{\mathbb{Z}}^{n+1}(G)})$. Let $\pi: \text{Aut}(G) \rightarrow \text{Out}(G)$ be the canonical projection. For $n \geq 1$, we set $B_n = \pi(G_n)$.

Theorem 7.8. *If G^{ab} is finitely generated, and $Z(\mathbb{F}_p \otimes \mathcal{L}_{\mathbb{Z}}(G)) = \{0\}$ for every prime number p , then $(B_n)_{n \geq 1}$ is a \mathbb{Z} -linear central filtration on B_1 . Furthermore, $(B_n)_{n \geq 1}$ is separating if and only if G satisfies the following condition:*

(IN(G)): *For every $\varphi \in \text{Aut}(G)$, if φ induces an inner automorphism of $\frac{G}{C_{\mathbb{Z}}^n(G)}$ for all $n \geq 1$, then φ is inner.*

Proof. Proved in [BL] (see Corollary 9.9). □

From now on, we assume that $G = G_\Gamma$ is a right-angled Artin group of rank r ($r \geq 1$). We shall show that G satisfies the conditions of Theorem 7.8. Since B_1 is precisely the Torelli group of G , Theorem 7.14 will then result from the following:

Theorem 7.9. *Let B be a group. Suppose that B admits a separating \mathbb{Z} -linear central filtration $(B_n)_{n \geq 1}$. Then B is residually torsion-free nilpotent.*

Proof. Proved in [BL] (see Theorem 6.1). □

We need to introduce the following notations. Let K be a commutative ring. We denote by M_Γ the monoid defined by the presentation

$$M_\Gamma = \langle V \mid vw = wv \text{ for all } \{v, w\} \in E \rangle,$$

by A_Γ the associative K -algebra of the monoid M_Γ , and by L_Γ the Lie K -algebra defined by the presentation

$$L_\Gamma = \langle V \mid [v, w] = 0 \text{ for all } \{v, w\} \in E \rangle.$$

The following theorem is due to Duchamp and Krob (see [DK1], Corollary II.16):

Theorem 7.10. *The K -module L_Γ is free.*

Thus, by the Poincaré–Birkhoff–Witt theorem, L_Γ can be regarded as a Lie subalgebra of its enveloping algebra, for which Duchamp and Krob established the following (see [DK1], Corollary I.2):

Theorem 7.11. *The enveloping algebra of L_Γ is isomorphic to A_Γ .*

Furthermore, Duchamp and Krob proved the following (see [DK2], Theorem 2.1), which generalizes a well-known theorem of Magnus:

Theorem 7.12. *Suppose that $K = \mathbb{Z}$. The graded Lie \mathbb{Z} -algebra associated to the lower central series of G is isomorphic to L_Γ .*

Set $Z = \bigcap_{v \in V} \text{star}(v)$. It follows from Servatius' Centralizer Theorem (see [S], Theorem 1) that the center $Z(G)$ of G is the special subgroup of G generated by Z .

Lemma 7.13. *Suppose that $Z(G) = \{1\}$. Then $Z(L_\Gamma) = \{0\}$.*

Proof. Let $g \in Z(L_\Gamma)$. Suppose that $g \neq 0$. Let $v \in V$. We have $[g, v] = 0$ (in L_Γ). Now, L_Γ can be regarded as a Lie subalgebra of A_Γ by Theorem 7.10 and Theorem 7.11. Thus, we have $gv = vg$ (in A_Γ). Therefore g belongs to the subalgebra of A_Γ generated by $\text{star}(v)$ (see [KR], Theorem 2). Since v is arbitrary, this leads to a contradiction with our assumption. □

From now on, we assume that $K = \mathbb{Z}$. We now turn to prove:

Theorem 7.14. *The Torelli group of a right-angled Artin group is residually torsion-free nilpotent.*

Proof. Let H be the special subgroup of G generated by $V \setminus Z$. Note that $Z(H) = \{1\}$. We have $G = H \times Z(G)$. First, we show that $\mathcal{T}(G) = \mathcal{T}(H)$. Let $\varphi: T(H) \rightarrow T(G)$ be the homomorphism defined by

$$\varphi(\alpha)(h, k) = (\alpha(h), k)$$

for all $\alpha \in T(H)$, $h \in H$, $k \in Z(G)$. Clearly, φ is well defined and injective. We shall show that φ is surjective. Let $\beta \in T(G)$. For $g \in G$, we set $\beta(g) = (\beta_1(g), \beta_2(g))$, where $\beta_1(g) \in H$ and $\beta_2(g) \in Z(G)$. Let $h \in H$. We denote by \bar{h} the canonical image of h in H^{ab} . Note that the canonical image of h in $G^{\text{ab}} = H^{\text{ab}} \times Z(G)$ is $(\bar{h}, 1)$. Since $\beta \in T(G)$, we have $(\bar{h}, 1) = (\beta_1(\bar{h}), \beta_2(h))$, and then $\beta_2(h) = 1$. Let $k \in Z(G)$. Since $\beta(k)$ lies in the center of G , we have $\beta_1(k) = 1$. Note that the canonical image of k in G^{ab} is $(1, k)$. As $\beta \in T(G)$, we have $\beta_2(k) = k$. Finally, we have

$$\beta(h, k) = (\beta_1(h), k)$$

for all $h \in H$ and $k \in Z(G)$. Applying the same argument to β^{-1} , we obtain that the restriction α of β_1 to H is an automorphism of H . Therefore $\beta = \varphi(\alpha)$. We deduce that φ is an isomorphism. Note that $\varphi(\text{Inn}(H)) = \text{Inn}(G)$. We conclude that $\mathcal{T}(G) = \mathcal{T}(H)$. Thus, up to replacing G by H , we can assume that $Z(G) = \{1\}$. As we noted above, $\frac{G}{C^n(G)}$ is torsion-free for all $n \geq 1$. Now, for all $n \geq 1$, $C^n(G) < C^n_{\mathbb{Z}}(G)$, and $\frac{C^n_{\mathbb{Z}}(G)}{C^n(G)}$ is the torsion subgroup of $\frac{G}{C^n(G)}$ by [BL], Proposition 7.2. It follows that $C^n_{\mathbb{Z}}(G) = C^n(G)$ for all $n \geq 1$, and that $\mathcal{L}_{\mathbb{Z}}(G) = L_{\Gamma}$ by Theorem 7.12. Since $Z(G) = \{1\}$, we have $Z(\mathbb{F}_p \otimes L_{\Gamma}) = \{0\}$ for every prime number p by Lemma 7.13. We deduce that $(B_n)_{n \geq 1}$ is a \mathbb{Z} -linear central filtration on $\mathcal{T}(G)$ by Theorem 7.8. Now, let $\varphi \in \text{Aut}(G)$ such that φ induces an inner automorphism on $\frac{G}{C^n(G)}$ for all $n \geq 1$. Let $g \in G$. Suppose that $\varphi(g)$ and g are not conjugate in G . Then it follows from the proof of Theorem 7.1 that there exists $n \geq 1$ such that the canonical images of $\varphi(g)$ and g in $\frac{G}{C^n(G)}$ are not conjugate in $\frac{G}{C^n(G)}$ – contradicting our assumption. Thus φ is conjugating. Therefore φ is inner by Proposition 7.2. We deduce that $(B_n)_{n \geq 1}$ is separating by Theorem 7.8. We conclude that $\mathcal{T}(G)$ is residually torsion-free nilpotent by Theorem 7.9. \square

Corollary 7.15. *The Torelli group of a right-angled Artin group is residually p -finite.*

Proof. Since $\mathcal{T}(G)$ is finitely generated by [D2], Theorem B, and residually torsion-free nilpotent by Theorem 7.14, it is residually p -finite by [G], Theorem 2.1. \square

It is known that residually torsion-free nilpotent groups are bi-orderable (see, for example, [CKM], Remark 2.6). Thus, Theorem 7.14 immediately yields:

Corollary 7.16. *The Torelli group of a right-angled Artin group is bi-orderable.*

A. Appendix

Let G be a group, and let H be a subgroup of G . Recall that the *normal core* of H , denoted by H_G , is defined to be the largest normal subgroup of G that is contained in H , i.e., $H_G = \bigcap_{g \in G} gHg^{-1}$. The following lemma is probably well known, though it does not seem to be in the literature. We include a proof for completeness.

Lemma A.1. *Let G be a group, and let H be a subgroup of G . Then H is open in the pro- p topology on G if and only if H is subnormal of p -power index.*

Proof. If H is open in the pro- p topology on G , then it contains a normal subgroup K of p -power index in G . Thus $[G : H]$ is a power of p . As $\frac{G}{K}$ is a finite p -group, every subgroup of it is subnormal. Therefore H is subnormal in G .

Conversely, if H is a subnormal subgroup of p -power index in G , then $[G : H_G]$ is a power of p (see, for example, [AF2], Lemma 3.3). Thus H contains an open subgroup of G , and hence is open itself. \square

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