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Groups, Geometry, and Dynamics

Centralizers in the R. Thompson group V_n

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Abstract. Let $n \geq 2$ and let $\alpha \in V_n$ be an element in the Higman–Thompson group V_n. We study the structure of the centralizer of $\alpha \in V_n$ through a careful analysis of the action of $\langle \alpha \rangle$ on the Cantor set $\mathfrak{C}$. We make use of revealing tree pairs as developed by Brin and Salazar from which we derive discrete train tracks and flow graphs to assist us in our analysis. A consequence of our structure theorem is that element centralizers are finitely generated. Along the way we give a short argument using revealing tree pairs which shows that cyclic groups are undistorted in V_n.

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Contents

1 Introduction . . . 821
2 Basic definitions . . . 826
3 Conjugation, roots, and centralizers . . . 831
4 Revealing pairs and related objects . . . 832
5 A partition of $\mathfrak{C}_n$ . . . 848
6 Centralizers over the set T_α . . . 849
7 Centralizers over the set Z_α . . . 853
8 Cyclic subgroup (non)distortion . . . 860
Appendix: An example of the finite generation of the centralizer of a pure torsion element . . . 862
References . . . 864

1. Introduction

In this paper, we produce a description of the structure of element centralizers within the Higman–Thompson groups V_n. As a corollary to our structure theorem we see that element centralizers in the V_n are finitely generated. Finally, we give a separate

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short argument which shows that all infinite-cyclic subgroups within V_n are embedded without distortion. (The groups V_n are first introduced by Higman in [14], where $V_n = G_{n,1}$ in his notation. The R. Thompson group denoted V in [9] is V_2 in our notation.)

For a given integer $n \geq 2$, our primary view of the group V_n is as a group of homeomorphisms acting on the Cantor set $\mathfrak{C}_n$ (seen as the boundary of the rooted regular infinite n-ary tree $\mathcal{T}_n$). This point of view informs most of our work, where we use a close study of the dynamics of subgroup actions on $\mathfrak{C}_n$ to derive our main result.

Centralizers in Thompson's group $F = F_2$ are first classified by Guba and Sapir in [13] as a consequence of their classification of element centralizers for diagram groups. In related but separate work, Brin and Squier in [6] describe roots and centralizers in $\mathrm{PL}_o(I)$, the group of orientation-preserving, piecewise linear homeomorphisms of the unit interval (from which it is also easy to describe the element centralizers of F_n, although Brin and Squier never formally do so). Guba and Sapir also show in [12] that element centralizers in F are embedded without distortion in F. In Chapter 8 of the thesis [17], Bleak, Kassabov, and Matucci classify centralizers in T_2 up to finite index. This paper can be seen as a continuation of the line of research leading to these results.

Higman's work in [14] also contains information about the structure of centralizers of elements in V_n (see Theorem 9.9 of [14]). If one reads Higman's proof of Theorem 9.9 carefully, one can derive with reasonable effort some of the information about the $\mathbb{Z}$ factors contained within the right-hand direct product in Theorem 1.1. However, our own result contains significantly more detail about the overall structure of element centralizers than is contained within [14].

Separately, Tuna Altinel and Alexey Muranov (see [1]) use model theory to analyze aspects of the groups $F_n < T_n < V_n$. In their work they compile some information with regard to element centralizers in these groups. Their results with regards to element centralizers are very similar to what is known from the work of Brin and Squier in [6] and of Kassabov and Matucci in [15], and appears to be contained within the results of Higman from [14].

Martínez-Pérez and Nucinkis [16] recently studied generalizations of the groups of Higman, Thompson, Stein and Brin and classified centralizers of finite subgroups in order to study finiteness properties of those groups via cohomology. Their result generalizes the one in [17] by Bleak, Kassabov, and Matucci and agrees with the one of the current paper when restricted to torsion elements.

The work in this paper uses in broad outline the approach of Bleak, Kassabov and Matucci to centralizers in Chapter 8 of [17], but we work in the more complex groups V_n, and thus we need to employ a more complex set of tools in our analysis. We chose to use Brin's revealing pair technology (see [5]) for our supporting calculations (from amongst a fairly long list of tools that provide similar data), and we developed discrete train tracks and our flow graph objects to further support our intuitive understanding of how elements of V_n act on $\mathfrak{C}_n$.

One advantage of considering a revealing pair (A, B, σ) representing an element $\alpha \in V_n$ over any random representative tree pair is that it is easier to understand the orbit structure of points in the Cantor set $\mathfrak{C}_n$ under the action of the group $\langle\alpha\rangle$. The paper [21] studies revealing pairs in depth, and includes a solution of the conjugacy problem for V_n using revealing pairs (Salazar gives all arguments in the context of V_2, but it is easy to see that her methods extend to n-ary trees).

We briefly describe some alternative technologies. As alluded above, one could use the seminormal and quasinormal forms from [14] to gather much of the information we obtained from revealing pairs. In fact, in response to early drafts of this article and conversations with the first author of this article, Nathan Barker has replicated and extended many of the results herein using Higman's technology and he has gone on to work on the simultaneous conjugacy problem in V_n [2]. Another technology is the strand diagrams of Belk and Matucci (see [3]), which themselves are refinements of Pride's pictures in [19], [20]. In turn, Pride's pictures are essentially dual objects to the Dehn diagrams from geometric group and semigroup theory (for instance, in this context, one can study the related analysis of conjugacy in F and other diagram groups by Guba and Sapir in [13]). In the end, these tools all provide access to similar content. We chose revealing pairs as we were comfortable with calculations using them, and because it was particularly easy to define our chief combinatorial objects, discrete train tracks and flow graphs, from a revealing pair.

The dynamical information described by discrete train tracks and their corresponding flow graphs, forms a key ingredient in the proof by Bleak and Salazar [4] of the perhaps surprising result that $\mathbb{Z}^2 * \mathbb{Z}$ does not embed in V. In particular, those authors make significant use of our flow graph technology in their analysis.

Sections 2 and 3, and Section 4.1, serve as a mostly expository introduction to calculations in the generalized R. Thompson groups F_n, T_n, and V_n. An informed reader in the area can likely skip ahead to Section 4.2, looking back to these sections on the rare occasions in which a new term appears.

Acknowledgments. The authors would like to thank Claas Röver and Martin Kassabov for interesting and helpful conversations relating to this work. We also owe a debt of thanks to Mark Sapir for suggesting we look into the distortion of cyclic subgroups in V_n. We thank Nathan Barker for interesting conversations with respect to our flow graph object; it was by these conversations that the authors were finally convinced of the need to formally define flow-graphs, instead of just using them as informal guides to intuition. We thank Ewa Bieniecka for a very careful reading of this paper which led to many suggestions to improve the writing. Finally, the sixth author would like to thank Conchita Martínez-Pérez for pleasant conversations about her work with Nucinkis [16] where they discussed our independent approaches to the question, and the mutual confirmation of each other's results.

The authors also thank Cornell University for hosting, Robert Strichartz for organizing, and the NSF for funding the REU program at which most of this work took place. We also thank the Centre de Recerca Matemàtica (CRM) in Barcelona, the

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1.1. A formal statement of results. If $\alpha \in V_n$, we set

$$C_{V_n}(\alpha) := \{\beta \in V_n \mid \alpha\beta = \beta\alpha\}.$$

We call the group $C_{V_n}(\alpha)$ the *centralizer in V_n of α*, as is standard.

Our primary theorem is the following.

Theorem 1.1. *Let $n > 1$ be a positive integer and suppose $\alpha \in V_n$. Then there are non-negative integers s, t, m_i, and r_i and groups K_{m_i}, G_{n,r_i}, A_j, and P_t, for $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, t\}$, so that*

$$C_{V_n}(\alpha) \cong \Big(\prod_{i=1}^{s} K_{m_i} \rtimes G_{n,r_i} \Big) \times \Big(\prod_{j=1}^{t} ((A_j \rtimes \mathbb{Z}) \wr P_{q_j}) \Big).$$

We now explain the statement of this theorem in a bit more detail.

The group $\langle\alpha\rangle$ acts on a subset of the nodes of the infinite n-ary tree. The number s represents the number of distinct lengths of finite cyclic orbits of nodes under this action. The value of s is easy to compute from any given revealing pair representing α.

For the action mentioned above, each r_i is determined as a minimal number of nodes carrying a fundamental domain (for the action of a conjugate version of $\langle\alpha\rangle$) in the set of nodes supporting the cycles of length m_i.

For each m_i, we have $K_{m_i} = (\mathrm{Maps}(\mathfrak{C}_n, \mathbb{Z}_{m_i}))^{r_i}$, where $\mathrm{Maps}(\mathfrak{C}_n, \mathbb{Z}_{m_i})$ is the group of continuous maps from $\mathfrak{C}_n$ to $\mathbb{Z}_{m_i}$ under point-wise multiplication, and where $\mathbb{Z}_{m_i}$ is the cyclic group $\mathbb{Z}/(m_i\mathbb{Z})$ under the discrete topology. We note that K_{m_i} is not finitely generated for $m_i > 1$.

The groups G_{n,r_i} are the Higman–Thompson groups from [14].

Given any element $\beta \in V_n$, one can associate an infinite collection of finite labeled graphs (which we call *flow graphs*). Flow graphs are labeled, directed, finite graphs and which describe structural meta-data pertaining to the dynamics of certain subsets of $\mathfrak{C}_n$ under the action of $\langle\alpha\rangle$. Flow graphs are themselves "quotient objects" coming from *discrete train tracks*, which are objects we introduce here to model dynamics in the Cantor set much as regular train tracks model dynamics in surface homeomorphism theory [22], [23], [18].

Components of a flow graph associated with α fall into equivalence classes $\{\mathrm{ICC}_i\}$ which model similar dynamics. The number t is the number of equivalence classes of components carrying infinite orbits under the action of $\langle\alpha\rangle$. This number happens to be independent of the representative flow graph chosen.

The right factor of the main direct product represents the restriction of the centralizer of α to elements which are the identity away from the closure of the region where $\langle\alpha\rangle$ acts with non-finite orbits.

For each j, the supports of the elements of ICC_j represent regions in the action of $\langle\alpha\rangle$ where the centralizers of α, with restricted actions to these regions, are isomorphic. For any such support of an element of ICC_j, the finite order elements of the restricted centralizer form a group. We take A_j to be a representative group from set of these isomorphic finite groups.

The group P_{q_j} is the full symmetric group on the q_j isomorphic flow graph components in ICC_j.

Recall now that by Higman in [14], the groups $G_{k,r}$ are all finitely presented. This fact, together with a short analysis of the nature of the actions in the left-hand semi-direct products, shows that each of the groups $K_{m_i} \rtimes G_{n,r_i}$ are finitely generated (see Corollary 6.2 below). As the groups on the right-hand side of the central direct product are manifestly finitely generated, we obtain the following corollary to Theorem 1.1.

Corollary 1.2. *Let $n > 1$ be an integer and $\alpha \in V_n$. The group $C_{V_n}(\alpha)$ is finitely generated.*

One could try to improve this last corollary to obtain a statement of finite-presentation, which may be feasible. Such a proof might be accomplished through a careful study of presentations of the Higman groups $G_{n,r}$. In this direction, we ask the following.

Question 1. Must the group $C_{V_n}(\alpha)$ be finitely presented for every $\alpha \in V_n$?

We have also found some evidence supporting the possibility that the answer to the following question is "Yes".

Question 2. Is it true that for each index j the group A_j is abelian?

It is not completely trivial to find an example element $\alpha \in V = V_2$ where any A_i is not cyclic. In Section 7 we give such an example where $t = 1$ and $A_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

A generator of each $\mathbb{Z}$ in the right-hand terms $A_i \rtimes \mathbb{Z}$ is given by a root of a restricted version of α which is restricted to act only on the support of an element in ICC_j. We know that A_i commutes with the restricted version of α by definition, but it is not clear that A_i will commute with any valid choice of a generator for the $\mathbb{Z}$ term.

Question 3. Is it possible to replace the right-hand terms $A_i \rtimes \mathbb{Z}$ with $A_i \times \mathbb{Z}$?

In the final section of the paper we prove the following theorem.

Theorem 1.3. *Suppose that $\alpha \in V_n$ so that $\langle\alpha\rangle \cong \mathbb{Z}$. The group $\langle\alpha\rangle$ is undistorted as a subgroup of V_n.*

1.2. Our general approach. We give a description of our approach to centralizers in the broadest terms.

Let $\alpha \in V_n$. Let $H = \langle \alpha \rangle$. Define the *fundamental domain of the action of* H to be the space $\mathfrak{C}_n/H$. Note that this space is generally not very friendly, i.e., it is typically not Hausdorff, but that fact will have almost no bearing on our work.

If an element commutes with α, it will induce an action on $\mathfrak{C}_n/H$.

Thus, we get a short exact sequence

$$1 \to \mathcal{K} \to C_{V_n}(\alpha) \to \mathcal{Q} \to 1.$$

Here $\mathcal{K}$ represent the elements in $C_{V_n}(\alpha)$ which act on $\mathfrak{C}_n$ in such a way that their induced action on $\mathfrak{C}_n/H$ is trivial. The group $\mathcal{Q}$ is the natural quotient of $C_{V_n}(\alpha)$ by the image of the inclusion map $\mathcal{K} \to C_{V_n}(\alpha)$. Loosely speaking, elements of $\mathcal{Q}$ are represented by elements of V_n which act in the "same" way on each "copy" of the fundamental domain in $\mathfrak{C}_n$ (this of course is imprecise; there may be no embedding of a fundamental domain in $\mathfrak{C}_n$).

The first author is indebted to Martin Kassabov for pointing out this general structural approach to analyzing centralizers in groups of homeomorphisms.

2. Basic definitions

Throughout this section, let us fix an integer $n > 1$ for our discussion. We will also establish other conventions later that will hold throughout the section, and not just in a particular subsection.

We assume the reader is familiar with [9], and with the definition of a tree pair representative of an element of $V = V_2$. Nonetheless, we give an abbreviated tour through those definitions for the non-experts (extending them to include the groups $F_n \leq T_n \leq V_n$) and we state some essential lemmas which either occur in that source or which are easily derived by the reader with full understanding of these definitions. We give some examples demonstrating most of that language, and we add some new language to the lexicon mostly in support of our own later definition of a flow graph. In general the reader experienced with R. Thompson group literature will find little new material in this section and is encouraged to skip ahead, returning only if he or she runs into an unfamiliar term in the later parts of the paper.

2.1. Trees and Cantor sets. The only material in this section that may be unfamiliar to readers conversant with R. Thompson group literature is some of the language describing *the Cantor set underlying a node of the tree* $\mathcal{T}_n$ and related concepts.

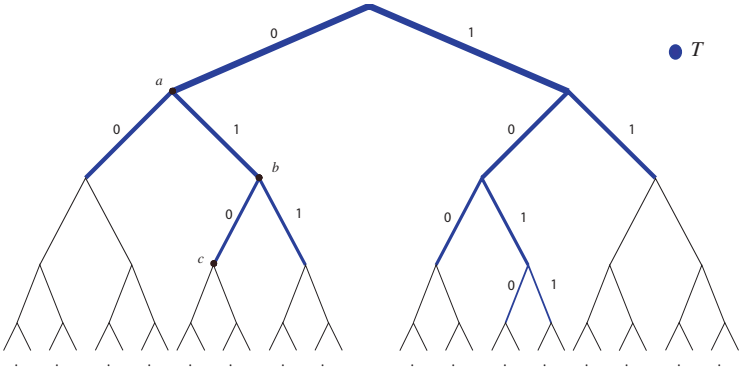
Our primary perspective will be to consider V_n as a group of homeomorphisms of the Cantor set $\mathfrak{C}$. In particular, V_n should be thought to act as a group of homeomorphisms of the Cantor set $\mathfrak{C}_n \cong \mathfrak{C}$. That is, the version $\mathfrak{C}_n$ of the Cantor set that is realized as the boundary of the standard infinite, rooted n-ary tree $\mathcal{T}_n$. While we

assume the reader understands that realization of the Cantor set, and also the terms "node" or "vertex", "child", "parent", "ancestor", "descendant", "leaf", and "n-caret" (or simply "caret"), and similar related language when referring to aspects of rooted n-ary trees, we give a short discussion below to establish some of our standard usage.

The left-to-right ordering of the children of a vertex in $\mathcal{T}_n$ allows us to give a name to each vertex in $\mathcal{T}_n$. Given a vertex p in $\mathcal{T}_n$, there is a unique order preserving bijection $\mathrm{Ord}_p : \mathrm{Children}(p) \to \{0, 1, 2, \ldots, n-1\}$. Let v be a vertex of $\mathcal{T}_n$ and let $(v_i)_{i=1}^m$ be the unique descending path in $\mathcal{T}_n$ starting at the root $r = v_1$ and ending at $v = v_m$, then we name the vertex v with the sequence $(\mathrm{Ord}_{v_i}(v_{i+1}))_{i=1}^{m-1}$. We will think of this sequence as a string (ordered from left-to-right).

Given names of two vertices, we may concatenate these strings to produce the name of a third vertex which will be a descendant of the first vertex

Below, we diagram an example of $\mathcal{T}_2$, with a finite tree T highlighted within it. The vertex c is a leaf of T, and in both T and $\mathcal{T}_2$, c is a child of b which is a child of a. The vertex a is an ancestor of c and c is a descendant of a. The name of the vertex labeled by c is 010.



We can view a finite rooted n-ary tree T as an instruction on how to partition the Cantor set $\mathfrak{C}_n$. Consider the natural embedding of T into the tree $\mathcal{T}_n$, where we send the root of T to the root of $\mathcal{T}_n$, and we preserve orders of children. For instance, as in the diagram above. Now, consider the set P_n of all infinite descending paths in $\mathcal{T}_n$ which start at the root of $\mathcal{T}_n$. If we consider each ordered n-caret in $\mathcal{T}_n$ as an instruction to pass through another inductive subdivision of the unit interval in the formation process of $\mathfrak{C}_n$, then each element in P_n can be thought of as limiting on an element of $\mathfrak{C}_n$. We thus identify P_n with $\mathfrak{C}_n$. The set P_n will now be considered to be topologized using the induced topology from the metric space topology of the unit interval. Now if we consider a vertex c of T, we can associate c with the subset of $\mathfrak{C}_n$ corresponding to the paths in P_n which pass through c. We will call this *the Cantor set under* c, and we will call any such subset of $\mathfrak{C}_n$ an *interval of* $\mathfrak{C}_n$. It is immediate that any interval in $\mathfrak{C}_n$ is actually homeomorphic with $\mathfrak{C}_n$. Given a node c, the Cantor set under c is also commonly called a cone neighborhood in $\mathfrak{C}_n$, and by definitions these sets form the standard basis for the product topology on

$P_n = \{0, 1, \ldots, n-1\}^\omega \cong \mathfrak{C}_n$. Thus, the leaves of T partition the set P_n, and they also partition $\mathfrak{C}_n$, into a set of open basis sets. We will call this *the partition of* $\mathfrak{C}_n$ *associated with the tree* T.

Extending the language of the previous paragraph, given a set $X \subset \mathfrak{C}_n$, we will call any node c of the universal tree $\mathcal{T}_n$ which has its underlying set contained in X, a *node of* X.

Of course, for all integers $m, n > 1$, we have that $\mathfrak{C}_m \cong \mathfrak{C}_n \cong \mathfrak{C}_2 = \mathfrak{C}$.

Using the example above, the interval of $\mathfrak{C}_2$ under c is $\mathfrak{C}_2 \cap [2/9, 7/27]$. In discussion, we will generally not distinguish between a vertex of $\mathcal{T}_n$ and the interval under it.

Remark 2.1. Any finite union of disjoint intervals in $\mathfrak{C}_n$ is homeomorphic with $\mathfrak{C}_n$.

2.2. Elements of V_n, T_n, and F_n. Some of the language in this subsection is unusual, although the general content will be familiar to all readers with knowledge of the R. Thompson groups.

An element of Homeo($\mathfrak{C}_n$) is *allowable* if it can be represented by an allowable triple (A, B, σ). The triple (A, B, σ) is *allowable* if there is a positive integer m so that A and B are rooted, finite, n-ary trees with the same number m of leaves, and $\sigma \in \Sigma_m$, the permutation group on the set $\{1, 2, 3, \ldots, m\}$. We explain below how to build the homeomorphism α which is represented by such an allowable triple (A, B, σ). We then call (A, B, σ) a *representative tree pair for* α. The group V_n consists of the set of all allowable homeomorphisms of $\mathfrak{C}_n$ under the operation of composition.

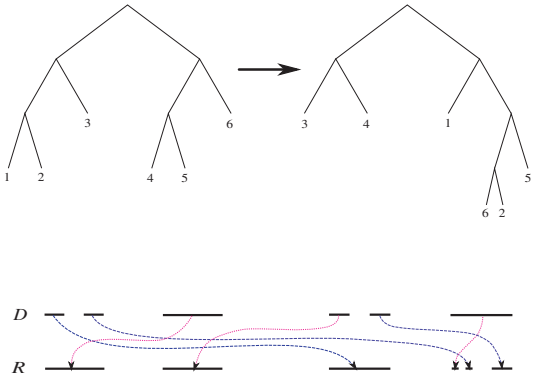
We are now ready to explain how an allowable triple (A, B, σ) defines an allowable homeomorphism α of $\mathfrak{C}_n$. Suppose that A and B both have m leaves, for some integer $m > 1$. We consider A to represent the domain of α, and B to represent the range of α. We take the leaves of A and B and number them in their natural left-to-right ordering from 1 to m. For each index i with $1 \leq i \leq m$, we map the interval under leaf i of A to the interval under the leaf $i\sigma$ of B using an orientation-preserving affine homeomorphism (that is, the homeomorphism of the two Cantor sets underlying these leaves defined by a restriction and co-restriction of an affine map with positive slope from the real numbers $\mathbb{R}$ to $\mathbb{R}$). Note for $\mathfrak{C}_n$ the slopes of such maps will be integral powers of $2n-1$. We will say that *the leaf* i *of* A *is mapped to the leaf* $i\sigma$ *of* B *by* α.

The next remark follows immediately from the discussion above:

Remark 2.2. Suppose that $v \in V_n$ and v is represented by a tree pair (A, B, σ). If η is a node in $\mathcal{T}_n$ so that η is either a leaf of A or a descendant node in $\mathcal{T}_n$ of a leaf of A, then v will carry the Cantor set underlying η affinely and bijectively in an order-preserving fashion to the Cantor set underlying some node τ of $\mathcal{T}_n$, where τ is either a leaf of B or a descendant of a leaf of B.

The diagram below illustrates an example for $V_2 = V$. The tree A is on the left, and the tree B is on the right. Note that we have re-decorated the leaves of B with the

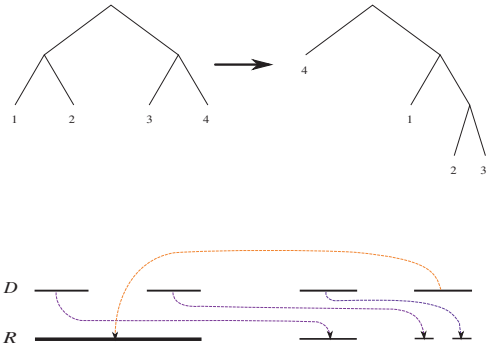
numbering from σ. The diagram beneath the tree pair is intended to indicate where intervals of $\mathfrak{C}_2$ are getting mapped, where the intervals D represent the domain and the intervals R represent the range. The lower diagram is superfluous when defining a general element of V.



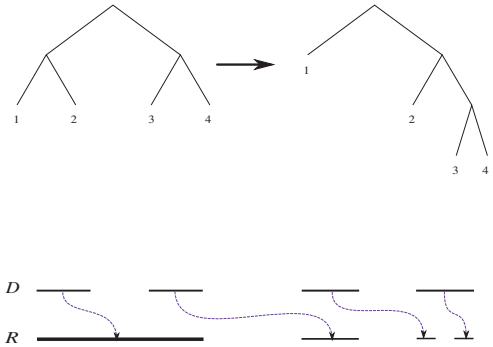
In general, as we can decorate our tree leaves with numbers to indicate the bijection, *we will now re-define the phrase "tree pair" throughout the remainder of the paper to mean an allowable triple*. We will still discuss the permutation of a tree pair as needed.

As mentioned in the introduction, there are groups $F_n \leq T_n \leq V_n$. If we only allow cyclic permutations, we get the group T_n. If our permutation is trivial, we get F_n. Thus, we can think of F_n as a group of piecewise-linear homeomorphisms of the interval $[0, 1]$, while T_n can be thought of as a group of piecewise-linear homeomorphisms of the circle S^1.

Here is an example element $\theta \in T = T_2$, which we will be considering again later.



Below is an example element from $F = F_2$.



2.3. Multiplication in V_n. There is nothing new in this subsection; readers familiar with the R. Thompson groups should skip ahead.

Naturally, the group operation of V_n is given by composition of functions. Building compositions using tree pairs is not hard. The process is enabled by the fact that there are many representative tree pairs for an element of V_n.

We give an example of what we mean by the last sentence. Given a representative tree pair (A, B, σ), one can use a leaf i of A to be a root of an extra n-caret, creating A', and one can build a tree B' from B by replacing the leaf $i\sigma$ with an n-caret. Label the leaves of A' in increasing order, and the leaves of B' using the induced labeling from the permutation σ on all leaves of B' that are also leaves of B. For the other leaves of B', use the labeling, in order, of the n-caret in A' that is not a n-caret of A. Let us call the permutation we have built from the leaves of A' to the leaves of B' by σ'. The process of replacing the tree pair (A, B, σ) by (A', B', σ') is called a *simple augmentation*.

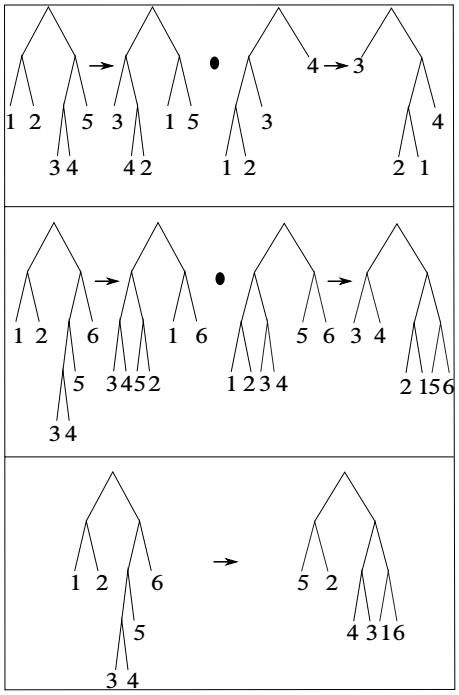
If the reverse process can be carried out (that is, deleting an n-caret from both the domain and range trees and re-labeling the permutation so that our initial two trees appear as a simple augmentation of our resulting tree pair) then we call this process a *simple reduction*. If we carry out either a simple augmentation or a simple reduction, we may instead say we have done a *simple modification* to our initial tree pair.

The following lemma is a straightforward consequence of the standard fact that any element in V_n has a unique tree pair representation that will not admit any simple reductions.

Lemma 2.3. *Any two representative tree pairs of a particular element of V_n are connected by a finite sequence of simple modifications.*

We are now ready to carry out multiplication of tree pairs. The essence of the idea is to augment the range tree of the first element and the domain tree of the second

element until they are the same tree (and carry out the necessary augmentations throughout both tree pairs), and then re-label the permutations, using the labeling of the range tree of the first element to seed the re-labeling of the permutation in the second element's tree pair. At this junction, the two inner trees are completely identical, and can be removed. The diagram below demonstrates this process.



Thus we now know what V_n is, and we can represent and multiply its elements.

3. Conjugation, roots, and centralizers

Recall that we will use right action notation to describe how V_n acts on $\mathfrak{C} = \mathfrak{C}_n$, as below.

The following discussion is completely basic and just carries out some straightforward points from permutation group theory.

Let $c \in \mathfrak{C}_n$ and $\alpha, \beta \in V_n$. In particular, $\alpha\colon \mathfrak{C}_n \to \mathfrak{C}_n$.

- We denote the image of c under α by $c\alpha$.
- For conjugation, we denote the conjugate of α by β by α^β and note therefore that $c\alpha^\beta = c\beta^{-1}\alpha\beta$.

We will often need to discuss what is moving under an action, so we need a definition as well. We define the support of an element $\alpha \in V_n$, denoted by $\operatorname{Supp}\alpha$,

as below.

$$\mathrm{Supp}(\alpha) = \{c \in \mathfrak{C}_n \mid c\alpha \neq c\}.$$

Note that this is distinct from the standard analysis version of support, which would extend the definition to include the closure of the set of points which are moving.

The following lemma is now standard from the theory of permutation groups.

Lemma 3.1. *We have* $\mathrm{Supp}(\alpha^\beta) = \mathrm{Supp}(\alpha)\beta$.

Recall that given $\alpha \in V_n$ we have

$$C_{V_n}(\alpha) = \{\beta \in V_n \mid \alpha\beta = \beta\alpha\}.$$

We point out the following obvious facts.

Remark 3.2. Suppose that α, β and $\gamma \in V_n$.

(1) If $\alpha\beta = \beta\alpha$, then $\alpha^\beta = \beta^{-1}\alpha\beta = \alpha$.

(2) If $\beta^k = \alpha$ for some integer k, then $\beta \in C_{V_n}(\alpha)$.

(3) We have $C_{V_n}(\alpha) \cong (C_{V_n}(\alpha))^\gamma = C_{V_n}(\alpha^\gamma)$.

We will use the third point above repeatedly to replace an element whose centralizer we are studying, by a conjugate element which admits a simpler tree-pair representative (simplifying our analysis without affecting centralizer structure).

4. Revealing pairs and related objects

Subsection 4.1 should be considered as mostly expository; it will contain definitions and lemmas from within [5] and [21]. We give very detailed examples of all of the concepts therein which are of use in our context. The following subsections on discrete train tracks, laminations and flow graphs, on the other hand, are entirely new.

Many tree pairs exist to represent a single element. Some tree pairs are more useful than others when it comes to discerning aspects of the dynamics of the element's action on the Cantor set. Consider the element $\theta \in T$ which was defined earlier as the second tree pair diagram in Subsection 2.2.

Thought of as a homeomorphism of the circle, θ has a rotation number, which, roughly stated, measures the average rotation of the circle under the action of θ. (Rotation numbers are a beautiful idea of Poincaré, and they are extremely useful in the analysis of circle maps). It is known, initially by work of Ghys and Sergiescu (see [11]), that the rotation number of any element of T is rational. Thus, θ has a rational rotation number p/q (in lowest terms). A lemma of Poincaré now shows that some point on the circle will have a periodic orbit, with period q. If the reader examines the

tree pair representative for θ, doubtless he or she will discover a point that travels on a finite orbit of length five. In fact, all points of the circle travel on their own periodic orbit of length five, as θ is torsion with order five. Under any reasonable definition of rotation number, θ must have rotation number $2/5$, since the reader can observe that after five iterations of θ, the circle will have rotated a total of two whole times around.

The previous example is intended to point to the fact that some tree pairs somehow hide the dynamics of an element's action.

In this section, we will describe revealing pairs, which are tree pairs that easily yield up all information about the orbit dynamics in $\mathfrak{C}_n$ under the action of a cyclic subgroup of V_n (or in the circle or the interval, in the cases of cyclic subgroups of T_n or F_n, respectively).

4.1. Revealing pair definitions. Throughout this section, we will work with some nontrivial $\alpha \in V_n$. We will assume that the tree pair (A, B, σ) represents α.

Consider A and B as finite rooted n-ary sub-trees (with roots at the root of $\mathcal{T}_n$) of $\mathcal{T}_n$.

We will call the set of vertices of $\mathcal{T}_n$ which are leaves of both A and B the *neutral leaves of* (A, B, σ). We will simply call these the neutral leaves, if the tree pair is understood.

As both A and B have a root and neither are empty, we can immediately form the tree $C = A \cap B$. It is immediate that the neutral leaves are leaves of C, but if $A \neq B$, then C will have other leaves as well.

We can make the sets $X = A - B$ and $Y = B - A$. The closures $\overline{X}$ and $\overline{Y}$ in $\mathcal{T}_n$ are both finite disjoint unions rooted n-ary trees (their roots are not sitting at the root of $\mathcal{T}_n$), where the number of carets in $\overline{X}$ is the same as the number of carets of $\overline{Y}$ (this number could be zero if $A = B$). We call $\overline{X}$ and $\overline{Y}$ *a difference of carets* for A and B.

In the remainder, when we write $D - E$, where D and E are trees, we actually want to take the difference of carets, so that our result will be a collection of rooted trees.

While $\langle\alpha\rangle$ acts on the Cantor set, it also induces a "partial action" on an infinite subset of the vertices of $\mathcal{T}_n$, as we explain in this paragraph. Since the interval under a leaf λ of A is taken affinely to the interval under a leaf of B (we will denote this leaf by $\lambda\alpha$), we see that α induces a map from the vertices of $\mathcal{T}_n$ under λ to the vertices of $\mathcal{T}_n$ under $\lambda\alpha$. In particular, the full sub-tree in $\mathcal{T}_n$ with root λ is taken to the full sub-tree in $\mathcal{T}_n$ with root $\lambda\alpha$ in order preserving fashion. We note in passing that we cannot extend this to a true action on the vertices of $\mathcal{T}_n$; if we consider a vertex η in $\mathcal{T}_n$ which is above a leaf of A, the map α may take the interval underlying η and map it in a non-affine fashion across multiple intervals in $\mathfrak{C}_n$.

As an example of the behavior mentioned above, consider the element θ again. The parent vertex of the domain leaves labeled 3 and 4 is mapped across multiple vertices of the range tree, in a non-affine fashion.

If η is a vertex of $\mathcal{T}_n$ we can now define a forward and backward orbit O_η of η in $\mathcal{T}_n$ under the action of $\langle\alpha\rangle$, to the extent that we restrict ourselves to powers of α that take the interval under η affinely bijectively to an interval under a vertex of $\mathcal{T}_n$. In more general circumstances, given any integer k and a vertex η of $\mathcal{T}_n$, we will use the notation $\eta\alpha^k$ to denote the subset of $\mathfrak{C}_n$ which is the image of the interval under η under the map α^k, and if that set happens to be the interval under a vertex τ in $\mathcal{T}_n$ (so that the restriction of α^k to the interval under η takes the interval underlying η affinely to the interval underlying τ), then we may denote the vertex τ by $\eta\alpha^k$ as well.

Let $L_{(A,B,\sigma)}$ denote the set of vertices of $\mathcal{T}_n$ which are either leaves of A or leaves of B, and let $\lambda \in L_{(A,B,\sigma)}$.

It is possible that λ is a neutral leaf whose vertex in $\mathcal{T}_n$ has orbit O_λ entirely contained in the neutral leaves of A and B. By Remark 2.2, and the fact that the neutral leaves are finite in number, we see that in this case the orbit of λ is periodic in $\mathcal{T}_n$. In this case, we call λ a *periodic leaf*.

Now suppose λ is not a periodic leaf of A. This implies that if we consider the forward and backward orbits of λ under the action of $\langle\alpha\rangle$, then in both directions, the orbit will exit the set of neutral leaves. In particular, there is a minimal integer $r \leq 0$ and a maximal integer $s \geq 0$ so that for all integers i with $r \leq i \leq s$ we have that $\lambda\alpha^i \in L_{(A,B,\sigma)}$. It is also immediate that $\lambda\alpha^r$ is a leaf of $A - B$ and $\lambda\alpha^s$ is a leaf of $B - A$, while for all values of i with $r < i < s$ we see that $\lambda\alpha^i$ is a neutral leaf.

Thus, we have the following lemma.

Lemma 4.1. *Suppose that $\alpha \in V$ is non-trivial and α is represented by a tree pair (A, B, σ). If $\lambda \in L_{(A,B,\sigma)}$, then either λ is*

(1) *a periodic neutral leaf, in which case there is a maximal integer $s \geq 0$ so that the iterated augmentation chain defined by* $\mathrm{IAC}(\lambda) := (\lambda\alpha^i)_{i=0}^{s}$ *is a sequence of neutral leaves so that $s+1$ is the smallest positive power so that $\lambda\alpha^{s+1} = \lambda$,*

(2) *a leaf of $A-B$, in which case there is a maximal integer $s > 0$ so that* $\mathrm{IAC}(\lambda) := (\lambda\alpha^i)_{i=0}^{s}$ *is a sequence of leaves of A or B, and furthermore, we then have $\lambda\alpha^s$ is a non-neutral leaf of B while $\lambda\alpha^i$ is a neutral leaf in $L_{(A,B,\sigma)}$ for all indices i with $0 < i < s$,*

(3) *a leaf of $B-A$, in which case there is a minimal integer $r < 0$ so that* $\mathrm{IAC}(\lambda) := (\lambda\alpha^i)_{i=r}^{0}$ *is a sequence of leaves of A or B, and furthermore, we then have $\lambda\alpha^r$ is a non-neutral leaf of A while $\lambda\alpha^i$ is a neutral leaf in $L_{(A,B,\sigma)}$ for all indices i with $r < i < 0$, or*

(4) *a neutral, non-periodic leaf, in which case, λ is a neutral leaf in a sequence* $\mathrm{IAC}(\eta)$ *for some vertex η which is a leaf of $A - B$, as discussed in point* (2)*. In this case we set* $\mathrm{IAC}(\lambda) := \mathrm{IAC}(\eta)$.

We now define and comment on some language from [5] and from [21]. Suppose we have the hypotheses of Lemma 4.1. The definition of iterated augmentation chain

IAC(λ) in Lemma 4.1 reflects the fact that we can augment the trees A and B at each vertex along an iterated augmentation chain, and end up with a new representative tree pair for α (the first vertex in an augmentation chain of type (2) or (3) can only be augmented in the domain tree A, while the last such vertex can only be augmented in the range tree B).

If λ is of type (2), with $\mathrm{IAC}(\lambda) = (\lambda\alpha^i)_{i=0}^{s}$ for some positive integer s, where $\lambda\alpha^s$ is an ancestor of λ, then we say that λ *is a repeller*. If λ is of type (3), with $\mathrm{IAC}(\lambda) = (\lambda\alpha^i)_{i=r}^{0}$ for some negative integer r, and where $\lambda\alpha^r$ is an ancestor of λ, then we say that λ *is an attractor*.

We are now ready to define what it means for a tree pair to be a revealing pair. Suppose that $\alpha \in V$ and the tree pair (A, B, σ) represents α. If every component of $A - B$ contains a repeller, and every component of $B - A$ contains an attractor, then we say that (A, B, σ) is a *revealing pair representing* α.

The discussion beginning Section 10.7 in [5] proves that every element of V_n admits a revealing pair. It is not hard to generate an algorithm which will transform any representative tree pair for an element of V_n into a revealing pair.

Given $\alpha \in V_n$, we will denote by R_α the set of all revealing pairs for α. We will use the symbol $\sim$ as a relation in the fashion $(A, B, \sigma) \sim \alpha$ denoting the fact that $(A, B, \sigma) \in R_\alpha$. In this case, we can further name leaves of $A - B$ and $B - A$. If λ is a leaf of $A - B$ and λ is not a repeller, then we say λ is a *source*. If λ is a leaf of $B - A$ and λ is not an attractor, then we say λ is a *sink*.

There are finitely many process types called "rollings" introduced by Salazar in [21]. Rollings are methods by which one can carry out a finite collection of simple expansions to a revealing tree pair (A, B, σ) to produce a new revealing pair (A', B', σ').

Below, we give the definitions and some discussion for rollings of type II. We give definitions for the other types of rollings in Section 4.2.

The tree pair (A', B', σ') is a *single rolling of type* II *from* (A, B, σ) if it is obtained from (A, B, σ) by adding a copy of a component U of $A - B$ to A at the last leaf in the orbit of the repeller in U and to B at its image; or, by adding a copy of a component W of $B - A$ to A at the first leaf in the orbit of the attractor (the leaf of A corresponding to the root node of W in B) and to B at its image.

The tree pair (A', B', σ') is a *rolling of type* II *from* (A, B, σ) if it is obtained from (A, B, σ) by a finite collection of single rollings of type II applied to the initial tree pair (A, B, σ) in some order.

We now state three lemmas about properties of revealing pairs. All of these properties are fairly straightforward to verify. In the cases of Lemma 4.2 and Lemma 4.3, the curious reader may also refer to the discussion in Sections 3.3 and 3.4 of [21] for alternative proofs.

Below, we slightly abuse the notion of a vertex name, by associating an infinite descending path with an infinite "name" string. This then represents a point in the Cantor set which is the boundary of the infinite tree.

Our first lemma discusses repellers, sources, and fixed points.

Lemma 4.2. *Suppose that* $\alpha \in V_n$ *and* $(A, B, \sigma) \sim \alpha$ *and that* λ *is a leaf of a component* C *of* $A - B$*, so that* $\mathrm{IAC}(\lambda) = (\lambda\alpha^i)_{i=0}^s$.

(1) *If* λ *is a repeller (so* $\lambda\alpha^s$ *is an ancestor of* λ*), set* γ_i *to be the name of the node* $\lambda\alpha^i$ *for each* $0 \leq i \leq s$*. Set* Γ *to be the string which is the suffix one would append to the name of the node* γ_s *to obtain the name of the node* γ_0 *i.e., the path from the ancestor to the repeller. We notate this by the expression* $\gamma_s\Gamma = \gamma_0$*. Thus the infinite descending path corresponding to the infinite string* $\gamma_i(\Gamma)^\infty$ *represents a unique repelling fixed point* p_{γ_i} *in the interval* X_i *of* $\mathfrak{C}_n$ *underneath* γ_i*, under the action of* $\langle\alpha^s\rangle$*, for each index* i *with* $0 \leq i < s$.

(2) *If* λ *is a source, then* $\lambda\alpha^s$ *is a sink.*

(3) $\langle\alpha\rangle \cong \mathbb{Z}$.

Proof. The inverse α^{-s} of α^s maps the Cantor set X_s under γ_s bijectively to the Cantor set X_0 under γ_0, where $X_0 \subset X_s$, in affine fashion. The only infinite descending path which is fixed by this map is the path terminating in Γ^∞, thus $p_{\gamma_0} = \gamma_0\Gamma^\infty = \gamma_s\Gamma\Gamma^\infty$ is the unique attracting fixed point of α^{-s} in X_s, and is thus the unique repelling fixed point of α^s within X_0 (and even within all of X_s) under the action of α^s.

We now show below why each of the points p_{γ_i} are also repelling fixed points of α^s.

We first obtain a new revealing tree pair $(A', B', \sigma') \sim \alpha$ which is a single rolling of type II from (A, B, σ), constructed as follows.

Glue a copy of C at λ_s in B to produce B', and a further copy of C at the leaf γ_{s-1} of A to produce A'. The resulting tree pair (A', B', σ') so obtained has all of the leaves of C of $A - B$ as neutral leaves (except in the case where $s = 1$, in which case A' is A with a copy of C attached at γ_0, a leaf of the original C).

In any case, the original copy of C, within A', is now contained in $A' \cap B'$. The new copy of C in A' is a complementary component of $A' - B'$ and contains the leaf $\gamma_{s-1}\Gamma$ as a repeller. There are no other new complementary components for the tree pair (A', B', σ'), which therefore must represent a revealing pair for α.

Now, by a minor adjustment to the argument in first paragraph, the point $p_{\gamma_{s-1}}$ is a fixed repelling point of α^s.

We can now continue inductively in this fashion to show that each point p_{γ_i} is the unique repelling fixed point of X_i under the action of α^s by building a revealing pair (A'', B'', σ'') for α with the point p_{γ_i} as a point in a repelling leaf of a complementary component of (A'', B'', σ'') with shape C and spine Γ rooted at γ_i.

We leave the second point of the lemma to the reader, while the third point is immediate from the first since some power of α has a repelling fixed point. □

We call each subset X_i a *basin of repulsion for* α, since all points in X_i eventually flow out of X_i under repeated iteration of α^s, never to return, except for p_{γ_i}, for each $0 \leq i \leq s$. We call the string Γ above the *spine of the repeller* γ_0 or the *spine of* C, as it describes the shape of the path in C from the root of C to the repeller γ_0.

We call each point p_{γ_i} a *periodic repelling point of* α for each index i with $0 \leq i < s$. We may also call these points *fixed repelling points of* α^s, noting in passing that not all periodic repelling points of α are necessarily fixed by α^s. In similar fashion we call the sequence of points $(p_{\gamma_i})_{i=0}^{s-1}$ a *periodic orbit of periodic repelling points for* α. We denote by $\mathcal{R}_\alpha$ the set of periodic repelling points of α, noting that it is a finite set.

We now give a similar lemma discussing attractors and sinks.

Lemma 4.3. *Suppose that* $\alpha \in V_n$ *and* $(A, B, \sigma) \sim \alpha$ *and* λ *is a leaf of a component* C *of* $B - A$, *so that* $\mathrm{IAC}(\lambda) = (\lambda\alpha^i)_{i=r}^{0}$, *for some negative integer* r.

(1) *If* λ *is an attractor (so* $\lambda\alpha^r$ *is an ancestor of* λ*), set* γ_i *to be the name of the node* $\lambda\alpha^i$ *for each* $r \leq i \leq 0$. *In this case the string* Γ *which has* $\gamma_r\Gamma = \gamma_0$ *has the property that the infinite descending path corresponding to the name* $\gamma_i(\Gamma)^\infty$ *represents a unique attracting fixed point* p_{γ_i} *in the interval* X_i *of* $\mathfrak{C}_n$ *underneath* γ_i, *under the action of* $\langle\alpha^r\rangle$, *for each index* i *with* $r \leq i < 0$.

(2) *If* λ *is a sink, then* $\lambda\alpha^r$ *is a source.*

(3) $\langle\alpha\rangle \cong \mathbb{Z}$.

Proof. This proof is similar to the proof of the previous lemma, where here α^{-1} has a tree pair with λ as a repeller, and each p_{γ_i} is a periodic repelling point of α^{-1}. □

We now extend the notation from Lemma 4.2 to apply to the sets named in Lemma 4.3 as below.

We call each subset X_i a *basin of attraction for* α *as indicated by* (A, B, σ), since all points in X_i eventually limit to p_{γ_i} under repeated iteration of α^{-r}, for each $r \leq i \leq 0$. We call the string Γ above the *spine of the attractor* γ_0 or the *spine of* C, as it describes the shape of the path in C from the root of C to the attractor γ_0.

We call each point p_{γ_i} a *periodic attracting point of* α for each index i with $r \leq i \leq 0$. We may also call these points *fixed attracting points of* α^{-r}. In similar fashion we call the sequence of points $(p_{\gamma_i})_{i=r}^{-1}$ a *periodic orbit of periodic attracting points for* α. We denote by $\mathcal{A}_\alpha$ the set of periodic attracting points of α, noting that it is a finite set.

In the previous two lemmas, if λ is a source or a sink (case two in each lemma), we refer to $\mathrm{IAC}(\lambda)$ as a *source-sink chain*.

The next lemma follows directly from the two above, and the classification of the orbits of the leaves of A and B. This lemma is a version of one result proved by Burillo, Cleary, Stein and Taback in their joint work [8] as Proposition 6.1. We give a new proof here, as the situation is greatly simplified through the use of revealing pairs.

Lemma 4.4. *Suppose that* $\alpha \in V_n$. *There is an integer* r *so that* α *has order* r *if and only if there is a tree pair* (A, B, σ) *representing* α *with* $A = B$.

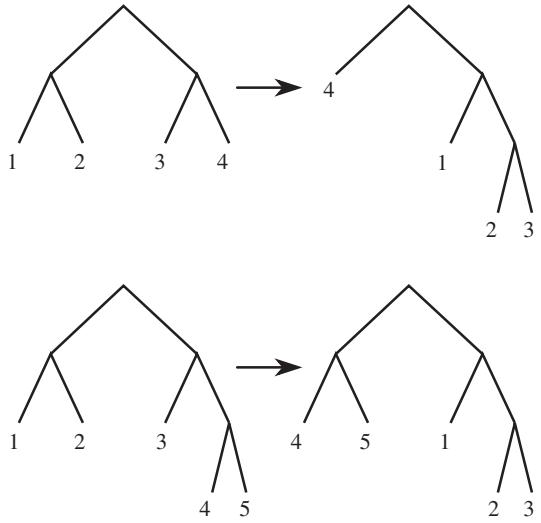
Proof. Suppose that α does not have infinite order, and let $(A, B, \sigma) \sim \alpha$ be a revealing pair representing α. We must have that (A, B, σ) admits no repellers or attractors, thus both $A - B$ and $B - A$ are empty, and so $A = B$.

Suppose instead that (A, B, σ) is a tree pair representing α with $A = B$. Then it is immediate from the definition of multiplication for tree pairs that the order of α is the order of the permutation σ. □

We denote by $\mathcal{P}_\alpha$ the points of $\mathfrak{C}_n$ which underlie the periodic neutral leaves of $L_{(A,B,\sigma)}$. Note that the set $\mathcal{P}_\alpha$ is independent of the choice of revealing pair used to represent α. We further denote by $\mathrm{Per}(\alpha)$ the set of all periodic points of α, that is

$$\mathrm{Per}(\alpha) = \mathcal{R}_\alpha \sqcup \mathcal{A}_\alpha \sqcup \mathcal{P}_\alpha.$$

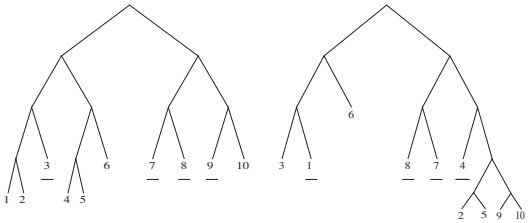
The following tree pairs represent the previously defined element $\theta \in T$ (we apply an augmentation to our first tree pair to produce a revealing pair representing θ).



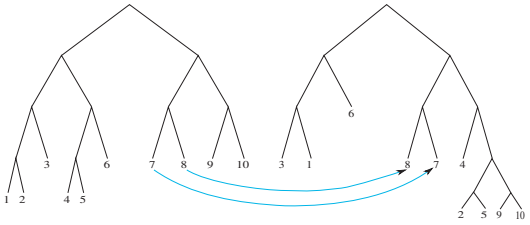
A quick examination of the second tree pair above should convince the reader that θ has order five and rotation number $2/5$.

We now give a series of diagrams for a revealing pair (A, B, σ) representing a particular non-torsion element $\alpha \in V$. This element has a revealing pair which contains many of the structures we have been discussing. In each diagram below we illustrate some of the particular aspects we have discussed above.

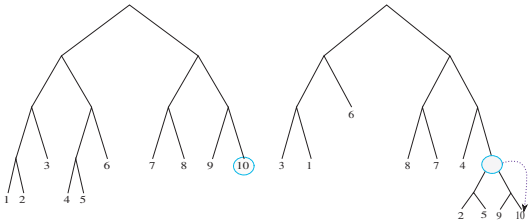
Below is an example of our revealing pair (A, B, σ), with the neutral leaves underlined. Recall that the left tree is A, and the right tree is B.



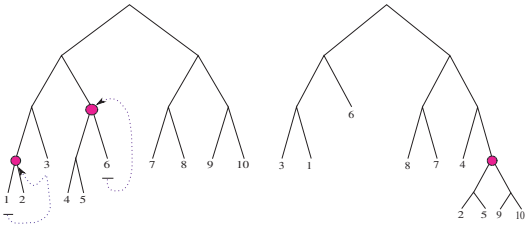
In the next diagram, we point out a periodic orbit of neutral leaves of length two.



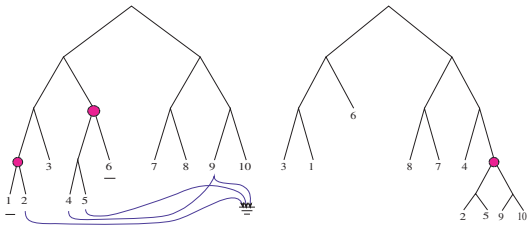
Below, the vertex 11111 in B is an attractor; its iterated augmentation chain is $(111, 11111)$, a sequence of length two. The fixed point of the attractor corresponds to the value 1 in the unit interval. The word Γ for this attractor is 11.



In the diagram to follow, the vertices 0000 and 011 represent repellers in A. The dotted paths track the orbits of the repellers along their iterated augmentation chains.



Finally, we highlight the fact that sources flow to sinks. Note how the lengths of the paths from sources to sinks are not uniform. In particular, the source 0100 first hops to 110 before next landing in the basin of attraction under the vertex 111.



4.2. Discrete train tracks and laminations. In this subsection we are concerned with modelling the dynamics in $\mathfrak{C}_n$ under the action of the group generated by an element v of V_n. Naturally, the relevant information is contained in any particular revealing pair $\rho = (C, D, \theta)$ for the element, but we have found that tools from surface topology afford us another way to visualize these dynamics. In particular, these dynamics can be usefully described by a combinatorial object introduced by Thurston (see [22], [23]) to aid in the study of surfaces, namely, a train track. Given a revealing pair, it is easy to draw a combinatorial train track which in turn "carries" a lamination in some compact surface with boundary. It is this lamination which models, in some sense, the movement of points in the Cantor set under iteration of the map v. As in the theory of laminations carried by a train track, one quickly realizes that most of the relevant visual information is actually contained in the train-track object (see for instance, Penner and Harer's book [18]). We call the train track object developed here a *discrete train track*, even though it is continuous in nature; the name is meant to emphasize that our train tracks model dynamics in a totally disconnected set under iterations of a fixed map.

Below we describe in detail our method for generating a discrete train track TT_ρ from the revealing pair ρ representing the general element $v \in V_n$ mentioned above, and briefly describe how to model the lamination it carries. We build an example discrete train track and lamination using the element α and the tree pair (A, B, σ) from the previous subsection. Finally, we discuss some of the utility of TT_ρ and some facts about how TT_ρ would change under basic operations applied to v (conjugation, then a choice of representative revealing pair) or ρ (Salazar's rollings). In the next subsection, we describe a derived object, the flow graph, which carries less information than the train track (although flow graphs in general still contain enough information to answer many dynamics questions). In our current practice, we find discrete train tracks and flow graphs to be helpful for understanding dynamics, while the generating revealing pairs tend to be helpful for any involved computations and for specifying elements with appropriate desired dynamics.

Note that any such discrete train track TT_ρ is a representative of an equivalence class of similar train tracks, where the equivalence class is determined by all the discrete train tracks for revealing pairs equivalent to ρ up to Salazar's rollings, choice of location for drawing the complementary trees along the orbits of repelling and attracting fixed points, and conjugacy in V_n. Thus we are picking a representative object which is in some sense less well chosen than some "minimal" train track in

this class. The result of applying Belk and Matucci's geometric conjugacy invariant, the closed strand diagram, is very close to what a "minimal" discrete train track for an element of V_n should be (leaving out the demarcations representing iterating the element, and of course, including some simplifications which result from the effects of conjugation (see [3])).

Here is how one draws a discrete train track from a revealing pair.

(1) List all of the iterated augmentation chains for the revealing pair.
(2) For each chain representing a non-trivial orbit of a periodic neutral leaf:
    (a) Draw a circle.
    (b) If the chain represents an orbit of length r, demarcate the circle into r equal subintervals (typically we end these intervals with dots), and label these with the names of the nodes carrying the neutral leaves, in a counter-clockwise order.
(3) For each chain of a repeller:
    (a) Draw a circle.
    (b) If the repelling periodic point travels an orbit of length r, then demarcate the circle into r equal subintervals as above, and label these with the names of the nodes carrying the orbit of the repeller in a counter-clockwise order.
    (c) For the segment corresponding to the interval of the repelling periodic point of the complementary tree, instead of the one label mentioned in the last point, we label the two ends of the segment with the top and bottom of the spine of the repeller (top (root node of complementary component) before bottom in counter-clockwise order).
    (d) Lay the complementary component of the repeller along the sub-arc of the circle with the spine labels, gluing the spine to the circle.
        (i) Scale the complementary component so that the spine is the length of the appropriate sub-arc of the circle.
        (ii) Smooth out the tree (and the spine in particular) and bend the spine so that the spine has the same shape as the sub-arc of the circle with the labels from the spine nodes (preserving the current length of the spine).
        (iii) Rigidly rotate the scaled, smoothed, and bent tree in the plane, and translate it so that the spine can be identified with the appropriate sub-arc of the circle.
(4) For each chain for an attractor:
    (a) Draw a circle.
    (b) If the attracting periodic point travels an orbit of length r, then demarcate the circle into r equal subintervals, and label these with the names of the nodes carrying the orbit of the attractor in a counter-clockwise order.
    (c) For the segment corresponding to the interval of the attracting periodic point

of the complementary tree, instead of the one label mentioned in the last point, we label the two ends of the segment with the top and bottom of the spine of the attractor (bottom (attracting leaf node of complementary component) before top in counter-clockwise order).

(d) Lay the complementary component of the attractor along the sub-arc of the circle with the spine labels, gluing the spine to the circle.

(i) Reflect the complementary component of the attractor across a vertical axis. Thus, the root of the component is now drawn at the bottom, while the left and right-hand sides are preserved, respectively, as left and right-hand sides.

(ii) Scale the complementary component so that the spine is the length of the appropriate sub-arc of the circle.

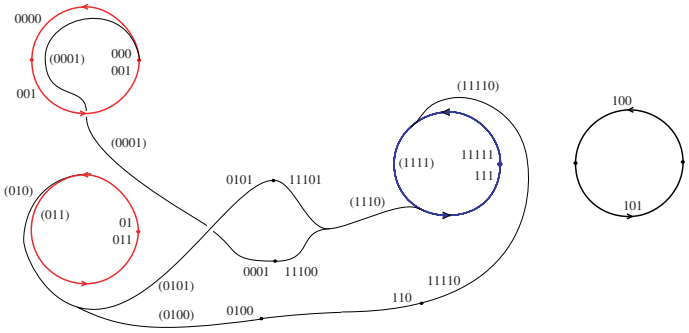
(iii) Smooth out the tree (and the spine in particular) and bend the spine so that the spine has the same shape as the sub-arc of the circle with the labels from the spine nodes (preserving the current length of the spine).

(iv) Rigidly rotate the reflected, scaled, smoothed, and bent tree in the plane, and translate it so that the spine can be identified with the appropriate sub-arc of the circle.

(5) For each source-sink chain, drawn a line connecting the appropriate sources and sinks (lines may have to cross each other in the case of V_n which is why the lamination carried by the train track can only be embedded in an surface with boundary; strips can pass "under" each other). Demarcate each line with dots representing the length of the source-sink chain, and label sub-arcs with appropriate node labels as above for other sorts of iterated augmentation chains.

(6) Add parenthetical labels for splittings of trees where the support of the whole tree will be mapped away by one application of the element. Add parenthetical labels anywhere else as desired to improve clarity. (This step is not strictly necessary, but we find it to be helpful.)

If we follow the process above for the element α, the diagram below is an example of what we may obtain (we include some drawn under-crossings following the methods from drawing knots from knot theory).



To visualize the lamination carried by a discrete train track, for each sub-arc in the demarcations of the discrete train tracks, one can draw a transverse copy of an I-fiber with an embedded copy of the Cantor set $\mathfrak{C}_n$. If we stabilize this by building a product with the interval I (which lines we draw parallel to any train track sub-arc), then one has a picture of the lamination carried by the train track. (Glue the ends of the I-fibers of the Cantor sets using the rules provided by the map v, as a local picture of a mapping cone on the I-fiber running transverse to any demarcation dot, allowing contraction and expansion in the transverse I-fibers near to any tree-splitting).

Note that by reflecting the attracting complementary components as directed, we are able to draw the resulting carried lamination on a compact orientable surface with boundary. (The "left" and "right" portions of a Cantor set underlying a node are correctly associated without any twisting.)

Separately, one can observe that the group homomorphism $\mathcal{S}$ of the penultimate subsection of this article is connected with holonomy measurements for the carried lamination along repelling cycles when applied to the element we are centralizing.

A local diagram of the lamination carried by this train track local to the repelling cycle in the upper left corner is included in the diagram below. One can see the rescaling near the periodic repeller and portions of the Cantor set moving away along flow lines.

Let TT_1 be a discrete train track drawn in accordance with the method above. We define the *support of the discrete train track* TT_1 to be given as the union of the Cantor sets underlying the node labels in the iterated augmentation chains of the (non-trivial) periodic orbits, the source-sink chains, and the orbits of the repellers and attractors used to create the drawing of TT_1. Seeing a discrete train track as a (possibly disconnected) graph, we define the components of TT_1 to be the connected components of TT_1 as a graph. Given any such component X, we describe the *support of the component* X to be the union of the Cantor sets underlying the node labels of X. We call a component a *torsion component* if the component is a circle generated from a periodic orbit of neutral leaves, otherwise we call a component a *non-torsion component*. Similar language will be defined for flow graphs, below.

We now mention some technical facts to do with the relationships amongst discrete train tracks drawn from distinct revealing pair representatives for an element of V_n, and also relationships which may arise as a consequence of conjugacy.

We give the basic definitions of Salazar's rollings below, with the intention of understanding of the variance of discrete train tracks across the set of all representative

revealing pairs for an element of V_n. The reader is encouraged to read Salazar's Section 3.5 in [21], where she defines rollings of various types and traces the various impacts of rollings on revealing tree pairs.

The tree pair (A', B', σ') is a *single rolling of type E from* (A, B, σ) if it is obtained from (A, B, σ) by adding an n-caret to A and B along each of the leaves of an iterated augmentation chain corresponding to one of two types of iterated augmentation chains; either to all of the leaves in A and B of a periodic orbit of neutral leaves, or to the initial source leaf in A for a source-sink chain, and then to all of the neutral leaves in A and B of that chain, and then to the leaf of B corresponding to the sink of that chain. (These are called *elementary rollings*.)

The tree pair (A', B', σ') is a *single rolling of type I* from (A, B, σ) if it is obtained from (A, B, σ) by adding a cancelling tree along all of the leaves of A along the orbit of a repeller, and at B at the image of these leaves under the map, or by adding a cancelling tree at all the leaves of A which appear in the reverse orbit of an attractor, and at the leaves of B to which these leaves of A are mapped. If W is the complementary component of $A - B$ or $B - A$ corresponding to the repeller or attractor in this discussion, and W is rooted at node Σ, then a tree C is a *cancelling tree for* W if it is obtained from W by first choosing a proper, non-empty prefix Δ of the spine Γ of W (so that $\Gamma = \Delta\Theta$ for some suffix Θ), and then taking C to be the maximal sub-tree of W which has root Σ and containing the node $\Sigma\Delta$ as a leaf. (Note that if one carries out this process, the corresponding complementary component W' created in $A' - B'$ or $B' - A'$ for the tree pair (A', B', σ') will now be rooted at $\Sigma\Delta$ and will have spine $\Theta\Delta$).

Finally, recall from Section 4.1 that a tree pair (A', B', σ') is a *single rolling of type* II *from* (A, B, σ) if it is obtained from (A, B, σ) by adding a copy of a component U of $A - B$ to A at the last leaf in the orbit of the repeller in U and to B at its image; or, by adding a copy of a component W of $B - A$ to A at the first leaf in the orbit of the attractor (the leaf of A corresponding to the root node of W in B) and to B at its image.

The following lemma lists some basic properties of discrete train tracks. The reader will not be required to use this lemma later in the paper, although it gives a separate view of some arguments.

Lemma 4.5. *Suppose that* $\tau_1 \sim P_1 = (D_1, R_1, \sigma_1)$ *and* $\tau_2 \sim P_2 = (D_2, R_2, \sigma_2)$ *are elements of* V_n, *and that* TT_1 *and* TT_2 *are the corresponding train tracks derived from the revealing pair representatives* P_1 *and* P_2 *of these elements. Then we have the following:*

(1) *Suppose that* $f \in V_n$ *so that* $\tau_1^f = \tau_2$. *Then:*
   (a) f *induces a* $1-1$ *correspondence between the components of* TT_1 *which describe dynamics around repelling/attracting orbits and the components of* TT_2 *which describe dynamics around repelling/attracting orbits.*
   (b) *This correspondence also guarantees that the individual cycles in these*

components are also carried to cycles of the same length and type (*repellers must move to repellers and attractors to attractors*).

(c) *This correspondence preserves the labelings of the spine of the complementary components for corresponding cycles.*

(2) *Suppose that* $\tau_1 = \tau_2$ *and* (D_1, R_1, σ_1) *differs from* (D_2, R_2, σ_2) *by an application of a rolling of type* X. *Then:*

(a) *If* X *is a rolling of type E, then a train track connecting a source node to a sink node will be split into n such parallel tracks along its length, or a periodic circle representing a periodic orbit of neutral leaves will be split into n copies of "parallel" periodic circles.*

(b) *If* X *is a single rolling of type I, then there is a suffix* Δ *of the spine so that the labels of all the sub-arcs in the relevant repelling or attracting circle will be modified by the addition of* Δ *as a suffix to all of the labels on the circle. Also, some source-sink chains which are incident on the affected circle will be lengthened by the length of the orbit of the repeller or the attractor.*

(c) *If* X *is a single rolling of type II, then the source-sink chains with sources or sinks beginning or ending in the relevant complementary component will all increase their lengths by one* (*each of these new arcs will need appropriate labels added*), *and the sub-arc of the relevant repelling* (*attracting*) *circle to which the spine of the complementary tree is glued will move one location backward* (*forward*) *in the cyclic ordering of the arcs in that circle* (*respectively*). *Finally, the labels of the affected arcs on the circle will change* (*the arc corresponding to the old spine will now be labelled by the old leaf label for the attractor or the repeller of that spine, while the arc corresponding to the new spine will have as root label its old label, and as leaf label, its old label concatenated with the word corresponding to the spine of the repeller of attractor*).

Proof. The latter points about rollings follow directly from the definitions of rollings, as given in Section 4.1.

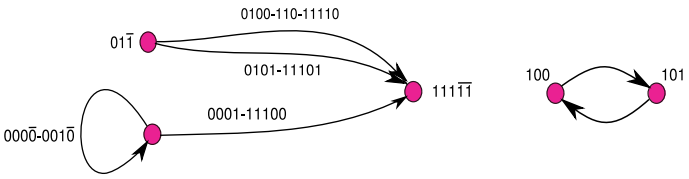
The first two sub-points of 1 are a result of the fact that V_n is a group of homeomorphisms, and topological conjugacy preserves the properties mentioned. The third sub-point of 1 follows from the fact that elements of V_n do not change infinite suffixes; so, the points in the finite orbit of a repelling periodic point or of an attracting periodic point all have the same infinite repeating suffix (that is, as described in Lemmas 4.3 and 4.2). Now the conjugating element f again cannot change this infinite suffix class, so the resulting orbit will consist of points with this same infinite suffix. (Note first that Belk and Matucci [3] also derive the infinite suffix of a finite repelling or attracting orbit as a conjugacy invariant for elements of $F < T < V$, using a method very similar to our discrete train tracks, although Belk and Matucci's definition is mildly different, and second that one can choose a representative revealing pair for the conjugate version τ_2 of τ_1 so that the spine is cyclically rotated, but by applying a

rolling of type II, this spine can be rotated back to the original spine, thus producing a discrete train track with the same spine labeling on the appropriate sub-arc of the circle (although, the prefixes of all nodes on the circle will be lengthened).) □

4.3. Flow graphs. The *flow graph of any revealing pair* $p = (A, B, \sigma) \sim \alpha \in V_n$ is a labeled directed graph which is, in some sense, a quotient object from a discrete train track (we lose the visual aspects of the branching of source sink lines as they leave the orbit of a repeller or join the orbit of an attractor, although even this information can be recovered from labels). We now describe how to build a flow graph from a revealing pair.

For each repeller of A, we draw a vertex. For each attractor of B, we draw a vertex. For each neutral leaf of $L_{(A,B,\sigma)}$ that is part of a periodic orbit of neutral leaves for p, we draw a vertex. We draw a directed edge from a repeller to an attractor for each source in the basin of repulsion of the repeller whose iterated augmentation chain terminates in a sink in the basin of attraction of the attractor (we call these *source-sink flow lines* or by similar language; they are in one-one correspondence with the set of source-sink chains for the pair p). We draw a directed edge from each repelling and attracting vertex to itself whenever the period of the corresponding repeller (or attractor) is greater than one (we call this a *repelling (or attracting) periodic orbit*). We draw an edge connecting two vertices representing periodic neutral leaves if a single iteration of α will take the first leaf to the second, whenever these leaves are not the same. We label all source-sink flow lines with the appropriate iterated augmentation chain. We label all repelling and attracting periodic orbits (even of length one) with the finite periodic orbit of the actual points in $\mathfrak{C}_n$, each such point labeled by its infinite descending path in P_n. (Note that one can detect the names of repelling and attracting basins by deleting the infinite "Γ^∞" portion of these labels.)

The diagram below is an example of a flow graph for the tree (A, B, σ) we have been examining. Strings of form Γ^∞ are indicated by overlines in the labels of the diagram.



The components of the flow graph for a revealing pair naturally decompose into two sets; components representing the flow (under the action of $\langle\alpha\rangle$) along the underlying sets of a periodic orbit of neutral leaves for the revealing pair, called *torsion components*, and components representing flows from basins of repulsion to basins of attraction, and characterizing the orbits of repelling and attracting periodic points, called *non-torsion components*.

Any flow line in the graph from a repeller to an attractor can be thought of as representing the complete bi-infinite forward and backward orbit in $\mathfrak{C}_n$ of the interval

underlying the source, under the action of $\langle\alpha\rangle$. The set so determined is called the *underlying set for the flow line*, and it limits on some of the periodic repelling points of α (on one end), and on some of the periodic attracting points of α (on the other end).

Given a component C of a flow graph for α, we can discuss the *underlying set for the component* C in $\mathfrak{C}_n$. We define this set as the closure of the underlying sets of the flow lines of the component if the component has a flow line (thus, we will capture the limiting repelling and attracting points in their finite orbits), and otherwise as the underlying set of the finite periodic orbit of an appropriate neutral leaf.

We observe in passing the following remark, which the reader can verify as a test of their understanding. The proof of this remark follows directly from the definition of flow graph or can be obtained as a consequence of Lemma 4.5.

Remark 4.6. Given any revealing pair $p = (A, B, \sigma)$ representing $\alpha \in V_n$, and $p' = (A', B', \sigma')$ the result of a single rolling of type II from (A, B, σ), the flow graph $\mathcal{F}_{\alpha,p'}$ for α generated by p' is identical to the flow graph $\mathcal{F}_{\alpha,p}$ for α generated by p, except for one pair (U, U') of corresponding components. The components U and U' are both non-torsion components which have the same underlying set and are isomorphic as graphs. However, the flow lines of U' will bear different labels from the flow lines of U, having had some of its iterated augmentation chains lengthened by one hop, and, one of the periodic orbits in the labeling of U', of either a repelling periodic point or of an attracting periodic point, may be cyclically permuted from the corresponding orbit as labeled in U.

A mildly more difficult statement is that in the following lemma.

Lemma 4.7. *Suppose that $\alpha \in V_n$ is represented by a revealing pair $p = (A, B, \sigma) \sim \alpha$. If z_1 and z_2 underlie distinct non-torsion components of $\mathcal{F}_{p,\alpha}$, then they cannot underlie the same component of a flow graph representing α^k for any integer k.*

Proof. Powers of α may split orbits of repelling (or attracting) periodic points under the action of $\langle\alpha\rangle$, but they can never move a point from the underlying set of one component of the flow graph to the underlying set of other components; these sets are themselves created as the unions of the images of the forward and backward orbits of the points underlying the interval portions of the fundamental domain of α which are carried in affine fashion by α. □

The next lemma is a direct consequence of the work of Salazar in [21].

Lemma 4.8. *Suppose that n is a positive integer and $\alpha \in V_n$ is represented by the revealing pair $p = (A, B, \sigma)$. For all cycle lengths m_k (with $1 \leq m_k$) of periodic orbits of neutral leaves of p, set c_k to be the number of such orbits with period m_k, and set r_k as the unique value with $1 \leq r_k < n$ so that $c_k \equiv r_k \mod (n-1)$.*

Then there is a conjugate $u = \alpha^f$ for some $f \in V_n$, where u has a representative tree pair (E, F, χ), so that the following hold.

(1) *The element u has r_k periodic orbits of neutral leaves with period k.*
(2) *For each $k \geq 0$, the number r_k is a conjugacy invariant of α.*
(3) *The number of components of $F - E$ is the same as the number of components of $B - A$, and these components have the same shape, and the same spine.*
(4) *Likewise, the number of components of $E - F$ is the same as the number of components of $A - B$, and these components have the same shape, and the same spine.*

Sketch of proof. The statements in the lemma above are for the most part stated in Corollary 1 in Section 4.4 of [21]. We explain our statements relating to the modular arithmetic involved in the reduction of the number of finite periodic orbits of the same length, as that is not given by Salazar's corollary. The fact that spines are preserved as well is not explicitly stated by Salazar, but it is an immediate consequence of her techniques.

One can describe any conjugation in V_n in terms of tree pairs by a process which we now sketch. Suppose that $u = \alpha^f$ and let f be represented by a tree pair (C, D, τ). There is an expansion (E', F', χ') of (A, B, σ) so that $C \subset (E' \cap F')$. One replaces the sub-tree C inside of $(E' \cap F')$ by D, producing a revealing tree pair (E, F, χ) for u (one has to remember to percolate out the effects of τ in both trees) with the same number of neutral leaves as for the (E', F', χ') tree pair.

This process of conjugation produces a revealing pair for α^f whose periodic neutral leaves are generated by expansions along the iterated augmentation chains of the original neutral cycles (together with some permuting of locations in the universal tree). Thus, the number of neutral periodic leaves of the result is in the same congruence class modulo $(n - 1)$ as for the tree pair for α.

One can now find a conjugate of α which has all finite periodic orbits of neutral leaves of the same period m_k adjacent in the universal tree $\mathcal{T}_n$. If there are more than n such orbits, one can find a conjugate such that n of the orbits travel in parallel, all carried in the m_k-orbit of an n-caret. One can also choose a conjugation so that one can build a new revealing pair for the resulting element with the same number of periodic orbit neutral leaves as in (A, B, σ). By simple reductions along the full orbit of the caret, one can reduce the current representative tree pair to another revealing pair with $m_k \cdot (n-1)$ fewer periodic neutral leaves, and can repeat this overall process until there are fewer than n orbits of neutral leaves for any period m_i. □

5. A partition of $\mathfrak{C}_n$

Given a positive integer n and some $\alpha \in V_n$ with representative revealing pair $(A, B, \sigma) \sim \alpha$, we can decompose $\mathfrak{C}_n$ as $T_\alpha \cup Z_\alpha$. Here we are using the following

notation:

- T_α: the subset of $\mathfrak{C}_n$ lying under the neutral leaves in A which are on cyclic orbits (possibly of length one).
- Z_α: the subset of $\mathfrak{C}_n$ underlying the root node of the complementary components of $A - B$ and $B - A$, as well as any neutral leaves which are part of source-sink chains.

The reader can easily verify the following.

Lemma 5.1. *If $\alpha \in V_n$ and $\beta \in C_{V_n}(\alpha)$, then*

(1) $T_\alpha \beta = T_\alpha$, *and*
(2) $Z_\alpha \beta = Z_\alpha$.

This lemma allows us to work to comprehend centralizers over each set, without regard to the behavior of these centralizers in the other regions.

From here to near the end of Section 7, we will assume that n and α are fixed, and α is the element whose centralizer in V_n we are analyzing, and that (A, B, σ) is a revealing pair representing α. We further assume through the use of the third point of Remark 3.2 and of Lemma 4.8 that for each periodic cycle length m_i of neutral leaves of A under the action of $\langle\alpha\rangle$, that there are precisely $r_i < n$ such cycles of periodic neutral leaves. These values m_i and r_i are the numbers which appear in the semi-direct product terms in the left-hand direct product in the statement of Theorem 1.1.

Suppose that $G \leq V_n$ and $X \subset \mathfrak{C}_n$. We define

$$G_X = \{v \in G \mid v|_{\mathfrak{C}_n \setminus X} = \mathrm{id}|_{\mathfrak{C}_n \setminus X}\}$$

so that G_X is the subgroup of elements of G which act as the identity **except on** the set X. Lemma 5.1, assures us that there are two commuting elements $\alpha_T \in V_{nT_\alpha}$ and $\alpha_Z \in V_{nZ_\alpha}$ of V_n, so that $\alpha|_{T_\alpha} = \alpha_T|_{T_\alpha}$, and $\alpha|_{Z_\alpha} = \alpha_Z|_{Z_\alpha}$. Thus, we see immediately that $\alpha = \alpha_T \alpha_Z$.

We will therefore restrict our attention to finding the centralizers $C_{V_{nT_\alpha}}(\alpha_T)$, and $C_{V_{nZ_\alpha}}(\alpha_Z)$. In fact, we have the following corollary to Lemma 5.1.

Corollary 5.2. *We have*

$$C_{V_n}(\alpha) \cong C_{V_{nT_\alpha}}(\alpha_T) \times C_{V_{nZ_\alpha}}(\alpha_Z)$$

This explains the central direct product in our statement of Theorem 1.1.

6. Centralizers over the set T_α

Taking advantage of the decomposition given by Corollary 5.2, we analyze the centralizer of α by restricting our attention to the set T_α, over which α acts as an element of torsion.

Below, when we refer to a cycle of α, we mean a cycle of neutral leaves for (A, B, σ).

Lemma 6.1. *The leaves of A over T_α can be partitioned into disjoint sets according to cycle lengths, denoted by S_{m_i}, where each S_{m_i} consists of all the leaves of A in a periodic orbit of length m_i. If $\beta \in C_{V_n}(\alpha)$, then β preserves the subsets of $\mathfrak{C}_n$ underlying the leaves in any particular set S_{m_i}.*

Proof. We first suppose T_α is not empty, and we further suppose $L_{(A,B,\sigma)}$ admits distinct neutral leaf cycles of length m_i and m_j under the action of α, where we chose our labels so that $m_i < m_j$. Finally, suppose $\beta \in C_{V_n}(\alpha)$ acts by mapping a point p' underlying a cycle of length m_j to a point $p = p'\beta$ underlying a cycle of α of length m_i. (If there is a $\gamma \in C_{V_n}(\alpha)$ which maps a point underlying a cycle of length m_i to a point underlying a cycle of length m_j, then γ^{-1} will match our requirements.) We now have the computation

$$p = p\alpha^{m_i} = p(\alpha^\beta)^{m_i} = p'\alpha^{m_i}\beta = q\beta,$$

where $q = p'\alpha^{m_i}$ is not p' since the orbit length for p' under α is m_j. However, we have just shown that $q\beta = p$ and by assumption $p'\beta = p$, so we have a contradiction. □

We suppose throughout the remainder that there are s distinct neutral leaf cycle lengths under the induced action of $\langle\alpha\rangle$ on the periodic neutral leaves in $L_{(A,B,\sigma)}$, namely $\{m_1, m_2, \ldots, m_s\}$.

We thus can focus on how a particular $\beta \in C_{V_k}(\alpha)$ can commute with α over the underlying set of the leaves in any particular set S_{m_i}. This is the reason for the left-hand direct product with s terms in our statement of Theorem 1.1. (Here we are following the same logic as used in the beginning of this section which allowed us to focus our attention on T_α based on the dynamical cause of the direct product decomposition of Corollary 5.2.)

For each leaf of S_{m_i}, we can consider its orbit in S_{m_i} under the induced action of $\langle\alpha\rangle$. Build a set $F_{m_i} \subset S_{m_i}$ by taking one leaf from each such orbit. Thus, F_{m_i} is a collection of r_i leaves (by the comment after Lemma 5.1). Let X_{m_i} denote the subset of $\mathfrak{C}_n$ underlying S_{m_i} and let E_{m_i} denote the subset of $\mathfrak{C}_n$ underlying the set F_{m_i}. By construction we see that a subset of the fundamental domain of α is $D_{m_i} := X_{m_i}/\langle\alpha\rangle \cong E_{m_i}$.

We now analyze the groups $G_{m_i} = C_{V_{nX_{m_i}}}(\alpha) = C_{V_n}(\alpha) \cap V_{nX_{m_i}}$, which are individually isomorphic to the terms in the left-hand direct product of Theorem 1.1.

As described in Section 1.2, G_{m_i} is an extension of its subgroup K_{m_i} consisting of the elements in G_{m_i} which have trivial induced action on D_{m_i} and which act trivially outside of X_{m_i}.

We now study K_{m_i}. Let β be an element of K_{m_i}, and fix β until we have finished our classification of K_{m_i}.

By the previous two paragraphs we see that β must carry each set underlying a leaf cycle in S_{m_i} (under the action of $\langle\alpha\rangle$) to itself. Consider a leaf $\lambda \in F_{m_i}$, and how β moves points from the set underlying λ (fix this choice of leaf λ for the remainder of the discussion leading to the classification of K_{m_i}). First, set $\lambda_r = \lambda\alpha^r$, the r'th leaf in the orbit of λ under the action of $\langle\alpha\rangle$ on S_{m_i}, for $r \in \{0, 1, \ldots, m_i - 1\}$, and denote by $\tau_{\lambda,r}$ the set underlying λ_r in $\mathfrak{C}_n$. If we take a point $p_0 \in \tau_{\lambda,0}$, its orbit under the action of $\langle\alpha\rangle$ is p_0, $p_1 = p_0\alpha$, $p_2 = p_0\alpha^2$, and etc., so that $p_r \in \tau_{\lambda,r}$. In order for $\langle\beta\rangle$ to have no induced action on D_{m_i}, we see that $p_0\beta = p_c$ for some index c. Now, in order to commute with the action of $\langle\alpha\rangle$, we must have $p_r\beta = p_{((r+c) \bmod m_i)}$ for any index $0 \leq r < m_i$. In particular, β must push the full orbit of p_0 under the action of $\langle\alpha\rangle$ forward by some constant index less than m_i. A consequence of this is that any point $p \in E_{m_i}$ is itself in an orbit of length less than or equal to m_i under the action of $\langle\beta\rangle$. Extending this discussion as possible via recalling our choices of F_{m_i} and λ, we see that $\beta^{(m_i!)}$ must act trivially over the whole set X_{m_i}, so β must be torsion.

This now leads to the conclusion of our classification of K_{m_i}. Suppose that (C, D, χ) is a revealing pair for β. Since β is torsion, we see that $C = D$. We assume (by taking a larger revealing pair to represent β if necessary), that λ is a node of C. Since α takes the set $\tau_{\lambda,r}$ to the set $\tau_{\lambda,((r+1) \bmod m_i)}$ in affine fashion, and β commutes with α in such a way as to have no induced action on D_{m_i}, it is straightforward to verify that the sub-tree $T_{\lambda,r}$ in C rooted at node λ_r is identical in shape to the sub-tree $T_{((r+1) \bmod m_i)}$ in D rooted at $\lambda_{((r+1) \bmod m_i)}$, for all indices r. Further, by the last sentence of the previous paragraph, for each leaf γ_0 of $T_{\lambda,0}$ there is an integer $t_{\gamma_0} \geq 0$ such that β will send the corresponding leaf γ_r of $T_{\lambda,r}$ to the corresponding leaf $\gamma_{((r+t_{\gamma_0}) \bmod m_i)}$ in $T_{\lambda,((r+t_{\gamma_0}) \bmod m_i)}$. Any map $\mathfrak{C}_n \to \mathfrak{C}_n$ which is the identity outside of X_{m_i} and which satisfies these properties can be found in $V_{nX_{m_i}}$, and a straightforward topological argument (using the compactness of $\mathfrak{C}_n$, and the basis cones of the topology on $\mathfrak{C}_n$. See Subsection 2.1 for the definition) shows that these maps form a subgroup which is isomorphic to the group $\mathrm{Maps}(\mathfrak{C}_n, \mathbb{Z}_{m_i})$. (That is, any continuous map from $\mathfrak{C}_n$ to $\mathbb{Z}_{m_i}$ can be described as a rooted, finite, labeled n-ary tree, where each label indicates where in $\mathbb{Z}_{m_i}$ to send the set underlying the labeled leaf and this represents the offset in the orbit under $\langle\alpha\rangle$ for the element β on the corresponding leaf.) Since the choice of map to $\mathbb{Z}_{m_i}$ on the set $\tau_{\lambda,0}$ has no bearing (for the definition of β) on the choice of map to $\mathbb{Z}_{m_i}$ for the sets underlying the other leaves in F_{m_i}, we see that

$$K_{m_i} \cong (\mathrm{Maps}(\mathfrak{C}_n, \mathbb{Z}_{m_i}))^{r_i}.$$

We now need to consider the structure of $G_{m_i}/K_{m_i} = Q_{m_i}$. Thus, we are modding out the subgroup of elements of $V_{nX_{m_i}}$ which commute with the action of $\langle\alpha\rangle$ by the subgroup K_{m_i}. In particular, we are looking at the elements of $V_{nX_{m_i}}$ which carry, for each index $0 \leq r \leq m_i - 1$, the sets underlying the r'th copy of the fundamental domain D_{m_i} to itself, where the map on the r'th copy of the fundamental domain is precisely the conjugate version (under the action of α^r) of the map on the 0'th copy E_{m_i}. Therefore, the group Q_{m_i} is isomorphically represented

by the restriction of the action of V_n to the domain E_{m_i}. In particular, $Q_{m_i} \cong V_{n\,E_{m_i}}$. Since E_{m_i} is given as the disjoint union of r_i distinct copies of $\mathfrak{C}_n$, we see that this group is precisely the finitely presented group G_{n,r_i} of Higman in [14]! That is, we have

$$Q_{m_i} \cong G_{n,r_i}.$$

Note that we can realize an isomorphic copy of Q_{m_i} in G_{m_i} as follows. Let β' be an element of $V_{n\,E_{m_i}}$, represented by the tree pair (C', D', χ'). The set S_{m_i} decomposes as m_i copies of the nodes of F_{m_i} in the universal tree $\mathcal{T}_n$ (take copy C_r as the nodes in the set $F_{m_i}\alpha^r$, using the induced action of $\langle\alpha\rangle$ on the set of subsets of the nodes in S_{m_i}, for each index $0 \leq r < m_i$, and fix this definition of the sets C_r for the remainder of this subsection). There is a revealing pair $(C'', D'', \chi'') \sim \beta'$ expansion of (C', D', χ') which has all of the nodes in the set S_{m_i} as leaves of C'' and D'' (excepting the nodes in F_{m_i}, which themselves are roots of a forest pair $f = (\mathcal{F}_d, \mathcal{F}_r, \theta)$ representing the element of G_{n,r_k} corresponding to β'). By simply gluing a copy of f to each C_r for $r > 0$ (the nodes in these C_r are leaves of C'' and D''), we can build a new revealing tree pair (C, D, χ) representing an element $\beta \in V_{nX_{m_i}}$ which acts on the set Γ_r underlying C_r as β' acts on Γ_0 under C_0, for each index r (fix this definition of the sets Γ_r for the remainder of the section as well). It is immediate by construction that the group $\widehat{Q}_{m_i}$ of elements β so constructed is isomorphic with Q_{m_i} and is a subgroup of G_{m_i} which splits the short exact sequence $K_{m_i} \hookrightarrow G_{m_i} \twoheadrightarrow Q_{m_i}$.

Thus, we have

$$G_{m_i} \cong K_{m_i} \rtimes Q_{m_i} \cong K_{m_i} \rtimes G_{n,r_i}.$$

We can complete our analysis of the centralizer of α over T_α by showing the following lemma, which concludes the proof of Corollary 1.2 from the introduction.

Lemma 6.2. *The group G_{m_i} is finitely generated.*

Proof. First, we recall that for all positive integer values $n > 1$ and r, Higman's group $G_{n,r}$ is finitely presented (Theorem 4.6 of [14]). Let us denote by $\langle A_{n,r} \mid R_{n,r}\rangle$ a finite presentation of $G_{n,r}$ for any such n and r.

We first describe our set of generators for G_{m_i}. For each generator $g' \in A_{n,r_i}$, we will take as a generator of G_{m_i} the element $g \in \widehat{Q}_{m_i}$ which duplicates the effect of g' on $E_{m_i} = \Gamma_0$ over each of the sets Γ_r underlying the copies C_r of the leaves C_0 over Γ_0, for each valid index r.

At this stage, our collection of generators generates the group $\widehat{Q}_{m_i}$, which is finitely presented still by carrying over the relations of G_{n,r_i} as well in corresponding fashion. We need to add only one further generator to generate the remainder of G_{m_i}. Let $g_1 \in K_{m_i}$ be the element represented by the revealing pair (S, T, θ). We define (S, T, θ) as follows. Let λ be a node of F_{m_i} (fix this choice and dependent derived

notation for the remainder of this subsection). Let $S = T$ be the minimal n-ary tree so that $\lambda_r = \lambda\alpha^r$ is a node which is a parent of n-leaves of S, for all index values $0 \leq r < m_i$. Let θ be the permutation from the leaves of S to the leaves of T which takes the first child of λ_r and sends it to the first child of $\lambda_{((r+1) \bmod m_i)}$, for all indices $0 \leq r < m_i$, and otherwise acts as the identity. The element g_1 so constructed is our last generator.

We now show that g_1, together with our other generators, is sufficient to generate K_{m_i}. Each element of K_{m_i} decomposes as a finite product of sub-node translations along the m_i orbit of F_{m_i} in S_{m_i} (under the action of $\langle\alpha\rangle$). That is, we choose a descendant node p of F_{m_i} in the universal tree $\mathcal{T}_n$, and a translation constant $0 \leq t \leq m_i$. Then we translate the full orbit of p under the action of $\langle\alpha\rangle$ forward cyclically by the constant t, while acting as the identity elsewhere. We denote this translation as p_t. By choosing a specific n-ary forest rooted at F_{m_i} and translating each leaf of the forest in such a fashion, we can obtain any element of K_{m_i}, as described above. Now, recall that G_{n,r_i} acts transitively on the set of nodes in the infinite n-ary forest descending from F_{m_i} which do not happen to represent the full domain of G_{n,r_i} (if $r_i = 1$, no element of $V_n = G_{n,1}$ can take a proper sub-node to the root node). Thus, given any particular descendant node p from a node of F_{m_i} and a translation distance t, we can find an element ρ of $\widehat{Q}_{m_i}$ taking the first descendant of λ_0 to p. It is now immediate by construction that $p_1 = g_1^{\rho}$, and $p_t = p_1^t$. In particular, the set consisting of g_1 and the generators of $\widehat{Q}_{m_i}$ together, is sufficient to generate G_{m_i}. □

7. Centralizers over the set Z_α

Let us fix a revealing pair $\mathfrak{p} = (A, B, \sigma) \sim \alpha$. Let $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_e\}$ represent the set of non-torsion flow graph components of the flow graph $\mathcal{F}_{\mathfrak{p},\alpha}$, where for each index i, we denote by X_i the component support of Γ_i. Let α_i represent the element in V_{nX_i} such that $\alpha_i|_{X_i} = \alpha|_{X_i}$, and suppose further that $\mathfrak{p}_i = (A_i, B_i, \sigma_i) \sim \alpha_i$ is a revealing tree pair that is identical to $\mathfrak{p}$ over the support X_i of γ_i, so that the flow graph $\mathcal{F}_{\mathfrak{p}_i,\alpha_i}$ is identical to Γ_i. Recall that by definition $Z_\alpha = \bigcup_i X_i$. For any $\beta \in V_n$ set

$$\mathcal{RA}_\beta := \mathcal{R}_\beta \sqcup \mathcal{A}_\beta.$$

Note that for any such β, we have that $\mathcal{RA}_\beta$ is a finite discrete set.

Lemma 7.1. *Let $g \in C_{V_n}(\alpha)$.*

(1) *The group $\langle g\rangle$ acts on the set $\mathcal{RA}_\alpha$.*
(2) *Given $r \in \mathcal{R}_\alpha$, there is a basin of repulsion U_r of α containing r so that $U_r g$ is contained in a basin of repulsion U_s for some repelling periodic point s of α.*
(3) *Given $r \in \mathcal{A}_\alpha$, there is a basin of attraction U_r of α containing r so that $U_r g$ is contained in a basin of attraction U_s for some attracting periodic point s of α.*

(4) *The group $\langle g \rangle$ acts bijectively on each of the sets $\mathcal{R}_\alpha$ and $\mathcal{A}_\alpha$ of repelling and attracting periodic points of α.*

Proof. Let $p \in \mathcal{RA}_\alpha$ have an orbit of size k under the action of $\langle \alpha \rangle$. We now have

$$pg = p\alpha^k g = pg\alpha^k.$$

Thus, pg is fixed by α^k, so that $\langle g \rangle$ bijectively preserves $\mathcal{RA}_\alpha$.

The second point follows from the continuity of g and the following computation. Let $r \in \mathcal{R}_\alpha$ so that $rg \neq r$ (if such a repelling periodic point fails to exist, we automatically have the second part of our lemma for the repelling periodic points). Choose $\widetilde{U}_r \subseteq \mathfrak{C}_n$ an interval neighborhood of r small enough so that g is affine on $\widetilde{U}_r$ and so that $\widetilde{U}_r$ is contained in a basin of repulsion for α. Expand the revealing tree pair (A, B, σ) representing α by a rolling of type II to create a new revealing tree pair (A', B', σ') representing α with a complementary component C rooted in some node whose underlying set is contained in $\widetilde{U}_r$. The root node U_r of C represents a basin of repulsion for α which is an interval neighborhood of r carried affinely by g to another interval of $\mathfrak{C}_n$. Assume that r is in a periodic orbit of length k under the action of $\langle \alpha \rangle$. Our result for repelling periodic points follows easily from the following limit:

$$(U_r)g\alpha^{-(nk)} = (U_r)\alpha^{-(nk)}g \to rg \quad (\text{as } n \to \infty).$$

Hence g takes a basin of repulsion neighborhood of r into a neighborhood N of some periodic repelling point $s = rg$ of α with orbit length dividing k where N limits on s under powers $-(nk)$ of α.

A similar argument shows the third point of the lemma, and the final point of the lemma is an immediate consequence of the previous three points. □

The following corollary is immediate from the first point of the lemma above, together with the fact that α only admits finitely many periodic points.

Corollary 7.2. *Let $g \in C_{V_n Z_\alpha}(\alpha)$, then $\mathcal{RA}_\alpha \subset \mathcal{RA}_g$.*

The following corollary now follows from Lemma 7.1, using the idea behind the proof from Section 6 that a centralizer of an element of torsion must carry the set of all finite orbits of length k (under the action of the torsion element) to itself.

Corollary 7.3. *Let $g \in C_{V_n}(\alpha)$, and let $r \in \mathcal{R}_\alpha$ or $r \in \mathcal{A}_\alpha$, with periodic orbit $(r_i = r\alpha^i)_{i=0}^{k-1}$ in $\mathcal{R}_\alpha$ or $\mathcal{A}_\alpha$ respectively, then there is a periodic orbit $(s_i)_{i=0}^{k-1}$ in $\mathcal{R}_\alpha$ or $\mathcal{A}_\alpha$ respectively such that $s_i = r_i g$.*

The next corollary depends on the proof of the second and third points of Lemma 7.1.

Corollary 7.4. *Suppose that $g \in C_{V_n}(\alpha)$ sends some point $z \in X_i$ to a point $zg \in X_j$ for some indices i and j, then g will send X_i bijectively to X_j.*

Proof. We first show that g will send all of the periodic repelling and attracting points α within X_i into X_j.

Set $P_i := X_i \cap \mathcal{RA}_\alpha$. If g takes two points in P_i to the supports of distinct components of the flow graph of α, then X_i must admit a flow line which has some point r_1 in its periodic repelling orbit vertex label sent to a periodic repelling point $r_1 g$ of one component Γ_k while having another point a_1 in its periodic attracting orbit vertex label sent to a periodic attracting point for α in a label of a distinct component Γ_m. But now there are basins of repulsion U_1 and attraction W_1 around r_1 and a_1 respectively which are carried by g into X_k and X_m respectively. This last is a contradiction as follows.

Recall that some non-zero power k of α fixes all of the repelling and attracting periodic points of α. Take $p_1 \in U_1 \backslash \{r_1\}$. It must be that for all integers z,

$$p_1 g \alpha^{kz} = p_1 \alpha^{kz} g,$$

however, for z large and negative, $p_1 \alpha^{kz}$ is near r_1 while for z large and positive, $p_1 \alpha^{kz}$ is near a_1. In particular α^k has a flow line connecting $r_1 g$ to $a_1 g$, which is not possible by Lemma 4.7.

If p_1 is in the support of a flow line, then by considering powers of α using similar arguments as above we can show the whole flow line is sent by g to a single flow graph component of α. □

We now define the function

$$\mathcal{S}(g) = \log_{2n-1}\Big(\prod_{r \in \mathcal{R}_\alpha} rg'\Big),$$

where rg' denotes the slope of g at the repeller r. This map is well defined by using the recognition that g is affine in small neighborhoods of points in $\mathrm{Per}(\alpha)$. It is not too hard to see that the function $\mathcal{S} : V_n \to \mathbb{Z}$ is not a homomorphism in general.

Lemma 7.5. *The map $\mathcal{S} : C_{V_n}(\alpha) \to \mathbb{Z}$ is a group homomorphism.*

Proof. Given $g_1, g_2 \in C_{V_n}(\alpha)$ we compute $\mathcal{S}(g_1 g_2)$ directly from the definition. We note that in small interval neighborhoods of the points in $\mathcal{R}_\alpha$, the maps g_i are differentiable, and so we can apply the chain rule. Since g_i acts bijectively on the set $\mathcal{R}_\alpha$ (by Corollary 7.3) each of the two terms of the product appears exactly once and so $\mathcal{S}(g_1 g_2) = \mathcal{S}(g_1) + \mathcal{S}(g_2)$ □

We will now shift attention to the local behavior of $C_{V_n X_i}$ over the region X_i for any particular index $i \in \{1, 2, \ldots, t\}$. For each such index i, set

$$\mathcal{S}_i(g) = \log_{2n-1}\Big(\prod_{r \in \mathcal{R}_{\alpha_i}} rg'\Big).$$

By the previous lemma, $\mathcal{S}_i$ is a group homomorphism when we restrict its domain to either $C_{V_{nX_i}}(\alpha_i)$ or even to $C_{V_n}(\alpha)$, and further, if $\beta_i \in C_{V_{nX_i}}(\alpha_i)$ while $\beta \in C_{V_n}(\alpha)$, with $\beta_i|_{X_i} = \beta|_{X_i}$, then $S_i(\beta_i) = S_i(\beta)$. For our immediate purposes below, we will use $S_i : C_{V_{nX_i}}(\alpha_i) \to \mathbb{Z}$.

The previous lemma now immediately implies the existence of the following exact sequence:

$$0 \to \ker(\mathcal{S}_i) \hookrightarrow C_{V_{nX_i}}(\alpha_i) \twoheadrightarrow \mathrm{im}(\mathcal{S}_i) = \mathbb{Z} \to 0. \tag{1}$$

Lemma 7.6 (Stair algorithm). *Let $g_1, g_2 \in C_{V_{nX_i}}(\alpha_i)$ and r, s be periodic repelling points of α_i for some index i. If*

$$rg_1 = rg_2 = s$$

and the slope of g_1 at r is equal to the slope of g_2 at r , then $g_1 = g_2$.

Proof. There is a basin of repulsion U_r of α_i containing r so that $g_1 = g_2$ on U_r. Let $x \in U_r \backslash \{r\}$. Since

$$xg_1\alpha_i^n = xg_2\alpha_i^n = x\alpha_i^n g_1 = x\alpha_i^n g_2$$

for all integers n, we see that on the underlying support of any flow line L of α_i limiting on r we have $g_1|_L = g_2|_L$. But this now means that g_1 agrees with g_2 on all of the underlying support of the edges of the graph Γ_i. In particular, $g_1 = g_2$ on sets limiting to each of the attracting orbits on the other ends of the flows lines of Γ_i leading away from the repelling orbit of r. Since g_1 and g_2 are always affine in small neighborhoods of the repelling and attracting periodic points for α, we then see that g_1 and g_2 actually agree on small neighborhoods of the attracting periodic orbits which appear on the terminal ends of those flow lines leaving the orbit containing r.

We now repeat this argument moving away from the attracting orbits to new repelling orbits along new flow lines, where we again have that $g_1 = g_2$ along these flow lines. Now by the connectivity of Γ_i, $g_1 = g_2$ over X_i. □

Lemma 7.7. *There is an exact sequence*

$$0 \to \mathbb{Z} \to C_{V_{nX_i}}(\alpha_i) \xrightarrow{q} Q \to 0, \tag{2}$$

where $Q \leq \mathrm{Sym}(\mathcal{R}_{\alpha_i})$. In particular, $C_{V_{nX_i}}(\alpha_i)$ is virtually infinite cyclic.

Proof. Define $M := \{\beta \in C_{V_{nX_i}}(\alpha_i) \mid \beta(r) = r \text{ for all } r \in \mathcal{R}_{\alpha_i}\}$. Fix $r_1 \in \mathcal{R}_{\alpha_i}$ and define the map

$$\varphi \colon M \to \mathbb{Z}, \quad \beta \mapsto \log_{2n-1}(r_1\beta').$$

By Lemma 7.6, the map φ is injective and so $M \cong \mathbb{Z}$. Now we observe that M is the kernel of the action of $C_{V_{nX_i}}(\alpha_i)$ on $\mathcal{R}_{\alpha_i}$ and so we get a natural map $q \colon C_{V_{nX_i}}(\alpha_i) \to Q$ where $Q := C_{V_{nX_i}}(\alpha_i)/M \leq \mathrm{Sym}(\mathcal{R}_{\alpha_i})$. □

We recall a few elementary facts from group theory. The proofs are immediate, but we provide them for completeness.

Lemma 7.8. *Let G, H be groups such that the sequence*

$$0 \to H \to G \xrightarrow{\omega} \mathbb{Z} \to 0$$

is exact. Then $G \cong H \rtimes \mathbb{Z}$.

Proof. Let $z \in \omega^{-1}(1)$. Then, by construction, $\mathbb{Z} \cong \langle z \rangle \leq G$ and $H \cap \langle z \rangle = \{1_G\}$. If $g \in G$ and $\omega(g) = k$, then $z^k g^{-1} \in H$, and so $G = H\langle z \rangle$. □

By Lemmas 7.5 and 7.8 applied on the exact sequence (1) we obtain $C_{V_{nX_i}}(\alpha_i) \cong \ker(\mathcal{S}_i) \rtimes \mathbb{Z}$. We will now show that $\ker(\mathcal{S}_i)$ is a finite group and that it coincides with the set of torsion elements of $C_{V_{nX_i}}(\alpha_i)$.

Lemma 7.9. *Let G, Q, K, C be groups, where $C = \mathbb{Z}$ and Q is a finite group of order m. Assume that the two sequences*

$$0 \to C = \mathbb{Z} \xrightarrow{\varphi} G \xrightarrow{\psi} Q \to 0$$

and

$$0 \to K \xrightarrow{\tau} G \xrightarrow{\sigma} \mathbb{Z} \to 0$$

are exact. Then K is a finite group and $G \cong K \rtimes \mathbb{Z}$.

Proof. In this proof we write $M \leq_f N$ to denote that M is a finite index subgroup of a group N. Let $h \in \sigma^{-1}(1)$ and $I := \langle h \rangle \leq G$. By Lemma 7.8 we have $G \cong K \rtimes I \cong K \rtimes \mathbb{Z}$. We need to show that K is a finite group. By assumption, $C \leq_f G$ and so we observe that

$$\frac{I}{C \cap I} \cong \frac{IC}{C} \leq \frac{G}{C} \cong Q,$$

therefore implying that $C \cap I \leq_f I$. In particular, $C \cap I$ is a non-trivial group, hence $C \cap I \leq_f C$ is too. By definition, for every $g \in G$, we have $g^m \in C$. Since $C \cap I \leq_f C$, there is an integer k such that $g^{km} \in C \cap I$, for every $g \in G$.

If $g \in K$, we have that $g^{km} \in K \cap C \cap I \leq K \cap I = 0$. Therefore, K is a torsion subgroup of finite exponent, hence $K \cap C = 0$ and so the first exact sequence implies that $K \cong \psi(K) \leq Q$ and therefore it is finite. □

Applying Lemma 7.9 on the two exact sequences (1) and (2) we deduce the following result:

Corollary 7.10. *Let $i \in \{1, 2, \ldots, e\}$. The centralizer $C_{V_{nX_i}}(\alpha_i)$ is isomorphic to $\ker(\mathcal{S}_i) \rtimes \mathbb{Z}$ and $\ker(\mathcal{S}_i)$ is a finite group.*

The following is a fairly technical statement about roots which does not particularly assist us in our exploration of the centralizer of α, but which deserves to be stated. It represents a viewpoint on the underlying reason for the corollary on p. 68 of [14]. Of course, a separate argument can also be given simply by noting how the lengths of spines of complementary components for representative revealing pairs change when one takes an element to powers.

Corollary 7.11. *Suppose that $\alpha \in V_n$ is a non-torsion element with precisely one flow graph component Υ which has component support Y. Then the set of roots of α in V_{nY} is finite.*

Proof. Consider the exact sequence from Lemma 7.8, applied to $C_{V_{nY}}(\alpha)$. If $\alpha \in \omega^{-1}(k)$ for some positive integer k, then all of the roots of α in V_{nY} occur in the sets $\omega^{-1}(j)$ for integers j which divide k. Thus the roots all occur in a finite collection of finite sets. □

We note in passing that it may be the case that for all $\gamma \in \omega^{-1}(1)$ we have that $\gamma^k = \tau_\gamma \cdot \alpha$, where τ_γ is a non-trivial torsion element for each such γ.

We now analyze the kernel of our $\mathcal{S}_i$ homomorphism a bit further.

Lemma 7.12. *Let $i \in \{1, 2, \ldots, e\}$. The kernel* $\ker(\mathcal{S}_i) = \{g \in C_{V_{nX_i}}(\alpha_i) \mid g^k = \mathrm{id}$ *for some $k \in \mathbb{Z}\}$, that is, the kernel of $\mathcal{S}_i$ is the set of torsion elements in $C_{V_{nX_i}}(\alpha_i)$.*

Proof. Let T be the set of torsion elements in $C_{V_{nX_i}}(\alpha_i)$. A priori, this may not be a subgroup.

We observe that $T \subseteq \ker(\mathcal{S}_i)$, because the slope of $g \in T$ multiplies to one across the full cycle of every periodic orbit of neutral leaves of g, and the orbits of the repelling periodic orbits of α_i are carried to each other by the action of any $\langle g \rangle$ for any $g \in C_{V_{nX_i}}(\alpha_i)$. So, for $g \in T$, we have by the definition of $\mathcal{S}_i$ that $\mathcal{S}_i(g) = 0$.

By the previous corollary, $\ker(\mathcal{S})$ is a finite group, hence $\ker(\mathcal{S}) \subseteq T$. □

At this juncture, we have pushed our analysis of the centralizer of an element of V_n with a discrete train track (or flow graph) with one component, which represents a non-torsion component, as far as necessary for us to be able to support the structure described in the right-hand product of our Theorem 1.1.

Proof of the right-hand product structure of Theorem 1.1. We now partition the non-torsion flow graph components $\{\Gamma_i\}$ by the rule $\Gamma_i \sim \Gamma_j$ if there exists $f \in C_{V_n}(\alpha)$ such that $(X_i)f = X_j$. In this case we note that $\alpha_i^f = \alpha_j$ and therefore $C_{V_{X_i}}(\alpha_i) \cong C_{V_{X_j}}(\alpha_j)$. In particular, we have that $\ker(\mathcal{S}_i) \cong \ker(\mathcal{S}_j)$.

Let us call such supports (X_* which can be carried to each other by an $f \in C_{V_n}(\alpha)$) *supports of isomorphic connected components*. Let $\{\mathrm{ICC}_1, \mathrm{ICC}_2, \ldots, \mathrm{ICC}_t\}$ be the set of $\sim$-equivalence classes of isomorphic (flow graph) connected components (each of which is denoted by ICC_j, for some j), where q_j is the cardinality of the set ICC_j.

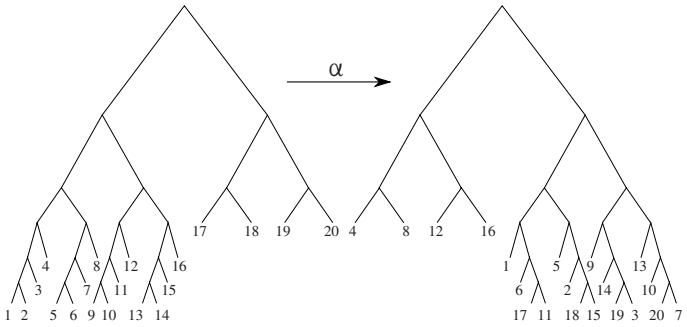
For each j, order the members of ICC_j and let $\Gamma_{i,j}$ be the i-th member of ICC_j. We re-index the X_i, α_i, $\mathcal{S}_i$ using the new double-index notation in corresponding fashion. Let us fix conjugators $\gamma_{1,k,j}$ in $C_{V_n}(\alpha)$ carrying $X_{1,j}$ to $X_{k,j}$ (acting as the identity elsewhere), and use these to generate a full permutation group P_{q_j} which acts on the underlying sets $X_{*,j}$ of the elements in ICC_j. Set $A_j := \ker(\mathcal{S}_{1,j})$ and note that $\ker(\mathcal{S}_{i,j}) = A_j^{\gamma_{1,i,j}}$.

For fixed j, we then have the centralizer of α over the set $\bigcup_i X_{i,j}$ consists of any self centralization on each individual $X_{i,j}$ (this group will be congruent to $B_{i,j} := \ker(\mathcal{S}_{i,j}) \rtimes \mathbb{Z}$), together with a product by any permutation of these components. Thus, this group is

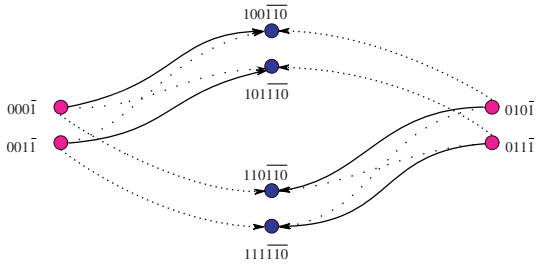
$$G_j \cong \Big(\prod_{i=1}^{q_j} B_{i,j} \Big) \rtimes P_{q_j} = (A_j \rtimes \mathbb{Z}) \wr P_{q_j}.$$

(Note that the product $\prod_{i=1}^{q_j} B_{i,j}$ is normal in G_j.) Now as non-isomorphic non-torsion components cannot be mapped onto each other by the action of f, we see that the action of f on the non-torsion components of α is describable as an element from the direct product of the individual groups G_j. Since t is the number of isomorphism classes $\{\mathrm{ICC}_j\}$, we obtain our statement of Theorem 1.1. □

Below is a revealing pair (A, B, σ) representing element α of V_2, which has centralizer congruent to $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{Z}$; the group corresponding to A_1 is the Klein 4-group. This example is included to answer a question of Nathan Barker which arose in conversation.

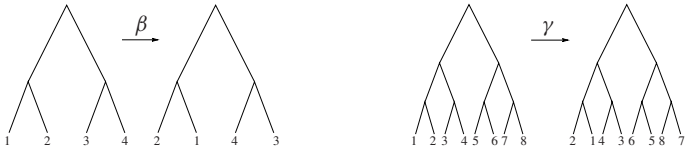


Below, we sketch the flow graph associated with (A, B, σ). We do not label the flow lines with their iterated augmentation chains, as that will only serve to clutter the essential aspects of this graph. The three different types of flow lines for this revealing pair are encoded by the different methods used in drawing the directed edges (solid, versus two distinct flavors of dashing).



In this graph, the repelling fixed points label the four vertices to the left and right, while the attracting fixed points are the four vertices in the middle.

Graph connectivity and the constraint on preserving flow-line types provide enough information to guarantee that if τ is a non-trivial torsion element commuting with α, then τ is completely determined by which repelling fixed point one chooses to send $000\overline{1}$ to. All such elements are order two, and the elements β and γ below generate the Klein 4-group K so determined by the orbit dynamics on the repelling fixed points. Thus, $A_1 \leq K$.



Since β and γ can be realized as a group of permutations of the leaves of the common tree $A \cap B$ which are in the repeller and attractor iterated augmentation chains, while preserving the orbit dynamics of $\langle\alpha\rangle$ over $\mathcal{RA}_\alpha$, we see that every element in K commutes with α. Note that as α has no roots, the generator of the $\mathbb{Z}$ factor is α, so $A_1 \rtimes \mathbb{Z} \cong A_1 \times \mathbb{Z}$, in this example.

8. Cyclic subgroup (non)distortion

We assume the reader is familiar with distortion of subgroups in a group. We use definitions consistent with their usage in [10].

The calculation of the (non)distortion of the cyclic subgroups in V_n is very close is spirit to the calculations in [7], [24], [8]. However, by using properties of revealing pairs, we remove many of the technical obstructions usually associated with counting carets and this shortens the arguments in those papers.

We believe that technology such as the revealing pair technology should generally simplify proofs of non-distortion of cyclic subgroups in the various families of generalized Thompson's groups following arguments similar to that which is given below. It would be interesting to see a general such tool developed for these families of groups.

We are indebted to M. Kassabov for a discussion of the spirit of this type of argument.

Proof of Theorem 1.3. Suppose that $\alpha \in V_n$ and that $\langle\alpha\rangle \cong \mathbb{Z}$. Suppose further that $X = \{x_1, x_2, \ldots, x_q\}$ is a finite generating set of V_n which is closed under inverses.

There is a minimal positive integer P_X so that for all $p \in \mathfrak{C}_n$, the slope of each x_j in small neighborhoods of p is $(2n-1)^{s_p}$ where $|s_p| < P_X$.

Suppose that $(A, B, \sigma) \sim \alpha$, and let r be a repeller for this tree pair. Suppose further that the iterated augmentation chain of r is $r = r_0$, $r_i = r_0\alpha^i$ for $0 \le i \le u$ (so that r_u is the root of the complementary component C_r of $A - B$ containing the repeller r). Let Γ_r be the spine of C_r, and suppose the length of Γ_r is L_r.

For each index $0 < i < u$, there is a jump $J_{(r,i)}$ in depth in the infinite binary tree, from the depth of r_i to r_{i+1}. Further, set $J_{(r,0)}$ to be the jump from the depth of r_u to the depth of r_1. (E.g., if r_u has depth 4, and r_1 has depth 3, then we set $J_{(r,0)}$ to be 1.) Given a positive integer z, set

$$S_{(r,z)} = \Sigma_{e=1}^{(z \bmod u)} J_{(r,(e-1))},$$

the partial sum of the first $z \bmod u$ jumps. (This sum may be negative.) Note that the sum of all u jumps is zero, so that

$$\Sigma_{e=1}^{z} J_{(r,\,(e-1) \bmod u)} = S_{(r,z)}.$$

Now fix a particular positive integer z. Set $w = \lfloor z/u \rfloor$, the largest positive integer less than or equal to z/u. Now, if y_i is the repelling periodic point under the leaf r_i for $0 \le i < u$, direct calculation shows that the slope of α^z at y_0 is

$$((2n-1)^{L_r})^{w+1} \cdot (2n-1)^{S_{(r,z)}}.$$

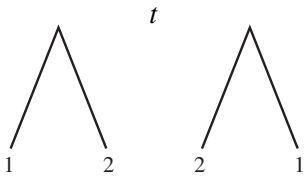
By the chain rule, we require at least $\lceil ((L_r \cdot w + L_r + S_{(r,z)})/P_X \rceil$ generators from X to create the element α^z (where $\lceil\cdot\rceil$ denotes the largest positive integer greater than or equal to its argument). This last function can be interpreted as a function in z that is bounded below by an affine function $g\colon \mathbb{N} \to \mathbb{Q}$ where $g(z) = (1/W) \cdot z + O$ with positive slope $1/W < (L_r/(u \cdot P_X))$ for some integer W and vertical offset $O \in \mathbb{Z}$ (for technical reasons, choose O so that $g(0) < 0$). Now the function $f_{\text{pos}} = g^{-1}$ (restricted and co-restricted to $\mathbb{N}$) is an affine distortion function for the positive powers of α in $\langle\alpha\rangle$ within V_n (α^z has minimal length z when expressed as an element in $\langle\alpha\rangle$ using the generating set $\{\alpha, \alpha^{-1}\}$, and by construction, $z < f_{\text{pos}}(m)$, where m is the minimal word length of α^z as expressed in X).

A similar argument produces an affine distortion function $f_{\text{neg}}\colon \mathbb{N} \to \mathbb{N}$ for the negative powers of α. Thus, $f := f_{\text{pos}} + f_{\text{neg}}$ will be an affine distortion function for the whole of $\langle\alpha\rangle$ in V_n. In particular, $\langle\alpha\rangle$ is undistorted in V_n. □

Appendix: An example of the finite generation of the centralizer of a pure torsion element

In this appendix we explore the generation of the centralizer of a torsion element of V, for a particular basic element, following the proof of Lemma 6.2 that the element centralizer $C_V(t)$ of a torsion element $t \in V$ is finitely generated.

We work with the most basic element t of order two as in the diagram below.

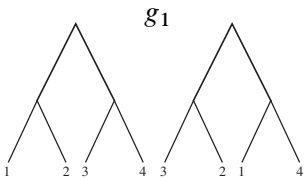


By Theorem 1.1, $C_V(t) \cong \mathrm{Maps}(\mathfrak{C}, \mathbb{Z}_2) \rtimes V$. This follows since there is only one orbit length, which is of length two. In our case, we have only one orbit to be concerned with, namely, the node "0" being carried to the node "1" and back again by the action of $\langle t \rangle$. Following the proof of Lemma 6.1 we have that we can choose F_2 to be the node "0", so that E_2 is the Cantor set underlying the node "0", namely,

$$E_2 = \{0x_1x_2x_3\cdots \in \mathfrak{C} = \{0,1\}^\omega\}.$$

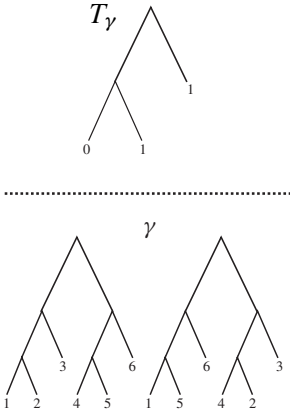
Now, $\widehat{Q}_2$ consists precisely of the elements of V which preserve the set E_2 as a set, and which have the same action on E_2 as they have (respectively) on the Cantor set underlying the node "1" It is immediate that this group is isomorphic to V.

The claim is that $\widehat{Q}_2$, together with a single particular non-trivial element g_1 of $\mathrm{Maps}(\mathfrak{C}, \mathbb{Z}_2)$, is sufficient to generate $C_V(t)$. The definition of g_1 is carefully worded. In our case, we follow the definition given in the proof of Lemma 6.2, and take $\lambda = \lambda_0$ to be the node "0", so that λ_1 is the node "1". Then g_1 takes the leftmost child-leaf of λ_0 and sends it to the leftmost child-leaf of λ_1, and also sends the leftmost child-leaf of λ_1 one position forward in the orbit cycle generated by t, which in this case closes the cycle by sending the leftmost child-leaf of λ_1 to the leftmost child-leaf of λ_0. In particular, the element g_1 of V is given in the diagram below.

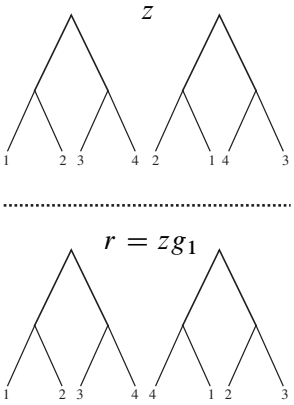


The claim is that $C_V(t) = \langle g_1, \widehat{Q}_2 \rangle$. What needs to be shown is that any map in $\mathrm{Maps}(\mathfrak{C}, \mathbb{Z}_2)$ can be built using these elements. So an element of $\mathrm{Maps}(\mathfrak{C}, \mathbb{Z}_2)$ is characterized by taking a finite tree and labeling its leaves with elements of $\mathbb{Z}_2$; each

number representing how far forward in the cycle generated by t we wish to send the corresponding leaves – all the corresponding descendent leaves of all the nodes in the orbit of F_2. An example diagram below takes such a tree T_γ, and gives the relevant element $\gamma \in C_V(t)$ characterizing the corresponding map in the centralizer of t.



We observe that we build this element by putting the tree T_γ at each node in the orbit of F_2, and then translating the appropriate leaves of these trees forward in the orbit under t by the amount listed on the leaves of T_γ. Note that the example element γ above can be built as the product of two conjugates of g_1, where the conjugating elements live in $\widehat{Q}_2$. In particular, let α be an element in $\widehat{Q}_2$ affinely sending the node"00" to the node "001" (and thus, simultaneously, α must also send "10" to "101", since α is in $\widehat{Q}_1$) and let β be an element of $\widehat{Q}_1$ sending the node "00" to the node "01", (and thus, also sending "10" to "11"). Then we have $\gamma = g_1^\alpha \cdot g_1^\beta$. In particular, we can build any appropriate map in Maps$(\mathbb{C}, \mathbb{Z}_2)$ corresponding fashion. Having such maps, together with the elements of $\widehat{Q}_2$, we can build anything in $C_V(t)$. Here is a particular example of building a square root r of t using only $\widehat{Q}_2$ and g_1; we find r as the result of following an element of $\widehat{Q}_2$ with an application of g_1 (in this example, $r^2 = t$).



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