

Hyperbolic hydra

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Abstract. We give examples of hyperbolic groups with finite rank free subgroups of huge (Ackermannian) distortion.

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1. Introduction

1.1. Our result. Hyperbolic groups are algorithmically tractable (their word and conjugacy problems are straightforward) and are characterised by a tree-like property that geodesic triangles in their Cayley graphs are close to tripods [5], [8]. The purpose of this article is to show that nonetheless some harbour extreme wildness within their subgroups – their finite rank free subgroups, even. We prove (the terminology is explained below):

Theorem 1.1. *There are hyperbolic groups Γ_k for all $k \geq 1$ with free rank $(k + 18)$ subgroups Λ_k whose distortion satisfies $\text{Dist}_{\Lambda_k}^{\Gamma_k} \geq A_k$, that is, grows at least like the k -th of Ackermann’s functions.*

A distortion function Dist_H^G measures the degree to which a subgroup $H \leq G$ folds in on itself within G by comparing the intrinsic word metric on H with the extrinsic word metric inherited from G . Suppose that S and T are finite generating sets for G and H , respectively. Then

$$\text{Dist}_H^G(n) := \max\{d_T(1, g) \mid g \in H \text{ with } d_S(1, g) \leq n\}.$$

Up to the following equivalence, capturing qualitative agreement of growth rates, Dist_H^G does not depend on S and T . For $f, g: \mathbb{N} \rightarrow \mathbb{N}$, we write $f \leq g$ when there

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exists $C > 0$ such that $f(n) \leq Cg(Cn + C) + Cn + C$ for all n . Define $f \simeq g$ when $f \leq g$ and $g \leq f$.

Ackermann’s $A_k: \mathbb{N} \rightarrow \mathbb{N}$ are a family of fast growing functions defined recursively:

$$A_0(n) = n + 2 \quad \text{for } n \geq 0,$$

$$A_k(0) = \begin{cases} 0 & \text{for } k = 1, \\ 1 & \text{for } k \geq 2, \end{cases}$$

and

$$A_{k+1}(n + 1) = A_k(A_{k+1}(n)) \quad \text{for } k, n \geq 0.$$

In particular, $A_1(n) = 2n$, $A_2(n) = 2^n$ and $A_3(n)$ is the n -fold iterated power of 2. They are representatives of the successive levels of the Grzegorzcyk hierarchy of primitive recursive functions – see, for example, [15].

1.2. The organisation of this article and an outline of our approach. Our groups Γ_k are elaborations of the *hydra groups*

$$G_k = \langle a_1, \dots, a_k, t \mid t^{-1}a_1t = a_1, t^{-1}a_it = a_ia_{i-1} \text{ (for all } i > 1) \rangle$$

explored by the second and third authors in [6]. These G_k are CAT(0), free-by-cyclic, biautomatic, and can be presented with only one relator, and yet the subgroups $H_k := \langle a_1t, \dots, a_k t \rangle$ are free of rank k and their distortion grows like the k -th of Ackermann’s functions: $\text{Dist}_{H_k}^{G_k} \simeq A_k$.

This extreme distortion stems from a phenomenon which can be described as a re-imagining of Hercules’ battle with the Lernaean Hydra. A *hydra* is a positive word w on the alphabet a_1, a_2, \dots . Hercules removes the first letter and then the creature regenerates in that each remaining a_i with $i > 1$ becomes a_ia_{i-1} . (Each remaining a_1 is unaffected.) This repeats and Hercules triumphs when the hydra is reduced to the empty word ε . The number of steps is denoted $\mathcal{H}(w)$. (Each step encompasses the removal of the first letter and then regeneration.) For example, $\mathcal{H}(a_2^3) = 7$:

$$a_2^3 \rightarrow (a_2a_1)^2 \rightarrow a_1a_2a_1^2 \rightarrow a_2a_1^3 \rightarrow a_1^3 \rightarrow a_1^2 \rightarrow a_1 \rightarrow \varepsilon.$$

In [6] it is shown that Hercules will be victorious whatever hydra he faces, but the number of strikes it takes can be huge: the functions \mathcal{H}_k , defined by $\mathcal{H}_k(n) = \mathcal{H}(a_k^n)$, grow like Ackermann’s functions: $\mathcal{H}_k \simeq A_k$.

The group G_k is not hyperbolic because it has the subgroup $\langle a_1, t \rangle \cong \mathbb{Z}^2$. We obtain Γ_k by combining G_k with another free-by-cyclic group, which is hyperbolic, in such a way that the hydra phenomenon persists in Γ_k , but the troublesome “Euclidean” relations $t^{-1}a_1t = a_1$ are replaced by something “hyperbolic”.

In Section 2 we will give two presentations P_k and Q_k for Γ_k and will prove they are equivalent. P_k is well suited to proving hyperbolicity: the associated Cayley

2-complex, suitably metrized, will be shown in Section 3 to be $\text{CAT}(0)$ and to contain no isometrically embedded copies of \mathbb{R}^2 and so is hyperbolic by the Flat Plane Theorem. Q_k places Γ_k in a class of free-by-cyclic groups which we show in Section 4 (for $k \geq 2$) contain free subgroups of rank $k + 18$ and distortion $\geq A_k$. (In the case $k = 1$, Theorem 1.1 is elementary: take Γ_1 to be a free group and Λ_1 to be Γ_1 .)

1.3. Background. Other heavily distorted free subgroups of hyperbolic groups have been exhibited by Mitra [12]: for all k , he gives an example with a free subgroup of distortion like a k -fold iterated exponential function and, more extreme, an example where the number of iterations grows like $\log n$. Barnard, the first author and Dani developed Mitra's constructions into more explicit examples that are also $\text{CAT}(-1)$ [3]. We are not aware of any example of a hyperbolic group with a finite rank free subgroup of distortion exceeding that of our examples. Indeed, we do not know of a hyperbolic group with a *finitely presented* subgroup of greater distortion. The Rips construction, applied to a finitely presentable group with unsolvable word problem yields a hyperbolic (in fact, $C'(1/6)$ small cancellation) group G with a finitely generated subgroup N such that Dist_N^G is not bounded above by any recursive function, but these N are not finitely presentable. (See [1], §3.4, [7], Corollary 8.2, [9], §3, $3.K_3''$, and [14].)

Whilst we will not call on it in this paper (as we will give the translation between the presentations P_k and Q_k explicitly), a result that lies behind how we came to our examples is that if a 2-complex admits an S^1 -valued Morse function all of whose ascending and descending links are trees, then its fundamental group is free-by-cyclic [2]. (The ascending link for our examples is visible in Figure 2 as the subgraph made up of all edges connecting pairs of negative vertices. The descending is that made up of all edges connecting pairs of positive vertices. Both are trees.)

1.4. Towards an upper bound on distortion. It seems likely that $\text{Dist}_{\Lambda_k}^{\Gamma_k} \simeq A_k$, but we do not offer a proof that $\text{Dist}_{\Lambda_k}^{\Gamma_k} \leq A_k$. The proof that $\text{Dist}_{H_k}^{G_k} \leq A_k$ in [6] may guide a proof that $\text{Dist}_{\Lambda_k}^{\Gamma_k} \leq A_k$, but that proof is technical and how to carry it over to $\text{Dist}_{\Lambda_k}^{\Gamma_k}$ is not readily apparent. We are content to present here just the lower bound, which we believe is the more significant.

1.5. Height and quasiconvexity. A finitely generated subgroup H of a finitely generated group G is *quasiconvex* when $\text{Dist}_H^G(n) \leq Cn$ for some constant C . An infinite subgroup H of a group G has *infinite height* when, for all n , there exist g_1, \dots, g_n such that $\bigcap_{i=1}^n g_i^{-1} H g_i$ is infinite and $H g_i \neq H g_j$ for all $i \neq j$.

As $\Lambda_k \leq \Gamma_k$, for $k \geq 2$, are new examples of non-quasiconvex finitely presented subgroups of hyperbolic groups, they are test cases for the question attributed to Swarup in [13]: if a finitely presented subgroup H of a hyperbolic group G has *finite height*, is H quasiconvex in G ? (We thank Ilya Kapovich for drawing our attention to this.)

Our $\Lambda_k \leq \Gamma_k$ do not resolve Swarup’s question as they have infinite height for all $k \geq 1$. We explain this using the notation of Section 4. It follows from Proposition 4.8 that $t^i \in \Lambda_k$ if and only if $i = 0$. So $\Lambda_k t^i \neq \Lambda_k t^j$ for all $i \neq j$. And $\bigcap_{i=1}^\infty t^{-i} \Lambda_k t^i$ is infinite since the rank l free group $\langle b_1, \dots, b_l \rangle$ is a subgroup of $t^{-i} \Lambda_k t^i$ for all i .

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2. Our examples

2.1. A CAT(0) presentation for Γ_k . This presentation P_k is well suited to establishing hyperbolicity (see Section 3):

generators: $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_8, \gamma_1, \dots, \gamma_8, \sigma, \tau,$

$$\alpha_i^{-1} \tau \alpha_i = \alpha_{i-1} \quad (1 < i \leq k),$$

relations: $\beta_i^{-1} \tau \beta_i = \beta_{i+1} \quad (1 \leq i \leq 7), \quad \beta_8 \tau \beta_8^{-1} = \beta_1,$

$$\gamma_i^{-1} \sigma \gamma_i = \gamma_{i+1} \quad (1 \leq i \leq 7), \quad \gamma_8 \sigma \gamma_8^{-1} = \gamma_1,$$

$$\gamma_3 \beta_5 = \beta_3 \gamma_5, \quad \alpha_1 \gamma_1 \tau = \tau \gamma_7 \alpha_1.$$

It is convenient to encode P_k as shown in Figure 1 (which displays the case $k = 6$). Each edge in the three labelled oriented trees (LOTs, see [10]) encodes a commutator relation, an edge labelled y from a vertex labelled x to a vertex labelled z corresponds to a relation $y^{-1}xy = z$. The square and hexagonal 2-cells represent the remaining two relations, $\gamma_3\beta_5 = \beta_3\gamma_5$ and $\alpha_1\gamma_1\tau = \tau\gamma_7\alpha_1$.

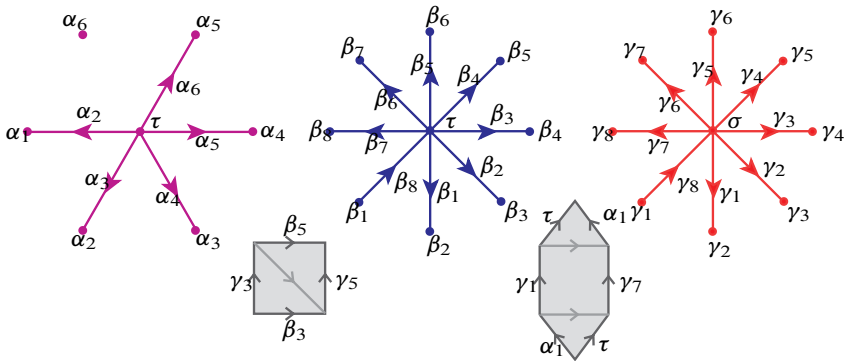


Figure 1. The defining relations of the presentation P_6 for Γ_6 displayed as three LOTs and two 2-cells.

If one removes the α_i and all the relations in which they appear from P_k , then one essentially gets groups studied by Meham and Muckerjee in [11]. These, in turn, are built from two copies of groups studied by Barnard and the first author in [2].

2.2. A presentation of Γ_k as a free-by-cyclic group. This presentation Q_k has

$$\begin{aligned} \text{generators: } & a_0, \dots, a_k, b_1, \dots, b_8, c_1, \dots, c_8, d, t, \\ \text{relations: } & t^{-1}a_it = \theta(a_i), \quad 0 \leq i \leq k, \\ & t^{-1}b_it = \phi(b_i), \quad 1 \leq i \leq 8, \\ & t^{-1}c_it = d\psi(c_5)\psi(c_i)\psi(c_5)^{-1}d^{-1}, \quad 1 \leq i \leq 8, \\ & t^{-1}dt = \phi^2(b_5)^{-1}d\psi(c_5c_3^{-1})\phi(b_3), \end{aligned}$$

where θ, ϕ and ψ are defined by

$$\begin{aligned} \theta(a_i) &= \begin{cases} ua_1v, & i = 0, \\ a_0, & i = 1, \\ a_ia_{i-1}, & 1 < i \leq k, \end{cases} \\ \phi(b_i) &= (b_i \dots b_7)b_1^{-1}b_8 \quad (1 \leq i \leq 8), \\ \psi(c_i) &= (c_i \dots c_8)c_1^{-1}c_8 \quad (1 \leq i \leq 8), \end{aligned}$$

and

$$u = t^{-k}c_7^{-1}td\psi(c_5)t^{k-1}, \quad v = t^{-(k-1)}\psi(c_5)^{-1}d^{-1}t^{-1}c_1t^k.$$

Lemma 2.1. Q_k presents a free-by-cyclic group

$$F(a_0, a_1, \dots, a_k, b_1, \dots, b_8, c_1, \dots, c_8, d) \rtimes \mathbb{Z},$$

where the \mathbb{Z} -factor is $\langle t \rangle$ and t acts as an automorphism.

Proof. First note:

- (i) u and v represent elements of the subgroup $\langle b_1, \dots, b_8, c_1, \dots, c_8, d \rangle$, and
- (ii) ϕ and ψ define automorphisms of $F(b_1, \dots, b_8)$ and $F(c_1, \dots, c_8)$, respectively, as would θ for $F(a_0, \dots, a_k)$ were $\theta(a_0)$ equal to a_1 rather than ua_1v .

The action of t by conjugation on

$$F(a_0, a_1, \dots, a_k, b_1, \dots, b_8, c_1, \dots, c_8, d)$$

apparent in the presentation Q_k is an automorphism because, as we will explain, the

following is a sequence of free bases:

$$\begin{aligned}
 &(a_0, a_1, \dots, a_k, b_1, \dots, b_8, c_1, \dots, c_8, d) \\
 &\xrightarrow{(1)} (a_1, t^{-1}a_1t, \dots, t^{-1}a_kt, b_1, \dots, b_8, c_1, \dots, c_8, d) \\
 &\xrightarrow{(2)} (t^{-1}a_0t, t^{-1}a_1t, \dots, t^{-1}a_kt, b_1, \dots, b_8, c_1, \dots, c_8, d) \\
 &\xrightarrow{(3)} (t^{-1}a_0t, t^{-1}a_1t, \dots, t^{-1}a_kt, t^{-1}b_1t, \dots, t^{-1}b_8t, \psi(c_1), \dots, \psi(c_8), d) \\
 &\xrightarrow{(4)} (t^{-1}a_0t, t^{-1}a_1t, \dots, t^{-1}a_kt, t^{-1}b_1t, \dots, t^{-1}b_8t, \psi(c_1), \dots \\
 &\qquad \qquad \qquad \dots, \psi(c_8), d\psi(c_5c_3^{-1})) \\
 &\xrightarrow{(5)} (t^{-1}a_0t, t^{-1}a_1t, \dots, t^{-1}a_kt, t^{-1}b_1t, \dots, t^{-1}b_8t, t^{-1}c_1t, \dots \\
 &\qquad \qquad \qquad \dots, t^{-1}c_8t, d\psi(c_5c_3^{-1})) \\
 &\xrightarrow{(6)} (t^{-1}a_0t, t^{-1}a_1t, \dots, t^{-1}a_kt, t^{-1}b_1t, \dots, t^{-1}b_8t, t^{-1}c_1t, \dots, t^{-1}c_8t, t^{-1}dt).
 \end{aligned}$$

This is because (1) $a_1, t^{-1}a_1t, \dots, t^{-1}a_kt$ is a free basis for $F(a_0, \dots, a_k)$ as per (ii) above; (2) $t^{-1}a_0t = ua_1v$, which is equivalent via transvections to a_1 by (i); (3) follows from (ii); (4) is via transvections; (5) conjugation by $\psi(c_5)^{-1}d^{-1} = \psi(c_5)^{-1}\psi(c_5c_3^{-1})\psi(c_5c_3^{-1})^{-1}d^{-1}$ is first conjugation by $\psi(c_5)^{-1}\psi(c_5c_3^{-1})$, which is an automorphism of $F(c_1, \dots, c_8)$, and then by $\psi(c_5c_3^{-1})^{-1}d^{-1}$; and (6) is via transvections as $t^{-1}b_1t, \dots, t^{-1}b_8t$ are a free basis for $F(b_1, \dots, b_8)$ and $\phi^2(b_5)^{-1}, \phi(b^3) \in F(b_1, \dots, b_8)$. □

The subgroup Λ_k of Theorem 1.1 will be

$$\langle a_0t, \dots, a_kt, b_1, \dots, b_8, c_1, \dots, c_8, d \rangle.$$

2.3. The equivalence of the presentations. We will prove:

Proposition 2.2. P_k and Q_k present the same groups.

As a first step we establish:

Lemma 2.3. Mapping $\tau \mapsto t^{-1}$ and $\beta_i \mapsto t^{-1}b_i$ for $1 \leq i \leq 8$ defines an isomorphism

$$\begin{aligned}
 &\langle \beta_1, \dots, \beta_8, \tau \mid \beta_i^{-1}\tau\beta_i = \beta_{i+1} \ (1 \leq i \leq 7), \ \beta_8\tau\beta_8^{-1} = \beta_1 \rangle \\
 &\rightarrow F(b_1, \dots, b_8) \rtimes_{\phi} \mathbb{Z} = \langle b_1, \dots, b_8, t \mid t^{-1}b_it = \phi(b_i) \ (1 \leq i \leq 8) \rangle.
 \end{aligned}$$

Proof. The given map translates the relations $\beta_i^{-1}\tau\beta_i = \beta_{i+1}$ ($1 \leq i \leq 7$) and $\beta_8\tau\beta_8^{-1} = \beta_1$ to the family

$$t^{-1}b_it = b_it^{-1}b_{i+1}t \ (1 \leq i \leq 7), \quad t^{-1}b_8t = b_1^{-1}b_8,$$

which is equivalent to $t^{-1}b_it = \phi(b_i)$ ($1 \leq i \leq 8$). □

Let P'_k and Q'_k be the presentation obtained from P_k and Q_k by removing all the generators α_i and a_i , respectively, and all the relations in which they occur.

Lemma 2.4. *The groups presented by P'_k and Q'_k are isomorphic via*

$$\begin{aligned}\tau &\mapsto t^{-1}, & \beta_i &\mapsto t^{-1}b_i \quad (1 \leq i \leq 8), \\ \sigma &\mapsto s^{-1}, & \gamma_i &\mapsto s^{-1}c_i \quad (1 \leq i \leq 8),\end{aligned}$$

where $s = td\psi(c_5)$.

Proof. As per Lemma 2.3, translate $\beta_1, \dots, \beta_8, \tau$ and associated relations to b_1, \dots, b_8, t and $\gamma_1, \dots, \gamma_8, \sigma$ and associated relations to c_1, \dots, c_8, s .

The given map converts the relation $\gamma_3\beta_5 = \beta_3\gamma_5$ to

$$s^{-1}c_3t^{-1}b_5 = t^{-1}b_3s^{-1}c_5.$$

This rearranges as

$$t^{-1}b_5c^{-5}sb_3^{-1}t = c_3^{-1}s$$

and then as

$$(t^{-1}b_5t)t^{-1}s(s^{-1}c_5^{-1}s)t(t^{-1}b_3^{-1}t) = s(s^{-1}c_3^{-1}s),$$

which is equivalent to

$$\phi(b_5)t^{-1}s\psi(c_5)^{-1}t\phi(b_3)^{-1} = s\psi(c_3)^{-1},$$

and so to

$$t^{-1}s\psi(c_5)^{-1}t = t^{-1}\phi(b_5)^{-1}s\psi(c_3)^{-1}\phi(b_3).$$

So, as $s = td\psi(c_5)$,

$$t^{-1}dt = t^{-1}\phi(b_5)^{-1}td\psi(c_5)\psi(c_3)^{-1}\phi(b_3),$$

which gives

$$t^{-1}dt = \phi^2(b_5)^{-1}d\psi(c_5c_3^{-1})\phi(b_3)$$

as per Q'_k . Next, as $s = td\psi(c_5)$, the relation $s^{-1}c_i s = \psi(c_i)$ is equivalent to

$$t^{-1}c_i t = d\psi(c_5)\psi(c_i)\psi(c_5)^{-1}d^{-1}$$

as per Q'_k . □

Inductively define words u_i and v_i for $i \geq 0$ by

$$\begin{aligned}u_0 &= \alpha_k, & u_{i+1} &= u_i^{-1}t^{-1}u_i \quad (i \geq 0), \\ v_0 &= a_k, & v_{i+1} &= v_i^{-1}t^{-1}v_i t \quad (i \geq 0).\end{aligned}$$

The following observation from [6] can be proved by inducting on i .

Lemma 2.5. *On substituting an a_k for each α_k in u_i , the words u_i and $t^{i-1}v_it^{-i}$ become freely equal for all $i \geq 1$.*

Proof of Proposition 2.2. By Lemma 2.4 there is a sequence of Tietze moves carrying the subpresentation P'_k of P_k to Q'_k and the remaining relations (those involving the α_i) to

$$\alpha_1 s^{-1} c_1 t^{-1} = t^{-1} s^{-1} c_7 \alpha_1, \quad \alpha_i^{-1} t^{-1} \alpha_i = \alpha_{i-1}, \quad 1 < i \leq k.$$

A sequence of Tietze moves eliminating $\alpha_1, \dots, \alpha_{k-1}$ transforms this family to the single relation

$$u_{k-1} s^{-1} c_1 t^{-1} = t^{-1} s^{-1} c_7 u_{k-1}.$$

Now substitute an a_k for each α_k . Then, by Lemma 2.5, this relation is equivalent to

$$(t^{k-2} v_{k-1} t^{-(k-1)}) s^{-1} c_1 t^{-1} = t^{-1} s^{-1} c_7 (t^{k-2} v_{k-1} t^{-(k-1)}),$$

which becomes

$$t^{-1} v_{k-1} t = (t^{-(k-1)} c_7^{-1} s t^{k-1}) v_{k-1} (t^{-(k-1)} s^{-1} c_1 t^{k-1})$$

on conjugating by t^{k-1} and rearranging. A sequence of Tietze moves introducing a_{k-1}, \dots, a_1 expands this to the family

$$t^{-1} a_1 t = t^{-(k-1)} c_7^{-1} s t^{k-1} a_1 t^{-(k-1)} s^{-1} c_1 t^{k-1}, \quad t^{-1} a_i t = a_i a_{i-1}, \quad 1 < i \leq k.$$

The first of these relations becomes $t^{-1} a_1 t = a_0$ when we introduce a_0 together with the new relation

$$a_0 = t^{-(k-1)} c_7^{-1} s t^{k-1} a_1 t^{-(k-1)} s^{-1} c_1 t^{k-1},$$

which becomes $t^{-1} a_0 t = u a_1 v$ on conjugating by t and eliminating the s and s^{-1} using $s = t d \psi(c_5)$. □

3. Hyperbolicity

We establish hyperbolicity using techniques employed in [2] and [11].

Consider the presentation 2-complex K_k for P_k assembled from Euclidean unit squares associated to each of the defining relations with the single exception of $\alpha_1 \gamma_1 \tau = \tau \gamma_7 \alpha_1$ for which we use a Euclidean hexagon made from one unit square and two equilateral triangles as shown in Figure 1.

The link in the case $k = 6$ is shown in Figure 2. All edges have length $\pi/2$ apart from the edges $\tau^+ - \alpha_1^+$ and $\tau^- - \alpha_1^-$ (shown in grey), which have length $\pi/3$, and the edges from $\gamma_1^+ - \tau^-$, $\gamma_7^+ - \alpha_1^-$, $\alpha_1^+ - \gamma_1^-$, and $\tau^+ - \gamma_7^+$ (shown in green), which have length $5\pi/6$. Inspecting the link we see that any simple loop in the graph has length at

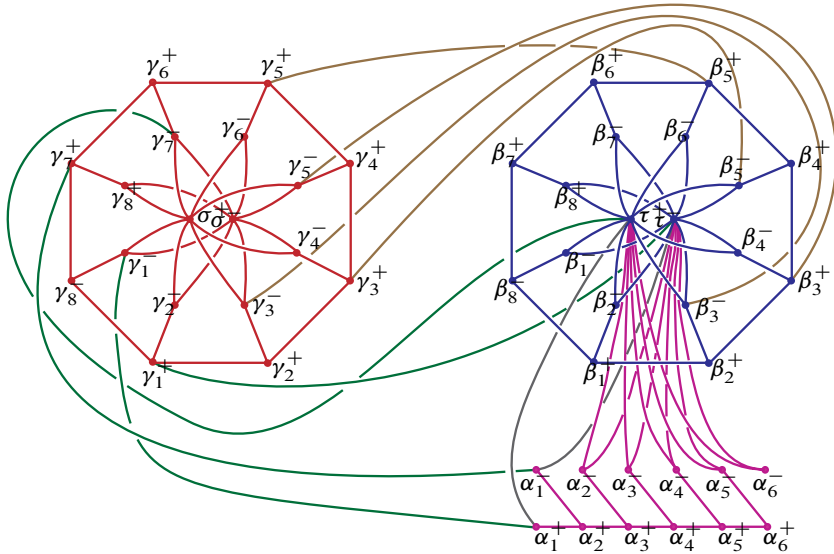


Figure 2. The link of the vertex in the presentation 2-complex associated to the presentation P_6 for Γ_6 given in Section 2. The two grey edges have length $\pi/3$, the four green edges have length $5\pi/6$, and all other edges have length $\pi/2$.

least 2π (separately considering the cases of monochrome and multicoloured simple loops in Figure 1 helps to check this) – that is, for all $k \geq 1$ the link is *large*. So K_k satisfies the *link condition* (see [5]) and its universal cover \widetilde{K}_k is therefore a CAT(0) space.

To establish that Γ_k is hyperbolic we will show that \widetilde{K}_k contains no subspace isometric to \mathbb{E}^2 and then appeal to the Flat Plane Theorem of [4], [8]. The link of a vertex in any isometric copy of \mathbb{E}^2 in \widetilde{K}_k would appear as a simple loop of length 2π in the link. But inspecting the link, we find that no edges of length $\pi/3$ or $5\pi/6$ (the grey and green edges) occur in a simple loop of length 2π . Next one can check the edges $\gamma_3^+ - \beta_5^-$, $\gamma_3^- - \beta_3^-$, $\gamma_5^- - \beta_3^+$ and $\gamma_5^+ - \beta_5^+$ (the brown edges in the figure) do not occur in a simple loop of length 2π . Then it becomes evident that edges occurring in simple loops of length 2π are precisely the edges

$$\begin{aligned} &\tau^+ - \alpha_i^-, \quad \tau^- - \alpha_i^- \quad (2 \leq i \leq k), \\ &\tau^+ - \beta_i^-, \quad \tau^- - \beta_i^- \quad (1 \leq i \leq 7), \quad \tau^+ - \beta_8^+, \quad \tau^- - \beta_8^+, \\ &\sigma^+ - \gamma_i^-, \quad \sigma^- - \gamma_i^- \quad (1 \leq i \leq 7), \quad \sigma^+ - \gamma_8^+, \quad \sigma^- - \gamma_8^+. \end{aligned}$$

So every corner of every 2-cell in an isometrically embedded \mathbb{E}^2 must give rise to one of the edges in this list. But, looking at the defining relations, we see that no 2-cell in \widetilde{K}_k has this property. Therefore there are no such \mathbb{E}^2 , and so Γ_k is hyperbolic.

4. Freeness and distortion

4.1. A family of free-by-cyclic groups. Fix an integer $l \geq 1$, words u and v on b_1, \dots, b_l , and an automorphism ϕ of $F(b_1, \dots, b_l)$. Then, for $k \geq 1$, define

$$\Psi_k := F(a_0, \dots, a_k, b_1, \dots, b_l) \rtimes_{\theta} \mathbb{Z},$$

where θ is the automorphism of $F(a_0, \dots, a_k, b_1, \dots, b_l)$ the restriction of which to $F(b_1, \dots, b_l)$ is ϕ and

$$\theta(a_i) = \begin{cases} ua_1v, & i = 0, \\ a_0, & i = 1, \\ a_i a_{i-1}, & 1 < i \leq k. \end{cases}$$

Let t denote a generator of the \mathbb{Z} -factor, so $t^{-1}a_it = \theta(a_i)$ and $t^{-1}b_jt = \theta(b_j)$ for all i and j .

The presentation Q_k in Section 2.2 shows Γ_k is an example of such a Ψ_k .

Our aim in the remainder of this section is to establish:

Proposition 4.1. *The subgroup*

$$\Lambda_k := \langle a_0t, \dots, a_k t, b_1, \dots, b_l \rangle$$

of Ψ_k is free of rank $k + l + 1$ and $\text{Dist}_{\Lambda_k}^{\Psi_k} \geq A_k$.

4.2. Towards a lower bound on distortion. In what follows, when, for a word $u = u(a_0, \dots, a_k, b_1, \dots, b_l)$, we refer to $\theta^r(u)$, we mean the freely reduced word that equals $\theta^r(u)$ in $F(a_0, \dots, a_k, b_1, \dots, b_l)$.

The extreme distortion in the hydra groups of [6] stemmed from the battle between Hercules and the hydra that we described in Section 1. When studying Ψ_k we will need the following more elaborate version of that battle. A *hydra* is now a word on

$$a_0, a_1, \dots, a_k, b_1, \dots, b_l$$

in which the a_i only appear with positive exponents. As before, Hercules fights a hydra by removing the first letter. But in this version, the hydra only regenerates after an a_i is removed, and that regeneration is: each remaining a_i and $b_j^{\pm 1}$ becomes $\theta(a_i)$ and $\theta(b_j^{\pm 1})$, respectively. Again, we consider Hercules victorious if, on sufficient repetition, the hydra is reduced to the empty word.

Reprising the example from Section 1, Hercules defeats a_2^3 as follows:

$$\begin{aligned} a_2^3 &\rightarrow (a_2a_1)^2 \rightarrow a_0a_2a_1a_0 \rightarrow a_2a_1a_0ua_1v \rightarrow a_0ua_1v\theta(u)a_0\theta(v) \\ &\rightarrow a_0\theta(v)\theta^2(u)ua_1v\theta^2(v) \rightarrow a_0\theta(v)\theta^3(v) \rightarrow \varepsilon. \end{aligned}$$

Here the steps in which Hercules removes a b_j are not shown; the arrows indicate the progression from when an a_i is about to be removed to when an a_i next appears at the front of the word or the hydra becomes the empty word.

The salient point is that a_0 and the b_j play no essential role in this battle; if we removed all b_j and replaced all a_0 by a_1 , we would have a battle of the original form. Thus we have the following lemma. (Recall that $\mathcal{H}(u)$ denotes the duration of the battle (of the original type from Section 1 and [6]) against the hydra u .)

Lemma 4.2. *Hercules wins against all hydra w and, in the battle, the number of times he removes an a_i equals $\mathcal{H}(\bar{w})$ where \bar{w} is the word obtained from w by removing all $b_j^{\pm 1}$ and replacing all a_0 by a_1 .*

Consideration of the original battle between Hercules and the hydra led to the result that, for all $k, n \geq 1$, there is a positive word $u_{k,n} = u_{k,n}(a_1t, \dots, a_kt)$ of length $\mathcal{H}_k(n)$ that equals $a_k^n t^{\mathcal{H}_k(n)}$ in G_k . (This is Lemma 5.1 in [6].) The reason is that the pairing off of a t with an initial a_i in a positive word on a_1, \dots, a_k corresponds to a decapitation, and the conjugation by t that moves that t into place from the right-hand end causes *regeneration* for the remainder of the word. For example $\mathcal{H}_2(3) = 7$ and

$$\begin{aligned} a_2^3 t^7 &= (a_2 t) t^{-1} a_2^2 t^6 \\ &= (a_2 t) (a_2 a_1)^2 t^6 \\ &= (a_2 t) (a_2 t) t^{-1} a_1 a_2 a_1 t^5 \\ &\vdots \\ &= (a_2 t) (a_2 t) (a_1 t) (a_2 t) (a_1 t) (a_1 t) (a_1 t) \\ &= u_{2,3}. \end{aligned}$$

In the corresponding calculation for Ψ_k , only the a_i get paired with t , and on each of the $\mathcal{H}_k(n)$ times that happens, the subsequent conjugation by t can increase length by a factor C which depends only on ϕ , $\ell(u)$ and $\ell(v)$. So:

Lemma 4.3. *There exists $C > 0$ such that for all $k, n \geq 1$ there is a word $\hat{u}_{k,n} = \hat{u}_{k,n}(a_0t, \dots, a_kt, b_1, \dots, b_l)$ that equals $a_k^n t^{\mathcal{H}_k(n)}$ in Ψ_k and has the properties that*

$$\mathcal{H}_k(n) \leq \ell(\hat{u}_{k,n}) \leq C^{\mathcal{H}_k(n)} n$$

and all the $(a_i t)$ it contains have positive exponents.

This and our next two lemmas will be components of a calculation that will yield Proposition 4.6 (the analogue of Proposition 5.2 in [6]), which will be the key to establishing a lower bound on the distortion of Λ_k in Ψ_k .

A simple calculation yields:

Lemma 4.4. $t^{-m}a_1t^{m+1} = \tau_m$ in Ψ_k for all $m \geq 1$ where

$$\tau_m := \begin{cases} a_0t & \text{for } m = 1, \\ \phi^{m-2}(u) \dots \phi^2(u)u(a_1t)\phi(v)\phi^3(v) \dots \phi^{m-1}(v) & \text{for even } m \geq 2, \\ \phi^{m-2}(u) \dots \phi^3(u)\phi(u)(a_0t)\phi^2(v)\phi^4(v) \dots \phi^{m-1}(v) & \text{for odd } m > 2. \end{cases}$$

This combines with

$$t^{-1}(a_1t^{-1})^n = (t^{-1}a_1t^2)(t^{-3}a_1t^4)(t^{-5}a_1t^6) \dots (t^{1-2n}a_1t^{2n})t^{-2n-1}$$

to give:

Lemma 4.5. *There is a constant $C > 0$, depending only on ϕ , $\ell(u)$ and $\ell(v)$, such that for all $n \geq 0$ there is a word $v_n = v_n(a_0t, b_1, \dots, b_l)$ such that $t^{-1}(a_1t^{-1})^n = v_n t^{-2n-1}$ in Ψ_k , the number of (a_0t) contained in v_n is n and all have positive exponent, and $n \leq \ell(v_n) \leq C^n$.*

Proposition 4.6. *For all $k \geq 2$ and $n \geq 1$, there is a reduced word of length at least $2\mathcal{H}_k(n) + 3$ on $a_0t, a_1t, \dots, a_kt, b_1, \dots, b_l$ that equals $a_k^n a_2 t a_1 a_2^{-1} a_k^{-n}$ in Ψ_k .*

Proof. After rewriting the relation $t^{-1}a_2t = a_2a_1$ as $a_2^{-1}ta_2 = ta_1^{-1}$, we see $a_2^{-1}t^{\mathcal{H}_k(n)}a_2 = (ta_1^{-1})^{\mathcal{H}_k(n)}$. So

$$a_k^n a_2 = \hat{u}_{k,n} a_2 (ta_1^{-1})^{-\mathcal{H}_k(n)}$$

for $\hat{u}_{k,n}$ as in Lemma 4.3. This gives the first of the equalities

$$\begin{aligned} a_k^n a_2 t a_1 a_2^{-1} a_k^{-n} &= \hat{u}_{k,n} a_2 (ta_1^{-1})^{-\mathcal{H}_k(n)} t a_1 (ta_1^{-1})^{\mathcal{H}_k(n)} a_2^{-1} \hat{u}_{k,n}^{-1} \\ &= \hat{u}_{k,n} (a_2 t) t^{-1} (ta_1^{-1})^{-\mathcal{H}_k(n)} t a_1 (ta_1^{-1})^{\mathcal{H}_k(n)} t (a_2 t)^{-1} \hat{u}_{k,n}^{-1} \\ &= \hat{u}_{k,n} (a_2 t) v_{\mathcal{H}_k(n)} t^{-2\mathcal{H}_k(n)-1} t a_1 t^{2\mathcal{H}_k(n)+1} v_{\mathcal{H}_k(n)}^{-1} (a_2 t)^{-1} \hat{u}_{k,n}^{-1} \\ &= \hat{u}_{k,n} (a_2 t) v_{\mathcal{H}_k(n)} \tau_{2\mathcal{H}_k(n)} v_{\mathcal{H}_k(n)}^{-1} (a_2 t)^{-1} \hat{u}_{k,n}^{-1}. \end{aligned}$$

The second is a free equality and the third and fourth are applications of Lemmas 4.5 and 4.4, respectively.

This calculation arrives at a word on $a_0t, a_1t, \dots, a_kt, b_1, \dots, b_l$ that equals $a_k^n a_2 t a_1 a_2^{-1} a_k^{-n}$ in Ψ_k . This word may not be freely reduced, but if we delete all the $b_j^{\pm 1}$ it contains, replace all $a_0^{\pm 1}$ by $a_1^{\pm 1}$, and then freely reduce (i.e., cancel away all $(a_i t)^{\pm 1} (a_i t)^{\mp 1}$ subwords), we get $u_{k,n}(a_2 t)(a_1 t)(a_2 t)^{-1} u_{k,n}^{-1}$, which has length $2\mathcal{H}_k(n) + 3$. □

4.3. Freeness and rank. The result of this section is:

Proposition 4.7. *The subgroup Λ_k is free of rank $k + l + 1$.*

It will be convenient to prove more. In the special case where w represents the identity, the following proposition tells us that there are no non-trivial relations between $a_0t, \dots, a_kt, b_1, \dots, b_l$ and so establishes Proposition 4.7.

Proposition 4.8. *If $w = w(a_0t, \dots, a_kt, b_1, \dots, b_l)$ represents an element of the subgroup $\langle t \rangle$ in Ψ_k , then w freely equals the empty word.*

We begin with an observation on how the groups Ψ_k nest.

Lemma 4.9. *For $1 \leq i \leq k$, the canonical homomorphism $\Psi_i \rightarrow \Psi_k$ is an inclusion.*

Proof. The free-by-cyclic normal forms – a reduced word on $a_0, \dots, a_k, b_1, \dots, b_l$ times a power of t – of an element of Ψ_i and its image in Ψ_k are the same. \square

We will prove Proposition 4.8 by induction, but first we give a corollary which will be useful in the induction step. We emphasise that when we say that $v(a_0t, \dots, a_kt, b_1, \dots, b_l)$ is *freely reduced* in the following we mean that there are no $(a_it)^{\pm 1}(a_it)^{\mp 1}$ or $b_j^{\pm 1}b_j^{\mp 1}$ subwords.

Corollary 4.10. *Suppose that $v(a_0t, \dots, a_kt, b_1, \dots, b_l)$ is a freely reduced word equalling $\hat{v}t^s$ in Ψ_k , where $s \in \mathbb{Z}$ and $\hat{v} = \hat{v}(a_0, \dots, a_k, b_1, \dots, b_l)$ is a word in which all the a_i that occur have positive exponents. Then all the (a_it) in v have positive exponents.*

Proof. When played out against $\hat{v}(a_0, \dots, a_k, b_1, \dots, b_l)$, the hydra battle described prior to Lemma 4.2 gives a word $v' = v'(a_0t, \dots, a_kt, b_1, \dots, b_l)$ and an integer s' such that $v' = \hat{v}t^{s'}$ in Ψ_k . Moreover, the exponents of all the (a_it) in v' are positive. Now $v^{-1}v' \in \langle t \rangle$ since $\hat{v} = vt^{-s} = v't^{-s'}$, and so v and v' are freely equal by Proposition 4.8. Therefore the exponents of all the (a_it) in v are positive. \square

Proof of Proposition 4.8. We induct on k . For the base case of $k = 1$, notice that defining $\bar{a}_0 := a_0t$ and $\bar{a}_1 := a_1t$, we can transform the presentation

$$\langle a_0, a_1, b_1, \dots, b_l, t \mid t^{-1}a_0t = ua_1v, t^{-1}a_1t = a_0, t^{-1}b_jt = \phi(b_j) \text{ for all } j \rangle$$

for Ψ_1 to

$$\langle \bar{a}_0, \bar{a}_1, b_1, \dots, b_l, t \mid t^{-1}\bar{a}_0t = u\bar{a}_1\phi(v), t^{-1}\bar{a}_1t = \bar{a}_0, t^{-1}b_jt = \phi(b_j) \text{ for all } j \rangle,$$

which is an alternative means of expressing Ψ_1 as a free-by-cyclic group from which the result is evident.

For the induction step, we consider a freely reduced word $w = w(a_0t, \dots, a_kt, b_1, \dots, b_l)$ representing an element of $\langle t \rangle$ in Ψ_k where $k \geq 2$. If no $(a_kt)^{\pm 1}$ are present in w we can deduce from the induction hypothesis and Lemma 4.9 that w

freely reduces to the empty word. For the remainder of our proof we suppose there are $(a_k t)^{\pm 1}$ present, and we seek a contradiction.

Consider shuffling the $t^{\pm 1}$ to the start of w using the defining relations – replacing each a_i and b_j passed by a $t^{\pm 1}$ with $\theta^{\pm 1}(a_i)$ and $\theta^{\pm 1}(b_j)$, respectively. The result will be a power of t times a word on $a_0, \dots, a_k, b_1, \dots, b_l$ which freely reduces to the empty word. Such is θ , no a_k are created or destroyed in this process of shuffling the $t^{\pm 1}$. So there is some expression $w_0(a_k t)^{\pm 1} u(a_k t)^{\mp 1} w_1$ for w such that $u = u(a_0 t, \dots, a_{k-1} t, b_1, \dots, b_l)$ and the $a_k^{\pm 1}$ and $a_k^{\mp 1}$ in the $(a_k t)^{\pm 1}$ and $(a_k t)^{\mp 1}$ buttressing u cancel after the shuffling and free reduction.

We will address first the case $w = w_0(a_k t)^{-1} u(a_k t) w_1$. Break down the shuffling process by first shuffling the $t^{\pm 1}$ out of w_0, u and w_1 , and then carrying the resulting powers to the front of the word:

$$\begin{aligned} w &= w_0(a_k t)^{-1} u(a_k t) w_1 \rightarrow t^{r_0} \hat{w}_0(a_k t)^{-1} t^r \hat{u}(a_k t) t^{r_1} \hat{w}_1 \\ &\rightarrow t^{r_0+r+r_1} \theta^{r+r_1}(\hat{w}_0) \theta^{r+r_1+1}(a_k^{-1}) \theta^{r_1+1}(\hat{u}) \theta^{r_1+1}(a_k) \hat{w}_1, \end{aligned}$$

where $r_0, r, r_1 \in \mathbb{Z}$ and

$$\begin{aligned} \hat{w}_0 &= \hat{w}_0(a_0, \dots, a_k, b_1, \dots, b_l), \\ \hat{u} &= \hat{u}(a_0, \dots, a_{k-1}, b_1, \dots, b_l), \\ \hat{w}_1 &= \hat{w}_1(a_0, \dots, a_k, b_1, \dots, b_l) \end{aligned}$$

are words such that $t^{r_0} \hat{w}_0 = w_0, t^r \hat{u} = u$ and $t^{r_1} \hat{w}_1 = w_1$ in Ψ_k . When we expand $\theta^{r+r_1}(a_k^{-1})$ and $\theta^{r_1+1}(a_k)$ as words on a_0, \dots, a_k , the former ends with an a_k^{-1} which must cancel with the a_k at the start of the latter. So $\theta^{r_1+1}(\hat{u})$, and therefore \hat{u} , freely equal the empty word. So u represents an element of $\langle t \rangle$ and, by induction hypothesis, freely reduces to the empty word, contrary to the initial assumption that $w(a_0 t, \dots, a_k t, b_1, \dots, b_l)$ is reduced.

In the case $w = w_0(a_k t) u(a_k t)^{-1} w_1$, the shuffling process is

$$\begin{aligned} w &= w_0(a_k t) u(a_k t)^{-1} w_1 \rightarrow t^{r_0} \hat{w}_0(a_k t) t^r \hat{u}(a_k t)^{-1} t^{r_1} \hat{w}_1 \\ &\rightarrow t^{r_0+r+r_1} \theta^{r+r_1}(\hat{w}_0) \theta^{r+r_1}(a_k) \theta^{r_1-1}(\hat{u}) \theta^{r_1}(a_k^{-1}) \hat{w}_1, \end{aligned}$$

where $t^{r_0} \hat{w}_0 = w_0, t^r \hat{u} = u$ and $t^{r_1} \hat{w}_1 = w_1$ in Ψ_k , as before. The first and last letters of $\theta^{r+r_1}(a_k) \theta^{r_1-1}(\hat{u}) \theta^{r_1}(a_k^{-1})$ are a_k and a_k^{-1} which cancel, so this subword must freely reduce to the empty word. So $\theta^r(a_k) \theta^{-1}(\hat{u}) a_k^{-1}$ also freely reduces to the empty word, that is, $\theta^{r+1}(a_k) \hat{u}$ freely equals $a_k a_{k-1}$.

If $r = 0$ then this says that \hat{u} freely equals the empty word and, as before, we have a contradiction. Suppose that $r > 0$. Then $\hat{u}^{-1} = (a_k a_{k-1})^{-1} \theta^{r+1}(a_k)$ would be a positive word on a_0, \dots, a_{k-1} were we to remove all the $b_l^{\pm 1}, \dots, b_1^{\pm 1}$ it contains. So, as $\hat{u}^{-1} t^{-r} = u^{-1}$, Corollary 4.10 applies and tells us that u^{-1} would be a positive word were we to remove all the $b_l^{\pm 1}, \dots, b_1^{\pm 1}$ it contains. But r is the exponent sum of the $(a_0 t)^{\pm 1}, \dots, (a_{k-1} t)^{\pm 1}$ in u , and so we deduce the contradiction $r \leq 0$. Finally we note that the case $r < 0$ also leads to a contradiction because if we replace w by w^{-1} it becomes the case $r > 0$. □

4.4. Conclusion. We deduce from Proposition 4.8 that the word posited in Proposition 4.6 is the *unique* reduced word on $a_0t, \dots, a_kt, b_1, \dots, b_l$ that equals $a_k^n a_2 t a_1 a_2^{-1} a_k^{-n}$ in Ψ_k . This establishes that $\text{Dist}_{\Lambda_k}^{\Psi_k} \geq \mathcal{H}_k$ for all $k \geq 2$. So, by Proposition 1.2 in [6], which says that $\mathcal{H}_k \simeq A_k$ for all $k \geq 1$, we have $\text{Dist}_{\Lambda_k}^{\Psi_k} \geq A_k$ for all $k \geq 2$. Added to Proposition 4.7, this completes the proof of Proposition 4.1.

Proposition 4.1 applies to the subgroup

$$\langle a_0t, \dots, a_kt, b_1, \dots, b_8, c_1, \dots, c_8, d \rangle$$

of Γ_k (presented as Q_k of Section 2) and so, as we established Γ_k to be hyperbolic in Section 3, Theorem 1.1 is proved.

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