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Geometric two-dimensional duality groups

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Abstract. We consider a finite, aspherical, 2-dimensional Cohen–Macaulay simplicial complex Δ and we find additional conditions that imply the universal cover $\widetilde{\Delta}$ has one end. In order to find these additional conditions we use a form of "Zeeman Duality". The context is an attempt to better understand duality groups.

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1. Introduction

1.1. The motivating question. Let Δ be a connected finite n-dimensional simplicial complex. If the link of each i-simplex is (PL homeomorphic to) the sphere S^{n-i-1} then the universal cover $\widetilde{\Delta}$ is an orientable PL n-manifold which therefore satisfies Poincaré duality: $H_c^i(\widetilde{\Delta}) \cong H_{n-i}(\widetilde{\Delta})$. The same holds if we only assume that the link of each i-simplex has the homology of S^{n-i-1}. In the case of interest in group theory, $\widetilde{\Delta}$ is assumed to be contractible, so that Δ is a finite $K(G,1)$-complex, where $G = \pi_1(\Delta, v)$. Then Poincaré duality implies $H_c^i(\widetilde{\Delta}) \cong 0$ when $i \neq n$ and $H_c^n(\widetilde{\Delta}) \cong \mathbb{Z}$. This group G is an example of an n-dimensional Poincaré duality group.

The motivation for this paper is the wish to provide an analogous set of statements for an n-dimensional *Cohen–Macaulay complex*, i.e., a connected finite n-dimensional simplicial complex Δ such that the link of each i-simplex has the homology of a non-trivial wedge of $(n-i-1)$-spheres. Again, in the case of interest in group theory, $\widetilde{\Delta}$ is assumed to be contractible, so that, again, Δ is a finite $K(G,1)$-complex. The question is: *What more must one know in order to deduce $H_c^i(\widetilde{\Delta}) \cong 0$ when $i \neq n$ and $H_c^n(\widetilde{\Delta})$ is torsion-free*? If the answer is positive then G is an n-dimensional duality group. For the definition of a duality group see [1] or [6]. (As the name implies, duality groups satisfy a generalization of Poincaré duality; we will have no need to make this duality explicit.)

Remark. (1) The hypothesis implies G is torsion-free and the requirement that $H_c^0(\widetilde{\Delta}) = 0$ is equivalent to G being infinite. From now on, we assume G is non-trivial.

(2) If an n-dimensional aspherical complex Δ satisfies $H_c^i(\widetilde{\Delta}) = 0$ when $i \neq n$, then it has to be the case that $H_c^n(\widetilde{\Delta}) \neq 0$ (see e.g. Theorem 13.10.1 of [6]), so that the geometric dimension of G is n. From now on we assume the geometric dimension of G is n.

Certainly, as we will see, further hypotheses are needed in order for G to be a duality group.

1.2. The 2-dimensional case. In this paper we only discuss the case $n = 2$. A connected finite 2-dimensional simplicial complex Δ is a 2-dimensional *Cohen–Macaulay complex* if the link of each vertex is non-empty and connected, and the link of each edge is non-empty. As before we write $G = \pi_1(\Delta, v)$ where v is a vertex. Since we are assuming $H_c^0(\widetilde{\Delta}) = 0$ and $H_c^2(\widetilde{\Delta}) \neq 0$, the geometric dimension of G must be 2. The question in Section 1.1 thus reduces to: *When is it true that* $H_c^1(\widetilde{\Delta}) = 0$? This is equivalent to asking: When does the 2-dimensional group G have one end? It is a theorem of Stallings that a finitely generated torsion-free group has one end if and only if it does not decompose as a non-trivial free product. But we are seeking something different: we are assuming that nothing is known about the group G, and we wish to understand when we can deduce from the Cohen–Macaulay property that $\widetilde{\Delta}$ has one end. (The strictly analogous statement for a closed 2-manifold of positive genus is that, by Poincaré duality, its universal cover always has one end; the fact that one knows the homeomorphism type is a result of geometry rather than of algebraic topology.)

Our strategy is to first consider when a general contractible locally finite infinite 2-dimensional Cohen–Macaulay complex X has one end. We will also require that the link of each vertex of X is 2-connected graph, i.e., it is a connected graph and remains connected when the open star of any vertex is deleted. The main result of this paper is Theorem 5.6. This theorem looks complicated and has many hypotheses, but in the case where the complex X is the universal cover of a connected finite 2-dimensional complex Δ it reduces to the following:

Theorem 1.1. *Let Δ be a connected finite 2-dimensional aspherical Cohen–Macaulay complex such that the link of every vertex is 2-connected (in the sense of graph theory) and Δ does not consist of only one 2-simplex. Let S_0 be the singular set of Δ (i.e., the full subgraph generated by the edges which are faces of more than two 2-simplexes). If each component of the boundary of a regular neighborhood of S_0 π_1-injects into Δ, then $\widetilde{\Delta}$ has one end. In particular, $\pi_1(\Delta, v)$ is a 2-dimensional duality group.*

That our task is non-trivial is shown by the following example:

Example 1.2 (Bestvina). Let Δ consist of two triangulated tori joined along a common edge. This complex is Cohen–Macaulay. The fundamental group of Δ is $\mathbb{Z}^2 * \mathbb{Z}^2$

which is not a duality group ($\widetilde{\Delta}$ is not one-ended). The links of the vertices of the common edge are two-petal roses, and the links of all other vertices are circles. Hence Δ is Cohen–Macaulay. The links of the vertices of the common edge are not 2-connected (if we remove the common vertex in the two-petal rose together with the edges that contain that vertex, the remaining part of the link is disconnected). The singular set S_0 in this complex consists only of one edge and neither component of $\partial N(S)$ π_1-injects into Δ.

Example 1.3. Let Δ consist of two triangulated tori joined along a common 2-simplex. This complex is Cohen–Macaulay, and $\pi_1(\Delta) \cong \mathbb{Z}^2 * \mathbb{Z}^2$, which is not one-ended. The link of each vertex is either a circle or the letter θ. Hence the link of each vertex of Δ is 2-connected. However, neither component of $\partial N(S)$ π_1-injects into Δ.

As a consequence of Theorem 1.1, we have Corollary 1.4 which provides necessary and sufficient conditions for $\widetilde{\Delta}$ to have one end. Let the singular set S_0 in Δ have components $S_1, S_2, \dots, S_{n_1}$. We write $N = \bigsqcup_{i=1}^{n_1} N_i$ where the N_i ($i = 1, 2, \dots, n_1$) are pairwise disjoint regular neighborhoods of $S_1, S_2, \dots, S_{n_1}$ respectively. The boundary of N in Δ consists of circles C_j ($j = 1, 2, \dots, n_2$). The closure in Δ of each component of $\Delta - N$ is a compact surface whose boundary consists of some of the circles C_j (each circle C_j occurs exactly once as a boundary of one surface). We denote these surfaces by M_k ($k = 1, 2, \dots, n_3$).

Corollary 1.4. *Let Δ be a finite, 2-dimensional, aspherical, Cohen–Macaulay complex such that the link of each vertex of Δ is 2-connected and Δ does not consist of only one 2-simplex. Assume that no surface M_k is a disk. Then the following are equivalent:*

(i) *Each C_j π_1-injects into N.*

(ii) *$\widetilde{\Delta}$ has one end and semistable fundamental group at infinity.*

Remark. In [2], Brady, McCammond, and Meier used Morse theory to deal with the case when Δ is a finite, n-dimensional, non-positively curved complex. For any simplex σ, they defined a *punctured link of σ at p*, denoted by $\mathrm{pkl}(\sigma, p)$, where p is any point of $\mathrm{lk}\,\sigma$, to be the set obtained by removing from $\mathrm{lk}\,\sigma$ all points within $\frac{\pi}{2}$ of p. The punctured link of σ is not in general a subcomplex of $\mathrm{lk}\,\sigma$, but deformation retracts onto a maximal subcomplex of $\mathrm{lk}\,\sigma$ that is contained in $\mathrm{plk}(\sigma, p)$. One of the results in [2] is: *Let Δ be a finite, non-positively curved complex of dimension n. If for each cell σ in X and for each $p \in \mathrm{lk}\,\sigma$, the spaces $\mathrm{lk}\,\sigma$ and $\mathrm{plk}(\sigma, p)$ are $(n - \dim\sigma - 2)$-acyclic, then $\pi_1(\Delta)$ is a duality group.* (Corollary 1.4. in [2])

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2. Homology with coefficients in a local cohomology stack

The exposition in this section is based on the work of E. C. Zeeman [12] and reworked in more detail by F. J. Fernandez-Lasheras in [5].

Let X be an oriented, n-dimensional, locally finite, simplicial complex; this means that an orientation has been chosen for each simplex of X. Let R be a ring.

We find it convenient to follow [6] (Section 12.1), calculating simplicial cohomology from chains (rather than cochains) and coboundaries.

The number $[\omega_\beta : \tau_\alpha]$, called the *incidence number* of the (oriented) simplices ω_β and τ_α, is defined as follows: If $\dim \tau_\alpha = i$, $\dim \omega_\beta = i + 1$, $\omega_\beta \succ \tau_\alpha$, and $i \geq 1$, then $[\omega_\beta : \tau_\alpha] = 1$ if the orientation of τ_α inherited from ω_β agrees with the orientation of τ_α. Otherwise $[\omega_\beta : \tau_\alpha] = -1$. If τ_α is not a simplex of ω_β, then $[\omega_\beta : \tau_\alpha] = 0$. This definition does not make sense for the case of edges and vertices. We will make the following convention: if e is an (oriented) edge and v is a vertex of e, then $[e : v] = 1$ if v is the terminal vertex of e, and $[e : v] = -1$ if v is the initial vertex of e; if v is not a vertex of e, then $[e : v] = 0$.

Let $R(\tau)$ denote the free left R-module generated by the (oriented) simplex $\tau \in X$. Let $C_q^\infty(X; R) = \prod_{\dim \tau = q} R(\tau)$. Elements of $C_q^\infty(X; R)$ are denoted by $\sum_{\dim(\tau_\alpha)=q} m_\alpha \tau_\alpha$ where $m_\alpha \in R$, and they are called *infinite simplicial q-chains* (or *locally finite q-chains*) in X with coefficients in R. The coboundary homomorphism for infinite chains is $\partial\colon C_q^\infty(X; R) \to C_{q+1}^\infty(X; R)$ defined by

$$\delta\Big(\sum_\alpha m_\alpha \tau_\alpha\Big) = \sum_\beta \Big(\sum_\alpha m_\alpha [\omega_\beta : \tau_\alpha]\Big)\omega_\beta$$

where $\dim \tau_\alpha = q$ and $\dim \omega_\beta = q + 1$.

For oriented simplicial pairs (X, A) let

$$C_q^\infty(X, A; R) := \big\{ \sum m_\alpha \tau_\alpha \in C_q^\infty(X; R) : m_\alpha = 0 \text{ whenever } \tau_\alpha \text{ is a } q\text{-simplex of } A \big\}.$$

The relative cohomology modules $H^*(X, A; R)$ are calculated from the cochain complex $(C_*^\infty(X, A; R), \delta)$. Of course $H^q(X, A; R) = Z^q(X, A; R)/B^q(X, A; R)$, where

$$Z^q(X, A; R) = \{c \in C_q^\infty(X; R) : \delta c = 0 \text{ and the coefficient in } c \text{ corresponding to each } q\text{-simplex of } A \text{ is } 0\}$$

and

$$B^q(X, A; R) = \{c \in C_q^\infty(X; R) : c = \delta d \quad \text{where } d \in C_{q-1}^\infty(X, A; R)\}.$$

This way of defining $H^*(X, A; R)$ can be found in [6] (Chapter 12).

The number of ends of X does not depend of the ring R (this follows from [6] Theorem 13.5.5). *Therefore, it is enough to work with $\mathbb{Z}_2$-coefficients and from now on $R = \mathbb{Z}_2$.*

By $\mathfrak{D}(X)$ we will denote the following category: the objects are the simplices $\sigma \in X$, and a morphism $\tau \to \sigma$ is the relation $\tau \succ \sigma$. The *star* of $\sigma \in X$ is the set $\mathrm{st}_X\, \sigma = \{\tau \in X : \tau \cup \sigma \in X\}$, the *link* of σ in X is the set $\mathrm{lk}_X\, \sigma = \{\tau \in \mathrm{st}_X\, \sigma : \tau \cap \sigma = \emptyset\}$, and the *open star* of σ in X is the set $\mathring{\mathrm{st}}_X\, \sigma = \{\tau \in X : \tau \succ \sigma\}$. We often omit the index X. Note that $\mathrm{lk}\, \sigma$ and $\mathrm{st}\, \sigma$ are subcomplexes of X, and $\mathring{\mathrm{st}}\, \sigma$ is not (in general).

Definition 2.1. The qth-*local cohomology stack* on X, $q \geq 0$ is the covariant functor $\mathfrak{L}^q : \mathfrak{D}(X) \to \mathcal{A}belian\,\mathcal{G}roups$ defined as follows:

$$\mathfrak{L}^q(\sigma) = H^q(X, X - \mathring{\mathrm{st}}(\sigma))$$

where σ is a simplex of X. If σ is a simplex of τ, i.e., if there is a morphism $\tau \to \sigma$, then $(\mathfrak{L}^q)^{\tau,\sigma} : \mathfrak{L}^q(\tau) \to \mathfrak{L}^q(\sigma)$ is induced by the inclusion $\sigma \subseteq \tau$.

Lemma 2.2. *If σ is a simplex of X then $H^q(X, X - \mathring{\mathrm{st}}\, \sigma) \cong \widetilde{H}^{q-\dim\sigma-1}(\mathrm{lk}\, \sigma)$.*

Proof. First we note that $(X - \mathring{\mathrm{st}}\, \sigma) \cap \mathrm{st}\, \sigma$ is homeomorphic to $\mathrm{bd}\sigma * \mathrm{lk}\, \sigma$, which is homeomorphic to $\sum^{\dim\sigma}(\mathrm{lk}\, \sigma)$, where $\sum^{\dim\sigma}$ stands for the $(\dim\sigma)^{\mathrm{th}}$-suspension of σ. Then, by excision, we have

$$H^q(X, X - \mathring{\mathrm{st}}\, \sigma)) \cong H^q(\mathrm{st}\, \sigma, (X - \mathring{\mathrm{st}}\, \sigma) \cap \mathrm{st}\, \sigma) \cong H^q\Big(\mathrm{st}\, \sigma, \sum^{\dim\sigma}(\mathrm{lk}\, \sigma)\Big).$$

Since $\mathrm{st}\, \sigma$ is contractible, we have

$$H^q(X, X - \mathring{\mathrm{st}}\, \sigma)) \cong H^{q-1}\Big(\sum^{\dim\sigma}(\mathrm{lk}\, \sigma)\Big) \cong \widetilde{H}^{q-\dim\sigma-1}(\mathrm{lk}\, \sigma). \qquad \square$$

Corollary 2.3. *If X is an n-dimensional Cohen–Macaulay complex then $\mathfrak{L}^q(\sigma) \cong 0$ for $q < n$ for any simplex $\sigma \in X$.*

Proof. If σ is a simplex of X, then $\widetilde{H}^{q-\dim\sigma-1}(\mathrm{lk}\, \sigma) \cong 0$ when $q - \dim\sigma - 1 < \dim \mathrm{lk}\, \sigma$. Since X is pure, i.e., all of its maximum faces have dimension $\dim X$, it follows that $\dim(\mathrm{lk}\sigma) = \dim X - \dim\sigma - 1$ ([4], Corollary 5.1.5, page 210), which implies $\widetilde{H}^{q-\dim\sigma-1}(\mathrm{lk}\, \sigma) \cong 0$ when $q < \dim \mathrm{lk}\, \sigma + \dim\sigma + 1 = n$. Therefore $\mathfrak{L}^q(\sigma) = H^q(X, X - \mathring{\mathrm{st}}\, \sigma) \cong 0$ if $q < n$. $\square$

We define a chain complex $J_*(X, \mathcal{L}^n)$ as follows:

Let

$$J_q(X, \mathcal{L}^n) := \bigoplus_{\dim \sigma = q} \mathcal{L}^n(\sigma), \quad q \geq 0.$$

Define $\partial_{\mathcal{L}} : J_q(X, \mathcal{L}^n) \to J_{q-1}(X, \mathcal{L}^n)$ as follows:

$$\partial_{\mathcal{L}}\Big(\sum_{i=1}^{m} a_i h_i\Big) = \sum_{i=1}^{m} \sum_{\tau_i \succ \sigma} a_i [\tau_i : \sigma] (\mathcal{L}^q)^{\tau_i, \sigma}(h_i)$$

where $h_i \in \mathcal{L}^q(\tau_i)$, $\sum_{i=1}^{m} a_i h_i \in J_q(X, \mathcal{L}^n)$, $\dim(\tau_i) = q$, $\dim(\sigma) = q - 1$. Then $(J_q(X, \mathcal{L}^n); \partial_{\mathcal{L}})$ is a chain complex. We denote its homology by $H_q(X, \mathcal{L}^n)$.

In the spirit of Zeeman [12], Lasheras [5] has proved the following generalization of Poincaré Duality:

Theorem 2.4. *Let X be an n-dimensional, locally finite, Cohen–Macaulay simplicial complex. Then for all $p \in \mathbb{Z}$*

$$H_p(X, \mathcal{L}^n) \cong H_c^{n-p}(X).$$

In Zeeman [12] there is a version of this theorem when X is finite complex.

3. About $H_*(X, \mathcal{L}^2)$

Let X be an oriented, 2-dimensional, path connected, locally finite Cohen–Macaulay complex. We will present an alternative way of viewing $H_*(X, \mathcal{L}^2)$.

Let K_2, K_1, and K_0 be the free abelian groups generated by all pairs (σ, σ), (σ, e), and (σ, v) respectively, where σ is a 2-simplex of X, e is an edge of σ and v is a vertex of σ. Then (K_*, ∂') is a chain complex, where the boundary is defined as follows:

$$\partial'(\sigma, \sigma) = (\sigma, \partial\sigma) := (\sigma, e_1) + (\sigma, e_2) + (\sigma, e_3),$$
$$\partial'(\sigma, e) = (\sigma, \partial e) := (\sigma, v_1) + (\sigma, v_2).$$

Here, ∂ is the boundary homomorphism for simplicial homology, $\partial\sigma = [\sigma : e_1]e_1 + [\sigma : e_2]e_2 + [\sigma : e_3]e_3$, and $\partial e = [e : v_1]v_1 + [e : v_2]v_2$.

We will use the following:

$$(\delta e, e) := \sum_{\sigma \succ e} [\sigma : e](\sigma, e), \quad (\delta e, v) := \sum_{\sigma \succ e} [\sigma : e](\sigma, v).$$

Here $(J_*, \partial_{\mathcal{L}})$ is the chain complex from Section 2 used to define $H_*(X, \mathcal{L}^2)$. If $[\omega] \in J_i = \bigoplus_{\dim \tau = i} H^2(X, X - \overset{\circ}{\mathrm{st}}\, \tau)$ we denote the τ-component by $[\omega]_\tau \in H^2(X, X - \overset{\circ}{\mathrm{st}}\, \tau)$ and all but finitely many $[\omega]_\tau$ are zero. If only the τ-component of $[\omega]$ is non-zero we abuse notation by writing $[\omega]_\tau \in J_i$.

First we will see how $\partial_{\mathcal{L}}$ works. Let $[\sigma]_\sigma \in J_2$ where σ is a 2-simplex. Using the definition for $\partial_{\mathcal{L}}$ we have

$$\partial_{\mathcal{L}}([\sigma]_\sigma) = \sum_{i=1}^{3} [\sigma : e_i][\sigma]_{e_i},$$

where e_1, e_2, and e_3 are the edges of σ.

Let $[\sigma]_e \in J_1$ where σ is a 2-simplex and e is an edge of σ. Then

$$\partial_{\mathcal{L}}([\sigma]_e) = [\sigma : v_1][\sigma]_{v_1} + [\sigma : v_2][\sigma]_{v_2},$$

where v_1, and v_2 are vertices of e.

The projection maps $\pi_i \colon K_i \to J_i$ are defined by $\pi_i(\sigma, \tau) = [\sigma]_\tau$, $i = 0, 1, 2$. Let $L_i := \ker \pi_i$. Then (L_*, ∂') is a chain complex.

We will consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & L_1 & \xrightarrow{\partial'} & L_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_2 & \xrightarrow{\partial'} & K_1 & \xrightarrow{\partial'} & K_0 & \longrightarrow & 0 \\
 & & \downarrow \pi_2 & & \downarrow \pi_1 & & \downarrow \pi_0 & & \\
0 & \longrightarrow & J_2 & \xrightarrow{\partial_{\mathcal{L}}} & J_1 & \xrightarrow{\partial_{\mathcal{L}}} & J_0 & \longrightarrow & 0
\end{array}$$

Figure 1

The columns are exact. The rows are chain complexes. We must show that the diagram commutes.

We will give a simpler description of the abelian groups L_1 and L_2.

Lemma 3.1. *The abelian groups L_0 and L_1 are generated by all pairs $(\delta e, v)$ and $(\delta e, e)$ respectively, where e is an edge of X and v is a vertex of e; and $L_2 \cong 0$.*

Proof. First we will show that $L_2 \cong 0$ by showing that $\ker \pi_2 = 0$.

Since $J_2 = \bigoplus_{\dim \sigma = 2} \mathcal{L}^2(\sigma) = \bigoplus_{\dim \sigma = 2} H^2(X, X - \mathring{\mathrm{st}}\,\sigma)$, for each 2-simplex σ we will calculate $H^2(X, X - \mathring{\mathrm{st}}\,\sigma)$ from the following cochain complex:

$$0 \to C_0^\infty(X, X - \mathring{\mathrm{st}}\,\sigma) \to C_1^\infty(X, X - \mathring{\mathrm{st}}\,\sigma) \to C_2^\infty(X, X - \mathring{\mathrm{st}}\,\sigma) \to 0;$$

$C_2^\infty(X, X - \mathring{\mathrm{st}}\,\sigma)$ is generated by σ, and $C_1^\infty(X, X - \mathring{\mathrm{st}}\,\sigma) = 0$. Hence $H^2(X, X - \mathring{\mathrm{st}}\,\sigma)$ is generated by $[\sigma]$, i.e., $H^2(X, X - \mathring{\mathrm{st}}\,\sigma) \cong \mathbb{Z}_2$. Let $x := \sum_{\sigma \in X} m_\sigma(\sigma, \sigma) \in$

K_2. Then $\pi_2(x) = \sum_{\sigma \in X} m_\sigma([\sigma]_\sigma)$ and $\pi_2(x) = 0$ if and only if $m_\sigma = 0$ for all 2-simplices σ. Hence, $\ker \pi_2 = 0$, and therefore $L_2 \cong 0$.

Next we will consider $J_1 = \bigoplus_{\dim e=1} \mathcal{L}^2(e) = \bigoplus_{\dim e=1} H^2(X, X - \overset{\circ}{\mathrm{st}}\, e)$. For each edge e, we will calculate $H^2(X, X - \overset{\circ}{\mathrm{st}}\, e)$ from the cochain complex

$$0 \to C_0^\infty(X, X - \overset{\circ}{\mathrm{st}}\, e) \to C_1^\infty(X, X - \overset{\circ}{\mathrm{st}}\, e) \to C_2^\infty(X, X - \overset{\circ}{\mathrm{st}}\, e) \to 0.$$

The abelian group $C_2^\infty(X, X - \overset{\circ}{\mathrm{st}}\, e)$ is generated by all 2-simplices σ such that e is an edge of σ. For each edge e, the free abelian group $C_1^\infty(X, X - \overset{\circ}{\mathrm{st}}\, e)$ is generated by e. Let $x = \sum m_{\sigma,e}(\sigma, e)$. Then $\pi_1(x) = \sum m_{\sigma,e}([\sigma]_e)$. For each edge e that appears in x, we will define x_e to be the sum of those pairs from x that have second coordinate e, i.e., $x_e := \sum_\sigma m_{\sigma,e}(\sigma, e)$. Then $\pi_1(x_e) = \sum_\sigma m_{\sigma,e}[\sigma]_e$. Hence $\pi_1(x_e) = [0]$ if and only if $\sum_\sigma m_{\sigma,e}[\sigma]_e = [0]$ (this summation is in $H^2(X, X - \overset{\circ}{\mathrm{st}}\, e)$). So $\sum_\sigma m_{\sigma,e}[\sigma]_e = [0]$ if and only if there is a constant m_e such that $\sum_{\sigma \succ e} m_{\sigma,e}\sigma = m_e \delta e$. Therefore, $x_e \in \ker \pi_1$ if and only if $x_e = \sum_{\sigma \succ e} m_{\sigma,e}(\sigma, e) = m_e(\delta e, e)$. Since $x = \sum_e x_e$, and $\pi_1(x) = \sum_e \pi_1(x_e)$, it follows that the elements of $\ker \pi_1$ are finite sums of pairs of type $(\delta e, e)$, i.e., L_1 is generated by all pairs $(\delta e, e)$ where e is an edge of X.

Similarly we can show that L_0 is generated by all pairs $(\delta e, v)$ where v is a vertex of X and e has v as a vertex. □

Using Lemma 3.1 it is straightforward to show that the diagram commutes. We omit the details.

Proposition 3.2. $H_1(K, \mathbb{Z}_2) \cong 0$.

Proof. Let $\beta = \sum_{\sigma \succ e} m_{(\sigma,e)}(\sigma, e) \in Z_1(K)$, i.e., $\partial'\beta = 0$ (only a finite number of coefficients $m_{\sigma,e}$ are 1). Let $m_{\sigma,e} = 1$ for some 2-simplex σ and some edge e of σ. Since $\partial'(\sigma, e) = (\sigma, v_1) + (\sigma, v_2)$ where v_1 and v_2 are vertices of e, we know that (σ, v_1) and (σ, v_2) are present (i.e., have non-zero coefficient) in $\partial'\beta$. We will denote by e_1 and e_2 the other two edges of σ.

The only way to cancel (σ, v_1) is if there is another term (σ, v_1) in $\partial'\beta$. We will get another term (σ, v_1) only if (σ, e_1) is present in β. Similarly, the only way to cancel (σ, v_2) is if (σ, e_2) is present in β. Hence, in β we have the following three terms (σ, e), (σ, e_1), and (σ, e_2) with coefficient 1. But $(\sigma, e) + (\sigma, e_1) + (\sigma, e_2) = \partial'(\sigma, \sigma)$. So $\beta \in B_1(K)$, i.e., each cycle in K_1 bounds. Therefore $H_1(K, \mathbb{Z}_2) \cong 0$. □

Theorem 3.3. *If X is an infinite, 2-dimensional, path-connected, Cohen–Macaulay complex then $H_1(X, \mathcal{L}^2_{\mathbb{Z}_2}) \cong \ker\{H_0(L, \mathbb{Z}_2) \to H_0(K, \mathbb{Z}_2)\}$.*

Proof. From the short exact sequence of chain complexes (Figure 1) we have the following long exact sequence:

$$\begin{aligned}0 \longrightarrow H_2(L,\mathbb{Z}_2) &\longrightarrow H_2(K,\mathbb{Z}_2) \longrightarrow H_2(J,\mathbb{Z}_2)\\ \longrightarrow H_1(L,\mathbb{Z}_2) &\longrightarrow H_1(K,\mathbb{Z}_2) \longrightarrow H_1(J,\mathbb{Z}_2)\\ \longrightarrow H_0(L,\mathbb{Z}_2) &\longrightarrow H_0(K,\mathbb{Z}_2) \longrightarrow H_0(J,\mathbb{Z}_2) \longrightarrow 0.\end{aligned}$$

By Proposition 3.2 we have $H_1(K,\mathbb{Z}_2) \cong 0$. Hence, we have the following long exact sequence

$$0 \longrightarrow H_1(J,\mathbb{Z}_2) \longrightarrow H_0(L,\mathbb{Z}_2) \longrightarrow H_0(K,\mathbb{Z}_2) \longrightarrow H_0(J,\mathbb{Z}_2) \longrightarrow 0.$$

Therefore $H_1(X,\mathcal{L}^2_{\mathbb{Z}_2}) = H_1(J,\mathbb{Z}_2) \cong \ker\{H_0(L,\mathbb{Z}_2) \to H_0(K,\mathbb{Z}_2)\}$. □

Because of Theorem 2.4 this gives a characterization of $H^1_c(X;\mathbb{Z}_2)$, i.e., the number of ends of X.

4. 2-connected links

We are assuming that X is an infinite, oriented, 2-dimensional, path connected Cohen–Macaulay complex. From now we also assume that X is acyclic with respect to $\mathbb{Z}_2$-coefficients, i.e., $\widetilde{H}_i(X;\mathbb{Z}_2) \cong 0$ for all integers i.

Our goal now is to determine what additional conditions on X are sufficient to ensure $H^1_c(X,\mathbb{Z}_2) \cong 0$, or, using Theorem 2.4, to ensure $H_1(X,\mathcal{L}^2_{\mathbb{Z}_2}) \cong 0$. This is equivalent to "X has one end".

Let Γ be a graph with at least three vertices and let w be a vertex of Γ. We say that a vertex $w \in \Gamma$ is *a cut vertex* of Γ if the removal of w and all edges containing w causes an increase in the number of connected components of Γ. A graph Γ that does not have a cut vertex is called a 2*-connected graph*. In Example 1.2, the links of the vertices v_1 and v_2 are not 2-connected and the links of all other vertices are 2-connected.

Lemma 4.1. *If the links of all vertices of X are* 2*-connected, then each edge of X is a face of at least two* 2*-simplices.*

Proof. Assume that there is an edge of X, denoted by e_1, that is a face of only one 2-simplex. We denote by σ the 2-simplex that has e_1 as an edge, we denote the vertices of σ by v_1, v_2, and v_3, and the other two edges by e_2 and e_3, where v_i is the vertex opposite e_i in σ. Then e_3 is the only edge in lk v_3 that has v_2 as a vertex, and v_1 is also a vertex of that edge. If e_2 is a face of at least two 2-simplices, then if we remove the vertex v_1 from lk v_3 together with the edges that have v_1 as a vertex, the remaining part of lk v_3 will not be connected; the vertex v_2 is not connected to the remaining part of the link. Hence v_1 is a cut vertex, but this is not possible because the link of each vertex of X is 2-connected (i.e., does not have any cut vertices).

Therefore e_2 is an edge of no other 2-simplex besides σ. Similarly we can conclude that e_3 is an edge of no other 2-simplex besides σ. Hence X consists of only one 2-simplex which is a contradiction since X is infinite and path connected. □

If X is a finite simplicial complex, from the proof of previous lemma we have the following corollary.

Corollary 4.2. *If X is a finite, path connected simplicial complex and the links of all vertices of X are 2-connected, then each edge of X is a face of at least two 2-simplices or X consists of only one 2-simplex.*

Now we will discuss the impact that the new condition "the link of each vertex of X is 2-connected" has on the structure of the complex X. We will consider once again the diagram in Figure 1. By Theorem 3.3 we are interested in $\ker\{H_0(L,\mathbb{Z}_2) \to H_0(K,\mathbb{Z}_2)\} \cong 0$.

We will find necessary and sufficient conditions (Theorem 4.5) for a chain α in K_0 to be in L_0.

Let $\alpha = \sum_{k=1}^{m} \sum_{\sigma_i \succ v_k} n_{ik}(\sigma_i, v_k)$. For each vertex v_k we define

$$\alpha_k := \sum_{\sigma_i \succ v_k} n_{ik}(\sigma_i, v_k).$$

It follows that $\alpha = \sum_k \alpha_k$.

Since X is 2-dimensional Cohen–Macaulay complex, the link of each vertex v_k is a connected graph. Let e be an arbitrary edge of $\mathrm{lk}\, v_k$. Let σ_i be the 2-simplex of $\mathring{\mathrm{st}}\, v_k$ that has e as a face (this 2-simplex is unique). We will relabel the edges from $\mathrm{lk}\, v_k$ in the following way: we will label the edge of σ_i opposite v_k by e_{ik}. Let

$$\bar{\alpha}_k := \sum_{\sigma_i \succ v_k} n_{ik} e_{ik}.$$

The difference between α_k and $\bar{\alpha}_k$ is that (σ_i, v_k) is replaced by e_{ik}.

Let v and w be two vertices in X. A *path* joining v and w is a sequence of distinct vertices (except that v may equal w) $\gamma : v = v_0, v_1, v_2, \ldots, v_{n-1}, v_n = w$ such that any two consecutive vertices are joined by an edge in X. If $v = w$, then the path is a *cycle*.

The proof of the following proposition can be found in [3] (Theorem 13.5.3).

Proposition 4.3. *The link of each vertex of X is 2-connected if and only if for each vertex $v \in X$, the link of v has the following property: for any two edges e and f of* $\mathrm{lk}\, v$ *there is a cycle in* $\mathrm{lk}\, v$ *that contains e and f.*

Lemma 4.4. *Let $\alpha = \sum_{k=1}^{m} \sum_{\sigma_i \succ v_k} n_{ik}(\sigma_i, v_k) \in K_0$. Then $\alpha \in L_0$ if and only if for each vertex v_k that appears in α, $\bar{\alpha}_k$ is a (1-dimensional) coboundary in* $\mathrm{lk}\, v_k$.

Proof. Suppose that $\alpha \in L_0$. Then

$$\alpha = \sum_{k=1}^{m} \sum_{e_j \succ v_k} n'_{jk}(\delta e_j, v_k)$$

where $n_{ik} = n'_{j_1 k} + n'_{j_2 k}$, and e_{j_1} and e_{j_2} are the edges of σ_i that have v_k as a vertex. For fixed v_k, $\bar{\alpha}_k = \sum_{k=1}^{m} \sum_{e_j \succ v_k} n'_{jk} \delta_{\mathrm{lk}\, v_k} v_{jk}$, where v_{jk} is the vertex of e_j different from v_k, and $\delta_{\mathrm{lk}\, v_k}$ denotes the coboundary in the cochain complex of lk v_k. Hence $\bar{\alpha}_k$ is a coboundary in lk v_k.

Suppose that $\bar{\alpha}_k$ is 1-dimensional coboundary in lk v_k for each vertex v_k that appears in α. Then, for fixed v_k, $\bar{\alpha}_k = \sum_{k=1}^{m} \sum_{e_j \succ v_k} n'_{jk} \delta_{\mathrm{lk}\, v_k} v_{jk}$. Hence, for fixed v_k, $\alpha_k = \sum_{k=1}^{m} \sum_{e \succ v_k} n'_{jk}(\delta e_j, v_k)$ which is in L_0. Since $\alpha = \sum_k \alpha_k$, it follows that $\alpha \in L_0$. □

Theorem 4.5. *Let the link of each vertex of X be 2-connected. Then $\alpha \in L_0$ if and only if for each v_k that appears in α and each cycle γ of* lk v_k *the sum of the coefficients of $\bar{\alpha}_k$ corresponding to the edges of γ is* 0.

Proof. Suppose that $\alpha \in L_0$, i.e., $\alpha = \sum_{k=1}^{m} \sum_{\sigma_i \succ v_k} n_{ik}(\sigma_i, v_k)$. Let v_k be a vertex appearing in α. By Lemma 4.4 , $\bar{\alpha}_k$ is a 1-dimensional coboundary in lk v_k. Hence $\bar{\alpha}_k = \sum_{k=1}^{m} \sum_{e_j \succ v_k} n'_{jk} \delta_{\mathrm{lk}\, v_k} v_{jk}$, where v_{jk} is the vertex of e_j different from v_k. Let γ be an arbitrary cycle of lk v_k. If a vertex v_{jk} that appears in $\bar{\alpha}_k$ lies on γ, then there are exactly two edges from γ that have v_{jk} as a vertex. Hence, the sum of the coefficients corresponding to those two edges in $\bar{\alpha}_k$ is 0. Therefore, for each vertex of γ that appears in $\bar{\alpha}_k$ we have exactly two edges of γ that have coefficient 1 in $\bar{\alpha}_k$. So the sum of the coefficients of $\bar{\alpha}_k$ corresponding to the edges of γ, and therefore on each cycle of lk v_k, is 0. This proves "only if".

Suppose now that, for each vertex v_k that appears in α, the sum of the coefficients from $\bar{\alpha}_k = \sum n_{ik} e_{ik}$ that correspond to the edges that lie on each cycle of lk v_k is 0. We will pick an edge from $\bar{\alpha}_k$ with non-zero coefficient, denoted by e_0 (the edges of X are already labeled; here we relabel them just for the purpose of this proof). Denote the vertices of e_0 by v_0 and v_1. Let $\gamma_0, \gamma_1, \dots, \gamma_s$ be all the cycles starting and ending at v_0 and containing e_0. The graph lk v_k is covered by these cycles, i.e., every edge of lk v_k lies on at least one cycle γ_i, $i = 0, 1, 2, \dots, s$ (this follows from Proposition 4.3). We will show that $\bar{\alpha}_k$ is a 1-dimensional coboundary in lk v_k by rewriting it using the cycles $\gamma_0, \gamma_1, \dots, \gamma_s$. We will start with the cycle γ_0 and we will write the terms from $\bar{\alpha}_k$ whose edges are on γ_0 as a coboundary in γ_0.

Let $e_0, e_1, e_2, \dots, e_{t(\gamma_0)}$ be the edges of γ_0 and $v_0, v_1, v_2, \dots, v_{t(\gamma_0)}$ be the vertices of γ_0 (as in Figure 2). If e_1 is present in $\bar{\alpha}_k$, then we can write $e_0 + e_1$ as $\delta_{\gamma_0} v_1$. If e_1 is not present in $\bar{\alpha}_k$, then we can add $e_1 + e_1$ to $\bar{\alpha}_k$ and we will have the previous case. Then we will consider the edge e_2, and "play the same trick" as with e_1, and so on. Since the sum of the coefficients of $\bar{\alpha}_k$ corresponding to γ_0 is 0, we can write

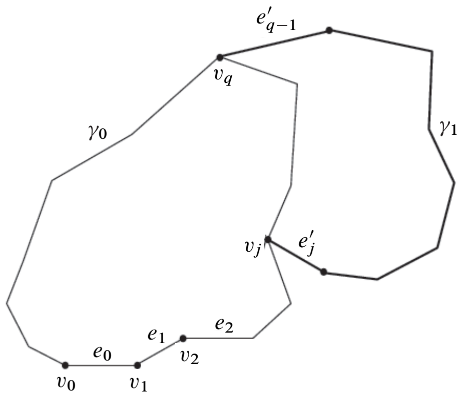

Figure 2

the terms of $\bar{\alpha}_k$ that correspond to edges of γ_0 as a coboundary of γ_0, i.e.,

$$\bar{\alpha}_k = \sum_{v \in \gamma_0} \delta_{\gamma_0} v + (\text{the rest of } \bar{\alpha}_k).$$

We will repeat the same process with γ_1 but our goal now is to write the terms of $\bar{\alpha}_k$ corresponding to γ_0 andγ_1 as a coboundary of $\gamma_0 \cup \gamma_1$. The cycle γ_1 has common edges with γ_0 (at least e_0 is a common edge). So we will start with the first term from $\bar{\alpha}_k$ that lies on γ_1 but not on γ_0. We will use the notation of Figure 2. If $\delta_{\gamma_0} v_j$ is not in $\bar{\alpha}_k$ then we will pick the first edge from γ_1 moving along γ_1 counterclockwise that has a non-zero coefficient in $\bar{\alpha}_k$ and using the same process as with the cycle γ_0 we will write $\bar{\alpha}_k = \sum_{v \in \gamma_0 \cup \gamma_1} \delta_{\gamma_0 \cup \gamma_1} v + (\text{the rest of } \bar{\alpha}_k)$. If $\delta_{\gamma_0} v_j$ is in $\bar{\alpha}_k$, then we have two possibilities: either e'_j is in $\bar{\alpha}_k$ or not. If e'_j is in $\bar{\alpha}_k$, then combining e'_j with $\delta_{\gamma_0} v_j$ we will get a term $\delta_{\gamma_0 \cup \gamma_1} v_j$ in $\bar{\alpha}_k$. If e'_j is not in $\bar{\alpha}_k$, then we will add $e'_j + e'_j$ to $\bar{\alpha}_k$ and get the case when e'_j is in $\bar{\alpha}_k$. We will continue "moving" along γ_1 and rewriting the terms of $\bar{\alpha}_k$. We need to consider a case when γ_1 meets γ_0 again. Let v_q be a vertex where γ_0 and γ_1 meet again. We must rule out the possibility that $\delta_{\gamma_0} v_q$ is in $\bar{\alpha}_k$ but e'_{q-1} is not (or vice versa). Consider the cycle that contains the parts from γ_0 and γ_1 between v_j and v_q. The sum of the coefficients of $\bar{\alpha}_k$ corresponding to this cycle is 0. Therefore $\delta_{\gamma_0} v_q$ is in $\bar{\alpha}_k$ if and only if e'_{q-1} is in $\bar{\alpha}_k$.

We repeat the process just described until we use all cycles of $\gamma_0, \gamma_1, \dots, \gamma_s$. Since each edge of $\operatorname{lk} v_k$ lies on some cycle, then we use all the edges of $\operatorname{lk} v_k$ with the algorithm described above. At the end we will find that $\bar{\alpha}_k$ is written as a coboundary, i.e., $\bar{\alpha}_k = \sum_{v_{jk} \in \operatorname{lk} v_k} n_{jk} \delta_{\operatorname{lk} v_k} v_{jk} \in L_0$. From Lemma 4.4 it follows that $\alpha \in L_0$. □

Let $\beta = \sum_{j=1}^{s} \sum_{\sigma_i \succ e_j} m_{ij}(\sigma_i, e_j) \in K_1$ and $\Gamma(\beta)$ be the graph spanned by all edges e that appear in β. The next theorem gives a geometric interpretation of the chains in K_1 whose boundaries are in L_0; this result will be used in proving the main result for the general case, Theorem 5.6. Again we refer to Figure 1.

Theorem 4.6. *Assume that the link of each vertex $v \in X$ is 2-connected, $\beta = \sum_{j=1}^{s} \sum_{\sigma_i \succ e_j} m_{ij}(\sigma_i, e_j) \in K_1$ does not contain terms of L_1 (terms of the form $(\delta e, e)$), and $\partial' \beta \in L_0$. Then the graph $\Gamma(\beta)$ is finite and has no vertices of valence* 1.

Proof. Suppose there is a vertex v_2 of valence 1 in $\Gamma(\beta)$. Let e be the edge of $\Gamma(\beta)$ that has v_2 as a vertex, and let v_1 be the other vertex of e (Figure 3). Let $\partial' \beta_{v_2} := \sum_{\sigma_i \succ v_2} n_{i2}(\sigma_i, v_2)$, i.e., $\partial' \beta_{v_2}$ is a sum only of those terms from $\partial' \beta$ that have v_2 as a second coordinate. Define $\overline{\partial' \beta}_{v_2} := \sum_{\sigma_i \succ v_2} n_{i2} e_{i2}$, where e_{i2} is the face of σ_i opposite v_2 (e_{i2} is an edge from lk v_2).

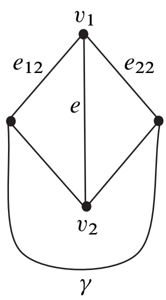

Figure 3

Since e is an edge of $\Gamma(\beta)$, there exists at least one term with non-zero coefficient in β whose second coordinate is e, and the first coordinate of that term is a 2-simplex that has e as an edge. Hence at least one edge of lk v_2 that has v_1 as a vertex has a non-zero coefficient in $\overline{\partial' \beta}_{v_2}$. Denote one of these edges by e_{12}. Since β does not contain any terms of form $(\delta e, e)$, this means that at least one edge of lk v_2 which has v_1 as a vertex is not present in $\overline{\partial' \beta}_{v_2}$, i.e., has a coefficient 0 in $\overline{\partial' \beta}_{v_2}$. Denote by e_{22} one of these edges. By Lemma 4.3, e_{12} and e_{22} lie on a cycle γ. Since the valence of v_2 in $\Gamma(\beta)$ is 1, all edges on γ other than e_{12} and e_{22} have coefficient 0 in $\overline{\partial' \beta}_{v_2}$, e_{22} has a coefficient 0 as well, and e_{12} has a coefficient 1. Therefore, the sum of the coefficients on γ is 1, which is not possible since $\partial' \beta \in L_0$ and, by Theorem 4.5, the sum of the coefficients on γ must be 0. Hence there exists at least one more edge in $\Gamma(\beta)$ that has v_2 as vertex different from e, i.e., the valence of v_2 is at least two. □

5. The Main Theorem

The following two examples suggest that we need further hypotheses on X.

Example 5.1. Let X be the simplicial complex obtained by triangulating $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\} \cup \{(x, y, z) \mid x^2 + y^2 \leq 1,\ z = 0\}$. This complex X is Cohen–Macaulay. The link of each vertex is either homeomorphic to either S^1 or the letter θ. Hence the link of each vertex is 2-connected. But $H^1_c(X, \mathbb{Z}_2) \ncong 0$, i.e., X is not one-ended.

Example 5.2. Let X be a simplicial complex obtained by gluing two triangulated planes along a line. This simplicial complex is Cohen–Macaulay and the link of each vertex is 2-connected. This complex is one-ended, i.e., $H^1_c(X, \mathbb{Z}_2) \cong 0$.

Recall that an edge e of X is singular if it is a face of more than two 2-simplices. By Lemma 4.1 the non-singular edges are those which are faces of exactly two 2-simplices. One of the differences between these two examples is that the simplicial complex in Example 5.1 has a cycle all of whose edges are singular, and the complex in Example 5.2 does not.

We will introduce a new hypothesis on X. We will assume that X is an infinite, 2-dimensional, acyclic, locally finite, path connected Cohen–Macaulay simplicial complex such that the link of each vertex is 2-connected and *each cycle has a non-singular edge.*

Our goal is to show that X has one end, i.e., $\ker\{H_0(L, \mathbb{Z}_2) \to H_0(K, \mathbb{Z}_2)\} \cong 0$, which by Theorem 3.3, is equivalent to $H_1(X, \mathcal{L}^2_{\mathbb{Z}_2}) \cong 0$. Let $i_*\colon H_0(L; \mathbb{Z}_2) \to H_0(K; \mathbb{Z}_2)$ be induced by the inclusion $i\colon L_0 \hookrightarrow K_0$. Let $[\alpha_0] \in H_0(L; \mathbb{Z}_2)$ and let $i_*([\alpha_0]) = [0]$ (Figure 1). This means that α_0 bounds in K, i.e., there exists a chain $\beta_0 \in K_1$ such that $\partial'\beta_0 = \alpha_0$. We want to show that α_0 bounds in L, i.e., $[\alpha_0] = [0]$. Therefore we need to show that there exists a chain $\beta \in L_1$ that is homologous to β_0 in K, i.e., $\partial'\beta_0 = \partial'\beta = \alpha_0$.

Let $\beta_0 = \sum_{j=1}^{s} \sum_{\sigma_i \succ e_j} m_{ij}(\sigma_i, e_j) \in K_1$. We may assume that β_0 is *reduced with respect to* L_1, i.e., $\beta_0 \neq \beta_0' + \beta_0''$ with $\beta_0'' \in L_1$ and each term in β_0' appears in β_0. In other words, no subset of terms in β_0 add up to an element of L_1. Since β_0 does not have any terms of form $(\delta e, e)$, Theorem 4.6 implies that the graph $\Gamma(\beta_0)$ consisting of all edges that appear in β_0 does not have vertices of valence 1.

Let γ_0 be a cycle in $\Gamma(\beta_0)$. There is a non-singular edge e_1 on γ_0. Let τ_0 be the unique 2-chain that bounds γ_0 (γ_0 is a 1-cycle in the sense of the simplicial homology theory; since X is 2-dimensional and acyclic, there is a unique 2-chain τ_0 that bounds γ_0). We will denote the support of τ_0 by $C(\gamma_0)$. Let $e_1 = \sigma_1 \cap \sigma_2$, where σ_1 and σ_2 are 2-simplices of X, $\sigma_1 \in C(\gamma_0)$, and $\sigma_2 \notin C(\gamma_0)$.

Our goal is to show that β_0 is homologous to an element of L_1. Let

$$\beta_0' := \beta_0 + \partial(\sigma_1, \sigma_1) = \beta_0 + (\sigma_1, e_1) + (\sigma_1, e_2) + (\sigma_1, e_3)$$

where e_1, e_2, and e_3 are edges of σ_1.

Let

$$\beta_1 := \beta_0' + (\text{terms of } \beta_0' \text{ that are in } L_1). \tag{5.3}$$

Since we work with $\mathbb{Z}_2$ coefficients, β_1 does not have any terms of L_1 and $\partial\beta_1 \in L_0$. By Theorem 4.6, the graph $\Gamma(\beta_1)$ does not have any vertices of valence 1.

Proposition 5.4. *The edge e_1 is not an edge of $\Gamma(\beta_1)$.*

Proof. Since e_1 is a non-singular edge of the graph $\Gamma(\beta_0)$ it follows that one of the terms (σ_1, e_1) or (σ_2, e_1) is in β_0, but not both.

Case 1. Let (σ_1, e_1) be in β_0. Then

$$\beta_0' = \beta_0 + \partial(\sigma_1, \sigma_1) = (\beta_0 + (\sigma_1, e_1)) + (\sigma_1, e_1) + (\sigma_1, e_1) + (\sigma_1, e_2) + (\sigma_1, e_3)$$

so

$$\beta_0' = (\beta_0 + (\sigma_1, e_1)) + (\sigma_1, e_2) + (\sigma_1, e_3).$$

Since (σ_1, e_1) is in β_0, the term (σ_1, e_1) is not in $(\beta_0 + (\sigma_1, e_1))$. Since β_0 does not contain (σ_2, e_1), it follows that β_0' does not have any terms with second coordinate e_1. Thus β_1 does not have a term with a second coordinate e_1. Hence e_1 is not an edge of $\Gamma(\beta_1)$.

Case 2. Let (σ_2, e_1) be in β_0. Then

$$\beta_0' = \beta_0 + \partial(\sigma_1, \sigma_1) = (\beta_0 + (\sigma_2, e_1)) + ((\sigma_2, e_1) + (\sigma_1, e_1)) + (\sigma_1, e_2) + (\sigma_1, e_3)$$

so

$$\beta_0' = (\beta_0 + (\sigma_2, e_1)) + (\delta e_1, e_1) + (\sigma_1, e_2) + (\sigma_1, e_3).$$

Since (σ_2, e_1) is in β_0, the term (σ_2, e_1) is not in $(\beta_0 + (\sigma_2, e_1))$. The only term with a second coordinate e_1 in β_0' is $(\delta e_1, e_1)$, and this term is not in β_1 since it is in L_1. Therefore β_1 does not contain any terms with a second coordinate e_1, so e_1 is not an edge of $\Gamma(\beta_1)$. □

We say that X is *strongly* 1*-acyclic* if for any finite subcomplex Y of X there exists a finite subcomplex F such that $Y \subseteq F$ and $H_1(F; \mathbb{Z}_2) \cong 0$.

Proposition 5.5. *Let X be an acyclic 2-dimensional simplicial complex that is strongly 1-acyclic. Then for each finite graph Γ in X there exists a unique minimal subcomplex $C(\Gamma)$ such that $\Gamma \subset C(\Gamma)$ and $H_1(C(\Gamma); \mathbb{Z}_2) \cong 0$.*

Proof. Since X is strongly 1-acyclic, there exists a subcomplex F such that $\Gamma \subset F$ and $H_1(F; \mathbb{Z}_2) \cong 0$. Let $C(\Gamma) := \bigcap_\alpha F_\alpha$ where the intersection is over all such subcomplexes F_α that contain Γ. It is clear that $\Gamma \subset C(\Gamma)$. We need to show that $H_1(C(\Gamma); \mathbb{Z}_2) \cong 0$. Let z be a 1-cycle in $C(\Gamma)$. Then z is a 1-cycle in F_α for each $\alpha \in A$. Assume that z bounds a 2-chain τ_{α_1} in F_{α_1} and a 2-chain τ_{α_2} in F_{α_2}. If $\tau_{\alpha_1} \neq \tau_{\alpha_2}$, then $\tau_{\alpha_1} + \tau_{\alpha_2}$ forms a non-zero 2-cycle in X which is not possible since X is acyclic and 2-dimensional. Hence $\tau_{\alpha_1} = \tau_{\alpha_2}$. □

If X is strongly 1-acyclic, we say an edge e of a finite subgraph Γ of X is *free* if only one of the 2-simplices that have e as an edge is in $C(\Gamma)$.

Now we are ready for the main result in the general case when X is an infinite complex. Again, we refer to Figure 1.

Theorem 5.6. *Let X be an infinite, 2-dimensional, locally finite, acyclic, Cohen–Macaulay simplicial complex with the following properties:*

(i) *The link of each vertex of X is 2-connected.*

(ii) *Any cycle in X has a non-singular edge.*

(iii) *X is strongly 1-acyclic.*

If for all $\beta \in K_1$ such that $\partial'\beta \in L_0$, β is reduced with respect to L_1 and $\Gamma(\beta)$ has a non-singular free edge, then $H_1(X, \mathcal{L}^2_{\mathbb{Z}_2}) \cong 0$ (equivalently, X has one end).

Proof. Let $\beta_0 \in K_1$ be such that $\partial'\beta_0 \in L_0$ and β_0 is reduced with respect to L_1. Let e_1 be a non-singular free edge of $\Gamma(\beta_0)$, let σ_1 and σ_2 be the 2-simplices of X that have e_1 as an edge with $\sigma_1 \in C(\Gamma(\beta_0))$ and $\sigma_2 \notin C(\Gamma(\beta_0))$. Let e_2 and e_3 be the other two edges of σ_1. We will define β_1 in the same way as we did in equation (5.3). We will show that $|C(\Gamma(\beta_1))|_2 < |C(\Gamma(\beta_0))|_2$ ($| * |_2$ denotes the number of 2-simplices in $*$).

From Proposition 5.4 it follows that e_1 is not in $\Gamma(\beta_1)$. Since $\sigma_1 \in C(\Gamma(\beta_0))$, it follows that $e_2, e_3 \in C(\Gamma(\beta_0))$. Hence, $C(\Gamma(\beta_0))$ contains $\Gamma(\beta_1)$. By the minimality of $C(\Gamma(\beta_1))$ it follows that $C(\Gamma(\beta_1)) \subseteq C(\Gamma(\beta_0))$.

We will show that $\sigma_1 \notin C(\Gamma(\beta_1))$. Since e_1 is a free edge of $C(\Gamma(\beta_0))$, i.e., $\sigma_1 \in C(\Gamma(\beta_0))$ and $\sigma_2 \notin C(\Gamma(\beta_0))$, by an elementary collapse operation on σ_1, $C(\Gamma(\beta_0))$ deformation retracts to a subcomplex $C(\Gamma(\beta_0))'$. Hence $C(\Gamma(\beta_0))' \subset C(\Gamma(\beta_0))$, $\sigma_1 \notin C(\Gamma(\beta_0))'$, and $|C(\Gamma(\beta_0))'|_2 < |C(\Gamma(\beta_0))|_2$. Since $C(\Gamma(\beta_0))$ is $\mathbb{Z}_2$-acyclic, it follows that $C(\Gamma(\beta_0))'$ is $\mathbb{Z}_2$-acyclic too. Since e_1 is not an edge of $\Gamma(\beta_1)$, $e_2, e_3 \in C(\Gamma(\beta_0))'$ and all other edges of $\Gamma(\beta_1)$ are in $C(\Gamma(\beta_0))'$ (they are edges of $\Gamma(\beta_0)$, so they are in $C(\Gamma(\beta_0))'$), it follows that $\Gamma(\beta_1)$ is contained in $C(\Gamma(\beta_0))'$. By the minimality of $C(\Gamma(\beta_1))$, and the fact that $H_1(C(\Gamma(\beta_0))'; \mathbb{Z}_2) \cong 0$, it follows that $C(\Gamma(\beta_1)) \subseteq C(\Gamma(\beta_0))'$, and $|C(\Gamma(\beta_1))|_2 \leq |C(\Gamma(\beta_0))'|_2 < |C(\Gamma(\beta_0))|_2$. Now we have

$$\beta_0 - \beta_1 = \partial' c_1 + l_1$$

where $c_1 \in K_2, l_1 \in L_1$ and $|C(\Gamma(\beta_1))|_2 < |C(\Gamma(\beta_0))|_2$.

We proceed by induction to find β_2, β_3, etc.

After the ith step we have

$$\beta_{i-1} - \beta_i = \partial' c_i + l_i \tag{5.7}$$

where $c_i \in K_2, l_i \in L_1$ and $|C(\Gamma(\beta_i))|_2 < |C(\Gamma(\beta_{i-1}))|_2$. Since $C(\Gamma(\beta_0))$ is finite, after some kth step we will get β_k such that $|C(\Gamma(\beta_k))|_2 = 0$. Hence $\Gamma(\beta_k)$ is the empty graph, or every component of $\Gamma(\beta_k)$ is a finite tree; the latter is not possible because $\Gamma(\beta_k)$ does not have vertices of valence 1. Hence $\Gamma(\beta_k)$ is the empty graph and $\beta_k = 0$.

If we add the equations (5.7) for $i = 1, 2, \ldots, k$ we will get

$$\beta_0 - \beta_k = \sum_{i=1}^{k} \partial'(c_i) + \sum_{i=1}^{k} l_i.$$

Let $\beta = \sum_{i=1}^{k} l_i$. Then $\beta \in L_1$. Since $\beta_k = 0$ it follows that

$$\beta_0 - \beta = \sum_{i=1}^{k} \partial'(c_i).$$

Let $i_*\colon H_0(L; \mathbb{Z}_2) \to H_0(K; \mathbb{Z}_2)$ be induced by the inclusion $i\colon L_0 \hookrightarrow K_0$. Let $[\alpha_o] \in H_0(L; \mathbb{Z}_2)$ and let $i_*([\alpha_0]) = [0]$. This means that α_0 bounds in K, i.e., there exists a chain $\beta_0 \in K_1$ such that $\partial'\beta_0 = \alpha_0$.

But we have shown that there exists a chain $\beta \in L_1$ that is homologous to β_0 in K, i.e., $\partial'\beta_0 = \partial'\beta' = \alpha_0$. Hence α_0 bounds in L, i.e., $[\alpha_0] = [0]$. Therefore $\ker i_* = 0$, and, by Lemma 3.3, it follows that $H_1(X, \mathcal{L}^2_{\mathbb{Z}_2}) \cong 0$. □

6. Application of the Main Theorem

We now apply Theorem 5.6 to the case where X is the universal cover of a finite, 2-dimensional, aspherical, Cohen–Macaulay simplicial complex such that the link of each vertex is 2-connected. We will denote this finite complex by Δ. We wish to understand when $\widetilde{\Delta}$ has one end. We assume that $\widetilde{\Delta}$ is a simplicial complex and that $p : \widetilde{\Delta} \to \Delta$, the universal covering projection, is simplicial. We will abuse notation by using the same notation for an abstract simplicial complex and for the associated polyhedron.

Let S be the subgraph of $\widetilde{\Delta}$ generated by all singular edges. If $S = \emptyset$, then $\widetilde{\Delta}$ is homeomorphic to $\mathbb{R}^2$ (Δ is Cohen–Macaulay, and the links of all vertices are connected). Hence, $\widetilde{\Delta}$ is one-ended. Therefore the case $S = \emptyset$ is trivial. From now on we will assume that $S \neq \emptyset$.

Lemma 6.1. *If $S \neq \emptyset$, then the following are equivalent:*

(i) *For every vertex $v_0 \in p(S)$, $i_\#\colon \pi_1(p(S), v_0) \to \pi_1(\Delta, v_0)$ is injective and $\pi_1(p(S), v_0)$ is non-trivial.*

(ii) *Each component of S is a non-compact tree.*

Proof. (i) $\Longrightarrow$ (ii): Let $S_{\tilde{v}_0}$ be a the path component of S, with base point $\tilde{v}_0$ over v. We have $\pi_1(S_{\tilde{v}_0}, \tilde{v}_0) \cong \ker i_\#$ (by Theorem 3.4.9 [6]). Since $i_\#$ is an injection, $\pi_1(S_{\tilde{v}_0}, \tilde{v}_0) \cong \{1\}$, which implies that each component of S is a tree. Since

$\pi_1(p(S), v_0)$ is non-trivial and $p(S)$ is a graph, this must be a free group, hence infinite. Hence $S_{\tilde{v}_0}$ is non-compact.

(ii) $\Longrightarrow$ (i): Let $\tilde{v}_0$ be a vertex over v_0 and let T be the component of S that contains $\tilde{v}_0$. By (ii), T is a non-compact tree. Since $p(T)$ is compact and T is a non-compact tree, $\pi_1(p(T), v_0) \neq \{1\}$. Let $[\omega] \in \ker i_\#$, where ω is a loop at v_0 in $p(T)$ that bounds in Δ. Then, ω lifts to a loop $\tilde{\omega}$ in the tree T (by Theorem 3.4.9 [6]) which therefore bounds in T. Hence, ω bounds in $p(T)$, and $\ker i_\# \cong \{1\}$. Therefore $i_\#$ is an injection. □

For the rest of the paper PL stands for "piecewise linear" and a general reference is [10].

If $P = |K|$ is a polyhedron and $x \in P$, the link of x in P, well defined up to PL homeomorphism, is $\mathrm{lk}_{K'}\, x$ where K' is a subdivision of K having x as a vertex. If the link of each point of $\widetilde{\Delta} - S$ is connected, then $\widetilde{\Delta} - S$ is an open 2-manifold.

Lemma 6.2. *If the link of each point of $\widetilde{\Delta} - S$ is connected, where S is the singular set of $\widetilde{\Delta}$, then $\widetilde{\Delta} - S$ is an open 2-manifold.*

Proof. Let $x \in \widetilde{\Delta} - S$. Let $\widetilde{\Delta}'$ be a subdivision of $\widetilde{\Delta}$ having x as a vertex. Then $\mathrm{lk}_{\widetilde{\Delta}'}\, x$ is a finite graph in which the link of each vertex consists of two points. Thus $\mathrm{lk}_{\widetilde{\Delta}'}\, x$ is a closed 1-manifold. Since the link of each point of $\widetilde{\Delta} - S$ is connected, it follows that $\mathrm{lk}_{\widetilde{\Delta}'}\, x$ is a circle. Hence $\widetilde{\Delta} - S$ is a 2-manifold without boundary.

If there were a component of $\widetilde{\Delta} - S$ that is compact, then that component would be closed and open in $\widetilde{\Delta}$ which is not possible since $\widetilde{\Delta}$ is connected and $S \neq \emptyset$. Therefore $\widetilde{\Delta} - S$ is an open 2-manifold. □

If L is a subcomplex of $\widetilde{\Delta}$, then the *regular neighborhood* $N(L)$ of L in $\widetilde{\Delta}$ is the simplicial neighborhood of $\mathrm{sd}\, L$ in $\mathrm{sd}\, \widetilde{\Delta}$, i.e., $N(L) = \bigcup_{v \in \mathrm{sd}\, L} \mathrm{st}_{\mathrm{sd}\, \widetilde{\Delta}}\, v$, where $\mathrm{sd}\, L$ and $\mathrm{sd}\, \widetilde{\Delta}$ denote the first barycentric subdivision of L and $\widetilde{\Delta}$ respectively.

Let M be a PL n-manifold. We say that a subpolyhedron $P \subset M$ is *collared* in M if there exists a closed neighborhood N of P in M such that (N, P) is PL homeomorphic to $(P \times [0, 1], P \times \{0\})$ and P is *bicollared* in M if there exists a closed neighborhood N of P in M such that (N, P) is PL homeomorphic to $(P \times [-1, 1], P \times \{0\})$. We recall that if M is a PL n-manifold with boundary, then ∂M is always collared. The following lemma is a standard PL topology exercise and we will omit the proof.

Lemma 6.3. *If $\widetilde{\Delta} - S$ is a PL 2-manifold and if N is a regular neighborhood of S, then $V := \mathrm{cl}_{\widetilde{\Delta}}(\widetilde{\Delta} - N)$ is a PL 2-manifold whose boundary $\partial V := \mathrm{bd}_{\widetilde{\Delta}}\, N$ is a bicollared 1-manifold without boundary in $\widetilde{\Delta}$.*

From now on we will write N for a neighborhood of S that has bicollared boundary, we write ∂N for $\mathrm{bd}_{\widetilde{\Delta}}\, N$, and $V := \mathrm{cl}_{\widetilde{\Delta}}(\widetilde{\Delta} - N)$.

Lemma 6.4. *Let M be a component of V. Then*

(i) *$\partial M \neq \emptyset$;*

(ii) *if any component of ∂M is a line then M is non-compact;*

(iii) *if every component of ∂M is a line then M is contractible.*

Proof. (i) Since $S \neq \emptyset$, by Lemma 6.3 any path from S to $\text{int}_{\widetilde{\Delta}}(M)$ meets ∂N, hence ∂M. Therefore $\partial M \neq \emptyset$.

(ii) is obvious.

(iii) We will show that $H_1(M;\mathbb{Z}) \cong 0$. Since $\widetilde{\Delta}$ is a simply connected, every 1-cycle in M bounds in $\widetilde{\Delta}$ and is therefore homologous to a 1-cycle in ∂M. Since every component of ∂M is a line, the only one 1-cycle in ∂M is 0. Hence, every 1-cycle in M is homologous to 0. By (i), M has the homotopy type of a connected graph, so $\pi_1(M)$ is free. Since $H_1(M;\mathbb{Z}) \cong 0$, and the rank of $\pi_1(M)$ is equal to the free abelian rank of $H_1(M;\mathbb{Z})$, it follows that $\pi_1(M) \cong \{1\}$.

Since $\partial M \neq \emptyset$, $H_2(M;\mathbb{Z}) \cong 0$, so by the Hurewicz Theorem, $\pi_2(M) \cong \{1\}$. Therefore M is contractible. □

By Lemma 6.3 ∂N is a 1-manifold without boundary. So each component of ∂N is a line or a circle. In the view of Lemma 6.4 from now on we will also assume that *every component of ∂N is a line.*

Corollary 6.5. *Each component of $N_0 := p(N)$, of $V_0 := p(V)$, and of $\partial V_0 := p(\partial V)$ π_1-injects in Δ.*

In this situation the components $N_0(\alpha)$, $V_0(\beta)$, and $\partial V_0(\alpha,\beta)$ of N_0, V_0, and ∂V_0 respectively are aspherical and each $\partial V_0(\alpha,\beta)$ meets one $N_0(\alpha)$ and one $V_0(\beta)$. Thus we have G decomposed as the fundamental group of a finite graph of groups, where the vertex groups are $G_\alpha = \pi_1(N_0(\alpha))$ and $H_\beta = \pi_1(V_0(\beta))$ and the edge groups are $K_{\alpha\beta} = \pi_1(\partial V_0(\alpha,\beta))$. Moreover, Δ is aspherical, i.e., $\widetilde{\Delta}$ is contractible. Thus G acts on a Bass–Serre tree T, and by the usual construction we have a "structure map" $g\colon \widetilde{\Delta} \to T$ where g^{-1} (each vertex) is a component of N or of V and g^{-1} (each edge) is a (component of ∂V) $\times[0,1]$ glued at one end to a component of N and at the other end to a component of V. This proves:

Lemma 6.6. *Let $N(\alpha)$ be a component of N, and let $V(\beta)$ be a component of V such that $\partial N(\alpha) \cap \partial V(\beta) \neq \emptyset$. Then $\partial N(\alpha)$ and $\partial V(\beta)$ have only one line in common.*

Lemma 6.7. *For any finite subcomplex Y of $\widetilde{\Delta}$ there exists a contractible subcomplex K of $\widetilde{\Delta}$ which contains Y.*

Proof. Let Y be a finite subcomplex of $\widetilde{\Delta}$. We will show that there exists a finite contractible subcomplex K of $\widetilde{\Delta}$ containing Y. In this proof N will denote a regular neighborhood of S. We may assume that $\widetilde{\Delta}$ is subdivided so that N, V, $Y \cap V$, and $Y \cap N$ are subcomplexes of $\widetilde{\Delta}$.

The subcomplex Y only has non-empty intersection with a finite number of components of V and of N. Let $V(\beta_1)$ have non-empty intersection with Y. Since $V(\beta_1)$ has countably many boundary components and Y is finite, it follows that Y may have non-empty intersection only with a finite number of them, say with $\partial V(\alpha_1)^{\beta_1}$, $\partial V(\alpha_2)^{\beta_1}, \ldots, \partial V(\alpha_k)^{\beta_1}$ (it may happen that Y does not intersect any of the boundary components of $V(\beta_1)$). Let $N(\alpha_1)^{\beta_1}$, $N(\alpha_2)^{\beta_1}, \ldots, N(\alpha_k)^{\beta_1}$ be the components of N having $\partial V(\alpha_1)^{\beta_1}$, $\partial V(\alpha_2)^{\beta_1}, \ldots, \partial V(\alpha_k)^{\beta_1}$ as boundary components respectively. By Lemma 6.6, $\partial V(\beta_1)$ has only one line in common with each of $\partial N(\alpha_i)^{\beta_1}$. Let $Y(\beta_1)$ be the subcomplex of Y that lies in $(\widetilde{\Delta} - S)(\beta_1) \cup S(\alpha_1)^{\beta_1} \cup S(\alpha_2)^{\beta_1} \cup \cdots \cup S(\alpha_k)^{\beta_1}$ where $(\widetilde{\Delta} - S)(\beta_1)$ is the component of $\widetilde{\Delta} - S$ that contains $V(\beta_1)$, and $S(\alpha_i)^{\beta_1}$ is the component of S that is in $N(\alpha_i)^{\beta_1}$ for all $i = 1, 2, \ldots, k$.

Let $Y(\beta_1)' := Y(\beta_1) \cap V(\beta_1)$ be a subcomplex of $V(\beta_1)$. There exists a contractible subcomplex $K(\beta_1)'$ of $V(\beta_1)$ that contains $Y(\beta_1)'$, and $K(\beta_1)' \cap \partial V(\alpha_i)^{\beta_1}$ is connected for all $i = 1, 2, \ldots, k$. Since $K(\beta_1)' \cap \partial V(\alpha_i)^{\beta_2}$ is connected, it collapses in the regular neighborhood $N(\alpha_1)^{\beta_1}$ onto a connected subtree $T(\alpha_i)^{\beta_1}$ of $S(\alpha_i)^{\beta_1}$. Let $K(\alpha_i)^{\beta_1}$ be a contractible subcomplex of the regular neighborhood $N(\alpha_i)^{\beta_1}$ containing $Y \cap N(\alpha_i)^{\beta_1}$ such that $K(\alpha_i)^{\beta_1} \cap S(\alpha_i)^{\beta_1}$ is connected, and also $K(\alpha_i)^{\beta_1} \cap \partial V(\alpha_i)^{\beta_1}$ is connected.

Since each intersection $K(\beta_1)' \cap K(\alpha_i)^{\beta_1}$ is contractible for all $i = 1, 2, \ldots, k$, $K(\beta_1) := K(\beta_1)' \cup K(\alpha_1)^{\beta_1} \cup K(\alpha_2)^{\beta_1} \cup \cdots \cup K(\alpha_k)^{\beta_1}$ is contractible and contains $Y(\beta_1)$.

Next we will do the same with the subcomplex $Y(\beta_2)$ of Y that lies in $(\widetilde{\Delta} - S)(\beta_2) \cup S(\alpha_1)^{\beta_2} \cup S(\alpha_2)^{\beta_2} \cup \cdots \cup S(\alpha_l)^{\beta_2}$ where $(\widetilde{\Delta} - S)(\beta_2)$ is the component of $\widetilde{\Delta} - S$ that contains $V(\beta_2)$, and $S(\alpha_i)^{\beta_2}$ is the component of S that is in $N(\alpha_i)^{\beta_2}$ for all $i = 1, 2, \ldots, l$. As with the construction of $K(\beta_1)$, there exists a contractible subcomplex $K(\alpha_2)$ that contains $Y(\alpha_2)$.

There is at most one component of N that has non-empty intersection with both $V(\beta_1)$ and $V(\beta_2)$ (this follows from Lemma 6.6). Suppose there exists a component $N(\alpha)$ such that $N(\alpha) \cap V(\beta_1) \neq \emptyset$ and $N(\alpha) \cap V(\beta_1) \neq \emptyset$, and let $S(\alpha)$ be the tree that has $N(\alpha)$ as a regular neighborhood. Then $K(\beta_1)$ and $K(\beta_2)$ are either disjoint or they intersect in a contractible subcomplex (their intersection is a connected subtree of $S(\alpha)$), so their union is contractible. If such $N(\alpha)$ does not exist then $K(\beta_1) \cup K(\beta_2)$ is a disjoint union of contractible subcomplexes. In either case, $K(\beta_1) \cup K(\beta_2)$ contains $Y(\beta_1) \cup Y(\beta_2)$.

We continue this way. The algorithm is finite because Y is finite. At the end we will get either a contractible subcomplex K that contains Y, or a disjoint union of contractible subcomplexes whose union contains Y. In the latter case, we will connect the components of K by segments, to get a contractible subcomplex K that contains Y. □

Corollary 6.8. $\widetilde{\Delta}$ *is strongly* 1*-acyclic.*

Lemma 6.9. *Let K be a finite 2-dimensional subcomplex of $\widetilde{\Delta}$. Then K has a free non-singular edge, i.e., there exists a non-singular edge e such that only one of the 2-simplices of which e is an edge is in K.*

Proof. Suppose that K does not have any free edges. Let τ be the 2-chain that is a sum of all 2-simplices from K, i.e., $\tau = \sum_{\sigma \in K} \sigma$ (we work with $\mathbb{Z}_2$-coefficients). For each non-singular edge e in K, both 2-simplices of which e is an edge are in τ because K does not have any free edges. Therefore, $\partial\tau$ does not contain any non-singular edges, i.e., all edges of $\partial\tau$ are singular. Each component of the singular set S is a tree. Since $\partial\tau$ is a 1-cycle, $\partial\tau$ is a sum of 1-cycles each supported in a component of $S \cap K$. But the only 1-cycle in a tree is the trivial one. Hence $\partial\tau = 0$ which is a contradiction because $\widetilde{\Delta}$ is a 2-dimensional contractible simplicial complex and there are no non-trivial 2-cycles in $\widetilde{\Delta}$. □

Our motivating question was: *Let Δ be a finite, 2-dimensional, aspherical Cohen–Macaulay simplicial complex. Under what hypotheses does $\widetilde{\Delta}$ have one end?*

The following lemma is true in a more general case, not necessarily a complex that is the universal cover of a finite complex.

Lemma 6.10. *Let X be an infinite, 2-dimensional, path connected, Cohen–Macaulay complex such that the link of each vertex is 2-connected and any cycle in X has a non-singular edge. Then each component of S is an (infinite) non-trivial tree without vertices of valence 1 in that tree.*

Proof. Let S' be a component of S. Suppose that v is a vertex of S' with valence 1 in S'. This implies that $\operatorname{lk} v$ is a graph with only one vertex, denoted by w, with valence greater than 2. If we remove w from the link together with the edges from $\operatorname{lk} v$ that have w as a vertex, we will disconnect $\operatorname{lk} v$ which is not possible since each vertex of X is 2-connected. Therefore S' does not have any vertices of valence 1. Since every cycle of X contains a non-singular edge, it follows that S' does not have any cycles. Hence S' is a tree. □

Lemma 6.11. *Let Δ be a finite, 2-dimensional, connected simplicial complex. Let S_0 be the singular set in Δ and assume that the link of each vertex of $\Delta - S_0$ is connected. If each component C of $\partial N(S_0)$ (which is homeomorphic to a circle) π_1-injects into Δ, then $i_\# \colon \pi_1(S_0, v_0) \to \pi_1(\Delta, v_0)$ is injective for every vertex $v_0 \in S_0$.*

Proof. Let $\omega \colon S^1 \to S_0$ be a loop in S_0 such that ω is homotopically trivial in Δ. Let $f : B^2 \to \Delta$ be a map (by a "map" we understand a "continuous function") such that the restriction $f_{|S^1} = \omega$. Let $S_{0,\omega}$ be the component of S_0 that contains ω. We think of (B^2, S^1) as a PL pair. We may assume that f is a PL map. Let $N(S_{0,\omega})$ be a regular neighborhood of $S_{0,\omega}$, and let $C_1, C_2, \ldots, C_k$ be all the boundary components of $\partial N(S_{0,\omega})$ ($C_1, C_2, \ldots, C_k$ are cycles). Since f maps the open 2-manifold $B^2 - f^{-1}(S_0)$ into the open 2-manifold $\Delta - S_0$, using transversality, we may make a small perturbation of f to get a map $f_1\colon B^2 \to \Delta$ such that $f_1^{-1}(C_1 \cup C_2 \cup \cdots \cup C_k)$ is a closed 1-manifold and $f_{|f^{-1}(S_0)} = f_{1_{|f^{-1}(S_0)}}$. Note $f_1^{-1}(C_1 \cup C_2 \cup \cdots \cup C_k) \cap f_1^{-1}(S_0) = \emptyset$.

Let $A \subset f_1^{-1}(C_1 \cup C_2 \cup \cdots \cup C_k)$ be a cycle. Let R be the closed region of $\mathring{B}^2$ that lies inside A (R is a disk by Schönflies Lemma). Assume $f_1(A) \subset C_1$. Since C_1 is π_1-injective in Δ, we may redefine f_1 so that $f_1(R) \subset C_1$. Let A_1 be a cycle in $\mathring{B}^2$ that is disjoint from A but very close to A, does not intersect any cycles from $f_1^{-1}(C_1 \cup C_2 \cup \cdots \cup C_k)$, and A is in the closed region R_1 that is inside A_1. We want to show that, with small changes on f_1, we can get $f_1(A_1) \cap C_1 = \emptyset$ (from now on we will use often "small" changes on f_1, but we will not use different notation; all "new" maps will be again denoted by f_1). C_1 is bicollared, hence $f_1(R_1)$ is on one side of C_1 in the bicollared neighborhood of C_1.

We will repeat the above on every cycle of $f_1^{-1}(C_1 \cup C_2 \cup \cdots \cup C_k)$. Then $f_1(B^2)$ is either in the regular neighborhood $N(S_{0,\omega})$ or outside $N(S_{0,\omega})$. Since $f_{1_{|S^1}} = \omega$, it follows that $f_1(B^2)$ is in the regular neighborhood $N(S_{0,\omega})$. The regular neighborhood $N(S_{0,\omega})$ strongly deformation retracts onto $S_{0,\omega}$, so by changing the map f_1 again, we can get that $f_1(B^2) \subset S_{0,\omega}$. Therefore ω is homotopically trivial in $S_{0,\omega}$, which implies that $S_{0,\omega}$ π_1-injects into Δ. $\square$

Theorem 6.12. *Let Δ be a finite, 2-dimensional, aspherical, Cohen–Macaulay complex such that the link of each vertex of Δ is 2-connected and Δ does not consist of only one 2-simplex. Let S_0 be the singular set in Δ. If each component C of the boundary of a regular neighborhood of S_0 π_1-injects into Δ, then $\widetilde{\Delta}$ has one end. In particular, $\pi_1(\Delta)$ is 2-dimensional duality group.*

Proof. The link of each point of $\widetilde{\Delta} - \widetilde{S}_0$ is connected because $\widetilde{\Delta}$ is Cohen–Macaulay.. By Lemma 6.11, it follows that $\pi_1(S_0, v_0)$ is non-trivial for every vertex $v_0 \in S_0$, and $i_\#\colon \pi_1(S_0, v_0) \to \pi_1(\Delta, v_0)$ is injective. Hence $\pi_1(\Delta, v_0)$ is non-trivial, and $\widetilde{\Delta}$ is infinite.

By Lemma 6.1, each component of $\widetilde{S}_0$ is a non-compact tree. This is equivalent with the property that every cycle in $\widetilde{\Delta}$ has a non-singular edge. Since the link of each vertex of $\widetilde{\Delta}$ is 2-connected, it follows that every edge of $\widetilde{\Delta}$ is a face of at least two 2-simplices (by Corollary 4.2). By Corollary 6.8 it follows that $\widetilde{\Delta}$ is strongly 1-acyclic. Therefore, the conditions (ii) and (iii) of Theorem 5.6 are satisfied.

It remains to check the last hypothesis of Theorem 5.6. Let β be a chain in K_1 that does not have any terms of L_1 and $\partial'\beta \in L_0$ (Figure 1). We will show that the graph $\Gamma(\beta)$ ($\Gamma(\beta)$ is the graph spanned by all edges that appear in β) has a non-singular free edge, i.e., there exists a non-singular edge of $\Gamma(\beta)$ that is a face of two 2-simplices σ_1 and σ_2 such that $\sigma_1 \in C(\Gamma(\beta))$ and $\sigma_2 \notin C(\Gamma(\beta))$.

Suppose that $\Gamma(\beta)$ does not have a non-singular free edge. Since $\Gamma(\beta)$ does not have any vertex with valence 1 (by Theorem 4.6), it follows that $\Gamma(\beta))$ has at least one cycle. Hence $C(\Gamma(\beta))$ is 2-dimensional. By Lemma 6.9, $C(\Gamma(\beta))$ has a non-singular free edge e. By assumption, the edge e is not in $\Gamma(\beta)$. Then, by an elementary collapse, we would get a subcomplex whose first homology is trivial, and contains $\Gamma(\beta)$, which contradicts the minimality of $C(\Gamma(\beta))$.

By Theorem 5.6, $H_1(\widetilde{\Delta}, \mathcal{L}^2_{\mathbb{Z}_2}) \cong 0$, and by Theorem 2.4 $H^1_c(\widetilde{\Delta}; \mathbb{Z}_2) \cong 0$. Hence $\widetilde{\Delta}$ has one end. □

Let $S_1, S_2, \ldots, S_{n_1}$ be the components of the singular set S_0 in Δ and N_i ($i = 1, 2, \ldots, n_1$) be the pairwise disjoint regular neighborhoods of $S_1, S_2, \ldots, S_{n_1}$ respectively. We write $N = \bigsqcup_{i=1}^{n_1} N_i$. The boundary of N in Δ consists of circles C_j ($j = 1, 2, \ldots, n_2$). The closure in Δ of each component of $\Delta - N$ is a compact surface whose boundary consists of some of the circles C_j (each circle C_j occurs exactly once as a boundary of one surface). We denote these surfaces by M_k ($k = 1, 2, \ldots, n_3$). Thus we express Δ as a graph of connected polyhedra; the vertex polyhedra are the N_is and the M_ks, the edge polyhedra are the C_js. Note that there is at least one M_k because Δ has one free edge (Lemma 6.9) and N collapses onto S. As a consequence of Theorem 6.12, we have the following corollary which provides necessary and sufficient conditions for $\widetilde{\Delta}$ to have one end.

Corollary 6.13. *Let Δ be a finite, 2-dimensional, aspherical, Cohen–Macaulay complex such that the link of each vertex of Δ is 2-connected and Δ does not consist of only one 2-simplex. Assume that no surfaces M_k is a disk. Then the following are equivalent:*

(i) *Each C_j π_1-injects into N.*

(ii) *$\widetilde{\Delta}$ has one end and semistable fundamental group at infinity.*

Proof. It is obvious that this is true when $S_0 = \emptyset$. Assume $S_0 \neq \emptyset$.

We choose a maximal tree in Δ , well behaved with respect to our graph-of-polyhedra decomposition. ([6]). Assuming (i), this gives a graph-of-groups decomposition of $\pi_1(\Delta)$ in which the vertex groups are (non-trivial) finitely generated free groups, and the edge groups are infinite cyclic (this is where we must rule out that any M_k is a disk). By Britton's Lemma each C_j π_1-injects into Δ. Thus Theorem 6.12 implies that $\widetilde{\Delta}$ has one end.

For the converse, we noted that there is at least one C_j. Assume some C_j does not π_1-inject into N; thus the inclusion $C_j \hookrightarrow N$ induces the trivial homomorphism on π_1. Then we have a decomposition of $\pi_1(\Delta)$ as the fundamental group of graph of groups similar to the previous one except that some of the edge groups are trivial instead of being infinite cyclic. The set S is non-empty, and Lemma 6.10 implies that each component of N has non-trivial (free) fundamental group. Let M be the surface having C_j in its boundary. Then the decomposition includes an edge whose vertex groups are non-trivial while the edge groups are trivial. This is enough to ensure that $\pi_1(\Delta)$ has more than one end.

We have proved that (i) is equivalent to:

(ii′) $\pi_1(\Delta)$ *is the fundamental group of a graph of groups where each vertex group is free* (*hence semistable at each end*) *and each edge group is infinite cyclic* (*hence finitely generated*).

By the Main Theorem of [9], the properties in parentheses in (ii′) are sufficient to imply that the group $\pi_1(\Delta)$ is semistable at infinity. □

Remark. Perrin Wright, in [11], has shown that every finite, 2-dimensional polyhedron can be deformed to a finite fake surface. Triangulated fake surfaces are Cohen–Macaulay and the link of each vertex of a fake surface is 2-connected. Hence, if we consider a finite 2-dimensional, aspherical simplicial complex K, then K can be deformed to a fake surface Δ. According to Theorem 6.12, $\widetilde{\Delta}$ has one end if each component of the boundary of a regular neighborhood of the singular set S_0 of Δ (which are circles) π_1-injects in Δ. The question that we are considering is *to identify the preimage of each of these circles in the simplicial complex* K.

A. Zeeman's spectral sequences

In this section we give a proof of Theorem 2.4. The proof is based on the work of F. J. F. Lasheras [5] where he generalized Theorem 1 from [12] for locally finite simplicial complexes (Theorem 1 in [12] is for finite simplicial complexes). In [12], Zeeman generalized the Poincaré duality for manifolds. The exposition in this section is based on [5], [8], and [7].

Let X be an n-dimensional, locally finite simplicial complex. The coefficients are in a commutative ring R with $1 \neq 0$ and will be omitted. We define $B^{r,s}$ to be a bigraded module in the following way: If $0 \leq s \leq r \leq n$ we define $B^{r,s}$ to be the free abelian group generated by all pairs of simplices (τ, σ) such that $\dim \tau = r$, $\dim \sigma = s$, and $\tau \succ \sigma$; we define $B^{r,s} := 0$ otherwise. We have:

$$d_1\colon B^{r,s} \to B^{r+1,s}, \qquad d_1((\tau,\sigma)) := (\delta\tau, \sigma),$$
$$d_2\colon B^{r,s} \to B^{r,s-1}, \qquad d_2((\tau,\sigma)) := (-1)^r(\tau, \partial\sigma),$$

$$d_1 \circ d_1 = 0 = d_2 \circ d_2,$$
$$d_1 \circ d_2 + d_2 \circ d_1 = 0.$$

Then $\{B^{r,s}; d_1, d_2\}$ is a double complex. We associate to this double complex a chain complex $\{B_*, d\}$ defined as follows:

$$d := d_1 + d_2\colon B_m \to B_{m-1}, \quad d((\tau,\sigma)) := (\delta\tau, \sigma) + (-1)^r(\tau, \partial\sigma)$$

where $B_m := \bigoplus_{s-r=m} B^{r,s}$.

There are two spectral sequences ${}_{\mathrm{I}}E^*$ and ${}_{\mathrm{II}}E^*$ such that

$${}_{\mathrm{I}}E^2_{p,q} = H^p(H_q(B; d_2); \bar{d}_1), \quad 0 \leq q \leq p \leq n,$$
$${}_{\mathrm{II}}E^2_{p,q} = H_p(H^q(B; d_1); \bar{d}_2), \quad 0 \leq p \leq q \leq n,$$

and the first spectral sequence converges to $H_{q-p}(B;d)$, and the second one converges to $H_{p-q}(B;d)$. The differentials $\bar{d}_1$ and $\bar{d}_2$ are induced by d_1 and d_2 respectively. (Theorem 2.15, [8])

Lemma A.1. *The spectral sequence ${}_{\mathrm{I}}E$ collapses to an isomorphism*

$$H_m(B) \cong H_c^{-m}(X).$$

Proof. Let $C^t := H_q(B^{t,*}; d_2)$. Then

$$C^t = H_q\Big(\bigoplus_{\dim \tau = t} B^{t,*}; d_2\Big) \cong \bigoplus_{\dim \tau = t} H_q(B_*^\tau; d_2).$$

The free abelian group B_*^τ is generated by all pairs (τ, σ) such that $\dim \tau = t$, $\dim \sigma = *$, and $\tau \succ \sigma$. Hence $H_q(B_*^\tau; d_2) \cong H_q(C(\tau); \partial)$ where $C(\tau)$ is the simplicial chain complex for τ. Then $C^t \cong \bigoplus_{\dim \tau = t} R$ when $q = 0$, and $C^t \cong 0$ when $q \neq 0$. Hence ${}_{\mathrm{I}}E^2_{p,0} = H^p(H_q(B; d_2); \bar{d}_1) \cong H^p(\{C^t\}_{t=0}^n; \delta) \cong H_c^p(X)$, and ${}_{\mathrm{I}}E^2_{p,q} \cong 0$ when $q \neq 0$. On the other hand

$$0 \cong {}_{\mathrm{I}}E^2_{-p+2,-1} \xrightarrow{d^2} {}_{\mathrm{I}}E^2_{-p,0} \xrightarrow{d^2} {}_{\mathrm{I}}E^2_{-p-2,1} \cong 0.$$

Therefore ${}_{\mathrm{I}}E^\infty_{-p,0} \cong {}_{\mathrm{I}}E^2_{-p,0}$. Since ${}_{\mathrm{I}}E^2_{p,q} \cong 0$ when $q \neq 0$, it follows that ${}_{\mathrm{I}}E^\infty_{p,q} \cong {}_{\mathrm{I}}E^2_{p,q} \cong 0$ when $q \neq 0$. Since ${}_{\mathrm{I}}E^2_{p,q}$ converges to $H_{q-p}(B;d)$, it follows that ${}_{\mathrm{I}}E^2_{-p,0} \cong {}_{\mathrm{I}}E^\infty_{-p,0} \cong H_p(B;d)$. Therefore $H_p(B;d) \cong H_c^{-p}(X)$. □

Lemma A.2. *The spectral sequence ${}_{\mathrm{II}}E$ runs:*

$${}_{\mathrm{II}}E^2_{p,q} \cong H_p(X; \mathcal{L}_X^q) \Longrightarrow H_c^{q-p}(X).$$

Proof. ${}_{\mathrm{II}}E^2_{p,q} = H_p(H^q(B; d_1); \bar{d}_2)$ converges to $H_{p-q}(B;d)$ which is isomorphic to $H_c^{q-p}(X)$ by the previous lemma.

Let

$$C_t := H^q(B^{*,t}; d_1) = H^q\Big(\bigoplus_{\dim \sigma = t} B_\sigma^*; d_1\Big) \cong \bigoplus_{\dim \sigma = t} H^q(B_\sigma^*; d_1)$$

where B_σ^* is the free abelian group generated all pairs (τ, σ), where $\dim \tau = *$, $\dim \sigma = t$, and $\tau \succ \sigma$. Hence $H^q(B_\sigma^*; d_1) \cong H^q(X, X - \mathring{\mathrm{st}}\, \sigma; \delta)$. Next

$${}_{\mathrm{II}}E^2_{p,q} = H_p\Big(\Big\{\bigoplus_{\dim \sigma = t} H^q(X, X - \mathring{\mathrm{st}}\, \sigma)\Big\}_{t=0}^n; d_2\Big) \cong H_p(X; \mathcal{L}_X^q).$$

Hence ${}_{\mathrm{II}}E^2_{p,q} \cong H_p(X; \mathcal{L}_X^q) \Longrightarrow H_c^{q-p}(X)$. □

A path connected, finite dimensional, locally finite simplicial complex Δ is *Cohen–Macaulay* if for all $\sigma \in \Delta$, $\widetilde{H}_i(\mathrm{lk}\, \sigma) \cong 0$ for all $i < \dim(\mathrm{lk}\, \sigma)$. It follows that if X is Cohen–Macaulay complex, then for any simplex $\sigma \in X$ we have $\widetilde{H}^i(\mathrm{lk}\, \sigma) \cong 0$ for all $i < \dim(\mathrm{lk}\, \sigma)$.

Next we will prove Theorem 2.4.

Theorem 2.4. *Let X be an n-dimensional, locally finite, Cohen–Macaulay simplicial complex. Then for all $p \in \mathbb{Z}$*

$$H_p(X, \mathcal{L}_X^n) \cong H_c^{n-p}(X).$$

Proof. Since X is Cohen–Macaulay, it follows that

$$H^q(X, X - \overset{\circ}{\mathrm{st}}\,\sigma) \cong \widetilde{H}^{q-\dim\sigma-1}(\mathrm{lk}\,\sigma) \cong 0$$

when $q \neq n$. Hence ${}_{\mathrm{II}}E^2_{p,q} \cong 0$ when $q \neq n$, and ${}_{\mathrm{II}}E^\infty_{p,q} \cong {}_{\mathrm{II}}E^2_{p,q} \cong 0$ when $q \neq n$. Then

$$0 \cong {}_{\mathrm{II}}E^2_{p+2,n-1} \xrightarrow{d^2} {}_{\mathrm{II}}E^2_{p,n} \xrightarrow{d^2} {}_{\mathrm{II}}E^2_{p-2,n+1} \cong 0.$$

Therefore ${}_{\mathrm{II}}E^\infty_{p,n} \cong {}_{\mathrm{II}}E^2_{p,n}$, i.e., $H_p(X, \mathcal{L}_X^n) \cong {}_{\mathrm{II}}E^2_{p,n} \cong H_c^{n-p}(X)$. □

References

[1] R. Bieri, *Homological dimension of discrete groups*. 2nd ed., Queen Mary College Mathematical Notes, Queen Mary College, Department of Pure Mathematics, London 1981. Zbl 0357.20027 MR 715779

[2] N. Brady, J. McCammond, and J. Meier, Local-to-asymptotic topology for cocompact CAT(0) complexes. *Topology Appl.* **131** (2003), 177–188. Zbl 1045.20037 MR 1981872

[3] R. A.Brualdi, *Introductory combinatorics*. 5th ed., Pearson Prentice Hall, Upper Saddle River, NJ, 2010. MR 2655770

[4] W. Bruns and J. Herzog, *Cohen-Macaulay rings*. Cambridge Stud. Adv. Math. 39, Cambridge University Press, Cambridge 1993. Zbl 0788.13005 MR 1251956

[5] F. J. Fernandez-Lasheras, Fake Surfaces, Thickenings and cohomology of groups. Dissertation, SUNY Binghamton, Binghamton 1996.

[6] R. Geoghegan, *Topological methods in group theory*. Graduate Texts in Math. 243, Springer, New York 2008. Zbl 1141.57001 MR 2365352

[7] P. J. Hilton and U. Stammbach, *A course in homological algebra*. Graduate Texts in Math. 4, Springer-Verlag, New York-Berlin 1971. Zbl 0238.18006 MR 0346025

[8] J. McCleary, *A user's guide to spectral sequences*. 2nd ed., Cambridge Stud. Adv. Math. 58, Cambridge University Press, Cambridge 2001. Zbl 0959.55001 MR 1793722

[9] M. L. Mihalik and S. T. Tschantz, Semistability of amalgamated products and HNN-extensions. *Mem. Amer. Math. Soc.* **98** (1992), no. 471. Zbl 0792.20027 MR 1110521

[10] C. P. Rourke and B. J. Sanderson, *Introduction to piecewise-linear topology*. Ergeb. Math. Grenzgeb. 69, Springer-Verlag, Berlin 1972. Zbl 0477.57003 MR 0350744

[11] P. Wright, Formal 3-deformations of 2-polyhedra. *Proc. Amer. Math. Soc.* **37** (1973), 305–308. Zbl 0253.57001 MR 0331397

[12] E. C. Zeeman, Dihomology III. A generalization of the Poincaré duality for manifolds. *Proc. London Math. Soc.* (3) **13** (1963), 155–183. Zbl 0109.41302 MR 0153011

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