Groups Geom. Dyn. 8 (2014), 143–155
DOI 10.4171/GGD/220

Groups, Geometry, and Dynamics

Rank gradient in co-final towers of certain Kleinian groups

Darlan Girão

Abstract. We prove that if the fundamental group of an orientable finite volume hyperbolic 3-manifold has finite index in the reflection group of a right-angled ideal polyhedron in $\mathbb{H}^3$ then it has a co-final tower of finite sheeted covers with positive rank gradient. The manifolds we consider are also known to have co-final towers of covers with zero rank gradient.

Mathematics Subject Classification (2010). 57M05, 57M10, 57M50.

Keywords. Kleinian groups, rank of groups.

1. Introduction

Let G be a finitely generated group. The *rank of* G is the minimal cardinality of a generating set, and is denoted by $\mathrm{rk}(G)$. If G_j is a finite index subgroup of G, the Reidemeister–Schreier process ([LS]) gives an upper bound on the rank of G_j:

$$\mathrm{rk}(G_j) - 1 \leq [G : G_j](\mathrm{rk}(G) - 1).$$

Recently Lackenby introduced the notion of *rank gradient* ([La1]). Given a finitely generated group G and a collection $\{G_j\}$ of finite index subgroups, the *rank gradient* of the pair $(G, \{G_j\})$ is defined by

$$\mathrm{rgr}(G, \{G_j\}) = \lim_{j\to\infty} \frac{\mathrm{rk}(G_j) - 1}{[G : G_j]}$$

We say that the collection of finite index subgroups $\{G_j\}$ is *co-final* if $\bigcap_j G_j = \{1\}$, and we call it a *tower* if $G_{j+1} < G_j$.

In some particular cases it is easy to determine rank gradient, for example:

(1) When G is a free group, the rank gradient of any pair $(G, \{G_j\})$ is positive.
(2) The same is true if G is the fundamental group of a closed surface S with $\chi(S) < 0$;
(3) If $G \twoheadrightarrow F_2$, where F_2 is the free group on two generators then, using (1), one can find a tower (not co-final) of subgroups with positive rank gradient;

(4) If G is virtually abelian or if G is the fundamental group of a virtually fibered 3-manifold then there are towers with zero rank gradient. In the latter case we consider the subgroups coming from the cyclic covers of the fibered manifold.
(5) $\mathrm{SL}(n, \mathbb{Z})$, $n > 2$, has zero rank gradient with respect to towers of congruence subgroups ([Ti], [La1]).

However, determining the rank gradient of a co-final tower is very hard in general. For example, the following question is the motivation for this note:

Question 1. Does there exist a torsion free finite covolume Kleinian group G with a co-final tower $\{G_j\}$ such that $\mathrm{rgr}(G, \{G_j\}) > 0$?

The main result of this note provides infinitely many such examples. To state it we introduce some notation.

If M_1 is an orientable finite volume hyperbolic 3-manifold, we call the family of covers $\{M_j \to M_1\}$ *co-final* (resp. a *tower*) if $\{\pi_1(M_j)\}$ is co-final (resp. a tower). By rank gradient of the pair $(M_1, \{M_j\})$, $\mathrm{rgr}(M_1, \{M_j\})$, we mean the rank gradient of $(\pi_1(M_1), \{\pi_1(M_j)\})$.

Theorem 3.1. *Let M_1 be an orientable finite volume hyperbolic 3-manifold whose fundamental group has finite index in the reflection group of a totally geodesic right-angled ideal polyhedron P_1 in $\mathbb{H}^3$. Then there exists a co-final tower of finite sheeted covers $\{M_j \to M\}$ with positive rank gradient.*

This theorem relates to the work of Abért and Nikolov ([AN]), and in particular to a question about *cost of group actions* ([Ga]).

Question 2. Let G be finitely generated and $\{G_j\}$ be a co-final tower of normal subgroups of G. Does $\mathrm{rgr}(G, \{G_j\})$ depend on the tower $\{G_j\}$?

We remark that the manifolds in Theorem 3.1 are known to have towers of covers with zero rank gradient. Agol proved in [Ag] that if the fundamental group of a 3-dimensional manifold satisfies an algebraic condition, called RFRS, then it virtually fibers. He also proved in [Ag] that the manifolds of the type considered in Theorem 3.1 are virtually RFRS. Therefore, given M_1 as in Theorem 3.1, it is possible to find a tower $\{\Gamma_j\}$ with $\mathrm{rgr}(\pi_1(M_1), \{\Gamma_j\}) = 0$.

The main idea of the proof of Theorem 3.1 is as follows: given P_1 as in the theorem, construct a collection of polyhedra $\{P_j\}$ whose reflection groups have finite index 2^{j-1} in the reflection group of P_1. If one is given an orientable hyperbolic 3-manifold M_1 whose fundamental group has finite index in the reflection group of P_1 then M_1 has at least as many cusps as the number of vertices of P_1. We may find manifold covers $M_j \to M_1$ so that M_j is a 2^{j-1}-sheeted covering and has at least as many cusps as the number of ideal vertices of P_j. We then show that the P_j can be chosen so that the number of its vertices is of the same magnitude as 2^j.

The paper will be organized as follows: Section 2 sets up notation and we recall a characterization of right-angled ideal polyhedra using Andreev's theorem ([An]). We then show how the construction of the family $\{P_j\}$ will be done. In Section 3 we prove Theorem 3.1. Section 4 contains all the technical results we need to estimate $\mathrm{rk}(\pi_1(M_j))$. In Section 5 we show how to construct $\{P_j\}$ so that the family $\{M_j\}$ is co-final. The idea for this appears in [Ag] (Theorem 2.2) and we include a proof here for completeness. Section 6 contains some final remarks and further questions.

Ackowledgements. I am very grateful to my thesis advisor, Alan Reid, for his extraordinary guidance and unwavering support. I am also thankful to Ian Agol, Grant Lakeland and Mark Norfleet for helpful conversations. The author was partially supported by CAPES/Fulbright Grant BEX 2411/05-9.

2. Set up

An *abstract polyhedron* $\mathcal{P}_1$ is a cell complex on S^2 which can be realized by a convex Euclidean polyhedron. A *labeling* of $\mathcal{P}_1$ is a map

$$\Theta\colon \mathrm{Edges}(\mathcal{P}_1) \to (0, \pi/2].$$

The pair $(\mathcal{P}_1, \Theta)$ is a labeled abstract polyhedron. A labeled abstract polyhedron is said to be *realizable* as a hyperbolic polyhedron if there exists a hyperbolic polyhedron P_1 such that there is a label preserving graph isomorphism between the 1-skeleton of P_1 with edges labeled by dihedral angles and the 1-skeleton of $\mathcal{P}_1$ with edges labeled by Θ.

Let P_1 be a totally geodesic right-angled ideal polyhedron in $\mathbb{H}^3$ (that is, faces of P_1 are contained in hyperplanes and all vertices of P_1 lie in the boundary at infinity S^2_∞, where we here consider the ball model for $\mathbb{H}^3$). We consider the 1-skeleton of P_1 as a graph $\Gamma_1 \subset S^2$ with labels $\theta_e = \pi/2$. Let Γ_1^* be its dual graph. A *k-circuit* is a simple closed curve composed of k edges in Γ_1^*. A *prismatic k-circuit* is a k-circuit γ so that no two edges of Γ_1 which correspond to edges traversed by γ share a vertex. Andreev's theorem for right-angled ideal polyhedra in $\mathbb{H}^3$ ([An], see also [At]) can be stated as:

Theorem 2.1. *Let $\mathcal{P}_1$ be an abstract polyhedron. Then $\mathcal{P}_1$ is realizable as a right-angled ideal polyhedron P_1 if and only if*

(1) *P_1 has at least 6 faces;*
(2) *vertices have valence 4;*
(3) *for any triple of faces of P_1, (f_i, f_j, f_k), such that $f_i \cap f_j$ and $f_j \cap f_k$ are edges of P_1 with distinct endpoints, $f_i \cap f_k = \emptyset$;.*
(4) *there are no prismatic 4-circuits.*

The above theorem implies that the 1-skeleton of P_1 is a 4-valent graph. The faces can therefore be checkerboard colored. Reflecting P_1 along a face f_1 gives a polyhedron P_2 which is also right-angled, ideal and totally geodesic with checkerboard colored faces (see figure below). We construct a sequence of polyhedra $P_1, P_2, \dots, P_j, \dots$ recursively, whereby P_{j+1} is obtained from P_j by reflection along a face f_j. The faces of P_{j+1} are colored accordingly with the coloring of the faces of P_j.

The notation for the remainder of the paper is as follows: the number of vertices in the face f_j is denoted by S_{f_j} and ϕ_{f_j} denotes the reflection along f_j. B_j and W_j represent the maximal number of ideal vertices on a black or white face of the polyhedron P_j, respectively. V_j denotes the total number of vertices on P_j.

Throughout, the construction of the polyhedra P_j will be done in an alternating fashion with respect to the color of the faces: P_{2j} is obtained from P_{2j-1} by refection along a black face and P_{2j+1} is obtained from P_{2j} by reflection along a white face.

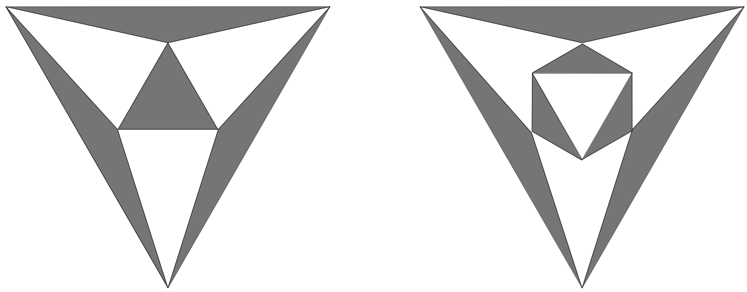

Figure 1. Polyhedron P_1 reflected along central black face yields P_2.

3. Main theorem

In this section we prove:

Theorem 3.1. *Let M_1 be an orientable finite volume hyperbolic 3-manifold whose fundamental group has finite index in the reflection group of a right-angled ideal polyhedron P_1 in $\mathbb{H}^3$. Then there exists a co-final tower of finite sheeted covers $\{M_j \to M\}$ with positive rank gradient.*

Our construction of the family $\{M_j\}$ was inspired by the proof of Theorem 2.2 of Agol's paper ([Ag]). The proof that this family can be made co-final is given in Section 5 (following [Ag]).

Proof of Theorem 3.1. Consider the family of polyhedra $\{P_j\}$ obtained from P_1 as described above. Denote by G_j the reflection group of P_j and observe that G_{j+1}

is a subgroup of G_j of index 2. G_1 acts on $\mathbb{H}^3$ with fundamental domain P_1. The orbifold $\mathbb{H}^3/G_1$ is non-orientable, and may be viewed as P_1 with its faces mirrored. The singular locus is the 2-skeleton of P_1. Each ideal vertex of P_1 corresponds to a cusp of $\mathbb{H}^3/G_1$.

Let M_1 be an orientable cusped hyperbolic 3-manifold such that $\pi_1(M_1)$ has finite index in G_1. Let $M_j \to M_1$ be the cover of M_1 whose fundamental group is $\pi_1(M_j) = \pi_1(M_1) \cap G_j$. Since $[G_j : G_{j+1}] = 2$, we must have $[\pi_1(M_j) : \pi_1(M_{j+1})] \leq 2$. Also note that since $\mathrm{vol}(P_j) = 2^{j-1}\mathrm{vol}(P_1)$, for all but finitely many j (at most $[G_1 : \pi_1(M_1)]$) we must have $[\pi_1(M_j) : \pi_1(M_{j+1})] = 2$. We may thus assume that $[\pi_1(M_j) : \pi_1(M_{j+1})] = 2$. By mirroring the faces of P_j, it may be regarded as a non-orientable finite volume orbifold (as described before). This implies that $M_j \to P_j$ is an orientable finite sheeted cover for $j = 1, 2, \ldots$.

Note that $[\pi_1(M_1) : \pi_1(M_j)] = 2^{j-1}$. Thus to show that the family $\{M_j \to M_1\}$ has positive rank gradient we will establish that $\mathrm{rk}(\pi_1(M_j))$ grows with the same magnitude as 2^j.

By "half lives half dies", an easy lower bound on the rank of the fundamental group of an orientable finite volume hyperbolic 3-manifold is the number of its cusps. Since the cusps of P_j correspond to its ideal vertices and the number of cusps does not go down under finite sheeted covers, it must be that M_j has at least as many cusps as the number of ideal vertices of P_j.

Recall that B_j and W_j are the maximal number of ideal vertices on a black or white face of the polyhedron P_j, respectively, and V_j is the total number of vertices on P_j. The claims below (proved in Section 4) give us the estimates we need for V_j in terms of V_1, B_1 and W_1.

Claim 1. $V_1 \geq B_1 + W_1 - 1$.

Claim 2. *For any* $j \geq 6$, $V_j \geq 2^{j-1}V_1 - 2^{j-1}(B_1 + W_1) + 2^{j-1} + 2^{j-2}$.

Given these, we argue as follows:

$$
\begin{aligned}
\mathrm{rgr}(M_1, \{M_j\}) &= \lim_{j\to\infty} \frac{\mathrm{rk}(\pi_1(M_j)) - 1}{[\pi_1(M_1) : \pi_1(M_j)]} \\
&\geq \lim_{j\to\infty} \frac{V_j - 1}{2^{j-1}} \\
&\geq \lim_{j\to\infty} \frac{2^{j-1}V_1 - 2^{j-1}(B_1 + W_1) + 2^{j-1} + 2^{j-2} - 1}{2^{j-1}} \\
&\geq \lim_{j\to\infty} \frac{2^{j-1}(B_1 + W_1 - 1) - 2^{j-1}(B_1 + W_1) + 2^{j-1} + 2^{j-2} - 1}{2^{j-1}} \\
&\geq \lim_{j\to\infty} \frac{2^{j-2} - 1}{2^{j-1}} = \frac{1}{2}
\end{aligned}
$$

which proves the theorem. □

4. Lower bounds on number of ideal vertices of P_j

We now proceed to prove Claims 1 and 2. This requires several preliminary results.

Lemma 4.1. *Let P_{j+1} be obtained from P_j by reflection along a face f_j. Then $V_{j+1} = 2V_j - S_{f_j}$.*

Proof. Here we abuse notation and write $v \in f_j$ if v is an ideal vertex of the face f_j and write $v \notin f_j$ otherwise. Note that if $v \notin f_j$, then v yields two vertices on P_{j+1}, namely, v and $\phi_{f_j}(v)$. If $v \in f_j$, then it yields a single vertex (v itself).

If $v \notin f_j$, then, by the observation above, v yields two ideal vertices on P_{j+1}. Since a total of S_{f_j} ideal vertices lie in f_j and $V_j - S_{f_j}$ do not, it must be that that

$$V_{j+1} = 2(V_j - S_{f_j}) + S_{f_j} = 2V_j - S_{f_j}. \qquad \square$$

Recall also that the construction of the family of polyhedra $\{P_j\}$ is made in an alternating fashion with respect to the color of the faces: P_{2j} is obtained from P_{2j-1} by refection along a black face and P_{2j+1} is obtained from P_{2j} by reflection along a white face.

Corollary 4.2. *For $j \geq 1$,*

(1) $V_{2j} \geq 2V_{2j-1} - B_{2j-1}$, *and*
(2) $V_{2j+1} \geq 2V_{2j} - W_{2j}$.

Proof. P_{2j} is obtained from P_{2j-1} by refection along a black face f_{2j-1}, thus $S_{f_{2j-1}} \leq B_{2j-1}$. By the lemma, $V_{2j} = 2V_{2j-1} - S_{f_{2j-1}}$ and therefore $V_{2j} \geq 2V_{2j-1} - B_{2j-1}$. The second inequality is similar. $\square$

With the notation established above we now find lower bounds for the V_j in terms of V_1, B_1 and W_1. First we need to find upper bounds for B_j and W_j in terms of B_1 and W_1. To do this in a way that will fit our purposes we establish two properties of the family $\{P_j\}$. As before, denote by ϕ_{f_j} the reflection along the face f_j.

Lemma 4.3. (1) *If P_j is reflected along a white (resp. black) face f_j, all black faces f_* adjacent to f_j yield new black faces $\tilde{f}_*$ on P_{j+1}. The number $S_{\tilde{f}_*}$ of ideal vertices on $\tilde{f}_*$ is $2S_{f_*} - 2$.*

(2) *A face f_* not adjacent to f_j yield two new faces, f_* itself and $\phi_f(f_*)$, both with S_{f_*} vertices.*

Proof. For the first property, reflecting f_* along f_j gives a face $\phi_{f_j}(f_*)$ in P_{j+1} adjacent to f_*. The dihedral angle between f_* and $\phi_f(f_*)$ is π. Thus, on P_{j+1}, they correspond to a single face denoted by $\tilde{f}_*$. The number of ideal vertices on $\tilde{f}_*$ is exactly $2S_{f_*} - 2$. The second property should be clear. See figure 1 for an illustration of these properties. $\square$

As an immediate consequence we have

Corollary 4.4.

(1) $\begin{cases} B_{2j} = B_{2j-1}, \\ W_{2j} \leq 2W_{2j-1} - 2. \end{cases}$

(2) $\begin{cases} B_{2j+1} \leq 2B_{2j} - 2, \\ W_{2j+1} = W_{2j}. \end{cases}$

We are now in position to estimate the values B_j and W_j in terms of B_1 and W_1.

Theorem 4.5. *With the notation as before we have*

(1) $W_{2j+1} = W_{2j} \leq 2^j W_1 - \sum_{l=1}^{j} 2^l$, *and*

(2) $B_{2j+2} = B_{2j+1} \leq 2^j B_1 - \sum_{l=1}^{j} 2^l$.

Proof. We proceed by induction. By Corollary 4.4 these statements are true for $j = 1$. Suppose they are also true for $j \leq n$. We now want to estimate $B_{2n+3} = B_{2n+4}$ and $W_{2n+2} = W_{2n+3}$. The hypothesis is that

$$W_{2j+1} = W_{2j} \leq 2^n W_1 - \sum_{l=1}^{n} 2^l$$

and

$$B_{2n+2} = B_{2n+1} \leq 2^n B_1 - \sum_{l=1}^{n} 2^l.$$

P_{2n+2} is obtained from P_{2n+1} by reflection along a black face, denoted by f. White faces on P_{2n+1} adjacent to f yield new white faces on P_{2n+2} with at most $2W_{2n+1}-2$ vertices, by Corollary 4.4. But

$$2W_{2n+1} - 2 \leq 2\Big[2^n W_1 - \sum_{l=1}^{n} 2^l\Big] - 2 = 2^{(n+1)} W_1 - \sum_{l=1}^{n+1} 2^l$$

which gives the desired result for W_{2n+2} and W_{2n+3}. Finally, P_{2n+3} is obtained from P_{2n+2} by a reflection along a white face, again denoted by f. Since black faces of P_{2n+2} have at most $B_{2n+2} (= B_{2n+1})$ vertices, black faces of P_{2n+3} will have at most $2B_{2n+1} - 2$ vertices, again by Corollary 4.4. But

$$2B_{2n+1} - 2 \leq 2\Big[2^n B_1 - \sum_{l=1}^{n} 2^l\Big] - 2 = 2^{(n+1)} B_1 - \sum_{l=1}^{n+1} 2^l.$$

This establishes the result for B_{2n+3} and B_{2n+4}. □

Theorem 4.6. *With the notation as before, and for* $j \geq 3$,

(1) $V_{2j} \geq 2^{2j-1} V_1 - B_1 \sum_{l=j-1}^{2j-2} 2^l - W_1 \sum_{l=j}^{2j-2} 2^l + \sum_{l=j+2}^{2j-1} 2^l + 2^j + 2$, *and*

(2) $V_{2j+1} \geq 2^{2j} V_1 - B_1 \sum_{l=j}^{2j-1} 2^l - W_1 \sum_{l=j}^{2j-1} 2^l + \sum_{l=j+2}^{2j} 2^l + 2.$

Proof. Lower bounds estimates for $V_1, \ldots, V_7$ are found recursively. V_1, V_2, V_3, V_4 and V_5 do not fit these formulas but V_6 and V_7 do. The statement is then true for $j = 3$. We now proceed by induction, using the previous proposition and Corollary 4.2. Suppose it is true for $j \leq n$, $n \geq 3$. We want to show this holds true for $j = n + 1$. By Corollary 4.2, $V_{2n+2} \geq 2V_{2n+1} - B_{2n+1}$. The hypothesis is that

$$V_{2n+1} \geq 2^{2n} V_1 - B_1 \sum_{l=n}^{2n-1} 2^l - W_1 \sum_{l=n}^{2n-1} 2^l + \sum_{l=n+2}^{2n} 2^l + 2.$$

We also know that

$$B_{2n+1} \leq 2^n B_1 - \sum_{l=1}^{n} 2^l$$

Thus

$$\begin{aligned} V_{2n+2} &\geq 2V_{2n+1} - B_{2n+1} \\ &\geq 2\Big[2^{2n} V_1 - B_1 \sum_{l=n}^{2n-1} 2^l - W_1 \sum_{l=n}^{2n-1} 2^l + \sum_{l=n+2}^{2n} 2^l + 2\Big] - \Big[2^n B_1 - \sum_{l=1}^{n} 2^l\Big] \\ &= 2^{2n+1} V_1 - B_1 \sum_{l=n}^{2n-1} 2^{l+1} - W_1 \sum_{l=n}^{2n-1} 2^{l+1} + \sum_{l=n+2}^{2n} 2^{l+1} + 2^2 + \sum_{l=1}^{n} 2^l \\ &= 2^{2n+1} V_1 - B_1 \sum_{l=n}^{2n} 2^l - W_1 \sum_{l=n+1}^{2n} 2^l + \sum_{l=n+3}^{2n+1} 2^l + 2^{n+1} + 2 \end{aligned}$$

which establishes (1) for $2(n + 1) = 2n + 2$.

We use the exact same idea and the estimate for V_{2n+2} to establish (2) for $2(n + 1) + 1 = 2n + 3$. □

Corollary 4.7. *For any* $j \geq 6$,

$$V_j \geq 2^{j-1} V_1 - 2^{j-1}(B_1 + W_1) + 2^{j-1} + 2^{j-2}.$$

Hence Claim 2 in the proof of Theorem 3.1 is proved. We now prove

Claim 1. $V_1 \geq B_1 + W_1 - 1.$

Proof. Let f_b and f_w be black and white faces of P_1 with maximal number of vertices, i.e., $S_{f_b} = B_1$ and $S_{f_w} = W_1$.

Case 1: *The faces f_b and f_w are not adjacent.* Here we get $V_1 \geq B_1 + W_1$ and the claim follows.

Case 2: *The faces f_b and f_w are adjacent.* Since f_b and f_w share exactly 2 vertices we see that $V_1 \geq B_1 + W_1 - 2$. Suppose we have equality. Then every vertex of P_1 must be a vertex of either f_b or f_w. Recall that we can visualize the 1-skeleton of P_1 as lying in S^2. Label the vertices of P_1 by $\{v_1, \dots, v_k\}$. The assumption is that all these vertices lie in the boundary of the disk $D = \overline{(f_b \cup f_w)} \subset S^2$. By Andreev's theorem, P_1 has at least 6 faces, every face is at least 3-sided and all vertices are 4-valent. Denoting by F_1 and E_1 the number of faces and edges of P_1 respectively we have the relation $V_1 - E_1 + F_1 = 2$. Since vertices are 4-valent we also have $E_1 = 2V_1$. From these relations and $F_1 \geq 6$, we get $V_1 \geq 4$. At two of the vertices, say v_1 and v_2, three of the emanating edges lie in D and one does not. Denote the ones that do not lie in D by e_1 and e_2, respectively. At all other v_i we have two edges that lie in D and two that do not. Denote the latter by e_i, e_i'. We have a total of $2(k-2) + 2 = 2k - 2$ edges not in D. The problem we have now is combinatorial:

Given the disk $D' = \overline{S^2 - D}$ and the points $v_1, \dots, v_k \in \partial D'$, $k \geq 4$, is it possible to subdivide D' by $2k - 2$ edges in a way that exactly one edge emanates from both v_1 and v_2 and exactly two edges emanate from $v_3, \dots, v_k$ in such a way that no pair of edges intersect and every face on the subdivision of D' is at least 3*-sided* (*here we also consider sides coming from the boundary*)?

A simple argument will show that the answer to this question is negative. Orient the boundary of D' counterclockwise. Starting at v_1, draw the edge e_1 emanating from it. The other endpoint of e_1 is some vertex v_{i_1}. Consider the vertices contained in the segment $[v_1, v_{i_1}] \subset \partial D'$ in the given orientation. If there are no vertices at all, then we must have a 2-sided face, which is not possible. Therefore, by relabeling, we may assume v_2 is the first vertex between v_1 and v_{i_1}. Observe that the edges emanating from v_2 are trapped between the edge e_1 and $\partial D'$. Draw an edge e_2 emanating from v_2 with the second endpoint v_{i_2}. It must be that v_{i_2} also lies in $[v_1, v_{i_1}]$, or else we find a pair of intersecting edges. As above, there must be a vertex in the segment $[v_2, v_{i2}]$. By repeating the above argument eventually we find a 2-sided face, which is not possible. Therefore it must be that $V_1 > B_1 + W_1 - 2$. □

5. Co-finalness

In this section we provide a way of choosing the black or white faces on the polyhedra P_j along which it is reflected in such a way that the resulting family $\{M_j\}$ of manifolds is co-final. The main result of this section, Theorem 5.1, appears as part of the proof of Theorem 2.2 of [Ag]. We include a proof here for completeness. To better describe this construction we need to change notation slightly by adding another index.

Start with P_1 and relabel it P_{11}. Reflect along a black face f_{11} obtaining P_{12}. Let $\phi_{f_{11}}$ represent such reflection. Observe that if f is adjacent to f_{11}, then $f \cup \phi_{f_{11}}(f)$ corresponds to a single face on P_{12}. We call f and $\phi_{f_{11}}(f)$ *subfaces* of $f \cup \phi_{f_{11}}(f)$. Next reflect P_{12} along a white face f_{12}, which is also a face of P_{11} or contains a face of P_{11} as a subface, obtaining P_{13}. We construct a subcollection $P_{11}, \dots, P_{1k_1}$ of polyhedra such that

(i) If P_{1j} is obtained from $P_{1(j-1)}$ by reflection along a white (black) face then $P_{1(j+1)}$ is obtained from P_{1j} by reflection along a black (white) face.
(ii) Whenever possible, the face f_{1j} must be a face of P_{11} or contain a face of P_{11} as a subface.
(iii) No faces of P_{11} are subfaces of P_{1k_1}.

Now set $P_{1k_1} := P_{21}$.

Suppose P_{n1} has been constructed. Construct the subcollection of polyhedra $P_{n1}, \dots, P_{nk_n}$ such that

(i) The reflections were performed in a alternating fashion with respect to the color of the faces;
(ii) Whenever possible, the face f_{nj} must be a face of P_{n1} or contain a face of P_{n1} as a subface.
(iii) No faces of P_{n1} are subfaces of P_{nk_n}.

Now set $P_{nk_n} := P_{(n+1)1}$. Inductively we obtain a collection of polyhedra

$$P_{11}, P_{12}, \dots, P_{1k_1} := P_{21}, \dots, P_{2k_2} := P_{31}, \dots, P_{nk_n} := P_{(n+1)1}, \dots$$

satisfying (i), (ii) and (iii) above.

Let G_{ij} be the reflection group of P_{ij} and let M_{ij} be the cover of M_{11} whose fundamental group is $\pi_1(M_{ij}) = \pi_1(M_{11}) \cap G_{ij}$. Co-finalness of the family $\{M_{ij} \to M_{11}\}$ is an immediate consequence of

Theorem 5.1. *Let G_{ij} be as above. Then $\bigcap_{ij} G_{ij} = \{1\}$.*

In order to prove this theorem we consider the base point for the fundamental group of each P_{ij} (viewed as orbifolds with their faces mirrored) to be the barycenter x_0 of P_{11}.

Proof of Theorem 5.1. Set $R_{ij} = \inf_\gamma\{\ell(\gamma)\}$, where γ is an arc with endpoints in faces (possibly edges) of P_{ij} going through x_0. Note that, by construction, $\lim_{i\to\infty} R_{ij} = \infty$. For a non-trivial element $g \in G_{11}$ set $R_g = \inf_{[\alpha]=g}\{\ell(\alpha)\}$, where α is a loop in P_{11} based at x_0 and $[\alpha]$ represents its homotopy class. Let α_g be a loop in P_{11} based at x_0 such that $[\alpha_g] = g$ and $\ell(\alpha_g) \le R_g + 1$.

We claim that for sufficiently large i one cannot have $g \in G_{ij}$. In fact, if α_{ij} is any nontrivial loop in P_{ij} based at x_0, then this loop bounces off faces of P_{ij}, yielding an arc γ_{ij} through x_0. Therefore $\ell(\alpha_{ij}) \geq \ell(\gamma_{ij}) \geq R_{ij}$. Since covering maps preserve length of curves, this implies that if i is large enough no such α_{ij} maps to α_g. Thus it is not possible to find a loop representative for g in P_{ij}. $\square$

6. Final remarks

Question 3. Is it possible, in our setting, to obtain a co-final tower of regular covers $\{M_j \to M_1\}$ with positive rank gradient?

A positive answer to this would be very relevant, as it implies that Question 2 has a negative answer. However, the tower constructed in Theorem 3.1 cannot consist of normal subgroups. To see this we argue as follows: using the main theorem in [Ma] we can find a sequence $\{\gamma_j\}$ of hyperbolic elements, $\gamma_j \in G_j$, whose translation lengths are bounded above by 2.634. Since there exist at most finitely many conjugacy classes of hyperbolic elements of bounded translation length in G_1, it must be that an infinite subsequence $\{\gamma_{j_k}\}$ lie in the same conjugacy class in G_1. Let γ be a representative of this class and $g_{j_k} \in G_1$ be such that $\gamma_{j_k} = g_{j_k}\gamma g_{j_k}^{-1}$. If the tower $\{G_j\}$ consists of normal subgroups, then $\gamma \in G_{j_k}$, contradicting the fact that $\{G_{j_k}\}$ is co-final. These covers are actually far from being normal: the Lück Approximation Theorem ([Lu]) implies that these covers do not even satisfy a weaker condition (called *Farber condition*). See [Fa] for details.

Question 3 is relevant also because of the following result (see [AN]):

Theorem (Abért–Nikolov). *Either the rank vs. Heegaard genus conjecture* (*see below*) *is false or Question 2 has a negative solution.*

If an orientable 3-manifold M is closed, a Heegaard splitting of M consists of two handlebodies H_1 and H_2 with their boundaries identified by some orientation preserving homeomorphism. Recall that the genus of, say, ∂H_1 gives an upper bound on the rank of $\pi_1(M)$. If M is not closed, these decompositions are given in terms of compression bodies, again denoted by H_1 and H_2. In order to obtain useful bounds on the rank of $\pi_1(M)$ we restrict ourselves to those decompositions in which H_1, for instance, is a handlebody. Note that if this is the case, then the genus of ∂H_1 is again an upper bound for the rank of $\pi_1(M)$. Recall that the *Heegaard genus* of M is the minimal genus of a Heegaard surface.

Another concept due to Lackenby is that of the *Heegaard gradient* ([La2]). Given a orientable 3-manifold M and a family $\{M_j\}$ of finite sheeted covers, we define the Heegaard gradient of $\{M_j \to M\}$ by

$$\mathrm{Hgr}(M, \{M_j\}) = \lim_{j\to\infty} \frac{-\chi(S_j)}{d_j}$$

where d_j is the degree of the cover $M_j \to M$ and S_j is a minimal genus Heegaard surface for M_j.

Note that if $\mathrm{rgr}(M, \{M_j\}) > 0$, then $\mathrm{Hgr}(M, \{M_j\}) > 0$. The problem that is related to the work of Abért and Nikolov is that of the growth ratio between rank and Heegaard genus:

Conjecture. *Let M be a finite volume hyperbolic 3-manifold and $\{M_i \to M\}$ a family of finite sheeted covers. Then*

$$\mathrm{rgr}(M, \{M_i\}) > 0 \quad \textit{if and only if} \quad \mathrm{Hgr}(M, \{M_i\}) > 0.$$

Our results provide examples for which this is true. In ([La2]) Lackenby showed that if $\pi_1(M)$ is an arithmetic lattice in $\mathrm{PSL}(2, \mathbb{C})$, then M has a co-final family of covers (namely, those arising from congruence subgroups) with positive Heegaard gradient. In [LLR] Long, Lubotzky and Reid generalize this result by proving that every finite volume hyperbolic 3-manifold has a co-final family of finite sheeted regular covers for which the Heegaard gradient is positive. These results were also motivation for this note.

We remark that very recently Tao Li ([Li]) announced examples of closed finite volume hyperbolic 3-manifolds for which the rank is smaller than its Heegaard genus. Whether such examples existed was a long standing question in hyperbolic 3-manifold theory.

A natural question that arises from our results is for which other categories of finite volume hyperbolic 3-manifolds they hold. For instance:

Question 4. Is it true that given a right-angled polyhedron P_1 (not necessarily ideal) and a manifold M_1 such that $\pi_1(M_1)$ has finite index in the reflection group of P_1, then there exists a co-final tower $\{M_j \to M_1\}$ of finite sheeted covers with positive rank gradient?

In our setting the ideal vertices played an important role as they were used to find lower bounds on the rank of the fundamental groups. If the polyhedron P_1 has vertices which are not ideal then we need to find another way of estimating the rank of the associated manifolds. Ian Agol has suggested a way for doing this. We are currently working on appropriate bounds for the rank in this case and will include it in a future work.

We also remark that Lackenby ([La1]) observed that very often it is possible to find towers with positive rank gradient. The problem here is that these towers are not co-final.

Theorem 6.1. *Let M be a compact irreducible 3-manifold with non-empty boundary, other than an I-bundle over a disk, annulus, torus or Klein bottle. Then $\pi_1(M)$ has a tower of finite index subgroups with positive rank gradient*

References

[AN] M. Abért and N. Nikolov, Rank gradient, cost of groups and the rank versus Heegaard genus problem. *J. Eur. Math. Soc.* (*JEMS*) **14** (2012), 1657–1677. Zbl 1271.57046 MR 2966663

[Ag] I. Agol, Criteria for virtual fibering. *J. Topol.* **1** (2008), 269–284. Zbl 1148.57023 MR 2399130

[An] E. M. Andreev, On convex polyhedra in Lobačevskiĭ spaces. *Mat. Sb.* (*N.S.*) **81 (123)** (1970), 445—478; English transl. *Math. USSR-Sb.* **10** (1970), 413–440. Zbl 0217.46801 MR 0259734

[At] C. K. Atkinson, Volume estimates for equiangular hyperbolic Coxeter polyhedra. *Algebr. Geom. Topol.* **9** (2009), 1225–1254. Zbl 1170.57012 MR 2519588

[De] J. DeBlois, Rank gradient of cyclic covers. Preprint 2010. http://www.pitt.edu/ jdeblois/posgrad_take2_4.pdf

[Fa] M. Farber, Geometry of growth: approximation theorems for L^2 invariants. *Math. Ann.* **311** (1998), 335–375. Zbl 0911.53026 MR 1625742

[Ga] D. Gaboriau, Coût des relations d'équivalence et des groupes. *Invent. Math.* **139** (2000), 41–98. Zbl 0939.28012 MR 1728876

[La1] M. Lackenby, Expanders, rank and graphs of groups. *Israel J. Math.* **146** (2005), 357–370. Zbl 1066.22008 MR 2151608

[La2] M. Lackenby, Heegaard splittings, the virtually Haken conjecture and property (τ). *Invent. Math.* **164** (2006), 317–359. Zbl 1110.57015 MR 2218779

[Li] T. Li, Rank and genus of 3-manifolds. *J. Amer. Math. Soc.* **26** (2013), 777–829. Zbl 1277.57004 MR 3037787

[Lu] W. Lück, *L^2-invariants: theory and applications to geometry and K-theory*. Ergeb. Math. Grenzgeb. (3) 44, Springer-Verlag, Berlin 2002. Zbl 1009.55001 MR 1926649

[LLR] D. D. Long, A. Lubotzky, and A. W. Reid, Heegaard genus and property τ for hyperbolic 3-manifolds. *J. Topol.* **1** (2008), 152–158. Zbl 1158.57018 MR 2365655

[LS] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*. Ergeb. Math. Grenzgeb. 89, Springer-Verlag, Berlin 1977. Zbl 0368.20023 MR 0577064

[Ma] J. D. Masters, Injectivity radii of hyperbolic polyhedra. *Pacific J. Math.* **197** (2001), 369–382. Zbl 1058.52004 MR 1815261

[Ti] J. Tits, Systèmes générateurs de groupes de congruence. *C. R. Acad. Sci. Paris Sér. A* **283** (1976), 693–695. Zbl 0381.14005 MR 0424966

Received September 21, 2011; revised February 15, 2012

D. Girão, Departamento de Matemática, Universidade Federal do Ceará, Av. Humberto Monte, s/n - Campus do PICI Bloco 914, Fortaleza, CE 60455-900, Brazil

E-mail: dgirao@mat.ufc.br