

The girth alternative for mapping class groups

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Abstract. The *girth* of a finitely generated group G is defined to be the supremum of the girth of its Cayley graphs. Let G be a finitely generated subgroup of the mapping class group Mod_Σ , where Σ is an orientable closed surface with a finite number of punctures and with a finite number of components. We show that G is either a non-cyclic group with infinite girth or a virtually free-abelian group; these alternatives are mutually exclusive. The proof is based on a simple dynamical criterion for a finitely generated group to have infinite girth, which may be of independent interest.

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1. Introduction

Let Σ be an orientable closed surface with a finite number of punctures and with a finite number of components. The *mapping class group* of Σ , denoted by Mod_Σ , is the group of isotopy classes of the group $\text{Homeo}^+(\Sigma)$ of orientation preserving homeomorphisms of Σ . Mapping class groups have been studied in complex analysis, low-dimensional topology, and geometric group theory for more than a century.

Mapping class groups share many properties with lattices in linear groups. One analogy between linear groups and mapping class groups can be seen in the following famous dichotomy regarding their subgroups, known as the *Tits-alternative* for linear groups and mapping class groups respectively.

Theorem ([30]). *Let \mathbb{k} be a field, and let G be a finitely generated subgroup of $\text{GL}(n, \mathbb{k})$. Then G either contains a non-abelian free subgroup or is virtually solvable; moreover, these alternatives are mutually exclusive.*

Theorem ([12], [20]). *Let Σ be an orientable closed surface with a finite number of punctures and with a finite number of components, and let G be a finitely generated subgroup of Mod_Σ . Then G either contains a non-abelian free subgroup or is virtually free-abelian; moreover, these alternatives are mutually exclusive.*

Some refinements and variations of the Tits alternative are known for linear groups and mapping class groups, such as the Margulis–Soifer theorem on maximal subgroups of linear groups [19] and the analogous result by Ivanov for mapping class groups [12]. This article demonstrates that the structural analogy between these groups can also be witnessed in the *girth* of a finitely generated subgroup.

Recall that the *girth* of a graph is the length of the shortest graph cycle, if any, in the graph. The girth of a Cayley graph has been studied extensively, particularly in its relationship to expansion properties; see, for example, [7], [8], [16], [18]. In [27], Schleimer introduced the *girth of a finitely generated group* G , denoted by $\text{girth}(G)$ in this article, to be the supremum of the girth of Cayley graphs of G over all finite generating sets. The qualitative difference between groups with finite girth and groups with infinite girth is evident in observations of Schleimer [27] and the work of Akhmedov [3]. In particular, Akhmedov gave the following *girth alternative* for linear groups, which shows that the dichotomy between finite girth and infinite girth coincides essentially with the dichotomy in the Tits alternative for linear groups.

Theorem ([3]). *Let \mathbb{k} be a field, and let G be a finitely generated subgroup of $\text{GL}(n, \mathbb{k})$. Then G is either a non-cyclic group with infinite girth or a virtually solvable group; moreover, these alternatives are mutually exclusive.*

Our result is the following analogous girth alternative for mapping class groups, showing that the dichotomy between subgroups with finite girth and the ones with infinite girth again coincides essentially with the dichotomy in the Tits alternative.

Theorem 1. *Let Σ be an orientable closed surface with a finite number of punctures and with a finite number of components, and let G be a finitely generated subgroup of Mod_Σ . Then G is either a non-cyclic group with infinite girth or a virtually free-abelian group; moreover, these alternatives are mutually exclusive.*

In the general context of finitely generated groups, the property of having infinite girth is neither stronger nor weaker than the property of containing non-abelian free subgroups; see §2 and references therein. Hence, the girth alternative is, *a priori*, neither stronger nor weaker than the Tits alternative; indeed, Akhmedov recently announced [4] that the girth alternative holds for subgroups of $\text{PL}^+(I)$, i.e., the group of orientation-preserving piecewise-linear homeomorphisms of a closed interval, although the Tits alternative fails for subgroups of $\text{PL}^+(I)$.

As we shall see, for mapping class groups, the girth alternative is a slightly more intricate manifestation of underlying structural properties that are responsible for the Tits alternative. Our proof of Theorem 1 capitalizes on Proposition 2 below, a criterion for a group to have infinite girth, which generalizes and reformulates the work of Akhmedov [3]; Proposition 2 is quite versatile and can be applied to many classes of groups for which the Tits alternative is known to hold [21].

Proposition 2. *Let G be a group acting on a set X , with a finite generating set $S := \{\gamma_1, \dots, \gamma_n\}$. Suppose there exist elements $\sigma, \tau \in G$, subsets $U_\sigma, U_\tau \subset X$, and a point $x \in X$, such that*

- (1) $x \notin (U_\sigma \cup U_\tau) \cup \bigcup_{\varepsilon=\pm 1} \bigcup_{i=1}^n \gamma_i^\varepsilon(U_\sigma \cup U_\tau)$,
- (2) $\sigma^k(\{x\} \cup U_\tau \cup \bigcup_{\varepsilon=\pm 1} \bigcup_{i=1}^n \gamma_i^\varepsilon(U_\tau)) \subset U_\sigma$ for all $k \in \mathbb{Z} - \{0\}$, and
- (3) $\tau^k(\{x\} \cup U_\sigma \cup \bigcup_{\varepsilon=\pm 1} \bigcup_{i=1}^n \gamma_i^\varepsilon(U_\sigma)) \subset U_\tau$ for all $k \in \mathbb{Z} - \{0\}$.

Then G is a non-cyclic group with $\text{girth}(G) = \infty$.

One difficulty in proving Theorem 1 is that Proposition 2 cannot be readily applied to a subgroup $G < \text{Mod}_\Sigma$ when Σ is *disconnected* or when G is *reducible*; see §4 for the definition of reducible subgroups. Moreover, unlike the Tits alternative, it is not sufficient to prove the statement of Theorem 1 for a finite index subgroup of G ; it is not known if a group G has infinite girth when G has a finite index subgroup with infinite girth. To work around these issues, we will study the structure of the subgroup $G < \text{Mod}_\Sigma$ carefully with Ivanov’s theory [13]; some aspects of his theory that were essential in the proof of the Margulis–Soifer theorem for Mod_Σ , which is much stronger than the Tits alternative, are crucial in our work as well.

For the special case of *irreducible* subgroups of Mod_Σ of a *connected* surface Σ , the girth alternative was obtained independently by Yamagata [31]; see §4 for the definition of an irreducible subgroup. Her result is equivalent to our Proposition 15 in §5, and her proof is based on a criterion for a group to have infinite girth, somewhat similar to, but weaker than, Proposition 2.

Outline. In §2, we review some results on the girth of finitely generated groups, and give the proof of Proposition 2, which is a reminiscent of the classical *ping-pong argument*. In §3 and §4, we review the properties of elements and subgroups of Mod_Σ from Thurston’s theory [29], [6], and Ivanov’s theory [13], and collect a few observations that are necessary for the application of Proposition 2 in the proof of Theorem 1. Finally, in §5, we prove Theorem 1.

Conventions. Throughout the article, a surface Σ will always be an orientable closed surface with a finite number of punctures and possibly with a finite number of components; for brevity, such a surface will be referred to as a *finite-type surface without boundary*. We emphasize that, by definition, an element of Mod_Σ may permute the punctures and the components of Σ .

A simple closed curve on Σ is said to be *peripheral* if it bounds a closed disk with one puncture in its interior. A *multi-loop* on Σ is a pairwise disjoint collection of

simple closed curves on Σ , and it is said to be *essential* if each component is neither null-homotopic, peripheral, nor parallel to another component.

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2. Girth of finitely generated groups

The *girth* of a graph \mathcal{G} , denoted by $\text{girth}(\mathcal{G})$, is the combinatorial length of the shortest nontrivial cycle in \mathcal{G} if there is such a cycle, and is set to be infinity if there is no nontrivial cycle in \mathcal{G} . Using the girth of Cayley graphs, Schleimer introduced in [27] the notion of the girth of a finitely generated group. Throughout the article, we will always assume that a group is finitely generated.

Definition. Let G be a group. For each generating set S of G , let $\mathcal{G}(G, S)$ be the Cayley graph of G with respect to S . The *girth* of the group G , denoted by $\text{girth}(G)$, is defined to be $\text{girth}(G) := \sup_S \{\text{girth}(\mathcal{G}(G, S))\}$, where the supremum is taken over all finite generating sets S of G .

Clearly, every free group has infinite girth, and every finite group has finite girth; it is also easy to see that an abelian group has finite girth unless it is infinite-cyclic. In this section, we discuss criteria for the girth of a group to be finite or infinite.

2.1. Subgroups and quotients. We first collect a few results from [27] and [3] on the relationship between the girth and homomorphisms.

Lemma 3 ([27]). *If a group G surjects onto a finite-kernel quotient G' with $\text{girth}(G') < \infty$, then $\text{girth}(G) < \infty$.*

Lemma 4 ([27]). *If a group G contains a finite-index subgroup $G'' < G$ with $\text{girth}(G'') < \infty$, then $\text{girth}(G) < \infty$.*

Proposition 5 ([3]). *If a group G surjects onto a non-cyclic quotient G' with $\text{girth}(G') = \infty$, then $\text{girth}(G) = \infty$.*

It should be noted that there exists a group with finite girth, containing a subgroup with infinite girth; more specifically, Akhmedov [2], §4, observed that Olshanskii's

result [25], §34–35, can be reformulated to produce a group with finite girth, containing an infinite-index non-abelian free subgroup. Akhmedov has also shown that a group has infinite girth if it contains a finite-index non-abelian free subgroup [3], §2. However, it is not known in general if a group has infinite girth when a finite-index subgroup has infinite girth. In the proof of Theorem 1, instead of studying the girth of finite-index subgroups, we look for a surjection of the given group onto another group with infinite girth and apply Proposition 5.

2.2. Criterion for finite girth. Generalizing the fact that every abelian group $G \not\cong \mathbb{Z}$ has finite girth, Schleimer obtained a useful criterion for a group to have finite girth [27]. Recall that a group G is said to *satisfy a law* if there is a word $w(x_1, \dots, x_n)$ on n letters such that $w(\gamma_1, \dots, \gamma_n) = 1$ in G for any $\gamma_1, \dots, \gamma_n \in G$.

Theorem 6 ([27]). *If a group G satisfies a law and $G \not\cong \mathbb{Z}$, then $\text{girth}(G) < \infty$.*

Since every solvable group satisfies a law, Theorem 6 and Proposition 4 together yield the following corollary which is relevant to the girth alternative.

Corollary 7. *If a group G is virtually solvable and $G \not\cong \mathbb{Z}$, then $\text{girth}(G) < \infty$.*

More generally, every uniformly amenable group satisfies a law [14]. There are also non-amenable groups satisfying some law, such as Tarski Monster groups [23], [24], many free Burnside groups [1], and Olshanskii–Sapir groups [26]. By Theorem 6, these groups all have finite girth. We note that there also exists a group with finite girth, satisfying no law; as mentioned in §2.1, there exists a group with finite girth, containing a non-abelian free subgroup, see [2], §4, and [25], §34–35, and such a group clearly satisfies no law.

2.3. Criterion for infinite girth. The proof of the Tits alternative for linear groups and mapping class groups, as well as for other classes of groups, use variations of the so-called *ping-pong lemma* to construct a non-abelian free subgroup; the classical lemma goes back to the work of Blaschke, Klein, Schottky, and Poincaré; see, for example, [15]. The following formulation was given in [30].

Proposition 8 ([30]). *Let G be a group acting on a set X . Suppose there exist elements $\sigma, \tau \in G$, subsets $U_\sigma, U_\tau \subset X$, and a point $x \in X$, such that*

- (1) $x \notin U_\sigma \cup U_\tau$,
- (2) $\sigma^k(\{x\} \cup U_\tau) \subset U_\sigma$ for all $k \in \mathbb{Z} - \{0\}$, and
- (3) $\tau^k(\{x\} \cup U_\sigma) \subset U_\tau$ for all $k \in \mathbb{Z} - \{0\}$.

Then $\langle \sigma, \tau \rangle$ is a non-abelian free subgroup of G .

Proof. Observe inductively that any nontrivial reduced word in $\sigma^{\pm 1}$ and $\tau^{\pm 1}$ takes $x \in X - (U_\sigma \cup U_\tau)$ into $U_\sigma \cup U_\tau$ via the action of $\langle \sigma, \tau \rangle$, showing that the word cannot represent the identity element of G . This is the classical *ping-pong argument*. \square

In the study of the girth of groups, it is instrumental to have a criterion for a group to have infinite girth. Generalizing and reformulating the ideas appearing in the work of Akhmedov [3], we give the following criterion, as stated in §1.

Proposition 2. *Let G be a group acting on a set X , with a finite generating set $S := \{\gamma_1, \dots, \gamma_n\}$. Suppose there exist elements $\sigma, \tau \in G$, subsets $U_\sigma, U_\tau \subset X$, and a point $x \in X$, such that*

- (1) $x \notin (U_\sigma \cup U_\tau) \cup \bigcup_{\varepsilon=\pm 1} \bigcup_{i=1}^n \gamma_i^\varepsilon(U_\sigma \cup U_\tau)$,
- (2) $\sigma^k \left(\{x\} \cup U_\tau \cup \bigcup_{\varepsilon=\pm 1} \bigcup_{i=1}^n \gamma_i^\varepsilon(U_\tau) \right) \subset U_\sigma$ for all $k \in \mathbb{Z} - \{0\}$, and
- (3) $\tau^k \left(\{x\} \cup U_\sigma \cup \bigcup_{\varepsilon=\pm 1} \bigcup_{i=1}^n \gamma_i^\varepsilon(U_\sigma) \right) \subset U_\tau$ for all $k \in \mathbb{Z} - \{0\}$.

Then G is a non-cyclic group with $\text{girth}(G) = \infty$.

Since $\sigma, \tau, U_\sigma, U_\tau$, and $x \in X$ in Proposition 2 satisfy the conditions in Proposition 8, we see that σ and τ generate a non-abelian free subgroup $\langle \sigma, \tau \rangle < G$. Proposition 2 applies to many groups which are known to contain non-abelian free subgroup by some version of the ping-pong argument; in the present work, we focus on the application of Proposition 2 to mapping class groups. The criterion can also be applied to non-elementary convergence groups, subgroups of $\text{Out}(F_n)$ containing strongly irreducible elements, many CAT(0) spaces, and some groups acting on trees; some results on the girth of these groups are collected in [21].

Although every group that satisfies Proposition 2 contains a non-abelian free subgroup, the property of having infinite girth and the property of containing a non-abelian free subgroup are generally independent of each other. As we have noted in §2.1, there exists a group with finite girth, containing a non-abelian free subgroup, [2], §4, [25], §34–35. On the other hand, there also exists a group with infinite girth, containing no non-abelian free subgroups [2], §2. Thompson's group F is another example of a group with infinite girth [9], [5], containing no non-abelian free subgroup [10].

Proof of Proposition 2. Let M be a positive integer, and we aim to find a new generating set \hat{S} for G such that $\text{girth}(\mathcal{G}(G, \hat{S})) \geq M$. Let $P = \{p_1, \dots, p_n\}$ be a set of positive integers such that $p_i > M$ for all i and $|p_i - p_j| > M$ for all distinct i, j . Let $\hat{\gamma}_i := \sigma^{p_i} \gamma_i \tau^{-p_i}$ for each i . The set $\hat{S} := \{\sigma, \tau, \hat{\gamma}_1, \dots, \hat{\gamma}_n\}$ clearly generates G .

Let w be a nontrivial reduced word in $\hat{S} \cup \hat{S}^{-1}$ of length less than M with respect to $\hat{S} \cup \hat{S}^{-1}$. We can write w as

$$w = u_1 \hat{\gamma}_{i_1}^{\varepsilon_1} u_2 \hat{\gamma}_{i_2}^{\varepsilon_2} \dots \hat{\gamma}_{i_s}^{\varepsilon_s} u_{s+1}$$

where $\varepsilon_\ell \in \{\pm 1\}$ and the subword $u_\ell = u_\ell(\sigma, \tau)$ is a (possibly empty) reduced word in $\{\sigma^{\pm 1}, \tau^{\pm 1}\}$. If u_ℓ is an empty word and $i_{\ell-1} = i_\ell$ for some ℓ , we must have $\varepsilon_{\ell-1} = \varepsilon_\ell$. For otherwise, a cancellation occurs and contradicts the assumption that w is a reduced word in $\hat{S} \cup \hat{S}^{-1}$.

Now, regarded as an element of G , w can be expressed as

$$\begin{aligned} w &= u_1 \hat{\gamma}_{i_1}^{\varepsilon_1} u_2 \hat{\gamma}_{i_2}^{\varepsilon_2} \dots \hat{\gamma}_{i_s}^{\varepsilon_s} u_{s+1} \\ &= v_1 \gamma_{i_1}^{\varepsilon_1} v_2 \gamma_{i_2}^{\varepsilon_2} \dots \gamma_{i_s}^{\varepsilon_s} v_{s+1} \end{aligned}$$

where $v_\ell = v_\ell(\sigma, \tau)$ is a reduced word in $\{\sigma^{\pm 1}, \tau^{\pm 1}\}$ for $\beta_{\ell-1} u_\ell \alpha_\ell$, with convention $\alpha_{s+1} = \beta_0 = 1$, and

$$\alpha_\ell = \begin{cases} \sigma^{p_{i_\ell}} & \text{if } \varepsilon_\ell = +1, \\ \tau^{p_{i_\ell}} & \text{if } \varepsilon_\ell = -1, \end{cases} \quad \beta_\ell = \begin{cases} \tau^{-p_{i_\ell}} & \text{if } \varepsilon_\ell = +1, \\ \sigma^{-p_{i_\ell}} & \text{if } \varepsilon_\ell = -1. \end{cases}$$

The idea of the proof is to apply the ping-pong argument to w to show that w cannot represent the identity element of G . Provided with suitable initial points in X , the ping-pong argument applies easily to the strings v_ℓ . What we need to show is that we can pass each $\gamma_{i_\ell}^{\varepsilon_\ell}$ in the ping-pong argument; in other words, we need to check that $\gamma_{i_\ell}^{\varepsilon_\ell}$ takes the terminal point from the ping-pong rally $v_{\ell+1}$ to a suitable initial point for the ping-pong rally v_ℓ . We will see that our choice of p_i prevents excessive cancellations, and we can indeed pass each $\gamma_{i_\ell}^{\varepsilon_\ell}$ under the conditions (2) and (3) in the statement of the proposition.

Claim 1. For each ℓ , v_ℓ is not an empty word. If $\varepsilon_\ell = +1$, then the last letter of v_ℓ is $\sigma^{\pm 1}$ and the first letter of $v_{\ell+1}$ is $\tau^{\pm 1}$. If $\varepsilon_\ell = -1$, then the last letter of v_ℓ is $\tau^{\pm 1}$ and the first letter of $v_{\ell+1}$ is $\sigma^{\pm 1}$.

Proof of Claim 1. Let us show that, if $\varepsilon_\ell = +1$, then v_ℓ is a non-empty word ending with $\sigma^{\pm 1}$. Since $\varepsilon_\ell = +1$, we have $\alpha_\ell = \sigma^{p_{i_\ell}}$ and $\beta_\ell = \tau^{-p_{i_\ell}}$. There are three cases to consider: (i) u_ℓ is an empty word; (ii) the last letter of u_ℓ is $\tau^{\pm 1}$; or (iii) the last letter of u_ℓ is $\sigma^{\pm 1}$.

Case (i): If u_ℓ is empty, then v_ℓ is the reduced word for $\beta_{\ell-1} \alpha_\ell$, and hence

$$v_\ell = \begin{cases} \sigma^{p_{i_\ell}} & \text{if } \ell = 1, \\ \tau^{-p_{i_{\ell-1}}} \sigma^{p_{i_\ell}} & \text{if } \ell \neq 1 \text{ and } \varepsilon_{\ell-1} = +1, \\ \sigma^{-p_{i_{\ell-1}} + p_{i_\ell}} & \text{if } \ell \neq 1 \text{ and } \varepsilon_{\ell-1} = -1. \end{cases}$$

In the last subcase, since $\varepsilon_\ell = +1 \neq -1 = \varepsilon_{\ell-1}$, we must have $i_\ell \neq i_{\ell-1}$ as noted before. Thus, we must have $|p_{i_\ell} - p_{i_{\ell-1}}| > M$, and it follows that v_ℓ is a nontrivial power of σ . In all subcases, v_ℓ is indeed a non-empty word ending with $\sigma^{\pm 1}$.

Case (ii): If the last letter of u_ℓ is $\tau^{\pm 1}$, then there is no cancellation between u_ℓ and $\alpha_\ell = \sigma^{p_{i_\ell}}$ as a word in $\{\sigma^{\pm 1}, \tau^{\pm 1}\}$. Hence, v_ℓ is again a non-empty word ending with $\sigma^{\pm 1}$.

Case (iii): Finally, suppose the last letter of u_ℓ is $\sigma^{\pm 1}$. If u_ℓ is not a power of σ , then $u_\ell = \dots \tau^q \sigma^p$ for some q and p . So, $u_\ell \alpha_\ell = \dots \tau^q \sigma^{p+p_{i_\ell}}$. Note that we must have $|p| < M$. For otherwise, the length of u_ℓ as a word in $\hat{S} \cup \hat{S}^{-1}$, and thus the length of w as a word in $\hat{S} \cup \hat{S}^{-1}$, is at least M ; this contradicts with the assumption on the length of w . Now, $|p| < M$ and $p_{i_\ell} > M$ together imply $p + p_{i_\ell} \neq 0$. Thus, the last letter of v_ℓ must be $\sigma^{\pm 1}$. If u_ℓ is a power of σ , say $u_\ell = \sigma^p$, then

$$v_\ell = \begin{cases} \sigma^{p+p_{i_\ell}} & \text{if } \ell = 1, \\ \tau^{-p_{i_{\ell-1}}} \sigma^{p+p_{i_\ell}} & \text{if } \ell \neq 1 \text{ and } \varepsilon_{\ell-1} = +1, \\ \sigma^{-p_{i_{\ell-1}}+p+p_{i_\ell}} & \text{if } \ell \neq 1 \text{ and } \varepsilon_{\ell-1} = -1. \end{cases}$$

In the first two subcases, v_ℓ ends with a nontrivial power of σ , because $|p| < M$ and $p_{i_\ell} > M$ imply $p + p_{i_\ell} \neq 0$. In the third subcase, we must have $i_\ell \neq i_{\ell-1}$, and hence $|p_{i_\ell} - p_{i_{\ell-1}}| > M$. It now follows from $|p| < M$ that $-p_{i_{\ell-1}} + p + p_{i_\ell} \neq 0$, and v_ℓ is again a nontrivial power of σ . Thus, in all subcases, v_ℓ is again a non-empty word ending with $\sigma^{\pm 1}$ as desired.

This concludes the proof that, if $\varepsilon_\ell = +1$, then v_ℓ is a non-empty word ending with $\sigma^{\pm 1}$. The analogous arguments show that, if $\varepsilon_\ell = -1$, then $v_{\ell+1}$ is a non-empty word beginning with $\tau^{\pm 1}$. The symmetric arguments show that, if $\varepsilon_\ell = -1$, then v_ℓ is a nonempty word ending with $\tau^{\pm 1}$ and $v_{\ell+1}$ is a non-empty word beginning with $\sigma^{\pm 1}$. \square

Claim 2. If the last letter of v_{s+1} is $\sigma^{\pm 1}$ and $y \in \{x\} \cup U_\tau \cup \bigcup_{\varepsilon=\pm 1} \bigcup_{i=1}^n \gamma_i^\varepsilon(U_\tau)$, or if the last letter of v_{s+1} is $\tau^{\pm 1}$ and $y \in \{x\} \cup U_\sigma \cup \bigcup_{\varepsilon=\pm 1} \bigcup_{i=1}^n \gamma_i^\varepsilon(U_\sigma)$, then $w(y) \in U_\sigma \cup U_\tau$.

Proof of Claim 2. We will prove the claim by induction on s . If $s = 0$, $w = v_1$ is merely a reduced word in $\sigma^{\pm 1}$ and $\tau^{\pm 1}$. In this case, $w(y) \in U_\sigma \cup U_\tau$ follows from the classical ping-pong argument as in the proof of the Free Subgroup Criterion.

Now, as the induction hypothesis, suppose that the claim is true for $s - 1 \geq 0$, and let $w = v_1 \gamma_{i_1}^{\varepsilon_1} v_2 \gamma_{i_2}^{\varepsilon_2} \dots \gamma_{i_s}^{\varepsilon_s} v_{s+1}$. Suppose $\varepsilon_s = +1$ for now, so that the first letter of v_{s+1} is $\tau^{\pm 1}$ by Claim 1. Then we have $v_{s+1}(y) \in U_\tau$ by the classical ping-pong argument, and we obtain $y' := \gamma_{i_s} v_{s+1}(y) \in \gamma_{i_s}(U_\tau)$. Now, also by Claim 1, the last letter of v_s is $\sigma^{\pm 1}$. Thus, applying the induction hypothesis to $w' := v_1 \gamma_{i_1}^{\varepsilon_1} v_2 \gamma_{i_2}^{\varepsilon_2} \dots \gamma_{i_{s-1}}^{\varepsilon_{s-1}} v_s$ and y' , we see that $w(y) = w'(y') \in U_\sigma \cup U_\tau$. The $\varepsilon_s = -1$ case is analogous. \square

Since $x \notin U_\sigma \cup U_\tau$ by the assumption and $w(x) \in U_\sigma \cup U_\tau$ by Claim 2, it follows that w cannot represent the identity element in G . Namely, any non-empty word in \hat{S} that represents the identity element of G must be of length at least M with respect to \hat{S} . Hence, $\text{girth}(G) \geq \text{girth}(\mathcal{G}(G, \hat{S})) \geq M$. \square

3. Elements of mapping class groups

We review some aspects of the Nielsen–Thurston classification of elements of Mod_Σ , for an orientable finite-type surface Σ with $\partial\Sigma = \emptyset$. An isotopy class \mathcal{A} of an essential multi-loop on Σ is a *reduction system* for $\sigma \in \text{Mod}_\Sigma$ if σ fixes \mathcal{A} . An element in Mod_Σ is *reducible* if it admits a *non-empty* reduction system \mathcal{A} , and is *irreducible* otherwise; it is *periodic* if its order is finite, and is *aperiodic* otherwise. There are three mutually exclusive types of elements in Mod_Σ : (i) periodic elements, (ii) aperiodic irreducible elements, and (iii) aperiodic reducible elements.

This trichotomy was studied by Nielsen for tori [22], and by Thurston more generally for other surfaces [29]. The theory was further developed by Birman, Lubotzky, and McCarthy [6], [20], and by Ivanov [12], [13], Chap. 2–8, leading to the proof of the Tits alternative for Mod_Σ . For the rest of this section, we assume that each component of Σ has negative Euler characteristic.

3.1. Canonical reduction of reducible elements. Suppose that \mathcal{A} is a reduction system for an element $\sigma \in \text{Mod}_\Sigma$, and let $\Sigma_{\mathcal{A}} := \Sigma - A$ where A is a multi-loop representing \mathcal{A} ; note that $\Sigma_{\mathcal{A}}$ is again an orientable finite-type surface with $\partial\Sigma_{\mathcal{A}} = \emptyset$, and σ induces a mapping class $\rho_{\mathcal{A}}(\sigma) \in \text{Mod}_{\Sigma_{\mathcal{A}}}$, called the *reduction of σ along \mathcal{A}* . We say that an element of Mod_Σ is *neat* if it takes each component of Σ to itself and each puncture of Σ to itself; adopting the terminology from [6], we say that an element $\sigma \in \text{Mod}_\Sigma$ is *adequately reduced* if there exists a neat nontrivial power σ^m whose restriction to each component of Σ is either (i) trivial or (ii) aperiodic and irreducible. A reduction system \mathcal{A} of an element $\sigma \in \text{Mod}_\Sigma$ is said to be an *adequate reduction system* if the reduction $\rho_{\mathcal{A}}(\sigma)$ is adequately reduced.

By Thurston’s classification theorem [29], Thm. 4, every element $\sigma \in \text{Mod}_\Sigma$ is either an adequately reduced element or an aperiodic reducible element with non-empty adequate reduction systems. For an aperiodic reducible element σ , there is a canonical choice of a non-empty adequate reduction system, [6], Thm. C, [13], §7.4, §7.11; it is indeed the unique minimal adequate reduction system for the element σ . We call such a system the *canonical reduction system* for σ , and denote it by \mathcal{C}_σ , or simply by \mathcal{C} if the reference to σ is clear from the context.

3.2. Adequately reduced elements. We recall the description of adequately reduced elements based on Thurston’s theory. In particular, we review the dynamical properties of aperiodic irreducible elements, acting on the spaces of *projective measured foliations* on a surface.

Connected surfaces. Suppose for now that Σ is connected, and write $g(\Sigma)$ and $n(\Sigma)$ for the genus and the number of punctures of Σ . It follows from the discussion in §3.1 that an element $\sigma \in \text{Mod}_\Sigma$ is adequately reduced if and only if it is (i) periodic or (ii) aperiodic and irreducible. We summarize Thurston’s characterization [29] of aperiodic irreducible elements; the details can be found in [11], Exp. 8–9.

Thurston introduced the space $\mathcal{PMF}(\Sigma)$ of *projective measured foliations* on Σ , homeomorphic to a sphere of dimension $6g(\Sigma) + 2n(\Sigma) - 7$. The mapping class group Mod_Σ acts on $\mathcal{PMF}(\Sigma)$ as homeomorphisms. An element $\sigma \in \text{Mod}_\Sigma$ is said to be *pseudo-Anosov* if the fixed-point set $\text{Fix}(\sigma) \subset \mathcal{PMF}(\Sigma)$ of the action of σ on $\mathcal{PMF}(\Sigma)$ consists of precisely two points in $\mathcal{PMF}(\Sigma)$. An essential property of a pseudo-Anosov element σ is that its action on $\mathcal{PMF}(\Sigma)$ exhibits the *north-south dynamics* with one of the fixed points as an attractor, denoted by \mathcal{F}_σ^+ , and the other as a repeller, denoted by \mathcal{F}_σ^- ; more precisely, for any pair of disjoint neighborhoods U_σ^+ of \mathcal{F}_σ^+ and U_σ^- of \mathcal{F}_σ^- , we have $\sigma^{\pm N}(\mathcal{PMF}(\Sigma) - U_\sigma^\mp) \subset U_\sigma^\pm$ respectively for all sufficiently large N . It turns out that aperiodic irreducible elements of Mod_Σ are precisely pseudo-Anosov ones [29], Thm. 4. If $\sigma, \tau \in \text{Mod}_\Sigma$ are pseudo-Anosov elements, then $\text{Fix}(\sigma) = \text{Fix}(\tau)$ or $\text{Fix}(\sigma) \cap \text{Fix}(\tau) = \emptyset$ in $\mathcal{PMF}(\Sigma)$ [13], §5.11; we say that σ and τ are *independent* if $\text{Fix}(\sigma) \cap \text{Fix}(\tau) = \emptyset$.

Disconnected surfaces. Let us now allow $\Sigma = \bigsqcup_i \Sigma_i$ to be disconnected, where Σ_i denotes each component. With some care, the space $\mathcal{PMF}(\Sigma)$ of projective measured foliations on Σ can be defined as the *join* of $\mathcal{PMF}(\Sigma_i)$, [13], [20]. However, as in the work of Ivanov [13], Chap. 6, it is sufficient and more convenient for our purposes to consider the space

$$\mathcal{PMF}^\#(\Sigma) := \bigsqcup_i \mathcal{PMF}(\Sigma_i)$$

and the action of Mod_Σ on it as homeomorphisms. A neat element in Mod_Σ takes each component $\mathcal{PMF}(\Sigma_i) \subset \mathcal{PMF}^\#(\Sigma)$ to itself under this action.

An element of Mod_Σ is said to be *pseudo-Anosov* if there exists a neat power σ^m whose restriction to each component Σ_i is a pseudo-Anosov element in Mod_{Σ_i} . The action of such a power σ^m on each component $\mathcal{PMF}(\Sigma_i) \subset \mathcal{PMF}^\#(\Sigma)$ exhibits the north-south dynamics with respect to the fixed points $\mathcal{F}_{\sigma^m|\Sigma_i}^\pm \in \mathcal{PMF}(\Sigma_i)$. Moreover, σ fixes $\bigcup_i \mathcal{F}_{\sigma^m|\Sigma_i}^+$ and $\bigcup_i \mathcal{F}_{\sigma^m|\Sigma_i}^-$ setwise, respectively, and the action of σ on $\mathcal{PMF}^\#(\Sigma)$ exhibits the north-south dynamics with respect to these sets; more precisely, for any pair of disjoint neighborhoods U_σ^+ of $\bigcup_i \mathcal{F}_{\sigma^m|\Sigma_i}^+$ and U_σ^- of $\bigcup_i \mathcal{F}_{\sigma^m|\Sigma_i}^-$, we have $\sigma^{\pm N}(\mathcal{PMF}^\#(\Sigma) - U_\sigma^\mp) \subset U_\sigma^\pm$ for all sufficiently large N . If $\sigma, \tau \in \text{Mod}_\Sigma$ are pseudo-Anosov elements, then for each i , $\text{Fix}(\sigma) \cap \mathcal{PMF}(\Sigma_i) = \text{Fix}(\tau) \cap \mathcal{PMF}(\Sigma_i)$ or $\text{Fix}(\sigma) \cap \text{Fix}(\tau) \cap \mathcal{PMF}(\Sigma_i) = \emptyset$; we say that σ and τ are *independent* if $\text{Fix}(\sigma) \cap \text{Fix}(\tau) = \emptyset$, i.e., $\text{Fix}(\sigma) \cap \text{Fix}(\tau) \cap \mathcal{PMF}(\Sigma_i) = \emptyset$ for all i .

Recall that an element $\sigma \in \text{Mod}_\Sigma$ is adequately reduced if there exists a neat nontrivial power σ^m whose restriction to each component Σ_i is either (i) trivial or (ii) aperiodic and irreducible. Clearly, σ is periodic if the restriction of σ^m to each Σ_i is trivial; also, σ is pseudo-Anosov if the restriction of σ^m to each Σ_i is aperiodic and irreducible. Generally, an adequately reduced element may be neither periodic nor pseudo-Anosov; the restrictions of σ^m to some components can be

trivial while the restrictions of σ^m to other components are aperiodic and irreducible.

4. Subgroups of mapping class groups

We review some aspects of Ivanov’s classification [13], Chap. 5–9, of subgroups of Mod_Σ , for an orientable finite-type surface Σ with $\partial\Sigma = \emptyset$. An isotopy class \mathcal{A} of an essential multi-loop is a *reduction system* for $G < \text{Mod}_\Sigma$ if it is a reduction system for every element $\sigma \in G$, i.e., if every element $\sigma \in G$ fixes \mathcal{A} . A subgroup G is *reducible* if it admits a *non-empty* reduction system \mathcal{A} , and is *irreducible* otherwise. There are three mutually exclusive types of subgroups in Mod_Σ : (i) finite subgroups, (ii) infinite irreducible subgroups, and (iii) infinite reducible subgroups. This trichotomy parallels and generalizes the one appearing in the Nielsen–Thurston classification; an element $\sigma \in \text{Mod}_\Sigma$ is (i) periodic, (ii) aperiodic and irreducible, or (iii) aperiodic and reducible, if and only if the cyclic subgroup $\langle \sigma \rangle < \text{Mod}_\Sigma$ is (i) finite, (ii) infinite and irreducible, or (iii) infinite and reducible, respectively.

Ivanov’s classification theory for subgroups of Mod_Σ , strictly speaking, is much deeper than what is required in the proof of the Tits alternative for Mod_Σ . However, it played an essential role in his proof of stronger theorems such as the analogue of Margulis–Soifer theorem [12], [13]. In our proof of the girth alternative, we will also need to utilize Ivanov’s classification theory. For the rest of this section, we assume that each component of Σ has negative Euler characteristic.

4.1. Canonical reduction of reducible subgroups. A *reduction* of a subgroup of Mod_Σ can be defined in a manner analogous to the reduction of an element of Mod_Σ . If \mathcal{A} is a reduction system for $G < \text{Mod}_\Sigma$, then the reduction $\rho_{\mathcal{A}}(\sigma) \in \text{Mod}_{\Sigma_{\mathcal{A}}}$ is well defined for each $\sigma \in G$, where $\Sigma_{\mathcal{A}}$ is defined as in §3.1. The assignment $\sigma \mapsto \rho_{\mathcal{A}}(\sigma)$ indeed defines the *reduction homomorphism*

$$\rho_{\mathcal{A}}: G \rightarrow \text{Mod}_{\Sigma_{\mathcal{A}}}$$

whose kernel is a free-abelian group generated by Dehn twists along some components of \mathcal{A} ; the image $\rho_{\mathcal{A}}(G)$ is called the *reduction of G along \mathcal{A}* . We say a subgroup $G < \text{Mod}_\Sigma$ is *neat* if it consists of neat elements; pushing the analogy further, we say that a subgroup $G < \text{Mod}_\Sigma$ is *adequately reduced* if there exists a neat finite-index normal subgroup $G' \triangleleft G$ whose restriction to each component of Σ is either (i) trivial or (ii) infinite and irreducible. If G is an adequately reduced group with a neat finite index subgroup $G' \triangleleft G$ as in the definition, then, for any finite-index normal subgroup $G'' \triangleleft G$, a neat finite-index subgroup $G' \cap G'' \triangleleft G''$ guarantees that G'' is also adequately reduced.

Every subgroup $G < \text{Mod}_\Sigma$ indeed contains a neat finite-index normal subgroup. In practice, we almost always consider a particular family of such subgroups. For

each integer $m \geq 3$, we consider the natural homomorphisms

$$\text{Mod}_\Sigma \rightarrow \text{Aut}(H_1(\Sigma; \mathbb{Z})) \rightarrow \text{Aut}(H_1(\Sigma; \mathbb{Z}/m\mathbb{Z}))$$

and let $\text{Mod}_\Sigma(m)$ be the kernel of this composition of homomorphisms. $\text{Mod}_\Sigma(m)$ is clearly a finite-index normal subgroup of Mod_Σ , and a classical theorem of Serre [28] says that $\text{Mod}_\Sigma(m)$ is torsion-free. For each subgroup $G < \text{Mod}_\Sigma$, we set $G(m) := G \cap \text{Mod}_\Sigma(m)$; it is a torsion-free finite-index normal subgroup of G . Ivanov made finer observations [13], §1.2, §1.6, on elements of $\text{Mod}_\Sigma(m)$ and their reductions: every element $\sigma \in \text{Mod}_\Sigma(m)$ is a neat element whose restriction to each component of Σ is trivial or aperiodic; furthermore, for every reduction system \mathcal{A} of $\sigma \in \text{Mod}_\Sigma(m)$, the reduction $\rho_{\mathcal{A}}(\sigma)$ is a neat element in $\text{Mod}_{\Sigma_{\mathcal{A}}}$ whose restriction to each component of $\Sigma_{\mathcal{A}}$ is trivial or aperiodic. In particular, for every subgroup $G < \text{Mod}_\Sigma$, $G(m) \triangleleft G$ is a neat finite-index normal subgroup of G .

Lemma 9 (Cf. [13], §7.11). *Suppose $G < \text{Mod}_\Sigma$ is adequately reduced. Then for every integer $m \geq 3$ the restriction of $G(m)$ to each component of Σ is either (i) trivial or (ii) infinite and irreducible.*

Proof. If $G < \text{Mod}_\Sigma$ is adequately reduced, the finite-index normal subgroup $G(m)$ is also adequately reduced; hence, there exists a neat finite-index normal subgroup $G' \triangleleft G(m)$ such that the restriction of G' to each component is either (i) trivial or (ii) infinite and irreducible. Since $G(m)$ is neat and its restriction to each component Σ_i is either trivial or infinite, Lemma follows. \square

By Ivanov's classification theorem [13], §7.11, every subgroup $G < \text{Mod}_\Sigma$ is either an adequately reduced subgroup or an infinite reducible subgroup with non-empty adequate reduction system. For an infinite reducible subgroup G , there is a canonical choice of non-empty reduction system [13], §7.2–7.4; it is indeed the unique minimal adequate reduction system for the subgroup G [13], §7.18. We call such a system the *canonical reduction system* for G , and denote it by \mathcal{C}_G , or simply by \mathcal{C} if the reference to G is clear from the context.

4.2. Adequately reduced subgroups. We now give a description of adequately reduced subgroups based on Ivanov's theory. Some of the material is not entirely explicit in his exposition [13], Chap. 9; we shall extract and gather the relevant statements, and record a few consequences which we will utilize in the next section.

Connected surfaces. Suppose for now that Σ is connected. Ivanov showed that $G(m) \triangleleft G$ contains a pair of independent pseudo-Anosov elements σ and τ if $G(m)$ is irreducible and not infinite-cyclic [13], §5.12; high powers of σ and τ then generates a non-abelian free subgroup of $G(m) \triangleleft G$ in the proof of the Tits alternative.

Theorem 10 (Cf. [13], §5.12). *Fix an integer $m \geq 3$. Let Σ be a connected surface and let $G < \text{Mod}_\Sigma$ be an adequately reduced subgroup. Then one of the following statements holds:*

- (0) $G(m) \triangleleft G$ is trivial, and G is finite;
- (1) $G(m) \triangleleft G$ is infinite-cyclic, and G is virtually infinite-cyclic;
- (2) $G(m) \triangleleft G$ contains an infinite number of pairwise independent pseudo-Anosov elements.

Hence, if G is neither finite nor virtually infinite-cyclic, then, for any finite collection $\varphi_1, \dots, \varphi_n \in G(m) \triangleleft G$ of pseudo-Anosov elements, there exists a pseudo-Anosov element $\psi \in G(m) \triangleleft G$ such that ψ and φ_j are independent for all j .

Remark. Since Σ is connected, it follows from Lemma 9 that $G(m)$ is either (i) trivial or (ii) infinite and irreducible. If $G(m)$ is infinite and irreducible, $G(m)$ contains a pseudo-Anosov element [13], §5.9; so, if $G(m)$ is infinite-cyclic, and hence irreducible, it follows that it is generated by a pseudo-Anosov element.

Proof. We assume $G(m)$ is neither trivial nor infinite-cyclic; by Lemma 9, $G(m)$ is irreducible. Hence, by Ivanov’s result [13], §5.12, mentioned earlier, there exists a pair of independent pseudo-Anosov elements $\sigma, \tau \in G(m)$. For each $k \in \mathbb{Z}$, let $\psi_k := \sigma^k \tau \sigma^{-k} \in G(m)$; it is a pseudo-Anosov element with $\text{Fix}(\psi_k) = \sigma^k(\text{Fix}(\tau))$. For $k \neq 0$, $\text{Fix}(\sigma^k) \cap \text{Fix}(\tau) = \text{Fix}(\sigma) \cap \text{Fix}(\tau) = \emptyset$ and hence $\text{Fix}(\psi_k) \cap \text{Fix}(\tau) = \sigma^k(\text{Fix}(\tau)) \cap \text{Fix}(\tau) = \emptyset$. Then, more generally for $k \neq \ell$, $\text{Fix}(\psi_k) \cap \text{Fix}(\psi_\ell) = \sigma^k(\sigma^{\ell-k}(\text{Fix}(\tau))) \cap \sigma^\ell(\text{Fix}(\tau)) = \sigma^\ell(\text{Fix}(\psi_{k-\ell}) \cap \text{Fix}(\tau)) = \emptyset$; namely, $\Psi := \{\psi_k \mid k \in \mathbb{Z}\} \subset G(m)$ is a pairwise independent collection of pseudo-Anosov elements. Hence, for any finite collection $\varphi_1, \dots, \varphi_n \in G(m)$ of pseudo-Anosov elements, there exists $\psi \in \Psi$ such that $\text{Fix}(\varphi_j) \cap \text{Fix}(\psi) = \emptyset$ for all j . \square

Disconnected surfaces. We now allow the surface $\Sigma = \bigsqcup_i \Sigma_i$ to be disconnected, where Σ_i denotes each component. Ivanov showed that $G(m) \triangleleft G$ contains a pair of independent pseudo-Anosov elements σ and τ if the restriction of $G(m)$ to each component Σ_i is irreducible and not infinite-cyclic [13], §6.4. Generally, the restriction of $G(m)$ to each component of Σ falls into one of the three cases in Theorem 10. An adequately reduced group G can be regarded as a hybrid of these three cases; this gives rise to a partition of Σ into three subsurfaces.

Lemma 11 (Cf. [13], §9.10). *Fix an integer $m \geq 3$, and let $G < \text{Mod}_\Sigma$ be an adequately reduced subgroup. Consider a partition $\Sigma = \Sigma_{[0]} \sqcup \Sigma_{[1]} \sqcup \Sigma_{[2]}$ where each subsurface $\Sigma_{[\ell]}$ is defined as follows:*

- (0) *the subsurface $\Sigma_{[0]}$ is the union of all components Σ_i such that the restriction of $G(m)$ to Σ_i is trivial;*
- (1) *the subsurface $\Sigma_{[1]}$ is the union of all components Σ_i such that the restriction of $G(m)$ to Σ_i is infinite-cyclic;*

- (2) *the subsurface $\Sigma_{[2]}$ is the union of all components Σ_i such that the restriction of $G(m)$ to Σ_i is neither trivial nor infinite cyclic.*

Then every element $\sigma \in G$ preserves this partition, i.e., $\sigma(\Sigma_{[\ell]}) = \Sigma_{[\ell]}$ for each ℓ ; hence, the restriction of G to $\Sigma_{[\ell]}$ is well defined for each ℓ .

The partition in Lemma 11 played no role in proving the Tits alternative for Mod_Σ , while it serves as a critical step in proving the Margulis–Soifer theorem for Mod_Σ [13], Chap. 9. This partition is also essential in the proof of Theorem 1.

Proof. The proof we present here is essentially contained in [13], §9.10. Since $G(m) \triangleleft G$, the conjugate of an element $\tau \in G(m)$ by an element $\sigma \in G$ must again belong to $G(m)$; if s and t are homeomorphisms representing σ and τ respectively, then $s \circ t \circ s^{-1}$ represents the element $\sigma\tau\sigma^{-1} \in G(m)$. For each component Σ_i , the restriction $\tau|_{\Sigma_i}$ is represented by $t|_{\Sigma_i}$, and the restriction $\sigma\tau\sigma^{-1}|_{\sigma(\Sigma_i)}$ is represented by $s \circ (t|_{\Sigma_i}) \circ s^{-1}$. Thus, the conjugation by σ defines a homomorphism from the restriction $G(m)|_{\Sigma_i}$ to the restriction $G(m)|_{\sigma(\Sigma_i)}$; it is clearly an isomorphism with the inverse being the conjugation by σ^{-1} . In particular, if the restrictions $G(m)|_{\Sigma_i}$ and $G(m)|_{\Sigma_j}$ are not isomorphic, e.g. if Σ_i and Σ_j belong to different parts in the partition $\Sigma = \Sigma_{[0]} \sqcup \Sigma_{[1]} \sqcup \Sigma_{[2]}$, then no element of G can take Σ_i to Σ_j . \square

Lemma 12. *Fix an integer $m \geq 3$. Let $\Sigma = \Sigma_{[0]} \sqcup \Sigma_{[1]} \sqcup \Sigma_{[2]}$ be the partition in Lemma 11. For each ℓ , let $G_{[\ell]}$ and $G(m)_{[\ell]}$ be the restrictions of G and $G(m)$ to $\Sigma_{[\ell]}$ respectively, and set $G_{[\ell]}(m) := G_{[\ell]} \cap \text{Mod}_{\Sigma_{[\ell]}}(m)$. Then $G(m)_{[\ell]}$ is a finite-index normal subgroup of $G_{[\ell]}(m)$.*

Proof. By definition, $G_{[\ell]}(m)$ consists of all elements in $G_{[\ell]}$ which acts trivially on $H_1(\Sigma_{[\ell]}; \mathbb{Z}/m\mathbb{Z}) < H_1(\Sigma; \mathbb{Z}/m\mathbb{Z})$. Since $G(m)$ acts trivially on the entire $H_1(\Sigma; \mathbb{Z}/m\mathbb{Z})$, its restriction $G(m)_{[\ell]}$ to $\Sigma_{[\ell]}$ acts trivially on $H_1(\Sigma_{[\ell]}; \mathbb{Z}/m\mathbb{Z})$; hence, $G(m)_{[\ell]} < G_{[\ell]}(m) \triangleleft G_{[\ell]}$. Since $G(m)$ is a finite-index normal subgroup of G , $G(m)_{[\ell]}$ must be a finite-index normal subgroup of $G_{[\ell]}$. Hence, we conclude that $G(m)_{[\ell]}$ must also be a finite-index normal subgroup of the intermediate subgroup $G_{[\ell]}(m)$. \square

Theorem 13 (Cf. [13], §9.10). *Fix an integer $m \geq 3$, and let $G < \text{Mod}_\Sigma$ be an adequately reduced subgroup. Let $\Sigma = \Sigma_{[0]} \sqcup \Sigma_{[1]} \sqcup \Sigma_{[2]}$ be the partition in Lemma 11, and set $G_{[\ell]}$ and $G_{[\ell]}(m)$ as in Lemma 12. Then the following statements hold:*

- (0) $G_{[0]}(m) \triangleleft G_{[0]}$ is trivial, and $G_{[0]}$ is finite;
- (1) $G_{[1]}(m) \triangleleft G_{[1]}$ is free-abelian, and $G_{[1]}$ is virtually free-abelian;
- (2) $G_{[2]}(m) \triangleleft G_{[2]}$ contains an infinite number of pairwise independent pseudo-Anosov elements.

Hence, for any finite collection $\varphi_1, \dots, \varphi_n \in G_{[2]}(m) \triangleleft G_{[2]}$ of pseudo-Anosov elements, there exists a pseudo-Anosov element $\psi \in G_{[2]}(m) \triangleleft G_{[2]}$ such that ψ and φ_j are independent for all j .

Proof. First, we consider $G(m)_{[0]}$, $G_{[0]}(m)$, and $G_{[0]}$. $G(m)_{[0]}$ is trivial by the choice of $\Sigma_{[0]}$ in Lemma 11; so, $G_{[0]}(m)$ is finite by Lemma 12, and hence $G_{[0]}$ is finite. We observe that $G_{[0]}(m)$ is actually trivial, since $G_{[0]}(m)$ is torsion-free.

Next, we consider $G(m)_{[1]}$, $G_{[1]}(m)$, and $G_{[1]}$. The restriction of $G(m)_{[1]}$ to each component $\Sigma_i \subset \Sigma_{[1]}$ is infinite-cyclic by the choice of $\Sigma_{[1]}$ in Lemma 11; hence, the restriction of $G_{[1]}(m)$ to Σ_i must be virtually infinite-cyclic by Lemma 12, and it is indeed infinite-cyclic by Theorem 10. It follows that $G_{[1]}(m)$ must be free-abelian, and hence $G_{[1]}$ is virtually free-abelian.

Finally, we consider $G_{[2]}(m)$ and $G_{[2]}$. Note that it suffices to prove the statement (2) under the assumption $\Sigma = \Sigma_{[2]}$, and hence $G = G_{[2]}$, $G(m) = G_{[2]}(m)$. With this assumption, the restriction of $G(m)$ to each component $\Sigma_i \subset \Sigma_{[2]} = \Sigma$ is neither trivial nor infinite-cyclic by the choice of $\Sigma_{[2]}$ in Lemma 11, and is infinite and irreducible by Lemma 9. Hence, by Ivanov’s result [13], §6.4, mentioned earlier, there exists a pair of independent pseudo-Anosov elements $\sigma, \tau \in G(m)$. As in the proof of Theorem 10, for each $k \in \mathbb{Z}$, let $\psi_k := \sigma^k \tau \sigma^{-k} \in G(m)$; by the same argument, it follows that $\Psi := \{\psi_k \mid k \in \mathbb{Z}\} \subset G(m)$ is a pairwise independent collection of pseudo-Anosov elements. Hence, for any finite collection $\varphi_1, \dots, \varphi_n \in G(m)$ of pseudo-Anosov elements, there exists an element $\psi \in \Psi$ such that $\text{Fix}(\varphi_j) \cap \text{Fix}(\psi) = \emptyset$ for all j . □

5. Girth alternative

We now study the girth of subgroups of Mod_Σ , for an orientable finite-type surface Σ with $\partial\Sigma = \emptyset$. Our main result, the *girth alternative*, states that the dichotomy between the subgroups with infinite girth and the ones with finite girth coincides with the structural dichotomy of the Tits alternative, [12], [20].

Theorem 1. *Let Σ be an orientable finite-type surface with $\partial\Sigma = \emptyset$, and let G be a finitely generated subgroup of Mod_Σ . Then G is either a non-cyclic group with infinite girth or a virtually free-abelian group; moreover, these alternatives are mutually exclusive.*

Theorem 1 reduces to the special case, Theorem 14 below, where Σ admits a complete hyperbolic metric. We first prove Theorem 1, assuming Theorem 14.

Theorem 14. *Let Σ be an orientable finite-type surface with $\partial\Sigma = \emptyset$, which admits a complete hyperbolic metric, and let G be a finitely generated subgroup of Mod_Σ . Then G is either a non-cyclic group with infinite girth or a virtually free-abelian group; moreover, these alternatives are mutually exclusive.*

Proof of Theorem 1. Suppose for now that Σ is a union of tori and hyperbolic components. Since the mapping class groups of tori and once-punctured tori are isomorphic, we may replace the copies of tori in Σ with the same number of copies of once-punctured tori. In turn, we now realize G as a subgroup of the mapping class group of a hyperbolic surface; Theorem 1 follows from Theorem 14 as desired.

For the general case, let $\Sigma = \Sigma^+ \sqcup \Sigma^-$ where Σ^+ is the union of spheres with at most two punctures and Σ^- is the union of tori and hyperbolic components. Let G^- be the restriction of G to Σ^- ; we have shown above that G^- is either a non-cyclic group with infinite girth or a virtually free-abelian group. Note that the restriction G^- of the subgroup G to Σ^- is the image of the restriction homomorphism $G \rightarrow \text{Mod}_{\Sigma^-}$; hence, if G^- is a non-cyclic group with infinite girth, then so is G by Proposition 5. So, we assume that G^- is virtually free-abelian and let $A < G^-$ be a finite-index free-abelian subgroup.

The mapping class group is trivial for the sphere and the sphere with one puncture, and is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ for the sphere with two punctures; hence, Mod_{Σ^+} is finite. It follows that the kernel $K \triangleleft G$ of the restriction homomorphism $G \rightarrow \text{Mod}_{\Sigma^+}$ is a finite-index normal subgroup of G . Let $K^- \triangleleft G^-$ be the restriction of K to Σ^- . Then $K^- \cap A$ is a finite-index free-abelian subgroup of K^- . Now, observe that K and K^- are actually isomorphic since K acts trivially on Σ^+ . Hence, K contains a finite-index free-abelian subgroup $H < K$, corresponding to $K^- \cap A < K^-$. Since K is a finite-index subgroup of G , it follows that $H < K < G$ is a finite-index free-abelian subgroup of G ; in other words, G is virtually free-abelian as desired. \square

We prove Theorem 14 in the rest of this section. In §5.1, we treat adequately reduced subgroups with Propositions 15 for connected surfaces and with Proposition 17 for disconnected surfaces; in §5.2, we treat reducible subgroups with Proposition 19. In Proposition 17 and 19, we apply Proposition 2 to the image of the surjections, suitably chosen by the results from §4, when the given subgroup is not virtually free-abelian.

5.1. Adequately reduced subgroups. We first consider an adequately reduced subgroup $G < \text{Mod}_{\Sigma}$, where Σ is a connected surface.

Proposition 15. *Let Σ be an orientable connected finite-type surface with $\partial\Sigma = \emptyset$, which admits a complete hyperbolic metric, and let $G < \text{Mod}_{\Sigma}$ be an adequately reduced subgroup. Then G is either a non-cyclic group with infinite girth, a virtually infinite-cyclic group, or a finite group; these alternatives are mutually exclusive.*

Proof. Choose an integer $m \geq 3$. Suppose G is an adequately reduced subgroup of Mod_{Σ} , and let $S = \{\gamma_1, \dots, \gamma_n\}$ be a generating set of G . We assume that G is neither virtually infinite-cyclic nor finite, and aim to show that it has infinite girth.

In this case, the statement (2) of Theorem 10 is satisfied. We know that there is a pseudo-Anosov element $\sigma \in G(m) \triangleleft G$. For each $1 \leq j \leq n$ and $\varepsilon =$

± 1 , the conjugate $\gamma_j^\varepsilon \sigma \gamma_j^{-\varepsilon} \in G(m) \triangleleft G$ is again a pseudo-Anosov element with $\text{Fix}(\gamma_j^\varepsilon \sigma \gamma_j^{-\varepsilon}) = \gamma_j^\varepsilon(\text{Fix}(\sigma))$. Hence, applying the statement (2) of Theorem 10 to the collection $\{\sigma\} \cup \{\gamma_j^\varepsilon \sigma \gamma_j^{-\varepsilon} \mid 1 \leq j \leq n, \varepsilon = \pm 1\}$, we see that there is another pseudo-Anosov element $\tau \in G(m) \triangleleft G$ such that $\text{Fix}(\tau) \cap \text{Fix}(\sigma) = \emptyset$ and $\text{Fix}(\tau) \cap \text{Fix}(\gamma_j^\varepsilon \sigma \gamma_j^{-\varepsilon}) = \emptyset$ for all $1 \leq j \leq n$ and $\varepsilon = \pm 1$.

Note that, if U_σ is a neighborhood of $\text{Fix}(\sigma)$, then $\gamma_j^\varepsilon(U_\sigma)$ is a neighborhood of $\text{Fix}(\gamma_j^\varepsilon \sigma \gamma_j^{-\varepsilon})$ for each $1 \leq j \leq n$ and $\varepsilon = \pm 1$. It then follows that there are small enough neighborhoods $U_\sigma \supset \text{Fix}(\sigma)$ and $U_\tau \supset \text{Fix}(\tau)$ such that

$$U_\sigma \cap U_\tau = \emptyset \quad \text{and} \quad U_\tau \cap \gamma_j^\varepsilon(U_\sigma) = \emptyset$$

for each $1 \leq j \leq n$ and $\varepsilon = \pm 1$, or equivalently

$$U_\sigma \cap U_\tau = \emptyset \quad \text{and} \quad \gamma_j^\varepsilon(U_\tau) \cap U_\sigma = \emptyset$$

for each $1 \leq j \leq n$ and $\varepsilon = \mp 1$. Now, since σ and τ are pseudo-Anosov elements, we can take high enough powers $\tilde{\sigma} := \sigma^N$ and $\tilde{\tau} := \tau^N$ such that

$$\tilde{\sigma}^k(\mathcal{PMF}(\Sigma) - U_\sigma) \subset U_\sigma \quad \text{and} \quad \tilde{\tau}^k(\mathcal{PMF}(\Sigma) - U_\tau) \subset U_\tau$$

for all non-zero integer k . In particular, we have

$$\tilde{\sigma}^k \left(U_\tau \cup \bigcup_{\varepsilon=\pm 1} \bigcup_{j=1}^n \gamma_j^\varepsilon(U_\tau) \right) \subset U_\sigma \quad \text{and} \quad \tilde{\tau}^k \left(U_\sigma \cup \bigcup_{\varepsilon=\pm 1} \bigcup_{j=1}^n \gamma_j^\varepsilon(U_\sigma) \right) \subset U_\tau$$

for all non-zero integer k . Applying Proposition 2 to $\tilde{\sigma}$, $\tilde{\tau}$, U_σ , U_τ , and

$$x \in \mathcal{PMF}(\Sigma) - \left((U_\sigma \cup U_\tau) \cup \bigcup_{\varepsilon=\pm 1} \bigcup_{j=1}^n \gamma_j^\varepsilon(U_\sigma \cup U_\tau) \right)$$

we conclude that G must be a non-cyclic group with infinite girth. □

We now consider an adequately reduced subgroup $G < \text{Mod}_\Sigma$ where Σ is disconnected. The strategy is to take the partition $\Sigma = \Sigma_{[0]} \sqcup \Sigma_{[1]} \sqcup \Sigma_{[2]}$ from Lemma 11, and restrict the group G to $\Sigma_{[2]}$ when $\Sigma_{[2]} \neq \emptyset$; the non-emptiness of $\Sigma_{[2]}$ then guarantees the infinite girth. When $\Sigma_{[2]}$ is empty, the following lemma from the proof of the Tits alternative shows that G is virtually free-abelian.

Lemma 16 (Cf. [13], §8.7). *Let $m \geq 3$ be an integer, and let G be an adequately reduced group. Then G is virtually free-abelian if and only if the restriction of $G(m)$ to each component of Σ is either trivial or infinite-cyclic, i.e., $\Sigma_{[2]} = \emptyset$ in the partition from Lemma 11.*

Proposition 17. *Let Σ be an orientable finite-type surface with $\partial\Sigma = \emptyset$, which admits a complete hyperbolic metric, and let $G < \text{Mod}_\Sigma$ be an adequately reduced subgroup. Then G is either a non-cyclic group with infinite girth or a virtually free-abelian group; these alternatives are mutually exclusive.*

Proof. Consider the decomposition $\Sigma = \Sigma_{[0]} \sqcup \Sigma_{[1]} \sqcup \Sigma_{[2]}$ in Lemma 11. If $\Sigma_{[2]} = \emptyset$, then G is virtually free-abelian by Lemma 16. Hence, we may assume that $\Sigma_{[2]} \neq \emptyset$. Note that the restriction $G_{[2]}$ of the subgroup G to $\Sigma_{[2]}$ is the image of the restriction homomorphism $G \rightarrow \text{Mod}_{\Sigma_{[2]}}$; hence, if the restriction $G_{[2]}$ is a non-cyclic group with infinite girth, then so is G by Proposition 5; it remains to show that $G_{[2]}$ is indeed a non-cyclic group with infinite girth.

For this, we may as well assume that $\Sigma = \Sigma_{[2]}$ and hence $G = G_{[2]}$. Choose $m \geq 3$. Let $S = \{\gamma_1, \dots, \gamma_n\}$ be a generating set of G . By Theorem 13, we know that there is a pseudo-Anosov element $\sigma \in G(m) \triangleleft G$, and there is another pseudo-Anosov element $\tau \in G(m) \triangleleft G$ such that $\text{Fix}(\tau) \cap \text{Fix}(\sigma) = \emptyset$ and $\text{Fix}(\tau) \cap \text{Fix}(\gamma_j^\varepsilon \sigma \gamma_j^{-\varepsilon})$ for each $1 \leq j \leq n$ and $\varepsilon = \pm 1$. Here, the conjugates $\gamma_j^\varepsilon \sigma \gamma_j^{-\varepsilon}$ are pseudo-Anosov elements with $\text{Fix}(\gamma_j^\varepsilon \sigma \gamma_j^{-\varepsilon}) = \gamma_j^\varepsilon(\text{Fix}(\sigma))$ in $\mathcal{PMF}^\#(\Sigma)$. From the discussion in §3.2, we know that the action of pseudo-Anosov elements in Mod_Σ on $\mathcal{PMF}^\#(\Sigma)$ exhibits the north-south dynamics; hence, by the arguments identical to the proof of Proposition 15 – with the space $\mathcal{PMF}(\Sigma)$ replaced by $\mathcal{PMF}^\#(\Sigma)$ – we see that G is a non-cyclic group with infinite girth by Proposition 2. \square

5.2. Reducible subgroups. We now consider a reducible group $G < \text{Mod}_\Sigma$. To prove the girth alternative, we take the canonical reduction $\rho_{\mathcal{E}}(G) < \text{Mod}_{\Sigma_{\mathcal{E}}}$, for which the girth alternative holds by Proposition 17. The following lemma, extracted from the proof of the Tits alternative [13], §8.9, characterizes virtually free-abelian subgroups G in terms of its reduction $\rho_{\mathcal{E}}(G)$.

Lemma 18 (Cf. [13], §8.9). *A reducible subgroup $G < \text{Mod}_\Sigma$ is virtually free-abelian if and only if the canonical reduction $\rho_{\mathcal{E}}(G)$ is virtually free-abelian.*

Proposition 19. *Let Σ be an orientable finite-type surface with $\partial\Sigma = \emptyset$, which admits a complete hyperbolic metric, and let $G < \text{Mod}_\Sigma$ be a reducible subgroup. Then G is either a non-cyclic group with infinite girth or a virtually free-abelian group; these alternatives are mutually exclusive.*

Proof. The canonical reduction $\rho_{\mathcal{E}}(G)$ of G is adequately reduced, and hence it is either a non-cyclic group with infinite girth or a virtually free-abelian group by Proposition 17. Recall that the canonical reduction $\rho_{\mathcal{E}}(G)$ is indeed the image of the reduction homomorphism $\rho_{\mathcal{E}} : G \rightarrow \text{Mod}_{\Sigma_{\mathcal{E}}}$; hence, if $\rho_{\mathcal{E}}(G)$ is a non-cyclic group with infinite girth, so is G by Proposition 5. If $\rho_{\mathcal{E}}(G)$ is virtually free-abelian, so is G by Lemma 18. \square

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