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Groups, Geometry, and Dynamics

Harmonic cochains and K-theory for $\tilde{A}_2$-groups

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Abstract. If Γ is a torsion free $\tilde{A}_2$-group acting on an $\tilde{A}_2$ building Δ, and $\mathfrak{A}_\Gamma$ is the associated boundary C^*-algebra, it is proved that $K_0(\mathfrak{A}_\Gamma) \otimes \mathbb{R} \cong \mathbb{R}^{2\beta_2}$, where $\beta_2 = \dim_{\mathbb{R}} H^2(\Gamma, \mathbb{R})$.

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1. Introduction

Let Γ be an $\tilde{A}_2$-group acting on an $\tilde{A}_2$ building Δ of order q. The Furstenberg boundary Ω of Δ is the set of chambers of the spherical building at infinity, endowed with a natural compact totally disconnected topology. The topological action of Γ on Ω is encoded in the full crossed product C^*-algebra $\mathfrak{A}_\Gamma = C(\Omega) \rtimes \Gamma$, which is studied in [10], [11], [12]. This full crossed product is isomorphic to the reduced crossed product, since the action of Γ on Ω is amenable [10], Section 4.2. As the notation suggests, $\mathfrak{A}_\Gamma$ depends only on Γ [12]. Motivated by rigidity theorems of Mostow, Margulis and others, whose proofs rely on the study of boundary actions, it is of interest to determine the extent to which the boundary C^*-algebra $\mathfrak{A}_\Gamma$ determines the group Γ.

In [12], T. Steger and the author computed the K-theory of $\mathfrak{A}_\Gamma$ for many $\tilde{A}_2$-groups with $q \le 13$. The computations were done for all the $\tilde{A}_2$-groups in the cases $q = 2, 3$ and for several representative groups for each of the other values of $q \le 13$. If $q = 2$ there are precisely eight $\tilde{A}_2$-groups Γ, all of which embed as lattices in $\mathrm{PGL}(3, \mathbb{K})$, where $\mathbb{K} = \mathbb{F}_2((X))$ or $\mathbb{K} = \mathbb{Q}_2$. If $q = 3$ there are 89 possible $\tilde{A}_2$-groups, of which 65 are “exotic” in the sense that they do not embed naturally in linear groups. Exotic $\tilde{A}_n$-groups only exist if $n = 2$, since all locally finite Euclidean buildings of dimension ≥ 3 are associated to linear algebraic groups. This justifies, to some extent, the focus on $\tilde{A}_2$-groups.

For each $\tilde{A}_2$-group Γ, the C^*-algebra $\mathfrak{A}_\Gamma$ has the structure of a rank-2 Cuntz–Krieger algebra [11], Theorem 7.7. These algebras are classified up to isomorphism by their K-groups [11], Remark 6.5, and it was proved in Theorem 2.1 of [12] that

$$K_0(\mathfrak{A}_\Gamma) = K_1(\mathfrak{A}_\Gamma) = \mathbb{Z}^{2r} \oplus T, \tag{1}$$

where $r \geq 0$ and T is a finite group. The computations in [12] led to some striking observations. For example, the three torsion-free $\tilde{A}_2$ subgroups of $\mathrm{PGL}_3(\mathbb{Q}_2)$ are distinguished from each other by $K_0(\mathfrak{A}_\Gamma)$. There was also evidence for the conjecture that, for any torsion free $\tilde{A}_2$-group Γ, the integer r in the equation (1) is equal to the second Betti number of Γ. The purpose of this article is to prove that this is indeed the case.

Theorem 1.1. *If Γ is a torsion free $\tilde{A}_2$-group acting on an $\tilde{A}_2$ building Δ of order q, then*

$$K_0(\mathfrak{A}_\Gamma) \otimes \mathbb{R} \cong \mathbb{R}^{2\beta_2}, \tag{2}$$

where $\beta_2 = \dim_{\mathbb{R}} H^2(\Gamma, \mathbb{R}) = \frac{1}{3}(q-2)(q^2+q+1)$.

The article [12] identified the integer r in (1) with the rank of a certain finitely generated abelian group $C(\Gamma)$. Two new ideas lead to the proof of Theorem 1.1. The local structure of the building Δ, together with the fact that Γ has Kazhdan's property (T), is used to show that $C(\Gamma) \otimes \mathbb{R}$ is isomorphic to the space of Γ-invariant $\mathbb{R}$-valued cochains on Δ, in the sense of [1], [4]. Then, according to an isomorphism of Garland [5], this space is isomorphic to $H^2(\Gamma, \mathbb{R})$.

Remark 1.2. An $\tilde{A}_2$-group is a natural analogue of a free group, which acts freely and transitively on the vertex set of a tree (which is a building of type $\tilde{A}_1$). If the tree is homogeneous of degree $q+1$, with $q \geq 2$, then Γ is a free group on $\frac{1}{2}(q+1)$ generators and one can again form the full crossed product C^*-algebra $\mathfrak{A}_\Gamma = C(\Omega) \rtimes \Gamma$, where Ω is the space of ends of the tree. The analogue of Theorem 1.1 states that $K_0(\mathfrak{A}_\Gamma) \otimes \mathbb{R} \cong \mathbb{R}^{\beta_1}$, where $\beta_1 = \dim_{\mathbb{R}} H^1(\Gamma, \mathbb{R}) = \frac{1}{2}(q+1)$ [8], Theorem 1.

Remark 1.3. Another simple C^*-algebra associated with the $\tilde{A}_2$-group Γ is the reduced group C^*-algebra $C_r^*(\Gamma)$. It is shown in [9], Theorem 6.1, that $K_0(C_r^*(\Gamma)) = \mathbb{Z}^{\chi(\Gamma)}$. This is a consequence of the fact that $\tilde{A}_2$-groups belong to the class of groups for which the Baum–Connes conjecture is known to be true.

Remark 1.4. This paper is a sequel to the articles [11], [12]. The key results used are [11], Theorem 7.7, which shows that $\mathfrak{A}_\Gamma$ is isomorphic to a rank-2 Cuntz–Krieger algebra, and [12], Theorem 2.1, which shows that the K-theory of this algebra is given by equation (1).

What happens in the case of a torsion free $\tilde{A}_n$-group Γ ($n \geq 3$)? There seems to be no fundamental obstruction to generalising [11], Theorem 7.7, to identify the boundary crossed product algebra with a higher rank Cuntz–Krieger algebra, in the sense of [11]. The arguments of the present paper should also generalise, but additional conditions which are vacuous in the rank-2 case would need to be verified [1], Theorem 2.3 (C), (D). However it would be more difficult to generalise [12], Theorem 2.1. This is because the proof of that result uses a Kasparov spectral sequence [12], Proposition 4.1, whose limit is clear only in the rank-2 case.

2. $\tilde{A}_2$-groups

Consider a locally finite building Δ of type $\tilde{A}_2$. Each vertex v of Δ has a type $\tau(v) \in \mathbb{Z}/3\mathbb{Z}$, and each chamber (maximal simplex) of Δ has exactly one vertex of each type. Each edge e is directed, with initial vertex of type i and final vertex of type $i+1$. An automorphism α of Δ is *type rotating* if there exists $i \in \mathbb{Z}/3\mathbb{Z}$ such that $\tau(\alpha(v)) = \tau(v) + i$ for all vertices v of Δ.

Suppose that Γ is a group of type rotating automorphisms of Δ, which acts freely and transitively on the vertex set of Δ. Such a group is called an $\tilde{A}_2$-group. The theory of $\tilde{A}_2$-groups has been developed in [3] and some, but not all, $\tilde{A}_2$-groups embed as lattice subgroups of $\mathrm{PGL}_3(\mathbb{K})$. Any $\tilde{A}_2$-group can be constructed as follows [3], I, Section 3. Let (P, L) be a finite projective plane of order q. There are q^2+q+1 points (elements of P) and q^2+q+1 lines (elements of L). Let $\lambda\colon P \to L$ be a bijection and write $\lambda(\xi) = \bar{\xi}$. A *triangle presentation* compatible with λ is a set $\mathcal{T}$ of ordered triples (ξ_i, ξ_j, ξ_k) where $\xi_i, \xi_j, \xi_k \in P$, with the following properties.

(i) Given $\xi_i, \xi_j \in P$, then $(\xi_i, \xi_j, \xi_k) \in \mathcal{T}$ for some $\xi_k \in P$ if and only if ξ_j and $\bar{\xi}_i$ are incident, i.e. $\xi_j \in \bar{\xi}_i$.

(ii) $(\xi_i, \xi_j, \xi_k) \in \mathcal{T} \Rightarrow (\xi_j, \xi_k, \xi_i) \in \mathcal{T}$.

(iii) Given $\xi_i, \xi_j \in P$, then $(\xi_i, \xi_j, \xi_k) \in \mathcal{T}$ for at most one $\xi_k \in P$.

In [3] there is exhibited a complete list of triangle presentations for $q=2$ and $q=3$. Given a triangle presentation $\mathcal{T}$, one can form the group

$$\Gamma = \Gamma_{\mathcal{T}} = \big\langle P \mid \xi_i\xi_j\xi_k = 1 \text{ for } (\xi_i, \xi_j, \xi_k) \in \mathcal{T} \big\rangle. \tag{3}$$

The Cayley graph of Γ with respect to the generating set P is the 1-skeleton of a building Δ of type $\tilde{A}_2$. Vertices are elements of Γ and a directed edge of the form $(\gamma, \gamma\xi)$ with $\gamma \in \Gamma$ is labeled by the generator $\xi \in P$.

The link of a vertex γ of Δ is the incidence graph of the projective plane (P, L), where the lines in L correspond to the inverses in Γ of the generators in P. In other words, $\bar{\xi} = \xi^{-1}$ for $\xi \in P$.

For the rest of this article, Γ is assumed to be torsion free. Therefore Γ acts freely on Δ and $X = \Gamma\backslash\Delta$ is a 2-dimensional cell complex with universal covering Δ. Let X^k denote the set of oriented k-cells of X for $k = 0, 1, 2$. Thus X^1 may be identified with P and X^2 may be identified with the set of orbits of elements of $\mathcal{T}$ under cyclic permutations.

Let $\widehat{\Delta}^2$ be the directed version of Δ^2 in which each 2-simplex has a specified base vertex, so that $\mathbb{Z}/3\mathbb{Z}$ acts naturally on $\widehat{\Delta}^2$. Let $\widehat{X}^2 := \widehat{\Delta}^2/\Gamma$, the set of directed

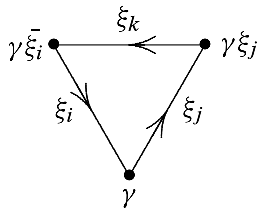


Figure 1. A chamber based at a vertex γ.

2-cells of X. Then $\widehat{X}^2$ may be identified with $\mathcal{T}$. From now on $a = \langle a_0, a_1, a_2\rangle$ will denote an element of $\mathcal{T}$, regarded as a directed 2-cell. Figure 2 illustrates the three directed 2-cells associated with an oriented 2-cell of X. In the diagram, the 2-cells are thought of as being directed upwards and the symbol • is placed opposite the "top" edge to indicate that direction.

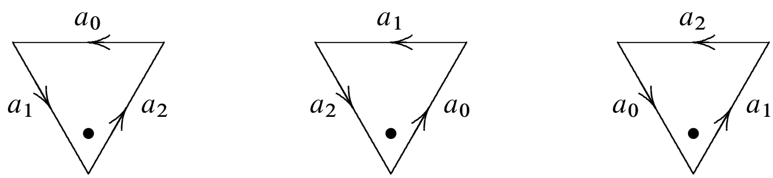


Figure 2. The directed 2-cells $\langle a_0, a_1, a_2\rangle$, $\langle a_1, a_2, a_0\rangle$, $\langle a_2, a_0, a_1\rangle$.

3. K-theory

Transition matrices $M = (m_{ab})_{a,b\in\widehat{X}^2}$ and $N = (n_{ab})_{a,b\in\widehat{X}^2}$ are defined as follows. If $a, b \in \widehat{X}^2$ then $m_{ab} = 1$ if and only if there are labeled triangles representing a, b in the building Δ which lie as shown on the right of Figure 3. If no such diagram is possible then $m_{ab} = 0$.

In terms of the projective plane (P, L), the matrix M is defined by

$$m_{ab} = 1 \iff b_2 \notin \bar{a}_2,\ \bar{b}_1 = a_0 \vee b_2.$$

It follows that each row or column of M has precisely q^2 nonzero entries.

Similarly, the matrix N is defined by

$$n_{ac} = 1 \iff a_1 \notin \bar{c}_1,\ c_2 = \bar{a}_0 \wedge \bar{c}_1.$$

as illustrated on the left of Figure 3.

Let r be the rank, and T the torsion part, of the abelian group $C(\Gamma)$ with generating set $\widehat{X}^2$ and relations

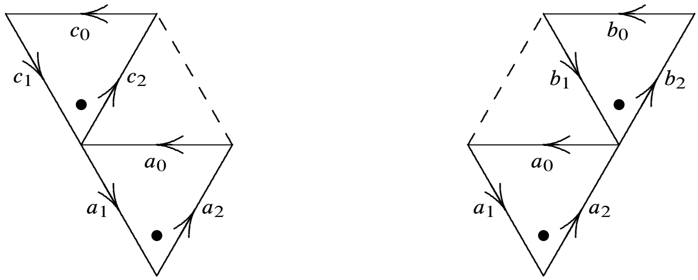


Figure 3. The relations $n_{ac} = 1$ and $m_{ab} = 1$.

$$a = \sum_{b \in \widehat{X}^2} m_{ab} b = \sum_{b \in \widehat{X}^2} n_{ab} b, \quad a \in \widehat{X}^2. \tag{4}$$

Thus $C(\Gamma) \cong \mathbb{Z}^r \oplus T$. The following result was proved in [12], Theorem 2.1.

Theorem 3.1. *Let Γ be an $\tilde{A}_2$-group, and let r be the rank, and T the torsion part of $C(\Gamma)$. Then*

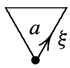
$$K_0(\mathfrak{A}_\Gamma) = K_1(\mathfrak{A}_\Gamma) = \mathbb{Z}^{2r} \oplus T. \tag{5}$$

Given $\xi \in P$, let

$$\langle \xi \rangle = \sum_{\substack{a \in \widehat{X}^2 \\ a_2 = \xi}} a \in C(\Gamma). \tag{6}$$

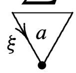
It is sometimes convenient to write such sums pictorially as

$$\langle \xi \rangle = \sum a. \tag{7}$$



Note that $a \in \widehat{X}^2$ with $a_2 = \xi$ if and only if $a = \langle a_0, a_1, a_2 \rangle$ where $a_2 = \xi$ and $a_0 \in \bar{\xi}$ (and a_1 is then uniquely determined). There are $q + 1$ such choices of a_0 and so there are $q + 1$ terms in the sum (6). Similar remarks apply to the element $\langle \bar{\xi} \rangle \in C(\Gamma)$ defined by

$$\langle \bar{\xi} \rangle = \sum_{\substack{a \in \widehat{X}^2 \\ a_1 = \xi}} a = \sum a. \tag{8}$$



In what follows the element

$$\varepsilon = \sum_{a \in \widehat{X}^2} a$$

plays a special role. An important observation, which is needed subsequently, is that ε has finite order in $C(\Gamma)$. The statement and its proof are very like [12], Proposition 8.2.

Lemma 3.2. *In the group* $C(\Gamma)$, $(q^2-1)\varepsilon = 0$.

Proof. Using relations (4) and the fact that each column of the matrix M has precisely q^2 nonzero entries,

$$\varepsilon = \sum_{a\in\widehat{X}^2} a = \sum_{a\in\widehat{X}^2}\sum_{b\in\widehat{X}^2} m_{ab}b = \sum_{b\in\widehat{X}^2}\Big(\sum_{a\in\widehat{X}^2} m_{ab}\Big)b = \sum_{b\in\widehat{X}^2} q^2 b = q^2\varepsilon. \qquad \square$$

Lemma 3.3. *If* $\langle a_0, a_1, a_2\rangle \in \widehat{X}^2$ *then, in the group* $C(\Gamma)$,

$$\langle a_1\rangle - \langle a_2, a_0, a_1\rangle = \langle \bar{a}_2\rangle - \langle a_1, a_2, a_0\rangle; \tag{9a}$$
$$\langle a_0\rangle + \langle a_1\rangle + \langle a_2\rangle = \varepsilon. \tag{9b}$$

Proof. Fix the base vertex $1 \in \Delta$. Any generator a for $C(\Gamma)$ has a unique representative directed chamber σ_a based at 1. The chamber σ_a has vertices $1, a_1^{-1}, a_2$. By [6], Section 15.4, each chamber based at 1, other than σ_a lies in a common apartment with σ_a, in exactly one of the five positions τ_2, τ_3, τ_4, τ_5, τ_6 in Figure 4. As before, directed chambers will be pointed.

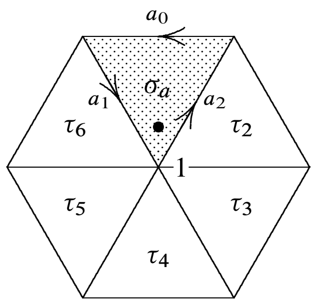


Figure 4

The left side of (9a) is equal to the sum of all the elements $b \in \widehat{X}^2$ represented by directed chambers σ_b in position τ_6, as illustrated in Figure 5 (a). Each such element b satisfies $b_2 = a_1$, and the relations (4) imply that

$$b = \sum_{c\in\widehat{X}^2} m_{bc}c.$$

That is, b is the sum of all the elements $c \in \widehat{X}^2$ with representative directed chambers σ_c lying in position τ_4, as illustrated in Figure 5 (a). Moreover, if σ_c is any directed chamber with base vertex 1, lying in position τ_4, then it arises in this way from a unique chamber σ_b in position τ_6. To see this, it is enough to take the convex hull of any such chamber σ_c with σ_a, which completely determines the whole hexagon in Figure 5 (a). Therefore the left side of (9a) is equal to the sum of all the elements $c \in \widehat{X}^2$ represented by directed chambers σ_c based at 1 which lie in position τ_4 of Figure 4.

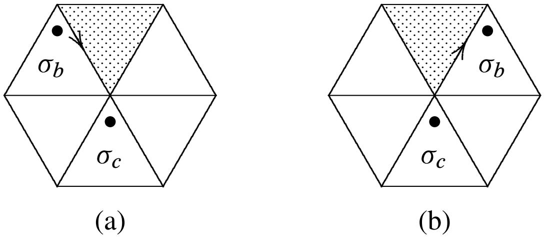


Figure 5

Similarly, the right side of (9a) is equal to the sum of all the elements $b \in \widehat{X}^2$ represented by directed chambers σ_b in position τ_2 as illustrated in Figure 5 (b). The relations (4) imply that, for each such chamber b,

$$b = \sum_{b \in \widehat{X}^2} n_{ac} c.$$

It follows that the right side of (9a) is also equal to the sum of all the elements $c \in \widehat{X}^2$ represented by directed chambers based at 1 which lie in position τ_4 of Figure 4. This proves that the left and right sides of (9a) are equal.

The next task is to prove (9b). Recall that ε is the sum of all the elements of $\widehat{X}^2$, and representative directed chambers for elements of this sum are σ_a together with all chambers lying in any of the five positions $\tau_2, \tau_3, \tau_4, \tau_5, \tau_6$ in Figure 4.

The set of chambers based at the vertex 1 representing the elements of the sum $\langle a_2 \rangle$ consists of σ_a together with all directed chambers lying in the position τ_2, as illustrated in Figure 6 (a). Here it may also be convenient to refer back to equation (7).

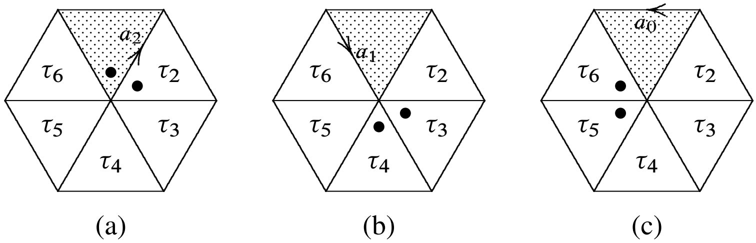


Figure 6

Using the relations (4), the sum $\langle a_1\rangle$ is equal to the sum of elements represented by chambers lying in the position τ_3 or τ_4, as in Figure 6 (b).

Finally, the sum $\langle a_0\rangle$ is equal to the sum of elements represented by directed chambers lying in the position τ_5 or τ_6, as in Figure 6 (c). For, cyclically permuting the indices in the equation (9a) gives

$$\langle a_0\rangle - \langle a_1, a_2, a_0\rangle = \langle \bar{a}_1\rangle - \langle a_0, a_1, a_2\rangle.$$

Therefore $\langle a_0\rangle = \langle a_1, a_2, a_0\rangle + \langle \bar{a}_1\rangle - \langle a_0, a_1, a_2\rangle$. Referring to Figure 6 (c), the relations (4) show that $\langle a_1, a_2, a_0\rangle$ is the sum of elements represented by directed chambers in position τ_5. Also $\langle \bar{a}_1\rangle - \langle a_0, a_1, a_2\rangle$ is the sum of elements represented by directed chambers in position τ_6.

This completes the proof that $\langle a_0\rangle + \langle a_1\rangle + \langle a_2\rangle = \varepsilon$. □

The next lemma is a major step in the proof of the main theorem. It depends on the fact that Γ has Kazhdan's property (T), which in turn depends only on the local structure of the building Δ. See, for example, the proof of [2], Theorem 5.7.7.

Lemma 3.4. *In the group $C(\Gamma) \otimes \mathbb{R}$, for all $\langle a_0, a_1, a_2\rangle \in \widehat{X}^2$ and $\xi \in P$,*

$$\langle a_0, a_1, a_2\rangle \otimes 1 = \langle a_1, a_2, a_0\rangle \otimes 1 = \langle a_2, a_0, a_1\rangle \otimes 1; \qquad (10a)$$

$$\langle \xi\rangle \otimes 1 = 0 = \langle \bar{\xi}\rangle \otimes 1. \qquad (10b)$$

Proof. By Lemma 3.2, ε has finite order in $C(\Gamma)$ and hence $\varepsilon \otimes 1$ is zero in $C(\Gamma) \otimes \mathbb{R}$. Therefore, by (9b),

$$\langle a_0\rangle \otimes 1 + \langle a_1\rangle \otimes 1 + \langle a_2\rangle \otimes 1 = 0,$$

for $\langle a_0, a_1, a_2\rangle \in \widehat{X}^2$. It follows from the presentation of Γ that the map $\xi \mapsto \langle \xi\rangle \otimes 1$, $\xi \in P$, induces a homomorphism θ from Γ into the abelian group $C(\Gamma) \otimes \mathbb{R}$.

The $\tilde{A}_2$-group Γ has Kazhdan's property (T), by [2], Theorem 5.7.7. It follows that the range of θ is finite [2], Corollary 1.3.6, and hence zero, since $C(\Gamma) \otimes \mathbb{R}$ is torsion free. Therefore $\langle \xi\rangle \otimes 1 = 0$, $\xi \in P$. Similarly, $\langle \bar{\xi}\rangle \otimes 1 = 0$, $\xi \in P$. This proves (10b). The relation (9a) then implies that $\langle a_1, a_2, a_0\rangle \otimes 1 = \langle a_2, a_0, a_1\rangle \otimes 1$ and the rest of (10a) follows by symmetry. □

Let $C_0(\Gamma)$ be the abelian group with generating set $\widehat{X}^2$ and the following relations:

$$\langle a_0, a_1, a_2\rangle = \langle a_1, a_2, a_0\rangle = \langle a_2, a_0, a_1\rangle, \qquad \langle a_0, a_1, a_2\rangle \in \widehat{X}^2; \qquad (11a)$$

$$\langle \xi\rangle = 0 = \langle \bar{\xi}\rangle, \qquad \xi \in P. \qquad (11b)$$

Lemma 3.5. *The relations (11) imply the relations (4).*

Proof. Let $a = \langle a_0, a_1, a_2\rangle \in \widehat{X}^2$. Then, using the relations (11), and referring to Figure 7,

$$\begin{aligned}
a &= \langle a_0, a_1, a_2\rangle = \langle a_1, a_2, a_0\rangle \qquad \text{[using (11a)]}\\
&= - \sum_{\substack{\langle c_2, b_1, a_0\rangle \in \widehat{X}^2 \\ c_2 \neq a_1}} \langle c_2, b_1, a_0\rangle \qquad \text{[using (11b), with } \xi = a_0]\\
&= - \sum_{\substack{\langle c_2, b_1, a_0\rangle \in \widehat{X}^2 \\ c_2 \neq a_1}} \Big(- \sum_{\substack{\langle b_0, b_1, b_2\rangle \in \widehat{X}^2 \\ b_0 \neq c_2}} \langle b_0, b_1, b_2\rangle \Big) \quad \text{[using (11b) again]}\\
&= \sum_{b \in \widehat{X}^2} m_{ab} b.
\end{aligned}$$

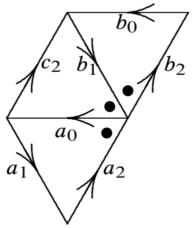


Figure 7

The proof of the relations $a = \sum_{b \in \widehat{X}^2} n_{ab} b$ in (4) is similar. □

Proposition 3.6. *If* Γ *is a torsion free* $\tilde{A}_2$*-group, then* $C(\Gamma) \otimes \mathbb{R} = C_0(\Gamma) \otimes \mathbb{R}$.

Proof. The groups have the same set of generators. By Lemmas 3.4 and 3.5, the relations in each group imply the relations in the other. The groups are therefore equal. □

4. Harmonic cochains and proof of the main result

A harmonic 2-cochain [4] is a function $c : \widehat{X}^2 \to \mathbb{R}$ satisfying the following conditions for all $a \in \widehat{X}^2$ and for all $\xi \in P$.

$$c(\langle a_0, a_1, a_2\rangle) = c(\langle a_1, a_2, a_0\rangle) = c(\langle a_2, a_0, a_1\rangle); \tag{12a}$$

$$c(\langle \xi \rangle) = c(\langle \bar{\xi} \rangle) = 0. \tag{12b}$$

Denote the set of harmonic 2-cochains by $C^2_{\text{har}}(\widehat{X}^2)$. Since the group Γ acts freely on Δ, $C^2_{\text{har}}(\widehat{X}^2)$ may be identified with the space of Γ-invariant harmonic cochains $c : \widehat{\Delta}^2 \to \mathbb{R}$, in the sense of [1]. Now $C^2_{\text{har}}(\widehat{X}^2)$ is the algebraic dual of $C_0(\Gamma) \otimes \mathbb{R}$. The next result is therefore an immediate consequence of Proposition 3.6.

Proposition 4.1. *$C^2_{\mathrm{har}}(\widehat{X}^2)$ is isomorphic to $C(\Gamma)\otimes\mathbb{R}$.*

The proof of Theorem 1.1 can now be completed. By Theorem 3.1, it is sufficient to show that the rank r of $C(\Gamma)$ is equal to $\beta_2 = \dim_{\mathbb{R}} H^2(\Gamma,\mathbb{R})$. Garland's isomorphism [1], Section 3.1, states that $H^2(\Gamma,\mathbb{R}) \cong C^2_{\mathrm{har}}(\widehat{X}^2)$. Note that the account of Garland's Theorem in [1] relates to the case where Γ is a lattice in $\mathrm{PGL}(3,\mathbb{K})$, but the proof applies without change to all torsion free $\tilde{A}_2$-groups.

It follows from Proposition 4.1 that $C(\Gamma)\otimes\mathbb{R} \cong H^2(\Gamma,\mathbb{R})$. Theorem 3.1 now implies that $K_0(\mathfrak{A}_\Gamma)\otimes\mathbb{R} \cong \mathbb{R}^{2\beta_2}$.

It remains to identify β_2 explicitly. The Euler characteristic of Γ is $\chi(\Gamma) = \frac{1}{3}(q-1)(q^2-1)$ [7], Section 4. Now $\chi(\Gamma) = \beta_0 - \beta_1 + \beta_2$ where $\beta_i = \dim_{\mathbb{R}} H_i(\Gamma,\mathbb{R})$. Since Γ has Kazhdan's property (T), the abelianisation $\Gamma/[\Gamma,\Gamma]$ is finite [2], Corollary 1.3.6, and so $\beta_1 = 0$. Also $\beta_0 = 1$. Therefore $\beta_2 = \chi(\Gamma) - 1 = \frac{1}{3}(q-2)(q^2+q+1)$. This completes the proof. □

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