

On the growth of Betti numbers in p -adic analytic towers

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Abstract. We study the asymptotic growth of Betti numbers in tower of finite covers and provide simple proofs of approximation results, which were previously obtained by Calegari and Emerton, in the generality of arbitrary p -adic analytic towers of covers. Further, we also obtain partial results about arbitrary pro- p towers.

Mathematics Subject Classification (2010). 58G12, 55P99.

Keywords. Asymptotic growth of Betti numbers, p -adic analytic groups.

1. Introduction and statement of results

This paper is mainly concerned with the asymptotic growth of Betti numbers in a tower of finite covers of a compact space X associated to a chain of subgroups of the fundamental group of X which gives rise to a p -adic analytic group. Both Betti numbers with coefficients in \mathbb{Q} and \mathbb{F}_p are considered. Especially the case of \mathbb{F}_p -coefficients received a lot of attention in recent years. We only name here the work of Calegari–Emerton [5], [6], which is motivated by the p -adic Langlands program, and the work of Lackenby [21], [22] in group theory, which is connected to property τ and 3-manifold theory.

1.1. Global setup. With the exception of Section 4, we retain the following setup throughout this paper. Let X be a connected compact CW-complex with fundamental group Γ . Let p be a prime, let n be a positive integer, and let $\phi: \Gamma \rightarrow \mathrm{GL}_n(\mathbb{Z}_p)$ be a homomorphism. The closure of the image of ϕ , which is denoted by G , is a p -adic analytic group admitting an exhausting filtration by open normal subgroups:

$$G_i = \ker (G \rightarrow \mathrm{GL}_n(\mathbb{Z}/p^i\mathbb{Z})).$$

Set $\Gamma_i = \phi^{-1}(G_i)$, and let X_i be the corresponding finite cover of X . Let \bar{X} be the cover of X corresponding to the kernel of ϕ and $\bar{\Gamma} = \Gamma/\mathrm{Ker}(\phi)$; note that $\bar{\Gamma}$ acts properly and freely on \bar{X} with quotient $\bar{\Gamma}\backslash\bar{X} = X$. Our main concern is the growth

of the Betti numbers

$$\begin{aligned} b_k(X_i) &= \dim_{\mathbb{Q}} H_k(X_i, \mathbb{Q}); \\ b_k(X_i, \mathbb{F}_p) &= \dim_{\mathbb{F}_p} H_k(X_i, \mathbb{F}_p), \end{aligned}$$

with coefficients in \mathbb{Q} and \mathbb{F}_p as functions of i .

1.2. Growth of Betti numbers in a p -adic analytic tower. W. Lück proved that for each integer k the sequence $b_k(X_i)/[\Gamma : \Gamma_i]$ always converges as $i \rightarrow \infty$, and the limit equals the k -th L^2 -Betti number $\beta_k(\bar{X}, \bar{\Gamma})$ of the action of $\bar{\Gamma}$ on \bar{X} . In that context we obtain the following result on the rate of convergence in terms of the dimension of G as a p -adic analytic group. We refer to [12], Theorem 8.36 on p. 201, for equivalent characterizations of the dimension of G .

Theorem 1.1. *Let $d = \dim(G)$. Then, for any integer k and as i tends to infinity, we have*

$$b_k(X_i) = \beta_k(\bar{X}, \bar{\Gamma})[\Gamma : \Gamma_i] + O([\Gamma : \Gamma_i]^{1-1/d}).$$

The novelty of Theorem 1.1 is obviously the error term. For more general covers, this has already been studied by Sarnak and Xue [28] and by Clair and Whyte [8] but they obtain much weaker results, in particular their results do not apply when 0 occurs in the L^2 -spectrum of \bar{X} .¹

Theorem 1.1 generalizes in the case of trivial coefficients the main theorem of Calegari–Emerton [5] which deals with arithmetic locally symmetric spaces. After this paper had been put on the ArXiv Frank Calegari informed us that Theorem 1.1 can be deduced from the method of [5]. In fact both proofs rest on a theorem of Harris, see Theorem 2.1 below, but we believe that our method of proof is somewhat simpler. We refer to Section 3 for more details on the relation to the work of Calegari–Emerton.

1.3. Growth of \mathbb{F}_p -Betti numbers in a p -adic analytic tower. Homological algebra over Iwasawa algebras and the theory of p -adic analytic groups provide important tools to study the asymptotic growth of Betti numbers in a p -adic analytic tower of covers. Whilst Iwasawa algebras are hidden in the proof of Theorem 1.1, they are essential even in the formulation of a corresponding result for \mathbb{F}_p -Betti numbers. The Iwasawa algebra of G over $R = \mathbb{F}_p$ or \mathbb{Z}_p is the completion of the group algebra $R[G]$:

$$R[[G]] = \varprojlim R[G/G_i].$$

The Iwasawa algebra is a right and left Noetherian domain. Further, if G is torsion-free, then $R[[G]]$ does not contain zero divisors and its non-zero elements satisfy the Ore condition, see §6 of [16]. This means that the ring of fractions $\mathcal{Q}(R[[G]])$ is a skew field, the Ore localization of $R[[G]]$. Hence there is a notion of rank:

¹We should note however that in the special case of lattices in $SU(2, 1)$ Sarnak and Xue produce a better exponent of $\frac{7}{12}$ in the error term for the first Betti number.

Definition 1.2. If G is torsion-free, we define the *rank* of a left $R[[G]]$ -module M as

$$\text{rank}_{R[[G]]}(M) = \dim_{Q(R[[G]])}(Q(R[[G]]) \otimes_{R[[G]]} M).$$

For general G we define the *rank* of M as

$$\text{rank}_{R[[G]]}(M) = \frac{1}{[G : G_0]} \text{rank}_{R[[G_0]]}(M),$$

where $G_0 < G$ is any uniform, hence torsion-free, subgroup, and M is regarded as an $R[[G_0]]$ -module by restriction.

Using the above rank, we define an analog of L^2 -Betti numbers in characteristic p . For a CW-complex Y the cellular chain complex will always be denoted by $C_*(Y)$. It is a consequence of the proof of Theorem 1.1 (see (2.10)) that, if you replace \mathbb{F}_p by \mathbb{Z}_p in the definition below, you obtain the L^2 -Betti numbers of \bar{X} .

Definition 1.3. The *mod p L^2 -Betti numbers* of the $\bar{\Gamma}$ -space \bar{X} are defined as

$$\beta_k(\bar{X}, \bar{\Gamma}; \mathbb{F}_p) = \text{rank}_{\mathbb{F}_p[[G]]}(H_k(\mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[\bar{\Gamma}]]} C_*(\bar{X}, \mathbb{F}_p))),$$

where $\mathbb{F}_p[[G]]$ is regarded as a right $\mathbb{F}_p[[\bar{\Gamma}]]$ -module via $\phi: \Gamma \rightarrow G$.

For these characteristic p analogs of L^2 -Betti numbers there is an approximation result similar to Theorem 1.1:

Theorem 1.4. Let $d = \dim(G)$. Then for any integer k and as i tends to infinity, we have

$$b_k(X_i; \mathbb{F}_p) = \beta_k(\bar{X}, \bar{\Gamma}; \mathbb{F}_p)[\Gamma : \Gamma_i] + O([\Gamma : \Gamma_i]^{1-1/d}).$$

In particular, the limit of the sequence $b_k(X_i; \mathbb{F}_p)/[\Gamma : \Gamma_i]$ exists and is equal to $\beta_k(\bar{X}, \bar{\Gamma}; \mathbb{F}_p)$.

Here again Calegari informed us that Theorem 1.4 can be deduced from his joint ongoing work with Emerton on *completed cohomology*. In fact one key feature of their theory is to set up the right framework to determine the growth rate of (mod p) Betti numbers even if the corresponding (mod p) L^2 -Betti number vanishes. Proving unconditional results seems difficult; we nevertheless point out that when X is 3-dimensional the main result of [6] implies in particular that the error term in Theorem 1.4 is the best possible in general. We also note that – in his PhD-thesis [30], Theorem 5.3.1 – Liam Wall had first constructed examples of p -adic analytic towers of covers of a finite volume hyperbolic 3-manifold which shows that one cannot replace the error term $O([\Gamma : \Gamma_i]^{1-1/d})$ by $O([\Gamma : \Gamma_i]^{1-1/d-\epsilon})$ for some $\epsilon > 0$.

We may have $\beta_k(\bar{X}, \bar{\Gamma}) \neq \beta_k(\bar{X}, \bar{\Gamma}; \mathbb{F}_p)$. An example is given in [23] (Example 6.2). One can even construct an example with X being a *manifold* (see Section 5):

Proposition 1.5. *There exists a link complement X and a sequence of p -covers X_i of X such that*

$$\lim_{i \rightarrow +\infty} \frac{\dim H_1(X_i, \mathbb{F}_p)}{[\Gamma : \Gamma_i]} \neq \lim_{i \rightarrow +\infty} \frac{\dim H_1(X_i, \mathbb{Q})}{[\Gamma : \Gamma_i]}.$$

We don't know of any example with \bar{X} being aspherical.

1.4. Beyond p -adic analytic groups. The following theorem about arbitrary pro- p towers is certainly known to some experts but we could find no proof in the literature except in degree one.

Theorem 1.6. *Let k be a field of characteristic $p > 0$. Let X be a compact connected CW-complex with Γ as fundamental group. Let $(\Gamma_i)_{i \geq 0}$ be a residual p -chain. We denote the finite cover of X associated to Γ_i by X_i . Then, for any $n \geq 0$, the sequence of normalized Betti numbers with k -coefficients*

$$\left(\frac{b_n(X_i; k)}{[\Gamma : \Gamma_i]} \right)_{i \geq 0}$$

is monotone decreasing and converges as $i \rightarrow \infty$.

We moreover prove that the limit is an integer in many situations, see Theorem 4.3 and the remark following it.

1.5. Acknowledgments. We thank the referee for a detailed and helpful report, especially for spotting an error in a previous version of the proof of Theorem 2.1, which is now corrected.

Work on this project was supported by the Leibniz Award of W.L. granted by the DFG. N.B. is a member of the Institut Universitaire de France. P.L. was partially supported by a grant from the NSA. R.S. thanks the Mittag-Leffler institute for its hospitality during the final stage of this project and acknowledges support by grant SA 1661/3-1 of the DFG.

2. Proof of Theorem 1.1 and 1.4

In the sequel we treat the cases $R = \mathbb{Z}_p$ and $R = \mathbb{F}_p$ simultaneously. Depending on which case, we denote by $\dim_R(M)$ either the dimension of a \mathbb{F}_p -vector space or the \mathbb{Z}_p -rank of a \mathbb{Z}_p -module, which equals the dimension of the \mathbb{Q}_p -vector space $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} M$.

As in the work of Calegari–Emerton and Emerton [5], [6] the following result of M. Harris ([18], Theorem 1.10) for which corrections appear in [19], is crucial.

Although not explicitly stated as such, a proof is also contained in the work of Farkas–Linnell [16]. We give a complete proof blending ideas from both Farkas–Linnell’s and Harris’ papers.

Theorem 2.1 (Harris). *Let $R = \mathbb{Z}_p$ or $R = \mathbb{F}_p$. Let M be a finitely generated $R[[G]]$ -module. Then*

$$\dim_R(R \widehat{\otimes}_{R[[G_i]]} M) = \text{rank}_{R[[G]]}(M)[G : G_i] + O([G : G_i]^{1-1/d}). \tag{2.1}$$

Here $\widehat{\otimes}$ denotes the completed tensor product.

Proof. Passing to a finite index subgroup of G we may, and shall, assume that G is uniform and torsion-free. The proof then proceeds through a sequence of reductions.

Reduction to the case of cokernels of elements in $R[[G]]$. We first show that it suffices to show the theorem for $R[[G]]$ -modules of the form

$$M = \text{coker}(R[[G]] \xrightarrow{-a} R[[G]]), \tag{2.2}$$

with $a \in R[[G]]$. Let N be an arbitrary finitely generated $R[[G]]$ -module. Since $R[[G]]$ is Noetherian, N is finitely presented and we can find a matrix $A \in M(r \times s, R[[G]])$ such that

$$N = \text{coker}(R[[G]]^r \xrightarrow{r_A} R[[G]]^s),$$

where r_A denotes the right multiplication

$$r_A(x_1, \dots, x_r) = (x_1, \dots, x_r) \cdot A$$

with A . Since the Ore localization $Q(R[[G]])$ is a skew field, by row and column reduction in $M_r(Q(R[[G]]))$ one can find invertible matrices $B \in M(r \times r, Q(R[[G]]))$ and $C \in M(s \times s, Q(R[[G]]))$ such that $D := BAC$ is a block matrix of the form

$$D = \begin{pmatrix} I_\delta & 0_{\delta, s-\delta} \\ 0_{r-\delta, \delta} & 0_{r-\delta \times s-\delta} \end{pmatrix} \in M(r \times s, R[[G]]), \tag{2.3}$$

where $I_\delta \in M(\delta \times \delta, R[[G]])$ is the identity matrix with $\delta \in \{0, \dots, s\}$ and the other blocks are suitable zero matrices. In other words, D describes the projection onto the first δ coordinates $(x_1, x_2, \dots, x_r) \mapsto (x_1, x_2, \dots, x_\delta, 0, 0, \dots)$. Since B, C are invertible, we have

$$\text{rank}_{R[[G]]}(N) = s - \delta.$$

There are nonzero $b, c \in R[[G]]$ such that bB, Cc are matrices over $R[[G]]$. We have

$$(bB)A(Cc) = \begin{pmatrix} bc \cdot I_\delta & 0_{\delta, s-\delta} \\ 0_{r-\delta, \delta} & 0_{\delta \times \delta} \end{pmatrix}. \tag{2.4}$$

Let $A_i, B_i,$ and C_i be the mod G_i reductions of $A, bB,$ and Cc . Because of $\ker(r_{A_i}) \subset \ker(r_{C_i} \circ r_{A_i}) = \ker(r_{A_i C_i})$ one obtains $\dim_R \ker(r_{A_i}) \leq \dim_R \ker(r_{A_i C_i})$. Because of $\text{im}(r_{B_i A_i C_i}) = \text{im}(r_{A_i C_i} \circ r_{B_i}) \subset \text{im}(r_{A_i C_i})$ we have $\dim_R \ker(r_{A_i C_i}) \leq \dim_R \ker(r_{B_i A_i C_i})$. Therefore: $\dim_R \ker(r_{A_i}) \leq \dim_R \ker(r_{B_i A_i C_i})$. Assuming the theorem is proved for modules as in (2.2), this implies that

$$\begin{aligned} \dim_R(R \widehat{\otimes}_{R[[G_i]]} N) &= \dim_R \text{coker}(r_{A_i}) \\ &= \dim_R \ker(r_{A_i}) - (r - s)[G : G_i] \\ &\leq \dim_R \ker(r_{B_i A_i C_i}) - (r - s)[G : G_i] \tag{2.5} \\ &= \dim_R \text{coker}(r_{B_i A_i C_i}) \\ &= (s - \delta) \cdot [G : G_i] + O([G : G_i]^{1-1/d}). \end{aligned}$$

To prove the assertion for N , under the assumption that it holds for modules as in (2.2), it remains to show that

$$\dim_R \text{coker}(r_{A_i}) \geq (s - \delta)[G : G_i] - O([G : G_i]^{1-1/d}). \tag{2.6}$$

Let $E \in M((r - \delta) \times r, R[[G]])$ be the matrix such that

$$R[[G]]^{r-\delta} \xrightarrow{rE} R[[G]]^r, \quad (y_1, y_2, \dots, y_{r-\delta}) \cdot E = (0, \dots, 0, y_1, y_2, \dots, y_{r-\delta}).$$

Let $F \in M((r - \delta) \times r, R[[G]])$ be the matrix $E \cdot (bB)$. The same argument as before leading to (2.5) but now applied to F shows

$$\dim_R \text{coker}(r_{F_i}) \leq \delta \cdot [G : G_i] + O([G : G_i]^{1-1/d}). \tag{2.7}$$

We have $0 = r_D \circ r_E = r_{ED}$, yielding $ED = 0$ and

$$FAC = E(bB)AC = E(bI_r)BAC = (bI_r)EBAC = (bI_{r-\delta})ED = 0.$$

Note that we used $E(bI_r) = (bI_{r-\delta})E$ here. As C is invertible this implies $FA = 0$ and $r_A \circ r_F = r_{FA} = 0$. In particular, $r_{A_i} \circ r_{F_i} = 0$ and hence $\text{im}(r_{F_i}) \subseteq \ker(r_{A_i})$. So $\dim_R \text{im}(r_{F_i}) \leq \dim_R \ker(r_{A_i})$. We compute

$$\begin{aligned} &\dim_R \text{coker}(r_{A_i}) + \dim_R \text{coker}(r_{F_i}) \\ &= s \cdot [G : G_i] - \dim_R \text{im}(r_{A_i}) + r \cdot [G : G_i] - \dim_R \text{im}(r_{F_i}) \\ &\geq s \cdot [G : G_i] - \dim_R \text{im}(r_{A_i}) + r \cdot [G : G_i] - \dim_R \ker(r_{A_i}) \\ &= s \cdot [G : G_i]. \end{aligned}$$

Now (2.6) follows from (2.7).

Reduction to the case $R = \mathbb{F}_p$. To prove the statement for a finitely generated $\mathbb{Z}_p[[G]]$ -module M we may assume that $\text{rank}_{\mathbb{Z}_p[[G]]}(M) = 0$ due to the reduction to the case (2.2) and the fact that $\mathbb{Z}_p[[G]]$ has no zero-divisors. For the following reason we may, in addition, assume that M has no p -torsion: Let $T = \{m \in M \mid$

$\exists_{d(m) \in \mathbb{N}} p^{d(m)} \cdot m = 0$ be its p -torsion part. Obviously, T is a $\mathbb{Z}_p[[G]]$ -submodule of M . One easily sees by additivity of dimension that

$$\dim_{\mathbb{Q}_p}(\mathbb{Q}_p \widehat{\otimes}_{\mathbb{Z}_p[[G_i]]} M/T) = \dim_{\mathbb{Q}_p}(\mathbb{Q}_p \widehat{\otimes}_{\mathbb{Z}_p[[G_i]]} M)$$

and

$$\text{rank}_{\mathbb{Z}_p[[G]]}(M) = \text{rank}_{\mathbb{Z}_p[[G]]}(M/T).$$

Hence we may assume that M has no p -torsion. We prove now that

$$\text{rank}_{\mathbb{F}_p[[G]]}(M/pM) = \text{rank}_{\mathbb{Z}_p[[G]]}(M), \tag{2.8}$$

hence both are zero. Since the ring $\mathbb{Z}_p[[G]]$ has finite projective dimension ([3], Section 5.1), and every projective $\mathbb{Z}_p[[G]]$ -module is free, see [32], Corollary 7.5.4 on p. 127, and $\mathbb{Z}_p[[G]]$ is Noetherian, the finitely generated $\mathbb{Z}_p[[G]]$ -module M possesses a finite resolution by finitely generated free $\mathbb{Z}_p[[G]]$ -modules:

$$0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_0 \rightarrow M \rightarrow 0.$$

Applying the functor $N \mapsto \mathbb{F}_p \otimes_{\mathbb{Z}} N \cong N/pN$ yields a resolution of M/pM by finitely generated, free $\mathbb{F}_p[[G]]$ -modules since M has no p -torsion:

$$0 \rightarrow \mathbb{F}_p \otimes_{\mathbb{Z}} F_n \rightarrow \dots \rightarrow \mathbb{F}_p \otimes_{\mathbb{Z}} F_0 \rightarrow M/pM \rightarrow 0.$$

Now equation (2.8) follows since the rank functions over $\mathbb{F}_p[[G]]$ and $\mathbb{Z}_p[[G]]$ are additive and the equation obviously holds for finitely generated free $\mathbb{Z}_p[[G]]$ -modules.

Let $N = \mathbb{Z}_p \widehat{\otimes}_{\mathbb{Z}_p[[G_i]]} M$. Because of $\text{rank}_{\mathbb{Z}_p[[G]]}(M) = 0$ it is enough to prove $\dim_{\mathbb{Z}_p}(N) = O([G : G_i]^{1-1/d})$. This follows from the $\mathbb{F}_p[[G]]$ -case, $\text{rank}_{\mathbb{F}_p[[G]]}(M/pM) = 0$, and the inequality

$$\dim_{\mathbb{Z}_p}(N) \leq \dim_{\mathbb{F}_p}(N/pN) = \dim_{\mathbb{F}_p}(\mathbb{F}_p \widehat{\otimes}_{\mathbb{F}_p[[G_i]]} M/pM).$$

So we reduced the proof of the theorem to the case $R = \mathbb{F}_p$ and henceforth assume $R = \mathbb{F}_p$.

Reduction to G being standard. We finally reduce the assertion to the case that G is standard in the sense of [12], §8.4. Being a p -adic analytic group, G has an open subgroup H which is standard with respect to the manifold structure induced from G , see Theorem 8.29 of [12]. Since H is open, we have $G_i < H$ for i greater than some i_0 . Being standard H has a preferred collection of open normal subgroups H_i which satisfy: $G_{i_0+i-1} \subset H_i \subset G_i$ ($i \geq 1$); see e.g., [12], Example 6, p. 168.

Recall that we may assume that $\text{rank}_{\mathbb{F}_p[[G]]}(M) = 0$ due to the reduction to the case (2.2) and the fact that $\mathbb{F}_p[[G]]$ has no zero-divisors.

Now if the assertion holds for H with respect to the H_i 's, then it follows that for $i > i_0$ the left-hand side of (2.1) is bounded by a constant times $\dim_R(R \widehat{\otimes}_{R[[H_{i-i_0+1]]}} M)$ and is therefore $O([H : H_i]^{1-1/d}) = O([G : G_i]^{1-1/d})$, so the assertion holds for

G as well. We assume from now on that G is standard and that $G_i = \psi^{-1}(p^i \mathbb{Z}_p^d)$ where ψ is the global atlas of G .

The remaining argument. Let M be as in (2.2). We may assume $a \neq 0$. By a fundamental result of Lazard, the graded ring $\text{gr } \mathbb{F}_p[[G]]$ with respect to the filtration $(\Delta^n)_{n \geq 0}$ by powers of the augmentation ideal $\Delta \subset \mathbb{F}_p[[G]]$ is a polynomial algebra $\mathbb{F}_p[X_1, \dots, X_d]$ with indeterminates $X_i = x_i - 1 + \Delta^2$ ([32], Theorem 8.7.10 on p. 160) where $\{x_1, \dots, x_d\} \subset G$ is a minimal generating set. Let $I_i \subset \mathbb{F}_p[[G]]$ be the closure of the ideal generated by elements $\lambda(h - 1)$ with $h \in G_i$. Note that $N/I_i N \cong \mathbb{F}_p \widehat{\otimes}_{\mathbb{F}_p[[G_i]]} N$ for any $\mathbb{F}_p[[G]]$ -module N . Since $\mathbb{F}_p[[G]]$ is a domain, $\text{rank}_{\mathbb{F}_p[[G]]}(M) = 0$. Now for each integer $i \geq 1$ (if $p > 2$) or $i \geq 2$ (if $p = 2$), the global atlas ψ of G induces an epimorphism $G_i \rightarrow p^i \mathbb{Z}_p^d / p^{i+1} \mathbb{Z}_p^d$ with kernel G_{i+1} . It therefore follows that $[G : G_i] = Cp^{id}$ for some rational constant $C > 0$ and we have to show that

$$\dim_{\mathbb{F}_p}(M/I_i M) = O(p^{(d-1)i}). \tag{2.9}$$

But it follows from Lemma 7.1 of [12] that there exists a positive integer m such that $\Delta^{mp^i} \subset I_i$ for all i . It therefore suffices to show (2.9) with I_i replaced by Δ^{mp^i} . Let $s \geq 0$ be such that $a \in \Delta^s \setminus \Delta^{s+1}$. Let $a_i : \mathbb{F}_p[[G]]/\Delta^{mp^i} \rightarrow \mathbb{F}_p[[G]]/\Delta^{mp^i}$ be the map induced by right multiplication with a . We have

$$\begin{aligned} \dim_{\mathbb{F}_p}(M/\Delta^{mp^i} M) &= \dim_{\mathbb{F}_p} \text{coker}(a_i) \\ &= \dim_{\mathbb{F}_p} \text{ker}(a_i) \\ &= \dim_{\mathbb{F}_p}(\Delta^{mp^i-s}/\Delta^{mp^i}). \end{aligned}$$

The last equality follows from the fact the graded ring is a polynomial ring. For the same reason the last number equals the number of monomials in a polynomial ring with d variables each of which has total degree in the interval $[mp^i - s, mp^i)$. The number of monomials of degree $< k$ is $\binom{d+k-1}{d}$. Hence

$$\dim_{\mathbb{F}_p}(M/\Delta^{mp^i} M) = \binom{d + mp^i - 1}{d} - \binom{d + mp^i - s - 1}{d}.$$

As a polynomial in p^i , each binomial coefficient has leading term $(mp^i)^d$. Their difference is a polynomial in p^i with degree at most $d - 1$. This implies (2.9). \square

Proofs of Theorems 1.1 and 1.4. We show for both cases $R = \mathbb{Z}_p$ and $R = \mathbb{F}_p$ simultaneously that

$$b_k(X_i; R) = \text{rank}_{R[[G]]}(H_k(R[[G]] \otimes_{R\bar{\Gamma}} C_*(\bar{X}))) \cdot [\Gamma : \Gamma_i] + O([\Gamma : \Gamma_i]^{1-1/d}). \tag{2.10}$$

The CW-structure on X lifts to a $\bar{\Gamma}$ -equivariant CW-structure on \bar{X} and to $\Gamma_i \setminus \Gamma$ -equivariant CW-structures on X_i . We may also view \bar{X} as a Γ -space via the quotient

map $\Gamma \rightarrow \bar{\Gamma}$. Let $C_*(\bar{X})$ be the cellular chain complex of \bar{X} . Each chain module $C_k(\bar{X})$ is a finitely generated free $\mathbb{Z}[\bar{\Gamma}]$ -module. The differentials in the chain complex $C_* = R \otimes_{\mathbb{Z}} C_*(\bar{X})$ are denoted by ∂_* . Note that $R \otimes_{R[\Gamma_i]} C_*$ is isomorphic to the cellular chain complex $R \otimes_{\mathbb{Z}} C_*(X_i)$ as an $R[\Gamma_i \setminus \Gamma]$ -chain complex. In particular, we have

$$H_*(X_i, R) \cong H_*(R \otimes_{R[\Gamma_i]} C_*). \tag{2.11}$$

We write \hat{C}_* and $\hat{\partial}_*$ short for $R[[G]] \otimes_{R\bar{\Gamma}} C_*$ and its differentials. We denote the cycles and boundaries in the chain complexes \hat{C}_* and C_* by \hat{Z}_* , \hat{B}_* and Z_* , B_* , respectively. Let $r_n \in \mathbb{N}_0$ be the rank of the finitely generated free $R[[G]]$ -module \hat{C}_n . In each degree n we have the obvious exact sequence

$$0 \rightarrow \hat{Z}_n \rightarrow \hat{C}_n \xrightarrow{\hat{\partial}_n} \hat{C}_{n-1} \rightarrow \text{coker}(\hat{\partial}_n) \rightarrow 0.$$

By additivity of $\text{rank}_{R[[G]]}$ we obtain that

$$\begin{aligned} \text{rank}_{R[[G]]}(H_k(\hat{C}_*)) &= \text{rank}_{R[[G]]}(\hat{Z}_k / \hat{B}_k) \\ &= \text{rank}_{R[[G]]}(\hat{Z}_k) - \text{rank}_{R[[G]]}(\hat{B}_k) \\ &= r_k - r_{k-1} + \text{rank}_{R[[G]]}(\text{coker}(\hat{\partial}_k)) - \text{rank}_{R[[G]]}(\hat{B}_k) \\ &= r_k - r_{k-1} + \text{rank}_{R[[G]]}(\text{coker}(\hat{\partial}_k)) \\ &\quad - (r_k - \text{rank}_{R[[G]]}(\text{coker}(\hat{\partial}_{k+1}))) \\ &= \text{rank}_{R[[G]]}(\text{coker}(\hat{\partial}_k)) + \text{rank}_{R[[G]]}(\text{coker}(\hat{\partial}_{k+1})) - r_{k-1}. \end{aligned} \tag{2.12}$$

By (2.11), a similar argument as above, and right-exactness of the tensor product, we obtain that

$$\begin{aligned} \dim_R(H_k(X_i, R)) &= \dim_R(H_k(R \otimes_{R[\Gamma_i]} C_*)) \\ &= \dim_R(\text{coker}(R \otimes_{R[\Gamma_i]} \partial_k)) \\ &\quad + \dim_R(\text{coker}(R \otimes_{R[\Gamma_i]} \partial_{k+1})) - [\Gamma : \Gamma_i] \cdot r_{k-1} \\ &= \dim_R(R \otimes_{R[\Gamma_i]} \text{coker}(\partial_k)) \\ &\quad + \dim_R(R \otimes_{R[\Gamma_i]} \text{coker}(\partial_{k+1})) - [\Gamma : \Gamma_i] \cdot r_{k-1}. \end{aligned}$$

Hence,

$$\frac{b_k(X_i, R)}{[\Gamma : \Gamma_i]} = \frac{\dim_R(R \otimes_{R[\Gamma_i]} \text{coker}(\partial_k))}{[\Gamma : \Gamma_i]} + \frac{\dim_R(R \otimes_{R[\Gamma_i]} \text{coker}(\partial_{k+1}))}{[\Gamma : \Gamma_i]} - r_{k-1}. \tag{2.13}$$

The natural map

$$R \otimes_{R[\Gamma_i]} R[\Gamma] \xrightarrow{\cong} R \hat{\otimes}_{R[[G_i]]} R[[G]]$$

induced by $\phi: \Gamma \rightarrow G$ is a right $R[\Gamma]$ -module isomorphism (recall that we regard $R[[G]]$ as a right $R[\Gamma]$ -module via ϕ). The inverse is obtained as follows: Since

$\Gamma_i \backslash \Gamma \cong G_i \backslash G$, there is a natural continuous homomorphism from G to the invertible elements of the R -algebra $R \otimes_{R[\Gamma_i]} R[\Gamma]$. By the universal property of the completed group algebra there is a continuous homomorphism $R[[G]] \rightarrow R \otimes_{R[\Gamma_i]} R[\Gamma]$ which descends to the desired inverse. As a consequence we get isomorphisms

$$\begin{aligned} R \widehat{\otimes}_{R[[G_i]]} \operatorname{coker}(\hat{\partial}_k) &\cong R \otimes_{R[[G_i]]} R[[G]] \otimes_{R[\bar{\Gamma}]} \operatorname{coker}(\partial_k) \\ &\cong R \otimes_{R[[G_i]]} R[[G]] \otimes_{R[\Gamma]} \operatorname{coker}(\partial_k) \\ &\cong R \otimes_{R[\Gamma_i]} \operatorname{coker}(\partial_k) \end{aligned}$$

and, thus,

$$\begin{aligned} \frac{b_k(X_i, R)}{[G : G_i]} &= \frac{\dim_R(R \widehat{\otimes}_{R[[G_i]]} \operatorname{coker}(\hat{\partial}_k))}{[G : G_i]} \\ &\quad + \frac{\dim_R(R \widehat{\otimes}_{R[[G_i]]} \operatorname{coker}(\hat{\partial}_{k+1}))}{[G : G_i]} - r_{k-1}. \end{aligned} \tag{2.14}$$

Now (2.10) follows from (2.12), (2.14), and Theorem 2.1. Note that (2.10) is exactly the statement of Theorem 1.4 in the case $R = \mathbb{F}_p$. Next we explain how Theorem 1.1 follows from (2.10) when $R = \mathbb{Z}_p$. Since \mathbb{Q}_p has characteristic zero, we have $b_k(X_i) = b_k(X_i, \mathbb{Q}_p) = b_k(X_i, \mathbb{Z}_p)$. Since $b_k(X_i)/[\Gamma : \Gamma_i] \rightarrow \beta_k(\bar{X}, \bar{\Gamma})$ as $i \rightarrow \infty$ [24], we conclude

$$\beta_k(\bar{X}, \bar{\Gamma}) = \operatorname{rank}_{\mathbb{Z}_p[[G]]}(H_k(\mathbb{Z}_p[[G]] \otimes_{\mathbb{Z}_p \bar{\Gamma}} C_*)). \quad \square$$

3. Relation with the completed homology

Calegari and Emerton [7], [6] have introduced the completed homology groups:

$$\tilde{H}_k = \varprojlim H_k(X_i, \mathbb{Z}_p) \quad \text{and} \quad \tilde{H}_k(\mathbb{F}_p) = \varprojlim H_k(X_i, \mathbb{F}_p).$$

These modules carry continuous actions of G and may therefore be considered as $\mathbb{Z}_p[[G]]$ -modules or $\mathbb{F}_p[[G]]$ -modules, respectively. In this section we want to clarify the relation of completed cohomology to (mod p) L^2 -Betti numbers.

Proposition 3.1. *Retaining the setup in Section 1.1 we have*

$$\beta_k(\bar{X}, \bar{\Gamma}; \mathbb{F}_p) = \operatorname{rank}_{\mathbb{F}_p[[G]]}(\tilde{H}_k(\mathbb{F}_p)).$$

Proof. Here again we may reduce to the case where G is torsion-free. Write $C_* = C_*(\bar{X}; \mathbb{F}_p)$. The claim is equivalent to

$$\tilde{H}_k(\mathbb{F}_p) = \varprojlim H_k(\mathbb{F}_p \otimes_{\mathbb{F}_p[\Gamma_i]} C_*)$$

and

$$H_k(\mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p\bar{\Gamma}} C_*) = H_k(\varprojlim(\mathbb{F}_p \otimes_{\mathbb{F}_p[\Gamma_i]} C_*))$$

having the same $\mathbb{F}_p[[G]]$ -rank. So the statement is equivalent to:

$$\begin{aligned} Q(\mathbb{F}_p[[G]]) \otimes_{\mathbb{F}_p[[\Gamma]]} H_k(\varprojlim(\mathbb{F}_p \otimes_{\mathbb{F}_p[\Gamma_i]} C_*)) \\ \cong Q(\mathbb{F}_p[[G]]) \otimes_{\mathbb{F}_p[[\Gamma]]} (\varprojlim H_k(\mathbb{F}_p \otimes_{\mathbb{F}_p[\Gamma_i]} C_*)). \end{aligned}$$

Since $\mathbb{F}_p \otimes_{\mathbb{Z}_p[\Gamma_i]} C_*$ is a tower of chain complexes of abelian groups satisfying the Mittag-Leffler condition, by Theorem 3.5.8 of [31] there is a short exact sequence

$$\begin{aligned} 0 \rightarrow \varprojlim^1 H_{k+1}(\mathbb{F}_p \otimes_{\mathbb{F}_p[\Gamma_i]} C_*) \rightarrow H_k(\varprojlim(\mathbb{F}_p \otimes_{\mathbb{F}_p[\Gamma_i]} C_*)) \\ \rightarrow \varprojlim H_k(\mathbb{F}_p \otimes_{\mathbb{F}_p[\Gamma_i]} C_*) \rightarrow 0. \end{aligned}$$

Moreover, since towers of finite dimensional vector spaces over a *field* satisfy the Mittag-Leffler condition, we conclude that

$$\varprojlim^1 H_{k+1}(\mathbb{F}_p \otimes_{\mathbb{F}_p[\Gamma_i]} C_*) = 0,$$

which yields the proposition. □

It follows from works of Calegari and Emerton that a similar result with \mathbb{F}_p replaced by \mathbb{Z}_p holds as well, that is,

$$\beta_k(\bar{X}, \bar{\Gamma}; \mathbb{Z}_p) = \text{rank}_{\mathbb{Z}_p[[G]]}(\tilde{H}_k), \tag{3.1}$$

and we want to indicate why. In [5], Theorem 3.2, it is shown for arithmetic congruence covers X_i of symmetric spaces that the so-called co-rank r_k of the completed cohomology \tilde{H}^k satisfies the equality in Theorem 1.1 with $\beta_k(\bar{X}, \bar{\Gamma})$ replaced by r_k . The proof in [5] is a consequence of a general result of Emerton [14], Theorem 2.1.5, and their Lemma 2.2. Both these results hold not only for arithmetic congruence covers but in our generality. Since the co-rank of \tilde{H}^k is the same as the rank of \tilde{H}_k , see [7], Theorem 1.1 (3), this implies (3.1).

It is somewhat harder to work with completed homology, see [5], [6]. We nevertheless want to emphasize that the latter contains a lot more information. It should be the right framework to determine the growth rate of (mod p) Betti numbers even if the corresponding (mod p) L^2 -Betti number vanishes. However it seems hard to extract the necessary information from completed homology, the only exception that we are aware of is in case X is 3-dimensional, see [6].

4. Approximation results for pro- p towers that are not p -adic analytic

The proof of Theorem 1.6 relies on the following well-known lemma (see also [15] for a proof).

Lemma 4.1. *Let k be a field of characteristic $p > 0$. Let Λ be a normal subgroup in a group Γ whose index is a p -power. Then*

$$\dim_k(k \otimes_{k[\Lambda]} M) \leq [\Gamma : \Lambda] \cdot \dim_k(k \otimes_{k[\Gamma]} M).$$

Proof. Because of the isomorphism

$$k \otimes_{k[\Gamma]} M \cong k \otimes_{k[\Gamma/\Lambda]} (k \otimes_{k[\Lambda]} M)$$

it suffices to prove the case where Λ is trivial and Γ is a finite p -group. Let

$$k[\Gamma]^m \xrightarrow{f} k[\Gamma]^n \rightarrow M \rightarrow 0$$

be a presentation of M . Then $\bar{f} = k \otimes_{k[\Gamma]} f$ is a presentation of $k \otimes_{k[\Gamma]} M$. Since $\dim_k(M) = |\Gamma| \cdot n - \dim_k(\text{im}(f))$ and $\dim_k(k \otimes_{k[\Gamma]} M) = n - \dim_k(\text{im}(\bar{f}))$, we have to show that

$$\dim_k(\text{im}(f)) \geq |\Gamma| \cdot \dim_k(\text{im}(\bar{f})).$$

Extend a k -basis $\{u_1, \dots, u_s\}$ of $\text{im}(\bar{f})$ to a k -basis $\{u_1, \dots, u_n\}$ of $k^n = k \otimes_{k[\Gamma]} k[\Gamma]^n$. Let x_1, \dots, x_n be lifts of the u_i to $k[\Gamma]^n$ such that $\{x_1, \dots, x_s\} \subset \text{im}(f)$. Since $k[\Gamma]$ is a local ring with the augmentation ideal as the unique maximal ideal ([32], Proposition 7.5.3), Nakayama’s lemma implies that $\{x_1, \dots, x_n\}$ generates $k[\Gamma]^n$ as a $k[\Gamma]$ -module. Since the k -dimension of the $k[\Gamma]$ -submodule generated by x_i is at most $|\Gamma|$ and $\dim_k k[\Gamma]^n = |\Gamma| \cdot n$, the k -dimension of the $k[\Gamma]$ -module generated by $\{x_1, \dots, x_i\}$ is $i|\Gamma|$. Because of $\{x_1, \dots, x_s\} \subset \text{im}(f)$,

$$\dim_k(\text{im}(f)) \geq |\Gamma| \cdot s = |\Gamma| \cdot \dim_k(\text{im}(\bar{f}))$$

follows. □

Proof of Theorem 1.6. It follows from Lemma 4.1 that, for any finitely presented $k[\Gamma]$ -module M , the sequence $(\dim_k(k \otimes_{k[\Gamma_i]} M)/[\Gamma : \Gamma_i])_{i \geq 0}$ is monotone decreasing. Let r_n be the number of n -cells in X . Let ∂_* denote the differentials in the $k[\Gamma]$ -complex $C_*(\tilde{X}; k)$. Exactly as in (2.13), one has

$$\frac{b_n(X_i; k)}{[\Gamma : \Gamma_i]} = \frac{\dim_k(k \otimes_{k[\Gamma_i]} \text{coker}(\partial_n))}{[\Gamma : \Gamma_i]} + \frac{\dim_k(k \otimes_{k[\Gamma_i]} \text{coker}(\partial_{n+1}))}{[\Gamma : \Gamma_i]} - r_{n-1},$$

from which monotonicity follows. Since Betti numbers are non-negative, the sequence converges. □

In the remainder of this section we study the question how we can express the limit

$$\left(\frac{b_n(X_i; k)}{[\Gamma : \Gamma_i]} \right)_{i \geq 0}$$

by some algebraic expression in very specific situations. For that we recall the notions of ordered group, the Malcev–Neumann power series ring, and the division closure.

An *ordered group* is a group with a strict total ordering of its elements which is invariant under left and right translations. A group which has such an ordering is called *orderable*. For example, residually torsion-free nilpotent groups are orderable, see [10], Proposition 1.2 on p. 274.

If Γ is an ordered group, then the set of formal power series $\sum_{\gamma \in \Gamma} a_\gamma \gamma$ with coefficients a_γ in a skew field k whose support $\{\gamma \in \Gamma \mid a_\gamma \neq 0\}$ is well-ordered becomes a skew field with the obvious ring structure extending the one of the group ring $k[\Gamma]$ (Corollary 15.10 on p. 95 of [9]) which is called the *Malcev–Neumann power series ring*. We denote it by $k((\Gamma))^2$.

Suppose R is a subring of a skew field K . Then $D(R, K)$ will denote the *division closure* of R in K , that is the smallest skew subfield of K that contains R . If M_1 and M_2 are the Malcev–Neumann power series rings of $k[\Gamma]$ with respect to two different orders and D_1, D_2 the division closures of $k[\Gamma]$ in M_1, M_2 , respectively, then there is a ring isomorphism $D_1 \cong D_2$ which is the identity on $k[\Gamma]$. This follows from the next theorem. As a consequence, the dimension of the $k((\Gamma))$ -vector space

$$k((\Gamma)) \otimes_{k[\Gamma]} M \cong k((\Gamma)) \otimes_{D(k[\Gamma], k((\Gamma)))} (D(k[\Gamma], k((\Gamma))) \otimes_{k[\Gamma]} M)$$

for a $k[\Gamma]$ -module M does not depend on the choice of the order on Γ .

Theorem 4.2. *Let k be a skew field, let Γ be an orderable group, and let M_1, M_2 be Malcev–Neumann power series rings for $k[\Gamma]$. Set $D_1 = D(k[\Gamma], M_1)$ and $D_2 = D(k[\Gamma], M_2)$. Then there is a ring isomorphism $\theta: D_1 \rightarrow D_2$ such that θ is the identity on $k[\Gamma]$.*

Proof. Recall [26], Corollary 1.4, that, being orderable, the group Γ is locally indicable, i.e., each finitely generated subgroup H surjects onto \mathbb{Z} . Let $J \triangleleft H$ be the kernel of this surjection and pick $t \in H$ so that $H = \langle J, t \rangle$. Let N_1 and N_2 denote the subrings of M_1 and M_2 respectively consisting of power series with supports in J . Then $D(kJ, D_1) \subseteq N_1$ and $D(kJ, D_2) \subseteq N_2$. Clearly $\{t^n \mid n \in \mathbb{N}\}$ is linearly independent over N_i in M_i for $i = 1, 2$. Theorem 4.2 therefore follows from a slight generalization of Hughes’ theorem [11], 7.1 Theorem; all that one needs to do is to modify the proof of 7.2 Lemma in [11]; this is done explicitly in [27], Hughes’ Theorem I 6.3. □

Theorem 4.3. *Let k be a field of characteristic $p > 0$. Let Γ be a finitely generated group, and let $\Gamma = \Gamma_0 > \Gamma_1 > \dots$ be a descending sequence of normal subgroups such that $\bigcap_i \Gamma_i = \{1\}$ and each Γ/Γ_i is torsion-free nilpotent. Set $H_i = \Gamma_i \Gamma^{p^i}$ (thus Γ/H_i is a finite p -group). If X is a compact connected CW-complex with fundamental group Γ and X_i the finite cover corresponding to H_i , then³, for every n ,*

$$\dim_{k((\Gamma))} (H_n(k((\Gamma)) \otimes_{k[\Gamma]} C_*(\tilde{X}, k))) = \lim_{i \rightarrow \infty} \frac{b_n(X_i; k)}{[\Gamma : H_i]}. \tag{4.1}$$

²We suppress the order in the notation for reasons to be seen below.

³Recall that the left-hand side of (4.1) does not depend on the order we choose to define $k((\Gamma))$.

In particular, the limit on the right-hand side is an integer.

Remark 4.4. A large collection of groups are known to be residually torsion-free nilpotent (RTFN): Carl Droms has proved in his PhD thesis [13] (Theorem 1.1 in Chapter III on page 58; see also [1]) that graph groups are RTFN. But this property is inherited by subgroups. It therefore follows from [17] and [4] that free groups, surface groups, reflection groups, right-angled Artin groups and arithmetic hyperbolic groups defined by quadratic forms are virtually RTFN. It follows from the recent breakthrough of Agol [2] and Wise [33] that the fundamental group of a closed hyperbolic 3-manifold is virtually RTFN. We finally note that the proof of Corollary 2.3 in [1] implies that direct and free products of RTFN groups are RTFN.

One deduces the preceding theorem from the following Proposition 4.5 by a similar, even easier, argument as used in the deduction of Theorem 1.1 from Theorem 2.1. More precisely: Similarly as in (2.12) and (2.13) one expresses the left- and right-hand side of (4.1) by the cokernels of the n -th and $(n + 1)$ -th differential of the complexes $k((\Gamma)) \otimes_{k[\Gamma]} C_*(\tilde{X}, k)$ and $C_*(X_i, k)$, respectively; then one applies the proposition below.

Proposition 4.5. *Let k be a field of characteristic $p > 0$. Let Γ be a finitely generated group, and let $\Gamma = \Gamma_0 > \Gamma_1 > \dots$ be a descending sequence of normal subgroups such that $\bigcap_i \Gamma_i = \{1\}$ and each Γ/Γ_i is torsion-free nilpotent. Set $H_i = \Gamma_i \Gamma^{p^i}$. If M is a finitely presented $k[\Gamma]$ -module, then*

$$\dim_{k((\Gamma))}(k((\Gamma)) \otimes_{k[\Gamma]} M) = \lim_{i \rightarrow \infty} \frac{\dim_k(k \otimes_{k[H_i]} M)}{[\Gamma : H_i]}.$$

In particular, the limit on the right-hand side is an integer.

A free group is residually torsion-free nilpotent. But even for a free group we cannot say anything for arbitrary residual p -chains. In particular, the following question remains open.

Question 4.6. Let F be a finitely generated free group, let k be a field of characteristic $p > 0$, and let $F = F_0 > F_1 > \dots$ be a descending sequence of normal subgroups with F/F_i a finite p -group for all i and $\bigcap_{i \in \mathbb{N}} F_i = 1$. Let M be a finitely presented kG -module. Can $\lim_{i \rightarrow \infty} |F/F_i|^{-1} \dim_k(k \otimes_{k[F_i]} M)$ be transcendental?

In the remainder of this section we are concerned with the proof of Proposition 4.5 for which we need the following lemma.

Lemma 4.7. *Let k be an arbitrary skew field, let M be a finitely presented $k[\Gamma]$ -module, and let $\Gamma = \Gamma_0 > \Gamma_1 > \dots$ be a descending sequence of normal subgroups with Γ/Γ_i torsion-free nilpotent for all i with $\bigcap_{i \in \mathbb{N}} \Gamma_i = 1$. Let D_i denote the skew*

field of fractions of $k[\Gamma/\Gamma_i]$, which is an Ore localization. Then there exists $m \in \mathbb{N}$ such that

$$\dim_{D_i}(D_i \otimes_{k[\Gamma]} M) = \dim_{k((\Gamma))}(k((\Gamma)) \otimes_{k[\Gamma]} M) \quad \text{for all } i \geq m.$$

Proof. Choose a nonprincipal ultrafilter ω on \mathbb{N} and let D denote the ultraproduct of the D_i with respect to the ultrafilter ω . Then D is a skew field and $k[\Gamma]$ embeds in D . Consider a nontrivial finitely generated subgroup H of Γ . Let n be the least positive integer such that H is not contained in Γ_n . Then $H/H \cap \Gamma_n$ is a nontrivial finitely generated torsion-free nilpotent group, so there exists $N \triangleleft H$ such that H/N is infinite cyclic. Choose $t \in H \setminus N$ so that $H = \langle N, t \rangle$. Then $\{t^i \mid i \in \mathbb{N}\}$ is linearly independent over $k((N))$ and it follows that $\{t^i \mid i \in \mathbb{N}\}$ is linearly independent over $D(k[N], k((\Gamma)))$. Next let E_i denote the skew field of fractions of $k[N/N \cap \Gamma_i]$ and form the ultraproduct E of the E_i with respect to ω . Then E is a skew field contained in D , and $\{t^i \mid i \in \mathbb{N}\}$ is linearly independent over E . Therefore $\{t^i \mid i \in \mathbb{N}\}$ is linearly independent over $D(k[N], D)$ and we deduce from [27], Hughes' Theorem I 6.3, that there is an isomorphism $\theta: D(k[\Gamma], k((\Gamma))) \rightarrow D(k[\Gamma], D)$ such that θ is the identity on $k[\Gamma]$. Therefore $\dim_{k((\Gamma))} k((\Gamma)) \otimes_{k[\Gamma]} M = \dim_D D \otimes_{k[\Gamma]} M$. Also $\dim_D D \otimes_{k[\Gamma]} M = \lim_{\omega} \dim_{D_i} D_i \otimes_{k[\Gamma]} M$ and the result follows. \square

Proof of Proposition 4.5. For each $i \in \mathbb{N}$, let D_i denote the skew field of fractions of $k[\Gamma/\Gamma_i]$. By Lemma 4.7, there exists $m_0 \in \mathbb{N}$ such that $\dim_{D_m} D_m \otimes_{k[\Gamma]} M = \dim_{k((\Gamma))} k((\Gamma)) \otimes_{k[\Gamma]} M$ for all $m \geq m_0$. For $i, m \geq m_0$, set $K_i^m = \Gamma_m \Gamma^{p^i}$. For any $m \geq m_0$ we have

$$\begin{aligned} \lim_{i \rightarrow \infty} |\Gamma/K_i^m|^{-1} \dim_k(k \otimes_{k[K_i^m]} M) &= \dim_{D_m}(D_m \otimes_{k[\Gamma]} M) \\ &= \dim_{k((\Gamma))} k((\Gamma)) \otimes_{k[\Gamma]} M \end{aligned}$$

by Theorem 6.3 of [23]. The result follows from the monotonicity Lemma 4.1. \square

5. Link complements

Suppose that $\Gamma \twoheadrightarrow \mathbb{Z}^d$. We embed $\mathbb{Z}^d \hookrightarrow \mathbb{Z}_p^d =: G$ and consider the homomorphism $\phi: \Gamma \rightarrow G$. The $\mathbb{Z}[\mathbb{Z}^d]$ -module $H_1(\bar{X}, \mathbb{Z})$ is the Alexander invariant of X . There is a natural map $\mathbb{Z}[\mathbb{Z}^d] \rightarrow \mathbb{F}_p[[\mathbb{Z}_p^d]]$. Since \mathbb{Z}^d is amenable, the group ring $\mathbb{F}_p[\mathbb{Z}^d]$ is an Ore domain and it follows from [23] that

$$\begin{aligned} \beta_k(\bar{X}, \bar{\Gamma}; \mathbb{F}_p) &= \text{rank}_{\mathbb{F}_p[[\mathbb{Z}_p^d]]} H_k(\bar{X}, \mathbb{F}_p[[\mathbb{Z}_p^d]]) \\ &= \text{rank}_{\mathbb{F}_p[\mathbb{Z}^d]} H_k(\bar{X}, \mathbb{F}_p[\mathbb{Z}^d]) \\ &= \text{rank}_{\mathbb{F}_p[t_1^{\pm 1}, \dots, t_d^{\pm 1}]} H_k(\bar{X}, \mathbb{F}_p[t_1^{\pm 1}, \dots, t_d^{\pm 1}]). \end{aligned}$$

This last expression is zero if and only if the (first) Alexander polynomial Δ of X is non zero modulo p .

The above remark in particular applies when X is a 3-manifold with boundary a union of d tori. If X is knot complement (in which case $d = 1$) the Alexander polynomial Δ is nonzero and its coefficients are relatively prime. Hence

$$\beta_1(\bar{X}, \bar{\Gamma}; \mathbb{F}_p) = \text{rank}_{\mathbb{F}_p[[\mathbb{Z}'_p]]} H_1(\bar{X}, \mathbb{F}_p[[\mathbb{Z}'_p]]) = 0 \tag{5.1}$$

and (see also [29], Corollary 4.4)

$$\lim_{i \rightarrow +\infty} \frac{\dim H_1(X_i, \mathbb{F}_p)}{[\Gamma : \Gamma_i]} = 0.$$

In that case one may even deduce from (5.1) that $H_1(X_i, \mathbb{F}_p) = \mathbb{F}_p$ for all i , see Lemma 5.4 of [6].

We may as well consider the case of a link complement $l = l_1 \cup \dots \cup l_d$ in \mathbb{S}^3 . Recall that, in the case $d = 2$, $\Delta(1)$ is equal to the linking number $\text{Lk}(l_1, l_2)$ of the two components of the link. The same proof as in the knot case then shows that if p is a prime that does not divide $\text{Lk}(l_1, l_2)$, then

$$\lim_{i \rightarrow +\infty} \frac{\dim H_1(X_i, \mathbb{F}_p)}{[\Gamma : \Gamma_i]} = 0.$$

One may wonder if the proof may be extended to show that $H_1(X_i, \mathbb{F}_p) = \mathbb{F}_p^2$ for all i as is true according to [29], Theorem 5.11.

We conclude this note by the proof of Proposition 1.5.

Proof of Proposition 1.5. It follows from [20] that there exists a link l with 2 components such that $\Delta(t, t) = p$. Now note that $\Delta(t, t)$ is the Alexander polynomial associated to the abelian cover of X corresponding to the map $\Gamma \rightarrow \mathbb{Z}^2 \rightarrow \mathbb{Z}^2 / \langle a-b \rangle \cong \mathbb{Z}$. Since $\Delta(t, t)$ is non zero,

$$\lim_{i \rightarrow +\infty} \frac{\dim H_1(X_i, \mathbb{Q})}{[\Gamma : \Gamma_i]} = 0.$$

But since $\Delta(t, t)$ is zero modulo p , we have

$$\lim_{i \rightarrow +\infty} \frac{\dim H_1(X_i, \mathbb{F}_p)}{[\Gamma : \Gamma_i]} \neq 0. \tag{□}$$

Other examples of closed finite CW -complexes with a chain

$$\pi_1(X) = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \dots$$

of normal subgroups of finite index in $\pi_1(X)$ such that $\lim_{i \rightarrow +\infty} \frac{\dim H_1(X_i, \mathbb{F}_p)}{[\Gamma : \Gamma_i]} \neq \lim_{i \rightarrow +\infty} \frac{\dim H_1(X_i, \mathbb{Q})}{[\Gamma : \Gamma_i]}$ holds can be found in [15] and [25]. One can additionally arrange $X = B\Gamma$ or $\bigcap_{i \geq 0} \Gamma_i = \{1\}$. However, the problem is still open to find an example with both $X = B\Gamma$ and $\bigcap_{i \geq 0} \Gamma_i = \{1\}$.

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Received September 28, 2012; revised January 27, 2013

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