

A note on trace fields of complex hyperbolic groups

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Abstract. We show that if Γ is an irreducible subgroup of $SU(2, 1)$, then Γ contains a loxodromic element A . If A has eigenvalues $\lambda_1 = \lambda e^{i\varphi}$, $\lambda_2 = e^{-2i\varphi}$, $\lambda_3 = \lambda^{-1} e^{i\varphi}$, we prove that Γ is conjugate in $SU(2, 1)$ to a subgroup of $SU(2, 1, \mathbb{Q}(\Gamma, \lambda))$, where $\mathbb{Q}(\Gamma, \lambda)$ is the field generated by the trace field $\mathbb{Q}(\Gamma)$ of Γ and λ . It follows from this that if Γ is an irreducible subgroup of $SU(2, 1)$ such that the trace field $\mathbb{Q}(\Gamma)$ is real, then Γ is conjugate in $SU(2, 1)$ to a subgroup of $SO(2, 1)$. As a geometric application of the above, we get that if G is an irreducible discrete subgroup of $PU(2, 1)$, then G is an \mathbb{R} -Fuchsian subgroup of $PU(2, 1)$ if and only if the invariant trace field $k(G)$ of G is real.

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Introduction

Arithmetic methods are a powerful tool in the study of Kleinian groups, discrete subgroups of $PSL(2, \mathbb{C})$, especially of finite-covolume discrete groups, as was demonstrated in [21], see also an extensive bibliography there. A central theme in this theory is to understand the structure of the invariant trace field and the invariant (quaternion) algebra associated to a Kleinian group. In the case of complex hyperbolic geometry, that is, in the case of subgroups of $PU(n, 1)$ ($SU(n, 1)$) little known about these objects, see for instance [23], where the study of the invariant trace fields and the invariant algebras associated to subgroups of $SU(n, 1)$ was initiated. In particular, in this work the invariant trace field and the invariant algebra were introduced for subgroups of $SU(n, 1)$. An important problem here is to understand whether a subgroup of $SU(n, 1)$ can be realized over the field generated by the eigenvalues of its elements. In this paper, we prove that any irreducible subgroup Γ of $SU(2, 1)$ contains a loxodromic element A and it can be realized over the field generated by the trace field of Γ and the eigenvalues of A .

The main result of our paper is the following theorem:

Theorem A. *Let Γ be an irreducible subgroup of $SU(2, 1)$ and $A \in \Gamma$ be loxodromic with eigenvalues $\lambda_1 = \lambda e^{i\varphi}$, $\lambda_2 = e^{-2i\varphi}$, $\lambda_3 = \lambda^{-1} e^{i\varphi}$. Then Γ is conjugate in*

$SU(2, 1)$ to a subgroup of $SU(2, 1, \mathbb{Q}(\Gamma, \lambda))$, where $\mathbb{Q}(\Gamma, \lambda)$ is the field generated by the trace field $\mathbb{Q}(\Gamma)$ of Γ and λ .

As a corollary of this theorem, we get the following

Theorem B. *Let Γ be an irreducible subgroup of $SU(2, 1)$ such that $\mathbb{Q}(\Gamma)$ is a subset of \mathbb{R} , then Γ is conjugate in $SU(2, 1)$ to a subgroup of $SO(2, 1)$.*

We would like to stress that in Theorem A and Theorem B we do not assume that the group Γ is discrete.

Also, in this paper, we define an invariant trace field for subgroups of $PU(2, 1)$. Let G be a subgroup of $PU(2, 1)$ and $\Gamma = \pi^{-1}(G)$, where $\pi: SU(2, 1) \rightarrow PU(2, 1)$ is a natural projection. Then the *invariant trace field* of G , denoted by $k(G)$, is defined to be the field $\mathbb{Q}(\Gamma^3)$, where $\Gamma^3 = \langle \gamma^3 : \gamma \in \Gamma \rangle$. It follows from [23] that the invariant trace field is an invariant of the commensurability class.

We say that a subgroup G of $PU(2, 1)$ is an \mathbb{R} -subgroup if it leaves invariant a totally real geodesic 2-plane in $\mathbf{H}_{\mathbb{C}}^2$. A subgroup G of $PU(2, 1)$ is called \mathbb{R} -Fuchsian if it is a discrete \mathbb{R} -subgroup. A subgroup G is a \mathbb{C} -subgroup if it leaves invariant a complex geodesic in $\mathbf{H}_{\mathbb{C}}^2$. A subgroup G of $PU(2, 1)$ is called \mathbb{C} -Fuchsian if it is a discrete \mathbb{C} -subgroup.

By applying Theorem B, we get the following characterization of discrete non-elementary \mathbb{R} -subgroups of $PU(2, 1)$.

Theorem C. *Let G be an irreducible discrete subgroup of $PU(2, 1)$. Then G is an \mathbb{R} -Fuchsian if and only if the invariant trace field $k(G)$ of G is real.*

This implies, in particular, that if G is an irreducible discrete subgroup of $PU(2, 1)$ whose invariant trace field is real, then the invariant algebra associated to G is of dimension nine over $k(G)$. On the other hand, if G is a non-elementary \mathbb{C} -subgroup of $PU(2, 1)$ (G is reducible in this case) with real invariant trace field, then the invariant algebra associated to G is of dimension four.

As a corollary of Theorem C, we have the following.

Theorem D. *Let G be a discrete non-elementary subgroup of $PU(2, 1)$ such that the invariant trace field $k(G)$ of G is real. Then G is either \mathbb{R} -Fuchsian or \mathbb{C} -Fuchsian.*

We remark that Theorem D can be considered as a complex hyperbolic analog of a classical result due to B.Maskit, see Theorem G.18 in [22] and Corollary 3.2.5 in [21]. Finally, we would like to mention that some related questions were considered in [3], [14], [15], [24], [27], [28], [29], [30].

The article is organized as follows. In Section 1, we review some basic facts in complex hyperbolic geometry. In Section 2, we prove our main results.

1. Complex hyperbolic plane and its isometry group

Let V be a 3-dimensional \mathbb{C} -vector space equipped with a Hermitian form $\langle -, - \rangle$ of signature $(2, 1)$. We denote by $\mathbb{P}(V)$ the complex projectivization of V and by $\mathbb{P} : V \setminus \{0\} \rightarrow \mathbb{P}(V)$ a natural projection.

Let V_-, V_0, V_+ be the subsets of $V \setminus \{0\}$ consisting of vectors where $\langle v, v \rangle$ is negative, zero, or positive respectively. Vectors in V_0 are called *null* or *isotropic*, vectors in V_- are called *negative*, and vectors in V_+ are called *positive*. Their projections to $\mathbb{P}(V)$ are called *isotropic*, *negative*, and *positive* points respectively.

The projective model of the *complex hyperbolic plane* $\mathbf{H}_{\mathbb{C}}^2$ is the set of negative points in $\mathbb{P}(V)$, that is, $\mathbf{H}_{\mathbb{C}}^2 = \mathbb{P}(V_-)$. The boundary $\partial\mathbf{H}_{\mathbb{C}}^2 = \mathbb{P}(V_0)$ of $\mathbf{H}_{\mathbb{C}}^2$ is the 3-sphere formed by all isotropic points.

The Hermitian form $\langle -, - \rangle$ defines a metric, the Bergman metric, on $\mathbf{H}_{\mathbb{C}}^2$, see [16]. Let $U(V)$ be the unitary group corresponding to this Hermitian form. Then the holomorphic isometry group of $\mathbf{H}_{\mathbb{C}}^2$ is the projective unitary group $PU(V)$, and the full isometry group of $\mathbf{H}_{\mathbb{C}}^2$ is generated by $PU(V)$ and complex conjugation. We denote by $SU(V)$ the subgroup of linear transformations in $U(V)$ with determinant 1.

For our purposes it is convenient to work with a basis $e = \{e_1, e_2, e_3\}$ in V which has the following properties:

$$\begin{aligned} \langle e_1, e_1 \rangle &= 0, & \langle e_2, e_2 \rangle &= 1, & \langle e_3, e_3 \rangle &= 0, \\ \langle e_1, e_2 \rangle &= 0, & \langle e_2, e_3 \rangle &= 0, & \langle e_1, e_3 \rangle &= 1. \end{aligned}$$

So, in this basis e_1 and e_3 are isotropic and e_2 is positive. In what follows, we denote by $\mathbb{C}^{2,1}$ the vector space V equipped with this basis. If the vectors $z = (z_1, z_2, z_3)^T$ and $w = (w_1, w_2, w_3)^T$ in $\mathbb{C}^{2,1}$ are given by their coordinates in $e = \{e_1, e_2, e_3\}$, then the Hermitian product $\langle v, w \rangle$ is given by

$$\langle v, w \rangle = z_1 \bar{w}_3 + z_2 \bar{w}_2 + z_3 \bar{w}_1.$$

The use of this basis simplifies essentially matrix computations and it was successfully applied in a series of works, see, for instance [6], [7], [8], [9], [10], [11], [12], [13], [18], [19], [29]. Throughout this paper we will use this basis.

Let $U(2, 1)$ and $SU(2, 1)$ denote the representations of $U(V)$ and $SU(V)$ in the basis $e = \{e_1, e_2, e_3\}$.

If A is an element of $SU(2, 1)$, then the matrix A is defined by the following simple conditions:

$$\begin{aligned} \langle v_1, v_1 \rangle &= 0, & \langle v_2, v_2 \rangle &= 1, & \langle v_3, v_3 \rangle &= 0, \\ \langle v_1, v_2 \rangle &= 0, & \langle v_2, v_3 \rangle &= 0, & \langle v_1, v_3 \rangle &= 1, \end{aligned}$$

where v_1, v_2, v_3 denote the vectors defined by the rows of A .

Also, we have the following useful formula for the inverse of $A \in SU(2, 1)$:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} \bar{a}_{33} & \bar{a}_{23} & \bar{a}_{13} \\ \bar{a}_{32} & \bar{a}_{22} & \bar{a}_{12} \\ \bar{a}_{31} & \bar{a}_{21} & \bar{a}_{11} \end{bmatrix}.$$

It is seen that A^{-1} is the Hermitian anti-transpose of A .

The non-trivial elements of $\text{PU}(2, 1)$ fall into three general conjugacy types, depending on the number and location of their fixed points.

- *Elliptic elements* have a fixed point in $\mathbf{H}_{\mathbb{C}}^2$,
- *Parabolic elements* have a single fixed point on the boundary of $\mathbf{H}_{\mathbb{C}}^2$,
- *Loxodromic elements* have exactly two fixed points on the boundary of $\mathbf{H}_{\mathbb{C}}^2$.

This exhausts all possibilities, see [16] for details.

Let $\pi : \text{SU}(2, 1) \rightarrow \text{PU}(2, 1)$ be a natural projection. We call an element $A \in \text{SU}(2, 1)$ loxodromic (parabolic, elliptic) if its projectivization $\pi(A)$ is loxodromic (parabolic, elliptic). For instance, any loxodromic element $A \in \text{SU}(2, 1)$ is conjugate in $\text{SU}(2, 1)$ to an element of the following form

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix},$$

where $\lambda_1 = \lambda e^{i\varphi}$, $\lambda_2 = e^{-2i\varphi}$, $\lambda_3 = \lambda^{-1} e^{i\varphi}$, $\lambda > 0$, $\lambda \neq 1$, $\varphi \in (-\pi, \pi]$.

A parabolic element $g \in \text{PU}(2, 1)$ is *unipotent* if it can be represented by a unipotent element of $\text{SU}(2, 1)$, that is, a matrix having 1 as its only eigenvalue. Otherwise, g is *ellipto-parabolic*. In that case g can be represented by an element of $\text{SU}(2, 1)$ having a repeated non-real eigenvalue of norm 1. Also, g has a unique invariant complex geodesic, see below.

There are two types of totally geodesic submanifolds of $\mathbf{H}_{\mathbb{C}}^2$ of real dimension two:

- *Complex geodesics* (copies of $\mathbf{H}_{\mathbb{C}}^1$) have constant sectional curvature -1 .
- *Totally real geodesic 2-planes* (copies of $\mathbf{H}_{\mathbb{R}}^2$) have constant sectional curvature $-1/4$.

Any complex geodesic is the intersection of a complex projective line in $\mathbb{P}(V)$ with $\mathbf{H}_{\mathbb{C}}^2$, and it is uniquely defined by its polar point, which is positive [16]. We recall that a polar point to a complex projective line c in $\mathbb{P}(V)$ is the projectivization of the Hermitian orthogonal complement in V of $\mathbb{P}^{-1}(c)$.

Any totally real geodesic 2-plane is the intersection of a totally real projective plane in $\mathbb{P}(V)$ with $\mathbf{H}_{\mathbb{C}}^2$.

We recall that a subspace S of $V_{\mathbb{R}}$, where $V_{\mathbb{R}}$ is the real vector space underlying V , is *totally real* if and only if S and its image $\mathbb{J}(S)$ are orthogonal with respect to the Hermitian product $\langle -, - \rangle$, see [16]. It is easy to show that S is totally real if and only if the Hermitian product $\langle v, u \rangle$ is real for all $v, u \in S$. An example of a totally real subspace is the \mathbb{R} -linear span of $e = \{e_1, e_2, e_3\}$, the basis considered above. We call this subspace a *canonical* totally real subspace. The projectivization of this space is called a *canonical totally real* projective 2-plane. Any totally real 2-plane in $\mathbb{P}(V)$ is the image of the canonical totally real projective 2-plane under an element from the

group $\text{PU}(2, 1)$. The stabilizer of the canonical totally real subspace in $\text{SU}(2, 1)$ can be canonically identified with the group $\text{SO}(2, 1)$.

A chain is the boundary of a complex geodesic. An \mathbb{R} -circle is the boundary of a totally real geodesic 2-plane, see [16].

Let G be a subgroup of $\text{PU}(2, 1)$. We say that G is a \mathbb{C} -subgroup if it leaves invariant a complex geodesic. G is called an \mathbb{R} -subgroup if it leaves invariant a totally real geodesic 2-plane. A subgroup Γ of $\text{SU}(2, 1)$ is called \mathbb{C} -subgroup if its projectivization is a \mathbb{C} -subgroup of $\text{PU}(2, 1)$. Similarly, a subgroup Γ of $\text{SU}(2, 1)$ is called an \mathbb{R} -subgroup if its projectivization is an \mathbb{R} -subgroup of $\text{PU}(2, 1)$. We say that a subgroup G of $\text{PU}(2, 1)$ is \mathbb{C} -Fuchsian if it is a discrete \mathbb{C} -subgroup, and G is called \mathbb{R} -Fuchsian if it is a discrete \mathbb{R} -subgroup. Typical examples of \mathbb{C} -subgroups of $\text{SU}(2, 1)$ are subgroups of $\text{SU}(1, 1)$ canonically embedded into $\text{SU}(2, 1)$, and typical examples of \mathbb{R} -subgroups of $\text{SU}(2, 1)$ are subgroups of $\text{SO}(2, 1)$ canonically embedded into $\text{SU}(2, 1)$.

Recall that a subgroup Γ of $\text{SU}(2, 1)$ is called *reducible* if it has an invariant proper \mathbb{C} -subspace of V , and called *irreducible* otherwise. It is clear that a subgroup Γ of $\text{SU}(2, 1)$ is irreducible if and only if Γ has no invariant 1-dimensional \mathbb{C} -subspaces of V .

A subgroup G of $\text{PU}(2, 1)$ is called *reducible* if all elements of G have a common fixed point in their action on the projective space $\mathbb{P}(V)$. Otherwise, G is called *irreducible*.

A subgroup G of $\text{PU}(2, 1)$ is called *elementary* if it has a finite orbit in its action on $\mathbf{H}_{\mathbb{C}}^2 \cup \partial\mathbf{H}_{\mathbb{C}}^2$. Otherwise, G is *non-elementary*. A subgroup Γ of $\text{SU}(2, 1)$ is called *elementary* if its projectivization is elementary. Otherwise, Γ is *non-elementary*.

Clearly, any \mathbb{C} -subgroup G of $\text{PU}(2, 1)$ is reducible because the polar point to the invariant complex geodesic of G is a common fixed point for all elements of G in their action on $\mathbb{P}(V)$. Also, it is clear that a non-elementary \mathbb{R} -subgroup of $\text{PU}(2, 1)$ is irreducible.

2. Trace fields of complex hyperbolic groups

Let Γ be a subgroup of $\text{SU}(2, 1)$. Then the *trace field* of Γ , denoted by $\mathbb{Q}(\text{tr}\Gamma)$, is the field generated by the traces of all the elements of Γ over the base field \mathbb{Q} of rational numbers. For simplicity, we will denote the trace field of Γ by $\mathbb{Q}(\Gamma)$. Of course, $\mathbb{Q}(\Gamma)$ is a conjugacy invariant. We remark that the trace field $\mathbb{Q}(\Gamma)$ is also invariant under complex conjugation, since for $B \in \text{SU}(2, 1)$ we have $\text{tr}(B^{-1}) = \overline{\text{tr}(B)}$.

If Γ contains a loxodromic element A with eigenvalues $\lambda_1 = \lambda e^{i\varphi}$, $\lambda_2 = e^{-i2\varphi}$, $\lambda_3 = \lambda^{-1} e^{i\varphi}$, then $\mathbb{Q}(\Gamma, \lambda)$ denotes the field generated by $\mathbb{Q}(\Gamma)$ and λ . Also, for any field k we denote by $\text{SU}(2, 1, k)$ the intersection of $\text{SU}(2, 1)$ with $\text{M}(3, k)$.

Lemma 2.1. *Let $\Gamma = \langle A \rangle$ be the group generated by a loxodromic element $A \in \text{SU}(2, 1)$ with eigenvalues $\lambda_1 = \lambda e^{i\varphi}$, $\lambda_2 = e^{-2i\varphi}$, $\lambda_3 = \lambda^{-1} e^{i\varphi}$. Then $e^{i\varphi} \in \mathbb{Q}(\Gamma, \lambda)$.*

Proof. Since $\mathbb{Q}(\Gamma)$ is invariant under complex conjugation, it follows that $\text{Re}(\text{tr}(A))$ and $|\text{tr}(A)|^2$ are in $\mathbb{Q}(\Gamma)$. We have that

$$\text{tr}(A) = \lambda e^{i\varphi} + \lambda^{-1} e^{i\varphi} + e^{-i2\varphi}$$

and

$$\text{tr}(A^{-1}) = \overline{\text{tr}(A)} = \lambda e^{-i\varphi} + \lambda^{-1} e^{-i\varphi} + e^{i2\varphi}.$$

A direct computation shows that

$$\text{Re}(\text{tr}(A)) = \cos 2\varphi + (\lambda + \lambda^{-1}) \cos \varphi = 2 \cos^2 \varphi + (\lambda + \lambda^{-1}) \cos \varphi - 1$$

and

$$|\text{tr}(A)|^2 = (\lambda^2 + \lambda^{-2}) + 2(\lambda + \lambda^{-1}) \cos 3\varphi + 3.$$

Since $\lambda + \lambda^{-1} \neq 0$, the last formula implies that $\cos 3\varphi \in \mathbb{Q}(\Gamma, \lambda)$.

Looking at these formulae, one could expect that $\cos \varphi$ is in a proper extension of $\mathbb{Q}(\Gamma, \lambda)$, but the following trick shows that, in fact, $\cos \varphi$ is in $\mathbb{Q}(\Gamma, \lambda)$.

It is easy to see that the following formula

$$(4 \cos^2 \varphi + 2t \cos \varphi + t^2 - 3) \cos \varphi = 4 \cos^3 \varphi - 3 \cos \varphi + (2 \cos^2 \varphi + t \cos \varphi)t$$

is true for all $t \in \mathbb{R}$ and for all $\varphi \in (-\pi, \pi]$.

Also, it is elementary to check that

$$4 \cos^2 \varphi + 2t \cos \varphi + t^2 - 3 \neq 0$$

for all $t > 2$ and for all $\varphi \in (-\pi, \pi]$. This implies that for all $t > 2$ and for all $\varphi \in (-\pi, \pi]$ we have that

$$\cos \varphi = \frac{4 \cos^3 \varphi - 3 \cos \varphi + (2 \cos^2 \varphi + t \cos \varphi)t}{4 \cos^2 \varphi + 2t \cos \varphi + t^2 - 3}.$$

In particular, taking $t = \lambda + \lambda^{-1}$ and using the identity $\cos 3\varphi = 4 \cos^3 \varphi - 3 \cos \varphi$, we get the formula

$$\cos \varphi = \frac{\cos 3\varphi + (2 \cos^2 \varphi + (\lambda + \lambda^{-1}) \cos \varphi)(\lambda + \lambda^{-1})}{4 \cos^2 \varphi + 2(\lambda + \lambda^{-1}) \cos \varphi + (\lambda + \lambda^{-1})^2 - 3}.$$

One verifies that this can be re-written as

$$\cos \varphi = \frac{\cos 3\varphi + (\text{Re}(\text{tr}(A)) + 1)(\lambda + \lambda^{-1})}{2\text{Re}(\text{tr}(A)) + (\lambda + \lambda^{-1})^2 - 1}.$$

Therefore, the above implies that $\cos \varphi \in \mathbb{Q}(\Gamma, \lambda)$ for all $\varphi \in (-\pi, \pi]$.

Next, a direct computation shows that

$$i \sin \varphi = \frac{\operatorname{tr}(A) - \overline{\operatorname{tr}(A)}}{2(\lambda + \lambda^{-1} + 2 \cos \varphi)}.$$

All the above implies that $e^{i\varphi} \in \mathbb{Q}(\Gamma, \lambda)$ for all $\varphi \in (-\pi, \pi]$. □

Corollary 2.1. *Let $\Gamma = \langle A \rangle$ be the group generated by a loxodromic element $A \in \operatorname{SU}(2, 1)$ with eigenvalues $\lambda_1 = \lambda e^{i\varphi}$, $\lambda_2 = e^{-i2\varphi}$, $\lambda_3 = \lambda^{-1} e^{i\varphi}$. Then all these eigenvalues belong to $\mathbb{Q}(\Gamma, \lambda)$.*

Lemma 2.2. *Let Γ be a subgroup of $\operatorname{SU}(2, 1)$ containing a loxodromic element $A = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_1 = \lambda e^{i\varphi}$, $\lambda_2 = e^{-i2\varphi}$, $\lambda_3 = \lambda^{-1} e^{i\varphi}$. Then for any element $B = (b_{ij}) \in \Gamma$, the diagonal elements of B are in $\mathbb{Q}(\Gamma, \lambda)$.*

Proof. We write

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} \bar{\lambda}_3 & 0 & 0 \\ 0 & \bar{\lambda}_2 & 0 \\ 0 & 0 & \bar{\lambda}_1 \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}.$$

Then computations show that

$$\begin{aligned} \operatorname{tr}(B) &= b_{11} + b_{22} + b_{33} = t_1, \\ \operatorname{tr}(AB) &= \lambda_1 b_{11} + \lambda_2 b_{22} + \lambda_3 b_{33} = t_2, \\ \operatorname{tr}(A^{-1}B) &= \bar{\lambda}_3 b_{11} + \bar{\lambda}_2 b_{22} + \bar{\lambda}_1 b_{33} = t_3. \end{aligned}$$

Note that every t_i lies in $\mathbb{Q}(\Gamma)$. We consider these equalities as a system of linear equations in three unknowns b_{11}, b_{22}, b_{33} . Let us show that the matrix L of this system is non-singular. We write

$$L = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \bar{\lambda}_3 & \bar{\lambda}_2 & \bar{\lambda}_1 \end{bmatrix}.$$

Then a computation gives that

$$\begin{aligned} \det L &= (\lambda_2 \bar{\lambda}_1 - \lambda_3 \bar{\lambda}_2) - (\lambda_1 \bar{\lambda}_1 - \lambda_3 \bar{\lambda}_3) + (\lambda_1 \bar{\lambda}_2 - \lambda_2 \bar{\lambda}_3) \\ &= (\lambda^{-2} - \lambda^2) + 2(\lambda - \lambda^{-1}) \cos 3\varphi. \end{aligned}$$

The equality $\det L = 0$ is equivalent to $\cos 3\varphi = (\lambda + \lambda^{-1})/2$, which is impossible since $\lambda + \lambda^{-1} > 2$.

It follows from Corollary 2.1 that every coefficient of the system lies in $\mathbb{Q}(\Gamma, \lambda)$. Solving this system by Cramer's rule, we conclude that every b_{ii} lies in $\mathbb{Q}(\Gamma, \lambda)$. □

Lemma 2.3. *Let Γ be a subgroup of $SU(2, 1)$ containing a loxodromic element $A = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_1 = \lambda e^{i\varphi}$, $\lambda_2 = e^{-i2\varphi}$, $\lambda_3 = \lambda^{-1} e^{i\varphi}$. Then for any element $B = (b_{ij}) \in \Gamma$, the products $b_{12}b_{21}$, $b_{13}b_{31}$, $b_{23}b_{32}$ are in $\mathbb{Q}(\Gamma, \lambda)$.*

Proof. Let $C = BAB$. First, we compute the diagonal elements c_{ii} of the matrix C . They are

$$\begin{aligned} c_{11} &= \lambda_1 b_{11}^2 + \lambda_2 b_{12}b_{21} + \lambda_3 b_{13}b_{31}, \\ c_{22} &= \lambda_1 b_{12}b_{21} + \lambda_2 b_{22}^2 + \lambda_3 b_{23}b_{32}, \\ c_{33} &= \lambda_1 b_{13}b_{31} + \lambda_2 b_{23}b_{32} + \lambda_3 b_{33}^2. \end{aligned}$$

Then by applying Lemma 2.2, we have that c_{ii} is in $\mathbb{Q}(\Gamma, \lambda)$. Also, by the same reason, b_{ii} is in $\mathbb{Q}(\Gamma, \lambda)$. Therefore, we can re-write these equalities in the following form

$$\begin{aligned} \lambda_2 b_{12}b_{21} + \lambda_3 b_{13}b_{31} &= t_1, \\ \lambda_1 b_{12}b_{21} + \lambda_3 b_{23}b_{32} &= t_2, \\ \lambda_1 b_{13}b_{31} + \lambda_2 b_{23}b_{32} &= t_3, \end{aligned}$$

where t_1, t_2, t_3 are some elements of $\mathbb{Q}(\Gamma, \lambda)$.

We consider these equalities as a system of linear equations in three unknowns $x_1 = b_{12}b_{21}$, $x_2 = b_{13}b_{31}$, $x_3 = b_{23}b_{32}$. The matrix L of this system is

$$L = \begin{bmatrix} \lambda_2 & \lambda_3 & 0 \\ \lambda_1 & 0 & \lambda_3 \\ 0 & \lambda_1 & \lambda_2 \end{bmatrix}.$$

A short computation shows that $\det L = -2\lambda_1\lambda_2\lambda_3 = -2\det A \neq 0$. Solving this system by Cramer's rule, we conclude that every x_i lies in $\mathbb{Q}(\Gamma, \lambda)$. \square

Lemma 2.4. *Let Γ be a subgroup of $SU(2, 1)$ containing a loxodromic element $A = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_1 = \lambda e^{i\varphi}$, $\lambda_2 = e^{-i2\varphi}$, $\lambda_3 = \lambda^{-1} e^{i\varphi}$. Then for any element $B = (b_{ij}) \in \Gamma$, the products $b_{12}\bar{b}_{32}$, $b_{13}\bar{b}_{31}$, $b_{23}\bar{b}_{21}$ are in $\mathbb{Q}(\Gamma, \lambda)$.*

Proof. Let $C = BAB^{-1}$. Let us compute the diagonal elements c_{ii} of the matrix C . They are

$$\begin{aligned} c_{11} &= \lambda_1 b_{11}\bar{b}_{33} + \lambda_2 b_{12}\bar{b}_{32} + \lambda_3 b_{13}\bar{b}_{31}, \\ c_{22} &= \lambda_1 b_{21}\bar{b}_{23} + \lambda_2 b_{22}\bar{b}_{22} + \lambda_3 b_{23}\bar{b}_{21}, \\ c_{33} &= \lambda_1 b_{31}\bar{b}_{13} + \lambda_2 b_{32}\bar{b}_{12} + \lambda_3 b_{33}\bar{b}_{11}. \end{aligned}$$

By applying Lemma 2.2, we get that c_{ii} lies in $\mathbb{Q}(\Gamma, \lambda)$. Since λ is real, we have that the field $\mathbb{Q}(\Gamma, \lambda)$ is invariant under complex conjugation. Hence, $b_{11}\bar{b}_{33}$, $b_{22}\bar{b}_{22}$, $b_{33}\bar{b}_{11}$ are all in $\mathbb{Q}(\Gamma, \lambda)$. Therefore, we can re-write these equalities in the following form

$$\lambda_2 b_{12}\bar{b}_{32} + \lambda_3 b_{13}\bar{b}_{31} = t_1,$$

$$\begin{aligned}\lambda_1 b_{21} \bar{b}_{23} + \lambda_3 b_{23} \bar{b}_{21} &= t_2, \\ \lambda_1 b_{31} \bar{b}_{13} + \lambda_2 b_{32} \bar{b}_{12} &= t_3,\end{aligned}$$

where t_1, t_2, t_3 are some elements of $\mathbb{Q}(\Gamma, \lambda)$.

Now let $D = B^{-1}AB$. Again we compute the diagonal elements d_{ii} of this matrix. They are

$$\begin{aligned}d_{11} &= \lambda_1 b_{11} \bar{b}_{33} + \lambda_2 b_{21} \bar{b}_{23} + \lambda_3 b_{31} \bar{b}_{13}, \\ d_{22} &= \lambda_1 b_{12} \bar{b}_{32} + \lambda_2 b_{22} \bar{b}_{22} + \lambda_3 b_{32} \bar{b}_{12}, \\ d_{33} &= \lambda_1 b_{13} \bar{b}_{31} + \lambda_2 b_{23} \bar{b}_{21} + \lambda_3 b_{33} \bar{b}_{11}.\end{aligned}$$

By applying the above arguments, we re-write these equalities in the following form

$$\begin{aligned}\lambda_2 b_{21} \bar{b}_{23} + \lambda_3 b_{31} \bar{b}_{13} &= s_1, \\ \lambda_1 b_{12} \bar{b}_{32} + \lambda_3 b_{31} \bar{b}_{12} &= s_2, \\ \lambda_1 b_{13} \bar{b}_{32} + \lambda_2 b_{23} \bar{b}_{21} &= s_3,\end{aligned}$$

where s_1, s_2, s_3 are some elements of $\mathbb{Q}(\Gamma, \lambda)$.

We consider the first and the conjugate third equality defined by C and the conjugate first equality defined by D . So, we have the following equalities:

$$\begin{aligned}\lambda_2 b_{12} \bar{b}_{32} + \lambda_3 b_{13} \bar{b}_{31} &= t_1, \\ \bar{\lambda}_1 b_{13} \bar{b}_{31} + \bar{\lambda}_2 b_{12} \bar{b}_{32} &= \bar{t}_3, \\ \bar{\lambda}_2 b_{23} \bar{b}_{21} + \bar{\lambda}_3 b_{13} \bar{b}_{31} &= \bar{s}_1.\end{aligned}$$

These equalities define a system of linear equations in three unknowns $x_1 = b_{12} \bar{b}_{32}$, $x_2 = b_{13} \bar{b}_{31}$, $x_3 = b_{23} \bar{b}_{21}$. The matrix L of this system is

$$L = \begin{bmatrix} \lambda_2 & \lambda_3 & 0 \\ \bar{\lambda}_2 & \bar{\lambda}_1 & 0 \\ 0 & \bar{\lambda}_3 & \bar{\lambda}_2 \end{bmatrix}.$$

The determinant $\det L = \lambda_2(\bar{\lambda}_1 \bar{\lambda}_2) - \lambda_3 \bar{\lambda}_2^2 = e^{2i\varphi}(e^{-3i\varphi} \lambda - e^{3i\varphi} \lambda^{-1})$. It is seen that $\det L = 0$ if and only if $\lambda^2 = e^{6i\varphi}$. This is impossible since $\lambda > 0, \lambda \neq 1$. Solving this system by Cramer's rule, we conclude that every x_i lies in $\mathbb{Q}(\Gamma, \lambda)$. \square

The following proposition is crucial in the proof of our main result.

Proposition 2.1. *Let $\Gamma = \langle A, B \rangle$ be an irreducible subgroup of $SU(2, 1)$, where A is a loxodromic element with eigenvalues $\lambda_1 = \lambda e^{i\varphi}$, $\lambda_2 = e^{-2i\varphi}$, $\lambda_3 = \lambda^{-1} e^{i\varphi}$. Then Γ is conjugate in $SU(2, 1)$ to a subgroup of $SU(2, 1, \mathbb{Q}(\Gamma, \lambda))$.*

Proof. We will show that one needs at most two conjugations to get the result we need.

First, by applying a suitable conjugation in $SU(2, 1)$, we may assume that

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}.$$

By $v_1 = (b_{11}, b_{12}, b_{13})^\top$, $v_2 = (b_{21}, b_{22}, b_{23})^\top$, and $v_3 = (b_{31}, b_{32}, b_{33})^\top$ we denote the vectors defined by the rows of the matrix B .

Let $X \in SU(2, 1)$ be a loxodromic element (elliptic if $r = 1$) such that

$$X = \begin{bmatrix} re^{i\alpha} & 0 & 0 \\ 0 & e^{-2i\alpha} & 0 \\ 0 & 0 & r^{-1}e^{i\alpha} \end{bmatrix}.$$

Then $XAX^{-1} = A$ and XBX^{-1} is

$$XBX^{-1} = \begin{bmatrix} b_{11} & re^{3i\alpha}b_{12} & r^2b_{13} \\ r^{-1}e^{-3i\alpha}b_{21} & b_{22} & re^{-3i\alpha}b_{23} \\ r^{-2}b_{31} & r^{-1}e^{3i\alpha}b_{32} & b_{33} \end{bmatrix}.$$

If the entries b_{12} , b_{32} of B are all equal to 0, then the group Γ is not irreducible since in this case A and B have a common invariant complex line spanned by the vector $(0, 1, 0)^\top$. In this case, Γ is a \mathbb{C} -group. Therefore, at least one of the numbers b_{12} , b_{32} is not equal to 0. Let us suppose that $b_{12} \neq 0$. In this case, by normalizing the elements A and B using X , we may assume without loss of generality that $b_{12} = 1$.

From the above results we know that b_{ii} , $b_{12}b_{21}$, $b_{13}b_{31}$, $b_{23}b_{32}$, $b_{12}\bar{b}_{32}$, $b_{13}\bar{b}_{31}$, $b_{23}\bar{b}_{21}$ are all in $\mathbb{Q}(\Gamma, \lambda)$.

Since $b_{12} = 1$, we get that b_{21} and b_{32} are in $\mathbb{Q}(\Gamma, \lambda)$. Let us first consider the case $b_{21} = 0$. Then we have that $b_{31} \neq 0$. Indeed, if $b_{31} = 0$, then A and B have a common invariant complex line spanned by the vector $(1, 0, 0)^\top$. This is impossible because Γ is irreducible. Let us assume now that $b_{32} = 0$. Since the vectors v_2 and v_3 are orthogonal, we get that $\langle v_2, v_3 \rangle = b_{23}\bar{b}_{31} = 0$. So, in this case, $b_{23} = 0$. This implies that $b_{22} \neq 0$. Hence, $\langle v_1, v_2 \rangle = \bar{b}_{22} \neq 0$, a contradiction. Therefore, we have that $b_{32} \neq 0$. It follows that b_{23} lies in $\mathbb{Q}(\Gamma, \lambda)$. Since the vectors v_1 and v_2 are orthogonal, we have that $b_{23} \neq 0$. Then, by considering the equality $\langle v_2, v_3 \rangle = b_{22}\bar{b}_{32} + b_{23}\bar{b}_{31} = 0$, we conclude that b_{31} lies in $\mathbb{Q}(\Gamma, \lambda)$. Since $b_{31} \neq 0$, we get that b_{13} lies in $\mathbb{Q}(\Gamma, \lambda)$. This shows that all the entries of the matrix B are in $\mathbb{Q}(\Gamma, \lambda)$. Therefore, we proved the proposition in this case.

Now let us assume that $b_{21} \neq 0$. It follows immediately that b_{23} lies in $\mathbb{Q}(\Gamma, \lambda)$.

We write $\langle v_1, v_2 \rangle = b_{11}\bar{b}_{23} + \bar{b}_{22} + b_{13}\bar{b}_{21} = 0$. Since $b_{21} \neq 0$, this implies that b_{13} is in $\mathbb{Q}(\Gamma, \lambda)$. If $b_{13} \neq 0$, we get that b_{31} is in $\mathbb{Q}(\Gamma, \lambda)$. Suppose that $b_{13} = 0$. In this case, $b_{23} \neq 0$, since otherwise A and B have a common invariant complex line spanned by the vector $(0, 0, 1)^\top$. This is impossible because Γ is irreducible.

We write $\langle v_2, v_3 \rangle = b_{21}\bar{b}_{33} + b_{22}\bar{b}_{32} + b_{23}\bar{b}_{31} = 0$. Since $b_{23} \neq 0$, we get that b_{31} lies in $\mathbb{Q}(\Gamma, \lambda)$.

Summarizing everything, we get that in the case $b_{12} \neq 0$ after conjugation all the entries of the matrix B lie in $\mathbb{Q}(\Gamma, \lambda)$. The case $b_{12} = 0, b_{32} \neq 0$ is similar to the previous one. □

Next we prove that any irreducible subgroup of $SU(2, 1)$ contains a loxodromic element. First, we will prove the following lemma.

Lemma 2.5. *Let Γ be a non-elementary subgroup of $SU(2, 1)$. If Γ contains a parabolic element, then Γ contains a loxodromic element.*

Proof. Let $B \in \Gamma$ be parabolic. Then B has a unique invariant isotropic complex line in V . By normalizing Γ in $SU(2, 1)$, we may assume without loss of generality that this line is spanned by the vector $(1, 0, 0)^\top$. This implies that B has the following form:

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{bmatrix}.$$

As before, we denote by $v_1 = (b_{11}, b_{12}, b_{13})^\top, v_2 = (b_{21}, b_{22}, b_{23})^\top, v_3 = (b_{31}, b_{32}, b_{33})^\top$ the vectors defined by the rows of the matrix B .

We consider two cases: (1) $b_{12} = 0, (2) b_{12} \neq 0$.

First, we consider the case $b_{12} = 0$. We have that $\langle v_2, v_2 \rangle = |b_{22}|^2 = 1$. From this and the equality $\langle v_2, v_3 \rangle = 0$, we get that $b_{32} = 0$. Then the equality $\langle v_1, v_2 \rangle = b_{11}\bar{b}_{23} = 0$ implies that $b_{23} = 0$. From the equality $\langle v_1, v_3 \rangle = 1$, we get that $b_{11}\bar{b}_{33} = 1$. Since B is parabolic, we have that $b_{13} \neq 0$ and that $|b_{11}| = 1$. Hence $|b_{33}| = 1$. We write $b_{11} = e^{i\varphi}$ and $b_{13} = re^{i\theta}, r > 0$. Then the equality $\langle v_1, v_1 \rangle = 0$ implies that $b_{13} = rie^{i\varphi}$. Therefore, when $b_{12} = 0$, the element B after a suitable normalization of Γ has the following form:

$$B = \begin{bmatrix} e^{i\varphi} & 0 & rie^{i\varphi} \\ 0 & e^{-2i\varphi} & 0 \\ 0 & 0 & e^{i\varphi} \end{bmatrix}.$$

We remark that if $\varphi = 0$, then B is unipotent and B is ellipto-parabolic otherwise.

Next, we consider the case $b_{12} \neq 0$. Using the same arguments as in the first case, we get that $|b_{22}| = 1$ and $b_{32} = 0$. Since $b_{12} \neq 0$, by normalizing Γ using the element X defined in the proof of Proposition 2.1, we may assume without loss of generality that $b_{12} = 1$. The equality $\langle v_1, v_3 \rangle = 1$ implies that $b_{11}\bar{b}_{33} = 1$. Note that in this case B is always unipotent. Hence $b_{11} = 1$. This implies that $b_{22} = b_{33} = 1$. Then it follows from the equality $\langle v_1, v_2 \rangle = 0$ that $b_{23} = -1$. Now let us consider the equality $\langle v_1, v_1 \rangle = \bar{b}_{13} + b_{13} + 1 = 0$. It easy to see that this equality is true if and only if $b_{13} = -1/2 + si$, where s is real. Summarizing everything, we get

that in the case $b_{12} \neq 0$ after a suitable normalization of Γ the element B has the following form:

$$B = \begin{bmatrix} 1 & 1 & \tau \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix},$$

where $\tau = -1/2 + si$, $s \in \mathbb{R}$.

An easy induction shows that for any $n \in \mathbb{N}$ in the first case

$$B^n = \begin{bmatrix} e^{ni\varphi} & 0 & e^{ni\varphi}inr \\ 0 & e^{-2ni\varphi} & 0 \\ 0 & 0 & e^{ni\varphi} \end{bmatrix}$$

and

$$B^n = \begin{bmatrix} 1 & n & n\tau + f(n) \\ 0 & 1 & -n \\ 0 & 0 & 1 \end{bmatrix}$$

in the second case, where the function $f: \mathbb{N} \rightarrow \mathbb{Z}$ is defined by the following conditions: $f(1) = 0$ and $f(n+1) = f(n) - n$. It is easy to show that $f(n) = (1-n)n/2$.

Since Γ is non-elementary it contains an element C which does not leave invariant the complex line spanned by the vector $(1, 0, 0)^\top$. Let $C = (c_{ij})$. Then we have that $c_{31} \neq 0$. Easy computation shows that

$$\text{tr}(B^n C) = c_{11}e^{ni\varphi} + c_{31}e^{ni\varphi}inr + c_{22}e^{-2ni\varphi} + c_{33}e^{ni\varphi}$$

in the first case, and

$$\text{tr}(B^n C) = c_{11} + c_{21}n + c_{31}(n\tau + f(n)) + c_{22} - c_{32}n + c_{33}$$

in the second one.

Hence, in the first case

$$\begin{aligned} |\text{tr}(B^n C)| &= |c_{31}e^{ni\varphi}inr - (-c_{11}e^{ni\varphi} - c_{22}e^{-2ni\varphi} - c_{33}e^{ni\varphi})| \\ &\geq |c_{31}e^{ni\varphi}inr| - |(-c_{11}e^{ni\varphi} - c_{22}e^{-2ni\varphi} - c_{33}e^{ni\varphi})| \\ &= |c_{31}|nr - |(-c_{11}e^{ni\varphi} - c_{22}e^{-2ni\varphi} - c_{33}e^{ni\varphi})|. \end{aligned}$$

Since $c_{31} \neq 0$, and $r > 0$, this inequality implies that $|\text{tr}(B^n C)|$ tends to infinity when n tends to infinity.

In the second case, we have that

$$\begin{aligned} |\text{tr}(B^n C)| &= |c_{11} + c_{21}n + c_{31}(n\tau + f(n)) + c_{22} - c_{32}n + c_{33}| \\ &\geq |c_{31}f(n)| - |-c_{31}n\tau - c_{11} - c_{21}n - c_{22} + c_{32}n - c_{33}| \\ &= |c_{31}||f(n)| - |-c_{31}n\tau - c_{11} - c_{21}n - c_{22} + c_{32}n - c_{33}|. \end{aligned}$$

Since $c_{31} \neq 0$ and $f(n)$ is quadratic, this inequality implies that $|\text{tr}(B^n C)|$ tends to infinity when n tends to infinity.

Thus, in both cases, we have that there exists n_0 such that $|\text{tr}(B^n C)| > 3$ for all $n > n_0$. Then using Goldman’s classification of the elements in $\text{SU}(2, 1)$ [16], we get that the elements $B^n C$ are loxodromic for all $n > n_0$. This proves the lemma. \square

We are very grateful to John Parker for his help in the proof of this lemma.

In what follows, we will need the following fundamental result due to Chen and Greenberg [4]:

Proposition 2.2. *Let Γ be a subgroup of $\text{SU}(2, 1)$. Consider the natural action of Γ on $\mathbf{H}_{\mathbb{C}}^2 \cup \partial\mathbf{H}_{\mathbb{C}}^2$. If there is no point in $\mathbf{H}_{\mathbb{C}}^2 \cup \partial\mathbf{H}_{\mathbb{C}}^2$ or proper totally geodesic submanifold in $\mathbf{H}_{\mathbb{C}}^2$ which is invariant under Γ , then Γ is either discrete or dense in $\text{SU}(2, 1)$.*

By applying this result and Lemma 2.5, we prove the following proposition.

Proposition 2.3. *Let Γ be an irreducible subgroup of $\text{SU}(2, 1)$. Then Γ contains a loxodromic element.*

Proof. Let us consider the natural action of Γ on the projective space $\mathbb{P}(V)$ (or equally the action of its projectivization). We know that the only proper totally geodesic submanifolds of $\mathbf{H}_{\mathbb{C}}^2$ are either geodesics, or complex geodesics, or totally real geodesic 2-planes [4], [16]. Note that if there is a geodesic in $\mathbf{H}_{\mathbb{C}}^2$ which is invariant under Γ , then there is a complex geodesic c in $\mathbf{H}_{\mathbb{C}}^2$ which is also invariant under Γ (this complex geodesic c is spanned by the geodesic in question). Therefore, the polar point to c is invariant under Γ for its action on the projective space $\mathbb{P}(V)$. Hence, if Γ is irreducible, then either Γ has no invariant proper totally geodesic submanifolds or Γ is an \mathbb{R} -subgroup of $\text{SU}(2, 1)$.

First, we consider the case when Γ has no invariant totally real geodesic 2-planes. By applying Proposition 2.2, we get that Γ is either discrete or dense in $\text{SU}(2, 1)$. If Γ is dense in $\text{SU}(2, 1)$, it contains a loxodromic element since the set of loxodromic elements is open in $\text{SU}(2, 1)$ [16], and we are done. Now let us assume that Γ is a discrete subgroup of $\text{SU}(2, 1)$. Suppose that Γ has no loxodromic elements. If Γ contains only elliptic elements, then it is easy to see that Γ is finite, and then, using the arguments similar to those in [25], we conclude that there is a point in $\mathbf{H}_{\mathbb{C}}^2$ which is invariant under Γ , a contradiction. Therefore, Γ contains a parabolic element γ . Since Γ is irreducible, it contains an element not fixing the unique fixed point of γ . By applying Lemma 2.5, we get that Γ contains a loxodromic element, a contradiction.

Now, let Γ be an \mathbb{R} -subgroup. Then Γ is conjugate in $\text{SU}(2, 1)$ to a subgroup of $\text{SO}(2, 1)$. Since Γ irreducible, the result follows from the plane real hyperbolic geometry [2], [25]. \square

Corollary 2.2. *Let G be an irreducible subgroup of $\mathrm{PU}(2, 1)$. Then G is non-elementary.*

Proof. Assume that G is elementary. Let $p \in \mathbf{H}_{\mathbb{C}}^2 \cup \partial\mathbf{H}_{\mathbb{C}}^2$ be a point such that the orbit $G(p)$ of p is finite. If $p \in \mathbf{H}_{\mathbb{C}}^2$, then it follows from the proof of Proposition 2.3 that there is a point in $\mathbf{H}_{\mathbb{C}}^2$ which is invariant under Γ , a contradiction. Next, let us suppose that $p \in \partial\mathbf{H}_{\mathbb{C}}^2$. By applying Proposition 2.3, we have that G contains a loxodromic element. Since $G(p)$ is finite, this implies that $G(p)$ consists of two distinct points. Let $G(p) = \{p_1, p_2\}$ and c be the unique complex geodesic spanned by $\{p_1, p_2\}$. Then c is invariant under G , and, therefore, G leaves invariant its polar point, a contradiction. \square

Thus, we proved that any irreducible subgroup of $\mathrm{SU}(2, 1)$ contains a loxodromic element. With this, we now prove our main result: if Γ is an irreducible subgroup of $\mathrm{SU}(2, 1)$, then Γ is conjugate in $\mathrm{SU}(2, 1)$ to a subgroup of $\mathrm{SU}(2, 1, \mathbb{Q}(\Gamma, \lambda))$, where λ is the absolute value of an eigenvalue of any loxodromic element of Γ with $|\lambda| \neq 1$. In particular, this implies that Γ can be defined over the field $\mathbb{Q}(\Gamma, \lambda)$.

Theorem 2.1. *Let Γ be an irreducible subgroup of $\mathrm{SU}(2, 1)$. Let $A \in \Gamma$ be any loxodromic element with eigenvalues $\lambda_1 = \lambda e^{i\varphi}$, $\lambda_2 = e^{-2i\varphi}$, $\lambda_3 = \lambda^{-1} e^{i\varphi}$. Then Γ is conjugate in $\mathrm{SU}(2, 1)$ to a subgroup of $\mathrm{SU}(2, 1, \mathbb{Q}(\Gamma, \lambda))$.*

Proof. First, we show that Γ contains two loxodromic elements A_1 and A_2 having the same eigenvalues as A (in fact, we show that A_1 and A_2 are conjugate to A in Γ) such that the subgroup $\Gamma_0 = \langle A_1, A_2 \rangle$ generated by A_1 and A_2 is irreducible. In what follows, we consider the natural action of Γ on the projective space $\mathbb{P}(V)$. We remark that A has three fixed points p_1, p_2, p_3 for its action on the projective space: p_1, p_2 are isotropic and p_3 is positive, p_3 is the polar point to the complex projective line α spanned by p_1 and p_2 . If all the elements of Γ fix the point p_3 , then Γ is reducible. So, there exists an element B of Γ which does not fix p_3 . Let $C = BAB^{-1}$ and $q_3 = B(p_3)$. Then $C(q_3) = q_3$. Note that C is loxodromic, hence C does not fix p_3 . Let q_1 and q_2 be the isotropic fixed point of C . If the sets $\{p_1, p_2\}$ and $\{q_1, q_2\}$ are disjoint, taking $A_1 = A$ and $A_2 = C$ we are done, since in this case the elements A and C have no common fixed points for their action on $\mathbb{P}(V)$. So, let us consider the case when these sets have non-empty intersection, that is, when the elements A and C have a common isotropic fixed point. If $\{p_1, p_2\} = \{q_1, q_2\}$, then the complex projective line α is invariant under C , and, therefore, $C(p_3) = p_3$, a contradiction. This implies that A and C may have only one common isotropic fixed point. One may assume without loss of generality that $p_1 = q_1$ is a unique common isotropic fixed point of A and C . Then since Γ is irreducible, there exists an element $D \in \Gamma$ which does not fix the point p_1 . Let $E = DAD^{-1}$. We have that E is loxodromic and $E(p_1) \neq p_1$. If the invariant complex geodesic β of E is not equal to α , then taking $A_1 = A$ and $A_2 = E$, we are done. If $\alpha = \beta$, we take $A_1 = C$ and $A_2 = E$.

Now, by applying Proposition 2.1, we get that there exists $f \in \text{SU}(2, 1)$ such that $\Gamma_0^* = f\Gamma_0f^{-1}$ is a subgroup of $\text{SU}(2, 1, \mathbb{Q}(\Gamma_0, \lambda))$. Let $\Gamma^* = f\Gamma f^{-1}$. Then Γ_0^* is a subgroup of Γ^* . In order to show that Γ^* is a subgroup of $\text{SU}(2, 1, \mathbb{Q}(\Gamma, \lambda))$, we apply Burnside’s density theorem, stated as Theorem 16'' in I. Kaplansky’s book [20], and the trick which one can find in the proof of Theorem B in [20]. According to Kaplansky this trick is due to C. Procesi.

Since Γ_0^* is irreducible, by applying Burnside’s density theorem, we get that Γ_0^* contains a basis of $M(3, \mathbb{C})$ over \mathbb{C} . Let $S = \{S_1, S_2, \dots, S_9\}$ be such a basis. Then it follows that for any element $\gamma \in \Gamma^*$ there exist complex numbers c_1, c_2, \dots, c_9 such that

$$\gamma = c_1S_1 + c_2S_2 + \dots + c_9S_9.$$

Let $\text{Tr}(A, B) = (A, B)$ denote the trace form on $M(3, \mathbb{C})$ so that

$$(A, B) = \text{Tr}(A, B) = \text{tr}(AB).$$

It is well known that Tr is a non-degenerate symmetric bilinear form.

It follows from the equality

$$\gamma = c_1S_1 + c_2S_2 + \dots + c_9S_9$$

that

$$(\gamma, S_i) = c_1(S_1, S_i) + c_2(S_2, S_i) + \dots + c_9(S_9, S_i)$$

for all $i = 1, 2, \dots, 9$.

We consider these equalities as a system of linear equations in unknowns c_1, c_2, \dots, c_9 . We have that for all $i, j = 1, 2, \dots, 9$, the product (S_i, S_j) is in $\mathbb{Q}(\Gamma_0, \lambda)$, and the product (γ, S_i) is in $\mathbb{Q}(\Gamma, \lambda)$. Therefore, this implies that every coefficient of the system lies in $\mathbb{Q}(\Gamma, \lambda)$. Since the form Tr is non-degenerate, this system is non-singular. Solving this system by Cramer’s rule, we conclude that every c_i lies in $\mathbb{Q}(\Gamma, \lambda)$. This implies that every $\gamma \in \Gamma^*$ lies in $\text{SU}(2, 1, \mathbb{Q}(\Gamma, \lambda))$. From this, we conclude that Γ^* is a subgroup of $\text{SU}(2, 1, \mathbb{Q}(\Gamma, \lambda))$. \square

As a corollary of this theorem, we get the following.

Theorem 2.2. *Let Γ be an irreducible subgroup of $\text{SU}(2, 1)$ such that $\mathbb{Q}(\Gamma)$ is a subset of \mathbb{R} , then Γ is conjugate in $\text{SU}(2, 1)$ to a subgroup of $\text{SO}(2, 1)$.*

Proof. Since Γ is irreducible, it follows from Proposition 2.3 that Γ contains a loxodromic element A . Let $\lambda_1 = \lambda e^{i\varphi}$, $\lambda_2 = e^{-2i\varphi}$, $\lambda_3 = \lambda^{-1} e^{i\varphi}$ be its eigenvalues. It is easy to see that in this case all the eigenvalues of A are real: A is either hyperbolic or loxodromic whose elliptic part is of order 2. Since the field $\mathbb{Q}(\Gamma)$ is real, the field $\mathbb{Q}(\Gamma, \lambda)$ is also real. So, the result follows from Theorem 2.1. \square

We would like to stress that in Theorem 2.1 and Theorem 2.2 we do not assume that the group Γ is discrete.

Next we show that the conclusions of Theorem 2.1 and Theorem 2.2 are not true if Γ is reducible. Indeed, let us consider the embedding of $\mathrm{SL}(2, \mathbb{R})$ in $\mathrm{SU}(2, 1)$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} a & 0 & -ib \\ 0 & 1 & 0 \\ ic & 0 & d \end{bmatrix}.$$

This defines a faithful representation of $\mathrm{SL}(2, \mathbb{R})$ in $\mathrm{SU}(2, 1)$. Let Γ be the image of $\mathrm{SL}(2, \mathbb{R})$ under this embedding. Then Γ is reducible because the complex line spanned by the vector $(0, 1, 0)^\top$ is invariant under Γ . In fact, Γ is a \mathbb{C} -subgroup of $\mathrm{SU}(2, 1)$. Also, we have that the trace field of Γ is real. It is easy to see that Γ cannot be conjugate in $\mathrm{SU}(2, 1)$ to a subgroup of $\mathrm{SO}(2, 1)$. Moreover, if we consider the image of $\mathrm{SL}(2, \mathbb{Z})$ under this embedding, we get an example of a non-elementary discrete subgroup of $\mathrm{SU}(2, 1)$ whose trace field is \mathbb{Q} but which is not conjugate in $\mathrm{SU}(2, 1)$ to a subgroup of $\mathrm{SO}(2, 1)$.

Now we consider the case of subgroups of $\mathrm{PU}(2, 1)$. We would like to find conditions under which a subgroup G of $\mathrm{PU}(2, 1)$ leaves invariant a totally real geodesic 2-plane in $\mathbf{H}_{\mathbb{C}}^2$. It is natural to ask if it is possible or not to get an answer in terms of the traces of lifts of elements of G to $\mathrm{SU}(2, 1)$.

Let $\pi: \mathrm{SU}(2, 1) \rightarrow \mathrm{PU}(2, 1)$ be a natural projection. Let G be a subgroup of $\mathrm{PU}(2, 1)$. Then the *trace field* of G , denoted $\mathbb{Q}(G)$, is the field $\mathbb{Q}(\Gamma)$, where $\Gamma = \pi^{-1}(G)$. We remark that this field is never real.

The following example is useful to understand the problem. Let Γ be a subgroup of $\mathrm{SO}(2, 1)$ such that the restriction $\pi: \Gamma \rightarrow \mathrm{PU}(2, 1)$ is a monomorphism. Let $G = \pi(\Gamma)$. We have that Γ is a lift of G . One could define an “invariant” trace field of G as the trace field of Γ . And, in this case, this field is real! So, it would be natural to define an “invariant” trace field of a subgroup G of $\mathrm{PU}(2, 1)$ which has a lift to $\mathrm{SU}(2, 1)$ as the trace field of its lift. Unfortunately, the problem when a subgroup G of $\mathrm{PU}(2, 1)$ has a lift to $\mathrm{SU}(2, 1)$ is still open, and it seems very difficult. To our knowledge, the only results in this direction are in [17], in the case when G is isomorphic to the fundamental group of a closed orientable surface. Some examples when G has no lift, the reader can find in [1]. The lifting problem in the case of real hyperbolic geometry of dimension three was completely solved in [5].

In [26], [21], [28] the invariant trace field was defined for subgroups of $\mathrm{PSL}(2, \mathbb{C})$, see also [23], for the case of $\mathrm{SU}(n, 1)$. We follow their ideas to define an invariant trace field for subgroups of $\mathrm{PU}(2, 1)$.

Let G be a subgroup of $\mathrm{PU}(2, 1)$ and $\Gamma = \pi^{-1}(G)$. Then the *invariant trace field* of G , denoted by $k(G)$, is defined to be the field $\mathbb{Q}(\Gamma^3)$, where $\Gamma^3 = \langle \gamma^3 : \gamma \in \Gamma \rangle$. It follows from [23] that the invariant trace field is an invariant of the commensurability class.

If $A \in \text{SU}(2, 1)$, then we have the following trace identity:

$$\text{tr}(A^3) = (\text{tr}(A))^3 - 3 \text{tr}(A) \text{tr}(A^{-1}) + 3,$$

see, for instance, [24]. Since an element of $\text{PU}(2, 1)$ has three lifts to $\text{SU}(2, 1)$ which differ by a cube root of unity, it follows from this formula that if G is an \mathbb{R} -subgroup of $\text{PU}(2, 1)$, then the invariant trace field of G is real.

By applying Theorem 2.2, we get the following characterization of discrete \mathbb{R} -subgroups of $\text{PU}(2, 1)$.

Theorem 2.3. *Let G be an irreducible discrete subgroup of $\text{PU}(2, 1)$. Then G is an \mathbb{R} -subgroup if and only if the invariant trace field $k(G)$ of G is real.*

Proof. Let $\Gamma = \pi^{-1}(G)$. Then it is clear that Γ and Γ^3 are irreducible. Let $H = \pi(\Gamma^3)$. It follows from Theorem 2.2 that Γ^3 is conjugate in $\text{SU}(2, 1)$ to a subgroup of $\text{SO}(2, 1)$. This implies that H leaves invariant a totally real geodesic 2-plane L in $\mathbf{H}_{\mathbb{C}}^2$. Since Γ^3 is a normal subgroup of Γ , it follows that H is a normal subgroup of G . Since the group G is irreducible, it follows from Corollary 2.2 that G is non-elementary. Therefore, the normality implies that the limit set of H is equal to the limit set of G , see, for instance, [4]. We remark that the limit set of H is contained in the boundary of L . Moreover, using the fact that G is non-elementary, we have that the set of fixed points of loxodromic elements of G is dense in its limit set. This implies that L is invariant with respect to G . Therefore, G is an \mathbb{R} -subgroup of $\text{PU}(2, 1)$. \square

As an immediate corollary we get the following theorem.

Theorem 2.4. *Let G be a discrete non-elementary subgroup of $\text{PU}(2, 1)$ such that the invariant trace field $k(G)$ of G is real. Then G is either \mathbb{R} -Fuchsian or \mathbb{C} -Fuchsian.*

Proof. If G is irreducible, then it follows from Theorem 2.3 that G is \mathbb{R} -Fuchsian. Now let us assume that G is reducible. Let p be a common fixed point of all elements of G in their action on the projective space $\mathbb{P}(V)$. Since G is non-elementary, we have that p cannot be in $\mathbf{H}_{\mathbb{C}}^2 \cup \partial\mathbf{H}_{\mathbb{C}}^2$. So, p is positive. This implies that G leaves invariant the complex geodesic whose polar point is p . Therefore, the group G is \mathbb{C} -Fuchsian. \square

We remark that Theorem 2.4 can be considered as a complex hyperbolic analog of a classical result due to B.Maskit, see Theorem G.18 in [22] and Corollary 3.2.5 in [21].

We say that $g \in \text{PU}(2, 1)$ is a screw motion iff any lift of g to $\text{SU}(2, 1)$ has non-real trace. For instance, purely hyperbolic and unipotent parabolic elements are not screw motions. Geometrically, $g \in \text{PU}(2, 1)$ is not screw motion iff g has an invariant totally real geodesic 2-plane.

Corollary 2.3. *Let G be an irreducible discrete subgroup of $\mathrm{PU}(2, 1)$. Then G is an \mathbb{R} -subgroup if and only if G contains no screw motions.*

Corollary 2.4. *Let G be an irreducible discrete subgroup of $\mathrm{PU}(2, 1)$ whose limit set is not contained in an \mathbb{R} -circle. Then its invariant trace field is non-real extension of \mathbb{Q} .*

Corollary 2.5. *Let G be a discrete subgroup of $\mathrm{PU}(2, 1)$ of finite co-volume. Then its invariant trace field is non-real extension of \mathbb{Q} .*

Proof. It is clear that G is irreducible. Suppose that the invariant trace field $k(G)$ of G is real. By Theorem 2.3, G is an \mathbb{R} -subgroup of $\mathrm{PU}(2, 1)$. Therefore, the limit set of G is contained in an \mathbb{R} -circle. This implies that G cannot have finite co-volume. \square

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