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Groups, Geometry, and Dynamics

On the topology of $\mathcal{H}(2)$

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Abstract. The space $\mathcal{H}(2)$ consists of pairs (M,ω), where M is a Riemann surface of genus two, and ω is a holomorphic 1-form which has only one zero of order two. There exists a natural action of $\mathbb{C}^*$ on $\mathcal{H}(2)$ by multiplication to the holomorphic 1-form. In this paper, we single out a proper subgroup Γ of $\mathrm{Sp}(4,\mathbb{Z})$ generated by three elements, and show that the space $\mathcal{H}(2)/\mathbb{C}^*$ can be identified with the quotient $\Gamma\backslash\mathcal{J}_2$, where $\mathcal{J}_2$ is the Jacobian locus in the Siegel upper half space $\mathfrak{S}_2$. A direct consequence of this result is that $[\mathrm{Sp}(4,\mathbb{Z}):\Gamma]=6$. The group Γ can also be interpreted as the image of the fundamental group of $\mathcal{H}(2)/\mathbb{C}^*$ in the symplectic group $\mathrm{Sp}(4,\mathbb{Z})$.

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1. Introduction

In this paper we are concerned with translation surfaces in the stratum $\mathcal{H}(2)$. Each element of $\mathcal{H}(2)$ can be either considered as a translation surface having only one singularity of angle 6π together with a parallel line field, or a pair (M,ω), where M is a Riemann surface of genus two, and ω is a holomorphic 1-form having a single zero of order two on M. Using the latter viewpoint, we see that $\mathbb{C}^*$ acts naturally on $\mathcal{H}(2)$ by multiplication to the holomorphic 1-form. Note that, if ω has only one zero on M, then this zero must be a Weierstrass point of M. Therefore, the quotient $\mathcal{H}(2)/\mathbb{C}^*$ consists of pairs (M,W), where M is a Riemann surface of genus two, and W is a Weierstrass point of M, two pairs (M_1,W_1) and (M_2,W_2) are identified if there exists a conformal homeomorphism $\phi\colon M_1\to M_2$ such that $\phi(W_1)=W_2$.

The space $\mathcal{H}(2)$ is well known to be a complex orbifold of dimension 4. For any pair (M,ω), let $\gamma_1,\dots,\gamma_4$ be a basis of the group $H_1(M,\mathbb{Z})$. Then the *period mapping*

$$(M,\omega)\longmapsto\left(\int_{\gamma_1}\omega,\dots,\int_{\gamma_4}\omega\right)\in\mathbb{C}^4$$

gives a local chart for $\mathcal{H}(2)$ in a neighborhood of (M,ω) (see [2], [7], [14]). Consequently, we see that $\mathcal{H}(2)/\mathbb{C}^*$ can be endowed with a complex projective orbifold structure.

In Section 3, we introduce a construction of translations surface in $\mathcal{H}(2)$ from triples of parallelograms by a unique gluing model. This construction gives rise to the notion of *parallelogram decomposition* of surfaces in $\mathcal{H}(2)$. Actually, given a translation surface (M,ω) in $\mathcal{H}(2)$, there exist infinitely many parallelogram decompositions of (M,ω). From a fixed parallelogram decomposition, one can obtain other ones by applying some elementary moves, which are called T, S and R, these moves are realized by some homeomorphisms of the surface M whose actions on the group $H_1(M,\mathbb{Z})$ (in an appropriate basis) are given by the following matrices respectively:

$$T=\begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad S=\begin{pmatrix}0&1&0&1\\0&0&-1&0\\0&1&0&0\\-1&0&1&0\end{pmatrix},$$

and

$$R=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix}.$$

Let Γ be the group generated by the matrices T, S and R, then Γ is a proper subgroup of $\mathrm{Sp}(4,\mathbb{Z})$. The main result of this paper is the following:

Theorem 1.1. *There exists a homeomorphism from $\mathcal{H}(2)/\mathbb{C}^*$ to the quotient $\Gamma\backslash\mathcal{J}_2$, where $\mathcal{J}_2$ is the Jacobian locus of Riemann surfaces of genus two in the Siegel upper half space $\mathfrak{H}_2$. As a consequence, we have* $[\mathrm{Sp}(4,\mathbb{Z}):\Gamma]=6$.

Since every Riemann surface of genus two is a two-sheeted branched cover of the sphere $\mathbb{CP}^1$, there exists a natural identification between $\mathcal{H}(2)/\mathbb{C}^*$ and the moduli space of pairs $(\lambda_0,\underline{\lambda})$, where $\lambda_0\in\mathbb{CP}^1$, and $\underline{\lambda}$ is a subset of cardinal five of $\mathbb{CP}^1\backslash\{\lambda_0\}$ up to action of $\mathrm{Aut}(\mathbb{CP}^1)$.

Let $\mathrm{Mod}_{0,6}$ denote the mapping class group of the sphere with six punctures, and $\mathrm{Mod}^*_{0,5}$, the subgroup of index 6 in $\mathrm{Mod}_{0,6}$ that fixes one of the punctures. The space $\mathcal{H}(2)/\mathbb{C}^*$ can be identified with the quotient $\mathcal{T}_{0,6}/\mathrm{Mod}^*_{0,5}$, where $\mathcal{T}_{0,6}$ is the Teichmüller space of the sphere with six punctures. The group Γ can be then considered as the image of $\mathrm{Mod}^*_{0,5}$ in $\mathrm{Sp}(4,\mathbb{Z})$. Note that we have an isomorphism between $\mathrm{Mod}^*_{0,5}$ and $B_5/Z(B_5)$, where B_5 is the braid group of the closed disk with five punctures, and $Z(B_5)$ is the center of B_5.

It is well known that the Jacobian locus $\mathcal{J}_2$ is the quotient of $\mathcal{T}_{0,6}$ by the Torelli group $\mathcal{I}_2$. Thus we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{T}_{0,6} & \longrightarrow & \mathcal{J}_2 \cong \mathcal{T}_{0,6}/\mathcal{I}_2 \\ \downarrow & \swarrow & \\ \mathcal{H}(2)/\mathbb{C}^* \cong \mathcal{T}_{0,6}/\mathrm{Mod}^*_{0,5} & & \end{array}$$

Since the action of $-\mathrm{Id}$ on $\mathfrak{H}_2$ is trivial, a direct consequence of Theorem 1.1 is the following

Corollary 1.2. *We have the following exact sequence*

$$1 \to \mathcal{I}_2 \to \mathrm{Mod}^*_{0,5} \cong B_5/Z(B_5) \to \Gamma/\{\pm\mathrm{Id}\} \to 1. \qquad (1)$$

We refer to [11] for a general discussion on symplectic and unitary representations of braid groups.

The paper is organized as follows: in Section 2, we recall the basic properties of the Theta functions and give a brief explanation of how these functions allow us to compute the branched points of hyperelliptic coverings. In the following section, we introduce the notion of *parallelogram decomposition* for surfaces in $\mathcal{H}(2)$ and the three elementary moves on those decompositions. We then define the group Γ as the group generated by the homology action of the elementary moves. Note that there are parallelogram decompositions for which some elementary moves can not be carried out. This means that the set of parallelogram decompositions is not the right place to study the action of the group generated by the elementary moves. To fix this problem, in Section 4, we introduce the notion of *admissible decomposition*, which generalizes the one of parallelogram decomposition. For every admissible decomposition, all the elementary moves can be carried out. The key ingredient of the proof of Theorem 1.1 is the fact that the symplectic homology bases associated to two admissible decompositions are always related by an element of the group Γ. This fact is the content of Theorem 5.1, which is proven in Section 5. The proof Theorem 1.1 is then given in Section 6. In the Appendices, we give the proof of the fact that every surface in $\mathcal{H}(2)$ admits parallelogram decompositions, and we give an explicit family of Γ-right cosets in $\mathrm{Sp}(4,\mathbb{Z})$.

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2. Siegel upper half space and Theta functions

In this section, we recall the definition and some basic properties of the theta functions. Our main references are [13] and [4].

2.1. Siegel upper half space and Jacobian locus. For any integer $g \geqslant 1$, the Siegel upper half space $\mathfrak{H}_g$ is the space of complex symmetric matrices Z in $\mathfrak{M}(g, \mathbb{C})$ such that $\mathrm{Im}(Z)$ is positive definite. This space is the quotient of the real symplectic group $\mathrm{Sp}(2g, \mathbb{R})$ by the compact subgroup $\mathrm{U}(g)$. The integral symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$ is a lattice in $\mathrm{Sp}(2g, \mathbb{Z})$ which acts properly discontinuously on $\mathfrak{H}_g$. The quotient $\mathcal{A}_g = \mathrm{Sp}(2g, \mathbb{Z}) \backslash \mathfrak{H}_g$ is the moduli space of principally polarized Abelian varieties of dimension g. In this paper, by the real symplectic group $\mathrm{Sp}(2g, \mathbb{R})$ we mean the group of real $2g \times 2g$ matrices preserving the symplectic form

$$\begin{pmatrix} J & & 0 \\ & \ddots & \\ 0 & & J \end{pmatrix}, \quad \text{where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let M be a Riemann surface of genus g and let $\{a_1, b_1, \dots, a_g, b_g\}$ be a symplectic basis of $H_1(M, \mathbb{Z})$, that is,

$$\langle a_i, a_j \rangle = \langle b_i, b_j \rangle = 0 \quad \text{and} \quad \langle a_i, b_j \rangle = \delta_{ij},$$

where $\langle ., . \rangle$ is the intersection form of $H_1(M, \mathbb{Z})$. There exist g holomorphic 1-forms $(\phi_1, \dots, \phi_g)$ on M uniquely determined by the following condition:

$$\int_{a_j} \phi_i = \delta_{ij}.$$

The matrix $\Pi = (\pi_{ij})_{i,j=1,\dots,g}$, where $\pi_{ij} = \int_{b_j} \phi_i$, belongs to $\mathfrak{H}_g$, and we then have a mapping from the set of pairs $(M, \{a_1, b_1, \dots, a_g, b_g\})$ into $\mathfrak{H}_g$. The image of this mapping is called the *Jacobian locus* and is denoted by $\mathcal{J}_g$.

For the case $g = 2$, it is well known that the complement of $\mathcal{J}_2$ in $\mathfrak{H}_2$ is a union of countably many copies of $\mathfrak{H}_1 \times \mathfrak{H}_1$, where $\mathfrak{H}_1$ is the upper half plane $\mathfrak{H}_1 = \{z \in \mathbb{C} \ : \ \mathrm{Im} z > 0\}$ (see [6], [12]).

2.2. Theta function. Fix an integer $g \geqslant 1$ and let $\mathfrak{H}_g$ be the Siegel upper half space of genus g. The Riemann's *theta function* is a complex value function defined on $\mathbb{C}^g \times \mathfrak{H}_g$ by the following formula

$$\theta(z, \sigma) = \sum_{N \in \mathbb{Z}^g} \exp\left(2\pi\imath\left(\frac{1}{2}{}^t N\sigma N +{}^t Nz\right)\right).$$

The function θ is holomorphic on $\mathbb{C}^g \times \mathfrak{H}_g$. We also consider functions defined on $\mathbb{C}^g \times \mathfrak{H}_g$ by

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z,\sigma) = \sum_{N\in\mathbb{Z}^g} \exp\left\{2\pi\imath\left[\tfrac{1}{2}{}^t\left(N+\tfrac{\epsilon}{2}\right)\sigma\left(N+\tfrac{\epsilon}{2}\right) + {}^t\left(N+\tfrac{\epsilon}{2}\right)\left(z+\tfrac{\epsilon'}{2}\right)\right]\right\}$$

where ϵ, ϵ' are integer vectors. These functions are called first order theta functions with integer characteristic $\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right]$.

Proposition 2.1. *The first order theta function with integer characteristic* $\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right]$ *has the following properties:*

i) $\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z+e^{(k)},\sigma) = \exp 2\pi\imath\left[\frac{\epsilon_k}{2}\right]\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z,\sigma)$,

ii) $\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z+\sigma^{(k)},\sigma) = \exp 2\pi\imath\left[-z_k - \frac{\sigma_{kk}}{2} - \frac{\epsilon'_k}{2}\right]\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z,\sigma)$,

iii) $\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(-z,\sigma) = \exp 2\pi\imath\left[\frac{{}^t\epsilon\epsilon'}{2}\right]\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z,\sigma)$,

iv) $\theta\begin{bmatrix}\epsilon+2\nu\\ \epsilon'+2\nu'\end{bmatrix}(z,\sigma) = \exp 2\pi\imath\left[\frac{{}^t\epsilon\nu'}{2}\right]\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z,\sigma)$, *with* $\nu,\nu'\in\mathbb{Z}^g$,

where $e^{(k)}$ *and* $\sigma^{(k)}$ *are the* k*-th column of the matrices* Id_g *and* σ *respectively.*

By the Torelli theorem, we know that a closed Riemann surface M is uniquely determined by its Jacobian variety $J(M)$, or equivalently, by the period matrix associated to a symplectic basis of $H_1(M,\mathbb{Z})$. If M is hyperelliptic, then we can get more information from the period matrix by using theta functions. We have (cf. [4], VII.4)

Theorem 2.2. *The branched points of the two sheeted representation of a hyperelliptic Riemann surface are holomorphic functions of the period matrix. Furthermore, the hyperelliptic surface is completely determined by its period matrix.*

To illustrate the ideas of the proof, we will indicate below the method to compute some of the branched points; details of the calculations can be found in [4], VII.1 and VII.4.

Assume that M is the two-sheeted branched covering of $\mathbb{CP}^1$ ramified above $\lambda_1,\dots,\lambda_{2g+2}$. Let $s_1,\dots,s_{g+1}$ be $g+1$ simple arcs in $\mathbb{CP}^1$ such that the endpoints of s_i are λ_{2i-1}, λ_{2i}, for $k=1,\dots,g+1$, and $s_i\cap s_j=\varnothing$, for $i\neq j$. We can consider M as the Riemann surface obtained by gluing two copies of $\mathbb{CP}^1$ slit along $s_1,\dots,s_{g+1}$. Let z be the meromorphic function on M realizing the two-sheeted branched cover from M to $\mathbb{CP}^1$. Let P_i denote $z^{-1}(\lambda_i)$, $i=1,\dots,2g+2$, then $\{P_1,\dots,P_{2g+2}\}$ is the set of Weierstrass points of M.

Using PSL$(2,\mathbb{C})$, we may assume that $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_{2g+2} = \infty$. The surface M is then the curve defined by the equation

$$w^2 = z(z-1)\prod_{i=3}^{2g-1}(z-\lambda_i).$$

The function z is then characterized, up to a non zero multiplicative constant, by the property that it has a double zero at P_1, a double pole at P_{2g+2}, and it is holomorphic and non zero elsewhere. First, we specify a symplectic basis $\{a_1, b_1, \dots, a_g, b_g\}$ of $H_1(M,\mathbb{Z})$ as follows:

- $b_k = z^{-1}(s_k)$, $k = 1, \dots, g+1$. By construction, b_k is a simple closed curve containing P_{2k-1} and P_{2k}, and b_k is preserved by the hyperelliptic involution.
- Let α_k, $k = 1, \dots, g$, be g simple closed curves pairwise disjoint in $\mathbb{CP}^1$ satisfying

  - α_k intersects transversely s_k and s_{g+1},
  - $\mathrm{Card}\{\alpha_k \cap s_k\} = \mathrm{Card}\{\alpha_k \cap s_{g+1}\} = 1$,
  - $\alpha_k \cap s_j = \varnothing$ if $j \notin \{k, g+1\}$,
  - $\alpha_k \cap \{\lambda_1, \dots, \lambda_{2g+2}\} = \varnothing$.

  Let a_k be a connected component of $z^{-1}(\alpha_k)$. Note that a_k and its image under the hyperelliptic involution are disjoint.

It follows from the construction that the family $\{a_1, b_1, \dots, a_g, b_g\}$ is a symplectic basis of $H_1(M,\mathbb{Z})$. Let $\{\zeta_1, \dots, \zeta_g\}$ be the basis of $\mathscr{H}^1(M)$, the space of holomorphic 1-forms on M, dual to $\{a_1, \dots, a_g\}$, that is

$$\int_{a_j} \zeta_i = \delta_{ij}.$$

Let $\Pi = (\pi_{ij})_{i,j=1,\dots,g}$, with $\pi_{ij} = \int_{b_j} \zeta_i$, be the period matrix associated to $\{a_1, b_1, \dots, a_g, b_g\}$. By definition, the Jacobian variety $J(M)$ of M is the quotient $\mathbb{C}^g/\Lambda$, where Λ is the lattice in $\mathbb{C}^g$ which is generated by the column vectors of the $g \times 2g$ matrix (Id_g, Π). We denote by $e^{(k)}$ and $\pi^{(k)}$, $k = 1, \dots, g$, the k-th columns of Id_g and Π respectively. Let $\varphi\colon M \to J(M)$ be the map defined by

$$\varphi(P) = \left(\int_{P_1}^{P} \zeta_1, \dots, \int_{P_1}^{P} \zeta_g\right) \in \mathbb{C}/\Lambda,$$

where the integrals are taken along any path joining P_1 to P. From the construction of the basis $\{a_1, b_1, \dots, a_g, b_g\}$, one can explicitly compute $\varphi(P_i)$, $i = 1, \dots, 2g+2$, as functions of $(e^{(1)}, \dots, e^{(g}, \pi^{(1)}, \dots, \pi^{(g)})$. Namely, we have (see Figure 1)

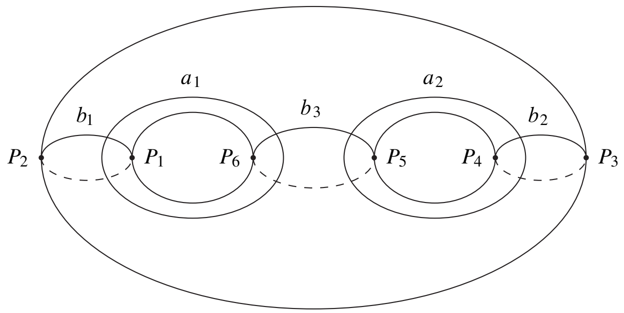

Figure 1. Symplectic basis on a hyperelliptic Riemann surface.

- $\varphi(P_1) = 0$,
- $\varphi(P_2) = \frac{1}{2}\pi^{(1)}$,
- $\varphi(P_3) = \frac{1}{2}(\pi^{(1)} + e^{(1)} + e^{(2)})$,
- $\cdots$
- $\varphi(P_{2k+1}) = \frac{1}{2}(\pi^{(1)} + \cdots + \pi^{(k)} + e^{(1)} + \cdots + e^{(k+1)})$,
- $\varphi(P_{2k+2}) = \frac{1}{2}(\pi^{(1)} + \cdots + \pi^{(k+1)} + e^{(1)} + \cdots + e^{(k+1)})$,
- $\cdots$
- $\varphi(P_{2g+1}) = \frac{1}{2}(\pi^{(1)} + \cdots + \pi^{(g)} + e^{(1)})$,
- $\varphi(P_{2g+2}) = \frac{1}{2}e^{(1)}$.

Since $\Pi \in \mathfrak{H}_g$, we can now consider the first order theta functions with integer characteristic $\theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](z, \Pi)$. From Proposition 2.1, we see that for any $\nu \in \Lambda$, $\theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](z + \nu, \Pi)$ differs from $\theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](z, \Pi)$ by a multiplicative factor. It turns out that the multiplicative behavior of the theta functions is such that

$$f(P) := \frac{\theta^2\left[\begin{smallmatrix}1&0&0&\cdots&0\\ 1&1&0&\cdots&0\end{smallmatrix}\right](\varphi(P), \Pi)}{\theta^2\left[\begin{smallmatrix}1&0&0&\cdots&0\\ 0&1&0&\cdots&0\end{smallmatrix}\right](\varphi(P), \Pi)}$$

is a meromorphic function on M with divisor $P_1^2 P_{2g+2}^{-2}$. Hence

$$f = cz \quad \text{where } c \in \mathbb{C}^*.$$

The constant c can be evaluated at P_2 since we have $f(P_2) = cz(P_2) = c$. Using the fact that $\varphi(P_2) = \frac{1}{2}\pi^{(1)}$, we have

$$c = f(P_2) = \frac{\theta^2\left[\begin{smallmatrix}1&0&0&\cdots&0\\ 1&1&0&\cdots&0\end{smallmatrix}\right](\frac{1}{2}\pi^{(1)}, \Pi)}{\theta^2\left[\begin{smallmatrix}1&0&0&\cdots&0\\ 0&1&0&\cdots&0\end{smallmatrix}\right](\frac{1}{2}\pi^{(1)}, \Pi)}.$$

It follows that

$$z(P) = \frac{\theta^2 \left[\begin{smallmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{smallmatrix}\right] (\frac{1}{2}\pi^{(1)}, \Pi) \theta^2 \left[\begin{smallmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \end{smallmatrix}\right] (\varphi(P), \Pi)}{\theta^2 \left[\begin{smallmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \end{smallmatrix}\right] (\frac{1}{2}\pi^{(1)}, \Pi) \theta^2 \left[\begin{smallmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{smallmatrix}\right] (\varphi(P), \Pi)}. \tag{2}$$

By setting $P = P_j$, for $j = 3, \dots, 2g+1$, we get a formula for λ_j. Note that this formula is only useful for $j = 3, 4, 6, .., 2g$, for other values of j, both the numerator and denominator vanish. By replacing P_1 by another Weierstrass point in the definition of φ, we can get similar formulae for the λ_j which can not be computed directly from (2).

3. The group Γ

In this section, we define the group Γ and prove some of its properties.

3.1. Construction of surfaces in $\mathcal{H}(2)$ by gluing parallelograms. Any parallelogram in $\mathbb{R}^2$ is determined by a pair of vectors in $\mathbb{R}^2$. In this section, we will represent any parallelogram by a pair of complex numbers (z_1, z_2) such that $\mathrm{Im}(\bar{z}_1 z_2) > 0$, $\mathrm{Im}(\bar{z}_1 z_2)$ is actually the area of the parallelogram. Note that by this convention, the pairs $(z_1, z_2), (z_2, -z_1), (-z_1, -z_2), (-z_2, z_1)$ represent the same parallelogram.

Let $\mathscr{P}^+$ denote the set $\{(z_1, z_2) \in \mathbb{C}^2 \ : \mathrm{Im} > 0(\bar{z}_1 z_2) > 0\}$. Given 4 complex numbers $(z_1, \dots, z_4)$ such that $(z_1, z_2), (z_2, z_3), (z_3, z_4)$ all belong to $\mathscr{P}^+$, let A, B, C denote the parallelograms determined by the pairs $(z_1, z_2), (z_2, z_3), (z_3, z_4)$ respectively. We can construct a translation surface in $\mathcal{H}(2)$ from A, B, C as follows:

- Glue two sides of A corresponding to z_1 together.
- Glue two sides of A corresponding to z_2 to two sides of B also corresponding to z_2.
- Glue two sides of B corresponding to z_3 to two sides of C also corresponding to z_3.
- Glue two sides of C corresponding to z_4 together.

It is easy to check that the surface M obtained from this construction is of genus 2, equipped with a flat metric structure with a single cone singularity, which arises from the identification of all the vertices of A, B, C, we denote this point by W. Since all the gluings are realized by translations, M is a translation surface. We also get naturally a holomorphic 1-form on M, considered as a Riemann surface, defined as follows: since translations of $\mathbb{R}^2$ preserve the holomorphic 1-form dz, the restrictions of dz into the parallelograms A, B, C are compatible with the gluings and give rise to a holomorphic 1-form ω on M with only one zero at W, which is necessarily of order two. Clearly, $(M, \omega) \in \mathcal{H}(2)$.

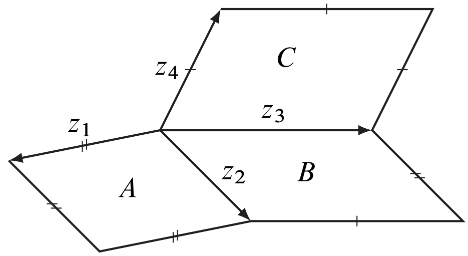

Figure 2. Construction of surfaces in $\mathcal{H}(2)$ from three parallelograms.

We know that M is a hyperelliptic surface, it is easy to visualize the hyperelliptic involution of M from its construction by gluing A, B, C. For each of the parallelograms A, B, C, consider the reflection through its center, one can easily check that these reflections agree with the gluing on the boundary of A, B, C. Thus, we have a conformal automorphism τ of M. One can check that $\tau^2 = \mathrm{Id}$, and the action of τ on $H_1(M, \mathbb{Z})$ is given by $-\mathrm{Id}$, therefore τ must be the hyperelliptic involution of M. We can also determine without difficulty the 6 fixed points of τ, which are the Weierstrass points of M: two of which are contained in A, two in C, one in the interior of B, and the last one is W.

3.2. Parallelogram decompositions and the group Γ. Recall that, on translation surfaces (M, ω), a *saddle connection* is a geodesic segment (with respect to the flat metric) which joins singularity to singularity, the endpoints of a saddle connection may coincide. A *cylinder* C is a subset of M which is isometric to $\mathbb{R}\times]0, h[/\mathbb{Z}$, where the action of $\mathbb{Z}$ is generated by $(x, y) \mapsto (x + \ell, y)$ with $\ell > 0$ and maximal with respect to this property (that is, C can not be embedded into a larger subset C' isometric to $\mathbb{R}\times]0, h'[/\mathbb{Z}$, with $h' > h$). In other words, C is the union of all simple closed geodesics in the same free homotopy class. The construction of translation surfaces in $\mathcal{H}(2)$ by gluing parallelograms with the model presented above suggests the following

Definition 3.1. Let (M, ω) be a pair in $\mathcal{H}(2)$. A *parallelogram decomposition* of (M, ω) is a family of six oriented saddle connections $\{a, b_1, b_2, c_1, c_2, d\}$ verifying the following conditions:

- The intersection of any pair of saddle connections in this family is the unique zero of ω.
- $b_1 \cup b_2$ (resp. $c_1 \cup c_2$) is the boundary of a cylinder which contains a (resp. d).
- The complement of $a \cup b_1 \cup b_2 \cup c_1 \cup c_2 \cup d$ has three components, each of which is isometric to an open parallelograms in $\mathbb{R}^2$.

- The orientations of the saddle connections in this family are chosen such that $\langle a, b_1\rangle = \langle b_1, c_1\rangle = \langle c_1, d\rangle = 1$, a goes from b_1 to b_2 and d goes from c_2 to c_1 (see Figure 3).

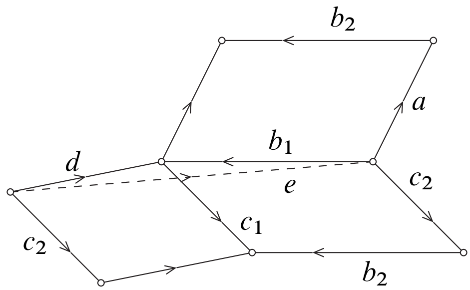

Figure 3. Parallelogram decomposition of surfaces in $\mathcal{H}(2)$.

If $\mathcal{D} = \{a, b_1, b_2, c_1, c_2, d\}$ is a parallelogram decomposition of (M, ω) then (a, b_1, c_1, d) is not a symplectic basis of $H_1(M, \mathbb{Z})$. Let b (resp. c) be a simple closed curve in the cylinder bounded by b_1 and b_2 (resp. by c_1 and c_2). Let e be a simple closed curved in the free homotopy class of the closed curve $d * (-b_1)$, that is $e = d - b$ in $H_1(M, \mathbb{Z})$, then (a, b, c, e) is a symplectic basis of $H_1(M, \mathbb{Z})$, we will call it the *symplectic basis associated to* $\mathcal{D}$.

Given a surface (M, ω) in $\mathcal{H}(2)$ which is obtained from three parallelograms $A = (z_1, z_2), B = (z_2, z_3), C = (z_3, z_4)$ as in the previous section, let $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ be the subsets of M which correspond to A, B, C respectively. By construction, $\widetilde{A}$ and $\widetilde{C}$ are two cylinders, while $\widetilde{B}$ is an embedded parallelogram. Let a (resp. d) denote the saddle connection in $\widetilde{A}$ (resp. $\widetilde{C}$) which corresponds to the z_1 sides of A (resp. z_4 sides of C). Let b_1, b_2 (resp. c_1, c_2) denote the boundary components of $\widetilde{A}$ (resp. $\widetilde{C}$). We choose the orientations of a, b_1, b_2, c_1, c_2, d in such a way that

$$\int_a \omega = z_1, \quad \int_{b_1} \omega = \int_{b_2} \omega = z_2, \quad \int_{c_1} \omega = \int_{c_2} \omega = z_3, \quad \int_d \omega = z_4.$$

We also choose the numbering of (b_1, b_2) (resp. (c_1, c_2)) such that the orientation of a goes from b_1 to b_2 and the orientation of d goes from c_2 to c_1. By definition $\{a, b_1, b_2, c_1, c_2, d\}$ is a parallelogram decomposition of (M, ω).

A surface (M, ω) in $\mathcal{H}(2)$ always admits a parallelogram decompositions (see Appendix A). The following operations allow us to get other decompositions from a particular one $\mathcal{D} = \{a, b_1, b_2, c_1, c_2, d\}$.

1. *The T move.* let a' be the saddle connection which is obtained from a by a Dehn twist in $\widetilde{A}$, then $\mathcal{D}' = \{a', b_1, b_2, c_1, c_2, d\}$ is another parallelogram decomposition of (M, ω) (see Figure 4). Here, we consider both "left" and "right" Dehn twists.

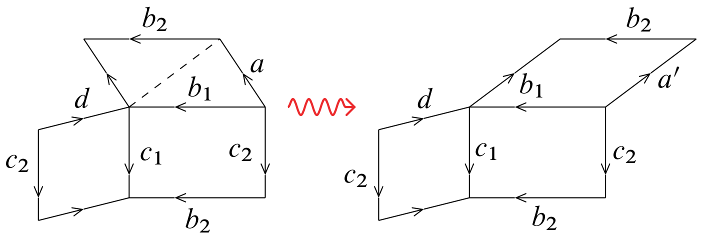

Figure 4. The T move.

2. *The S move.* If $\{a, b_1, b_2, c_1, c_2, d\}$ is a parallelogram decomposition of (M, ω) then $\mathcal{D}' = \{d, -c_2, -c_1, b_1, b_2, -a\}$ is also a parallelogram decomposition of (M, ω), the minus sign designates the same saddle connection with the inverse orientation (see Figure 5).

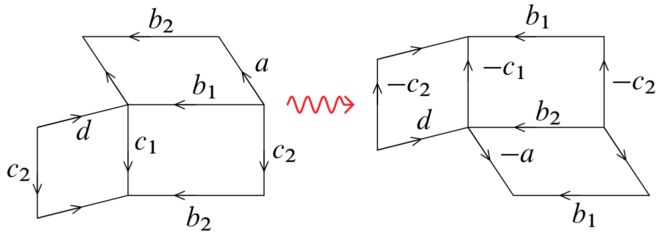

Figure 5. The S move.

3. *The R move.* Cut M along the saddle connections b_1, b_2 and d, we then obtain two annuli, one of which is $\widetilde{A}$, we denote the other by $\widetilde{A}'$. Let c'_1, c'_2 denote the images of c_1, c_2 respectively under a Dehn twist in $\widetilde{A}'$. Assume that c'_1, c'_2 can be made into saddle connections in $\widetilde{A}'$, then $\{a, b_1, b_2, c'_1, c'_2, d\}$ is a parallelogram decomposition of (M, ω) (see Figure 6).

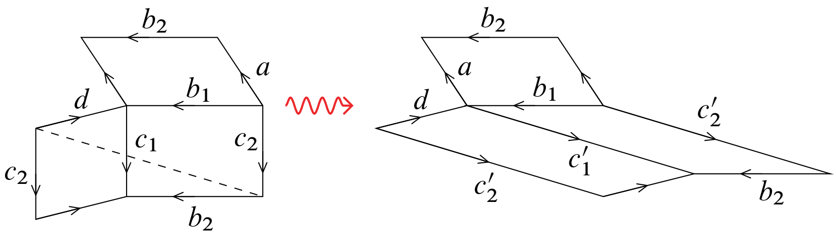

Figure 6. The R move.

Let $\mathcal{D}'$ be another parallelogram decomposition of (M,ω) which is obtained from $\mathcal{D}$ by one the moves presented above. Let (a,b,c,e) (resp. (a',b',c',e')) be the symplectic basis of $H_1(M,\mathbb{Z})$ associated to $\mathcal{D}$ (resp. $\mathcal{D}'$). The following lemma follows directly from the definitions.

Lemma 3.2. a) *If $\mathcal{D}'$ is obtained from $\mathcal{D}$ by a T move then*

$$\begin{pmatrix} a' \\ b' \\ c' \\ e' \end{pmatrix} = \begin{pmatrix} 1 & \pm 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \\ e \end{pmatrix}.$$

b) *If $\mathcal{D}'$ is obtained from $\mathcal{D}$ by an S move then*

$$\begin{pmatrix} a' \\ b' \\ c' \\ e' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \\ e \end{pmatrix}.$$

c) *If $\mathcal{D}'$ is obtained from $\mathcal{D}$ by an R move then*

$$\begin{pmatrix} a' \\ b' \\ c' \\ e' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \pm 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \\ e \end{pmatrix}.$$

We denote by T, S, R the matrices of basis change corresponding to the moves T, S, R respectively, to fix ideas, we can take

$$T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

and

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We denote by Γ the subgroup of $\mathrm{Sp}(4,\mathbb{Z})$ generated by those matrices.

3.3. Properties of Γ

Lemma 3.3. *We have*

(i) $S^2 = -\mathrm{Id}_4$.

(ii) $\begin{pmatrix} \mathrm{Id}_2 & 0 \\ 0 & \mathrm{SL}(2,\mathbb{Z}) \end{pmatrix} \subset \Gamma$.

(iii) $\Gamma \subsetneq \mathrm{Sp}(4,\mathbb{Z})$.

(iv) Γ *is not a normal subgroup of* $\mathrm{Sp}(4,\mathbb{Z})$.

Proof. (i) follows from direct calculation.

(ii) We have

$$S^{-1}TS = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

Since $\mathrm{SL}(2,\mathbb{Z})$ is generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

we deduce that $\begin{pmatrix} \mathrm{Id}_2 & 0 \\ 0 & \mathrm{SL}(2,\mathbb{Z}) \end{pmatrix}$ is contained in Γ.

(iii) The group $\mathrm{Sp}(4,\mathbb{Z})$ acts transitively on $(\mathbb{Z}/2\mathbb{Z})^4 \setminus \{0\}$, but Γ has two orbits: $\mathcal{O}_1$ containing $\mathbf{e}_1 = (1,0,0,0)$ and $\mathcal{O}_2$ containing $\mathbf{e}_2 = (0,1,0,0)$. As a matter of fact, we have

$$\mathcal{O}_1 = \left\{ \begin{matrix} (1,0,0,0) & (1,1,0,0) & \\ (0,1,1,0) & (0,1,0,1) & (0,1,1,1) \end{matrix} \right\}$$

and

$$\mathcal{O}_2 = \left\{ \begin{matrix} (0,1,0,0) & & \\ (1,0,1,0) & (1,0,0,1) & (1,0,1,1) \\ (1,1,1,0) & (1,1,0,1) & (1,1,1,1) \\ (0,0,1,0) & (0,0,0,1) & (0,0,1,1) \end{matrix} \right\}.$$

Here, we consider the action of $\mathrm{Sp}(4,\mathbb{Z})$ and Γ on $(\mathbb{Z}/2\mathbb{Z})^4$ by right multiplication.

(iv) Let

$$T' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remark that T' does not belong to Γ since it sends $\mathbf{e}_2$ to an element in $\mathcal{O}_1$. We have

$$T'^{-1}TT' = \begin{pmatrix} 2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Observe that $T'^{-1}TT'$ sends $\mathbf{e}_1$ to $\mathbf{e}_2$, thus it does not belong to Γ. We can then conclude that Γ is not a normal subgroup of $\mathrm{Sp}(4,\mathbb{Z})$. □

Lemma 3.4. *Set*

$$U = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then the integral symplectic group $\mathrm{Sp}(4,\mathbb{Z})$ *is generated by* T, S, R *and* U.

Proof. See Appendix B. □

4. Admissible decomposition

4.1. Definition. We have seen that for any surface (M,ω) in $\mathcal{H}(2)$, given a parallelogram decomposition, other decompositions of (M,ω) can be obtained by the elementary moves T, S and R. However, while T and S are always realizable, the R move is not, it is only realizable when $d * (-b_1) * c_2$ is homotopic to a saddle connection. In this section, we enlarge the set of decompositions so that the three elementary moves are always realizable. This leads us to the notion of *admissible decomposition* with respect to the pair (M, W). The definition of admissible decomposition is inspired from parallelogram decomposition and relies on the action of the hyperelliptic involution of M.

Throughout this section, M is a fixed closed Riemann surface of genus two, W is a Weierstrass point and τ is the hyperelliptic involution of M. For any closed curve γ with basepoint W, we denote by $[\gamma]$ the homotopy class of γ in $\pi_1(M, W)$.

Definition 4.1 (Admissible decomposition). Let $\{a, b_1, b_2, c_1, c_2, d\}$ be six oriented simple closed curves containing W. We say that $\{a, b_1, b_2, c_1, c_2, d\}$ is an *admissible decomposition* for the pair (M, W) if

- the intersection of any pair of curves in this family is $\{W\}$,
- $\tau(a) = -a, \tau(d) = -d$,
- $\tau(b_1) = -b_2$, $\tau(c_1) = -c_2$,
- $a \setminus \{W\}$ is contained in an open annulus A_b bounded by b_1 and b_2,
- $d \setminus \{W\}$ is contained in an open annulus A_c bounded by c_1 and c_2,
- $M \setminus (\overline{A}_b \cup \overline{A}_c)$ is homeomorphic to an open disk.

The orientations of $\{a, b_1, b_2, c_1, c_2, d\}$ are chosen such that

- $\langle a, b_1\rangle = \langle a, b_2\rangle = \langle c_1, d\rangle = \langle c_2, d\rangle = 1$, where $\langle ., .\rangle$ is the intersection form of $H_1(M,\mathbb{Z})$,
- in the annulus A_b, the orientation of a goes from b_1 to b_2,
- in the annulus A_c, the orientation of d goes from c_2 to c_1,

- the boundary of the disk $M \setminus (\overline{A}_b \cup \overline{A}_c)$ with the induced orientation is the concatenation $b_1 * c_1 * (-b_2) * (-c_2)$, which implies in particular that $[b_1][c_1][b_2]^{-1}[c_2]^{-1} = 1$ in $\pi_1(M, W)$.

Example. If $\mathcal{D} = \{a, b_1, b_2, c_1, c_2, d\}$ is a parallelogram decomposition for a pair (M, ω), where ω is a holomorphic 1-form with double zero at W, then $\mathcal{D}$ is an admissible decomposition for the pair (M, W). Note that on a fixed translation surface in $\mathcal{H}(2)$, there are admissible decompositions which can not be realized as parallelogram decompositions.

4.2. Projection to the sphere and associated homology basis. Let $\rho\colon M \to \mathbb{CP}^1$ be the two-sheeted branched covering from M onto $\mathbb{CP}^1$ ramified at the Weierstrass points of M. By definition, we have $\rho(P) = \rho(P')$ if and only if $P' \in \{P, \tau(P)\}$. Let $P_0, \dots, P_5$ denote the images of the Weierstrass points of M by ρ, with $P_0 = \rho(W)$. Let $\mathcal{D} = \{a, b_1, b_2, c_1, c_2, d\}$ be an admissible decomposition for the pair (M, W). The projections of the curves in $\mathcal{D}$ satisfy (see Figure 7).

- $\rho(a) = \bar{a}$ is a simple arc joining P_0 to another point in $\{P_1, \dots, P_5\}$. Without loss of generality, we can assume that the endpoints of $\bar{a}$ are P_0 and P_1.
- $\rho(b_1) = \rho(b_2) = \bar{b}$ is a simple closed curve with basepoint P_0, $\bar{b}$ is the boundary of an open disc D_1 which contains $\bar{a} \setminus \{P_0\}$ and two points of the set $\{P_1, \dots, P_5\}$. We can assume that $\mathrm{D}_1 \cap \{P_1, \dots, P_5\} = \{P_1, P_2\}$.
- $\rho(d) = \bar{d}$ is a simple arc disjoint from D_1, joining P_0 to a point in $\{P_3, \dots, P_5\}$. We can assume that the endpoints of $\bar{d}$ are P_0 and P_3.
- $\rho(c_1) = \rho(c_2) = \bar{c}$ is a simple closed curve with basepoint P_0 disjoint from D_1, $\bar{c}$ is the boundary of an open disc D_2 which contains $\bar{d} \setminus \{P_0\}$ and two points in the set $\{P_3, \dots, P_5\}$. We can assume that $\mathrm{D}_2 \cap \{P_1, \dots, P_5\} = \{P_3, P_4\}$.

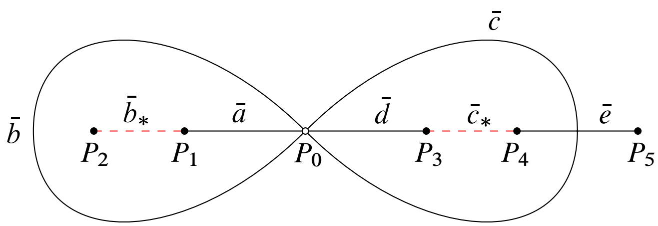

Figure 7. Projection of an admissible decomposition to $\mathbb{CP}^1$.

Remark. The orientations of the curves in $\mathcal{D}$ do not determine the orientations of the curves $\{\bar{a}, \bar{b}, \bar{c}, \bar{d}\}$.

Let $\bar{b}_*$ be a simple arc contained in D_1 which joins P_1 to P_2 such that $\bar{b}_* \cap \bar{a} = \{P_1\}$. Observe that $b = \rho^{-1}(\bar{b}_*)$ is a simple closed curve in M, freely homotopic to b_1 and b_2, and we have $\tau(b) = -b$. Similarly, let $\bar{c}_*$ be a simple arc in D_2 joining P_3 to P_4 such that $\bar{c}_* \cap \bar{d} = \{P_3\}$, then $c = \rho^{-1}(\bar{c}_*)$ is a simple closed curve freely homotopic to c_1, c_2 such that $\tau(c) = -c$.

Now, let $\bar{e}$ be a simple arc in $\mathbb{CP}^1$ which joins P_5 to P_4 such that $\bar{e} \cap (\bar{b} \cup \bar{d}) = \varnothing$ (see Figure 7), then $e = \rho^{-1}(\bar{e})$ is a simple closed curve such that $\tau(e) = -e$. Note that $\bar{e}$ is unique up to homotopy with fixed endpoints in $\mathbb{CP}^1 \setminus (\bar{b} \cup \bar{d})$.

Recall that a, d are already oriented, and we have an orientation for b which is induced by the orientation of b_1 and b_2. Choose the orientation of e such that $\langle c, e \rangle = 1$, then (a, b, d, e) is a symplectic basis of $H_1(M, \mathbb{Z})$. We will call (a, b, c, e) the basis associated to the decomposition $\mathcal{D}$. It is easy to see that this basis is also the one described in Section 2. Note also that if $\mathcal{D}$ is a parallelogram decomposition, then the two definitions of associated symplectic basis of $H_1(M, \mathbb{Z})$ agree (see Figure 8).

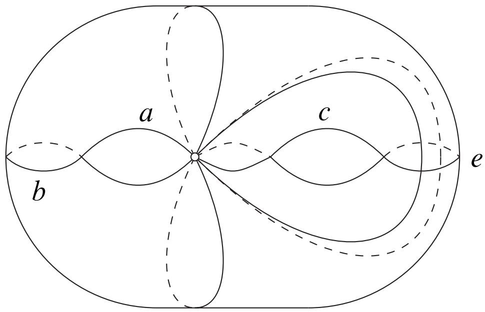

Figure 8. Symplectic homology basis associated to an admissible decomposition.

4.3. Elementary moves. Let $\mathcal{D} = \{a, b_1, b_2, c_1, c_2, d\}$ be an admissible decomposition for the pair (M, W). Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, D_1, D_2$ be as above.

- *The T move*. Let $\bar{a}'$ be a simple arc joining P_0 to P_2 such that $(\bar{a}' \setminus \{P_0\}) \subset D_1$ and $\bar{a}' \cap \bar{a} = \{P_0\}$. The preimage a' of $\bar{a}'$ in M is a simple closed curves contained in the annulus bounded by b_1, b_2 which satisfies $\tau(a') = -a'$. By choosing an appropriate orientation for a', we see that the family $\mathcal{D}' = \{a', b_1, b_2, c_1, c_2, d\}$ is an admissible decomposition for (M, W). The symplectic homology basis associated to $\mathcal{D}'$ is (a', b, c, e) and we have

$$\begin{pmatrix} a' \\ b \\ c \\ e \end{pmatrix} = \begin{pmatrix} 1 & \pm 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

We call this transformation the T move (see Figure 9).

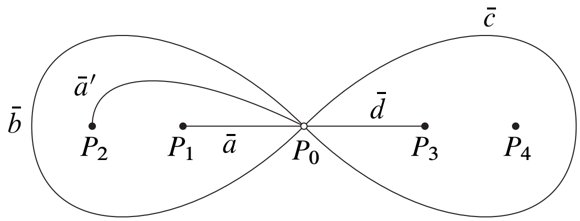

Figure 9. T move for an admissible decomposition.

- *The S move*. We define the S move for admissible decompositions in the same way as for parallelogram decompositions, that is, an S move transforms $\mathcal{D}$ into the family $\mathcal{D}' = \{d, -c_2, -c_1, b_1, b_2, -a\}$. Too see what happens to the curves in $\mathbb{CP}^1$, let $\bar{e}'$ be a simple arc in $\mathbb{CP}_1$ joining P_5 to P_2 such that $\bar{e}' \cap (\bar{a} \cup \bar{c}) = \varnothing$, then the symplectic homology basis associated to $\mathcal{D}'$ is $(d, -c, b, e')$ where $e' = \rho^{-1}(\bar{e}')$. It is easy to check that the symplectic homology bases associated to $\mathcal{D}$ and $\mathcal{D}'$ are related by the matrix S.

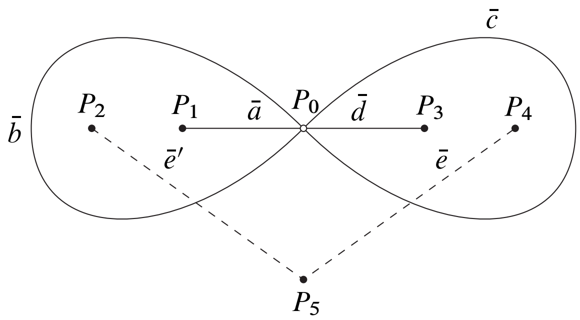

Figure 10. S move for an admissible decomposition.

- *The R move*. Let $\bar{c}'$ be a simple closed curve in $\mathbb{CP}^1$ with basepoint P_0 which is disjoint from D_1 and bounds an open disc D_2' such that (see Figure 11)

  - $\mathrm{D}_2' \cap \{P_1, \dots, P_5\} = \{P_3, P_5\}$,
  - $(\bar{d} \setminus \{P_0\}) \subset \mathrm{D}_2'$.

  The preimage of $\bar{c}'$ in M is the union of two simple closed curves c_1', c_2' with basepoint W. By choosing an appropriate orientation or c_1', c_2', we see that the family $\mathcal{D}' = \{a, b_1, b_2, c_1', c_2', d\}$ is an admissible decomposition. Let $\bar{c}_*'$ be a simple arc in D_2' joining P_3 to P_5 and let c' be the preimage of $\bar{c}_*'$ in M. Then (a, b, c', e) is the symplectic homology basis associated to $\mathcal{D}'$. One can easily check that

$$\begin{pmatrix} a \\ b \\ c' \\ e \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \pm 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \\ e \end{pmatrix}.$$

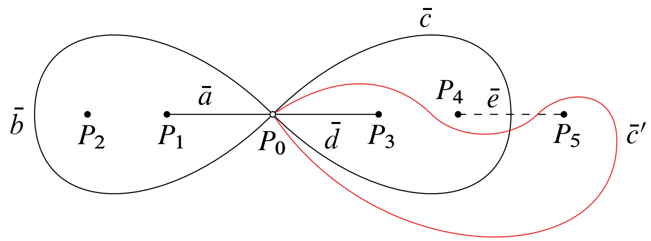

Figure 11. R move for an admissible decomposition.

The elementary moves T, S, R defined above transform an admissible decomposition for the pair (M, W) into another admissible decomposition for the same pair. Let us now introduce an elementary move which transforms an admissible decomposition for the pair (M, W) into an admissible decomposition for another pair (M, W').

Let $\bar{a}'$ be a simple arc in $\mathbb{CP}^1$ joining P_5 to P_4 such that $\bar{a}' \cap (\bar{b} \cup \bar{d}) = \varnothing$. Let $\bar{d}'$ be a simple arc joining P_5 to P_2 such that $\bar{d}' \cap (\bar{a} \cup \bar{c}) = \varnothing$. Let $\bar{b}'$ and $\bar{c}'$ be two simple closed curves with basepoint P_5 which are respectively the boundary of the open disks D'_1 and D'_2 satisfying (see Figure 12)

- $\mathrm{D}'_1 \cap \mathrm{D}'_2 = \varnothing$, $\mathrm{D'}_i \cap \mathrm{D}_i = \varnothing$, $i = 1, 2$,
- $\mathrm{D}'_1 \cap \{P_0, \dots, P_4\} = \{P_3, P_4\}$, $\mathrm{D}'_2 \cap \{P_0, \dots, P_4\} = \{P_1, P_2\}$,
- $\mathrm{int}(\bar{a}') \subset \mathrm{D}'_1$, $\mathrm{int}(\bar{d}') \subset \mathrm{D}'_2$.

Then the preimages of $\bar{a}', \bar{b}', \bar{c}', \bar{d}'$ is an admissible decomposition $\mathcal{D}'$ for the pair $(M, \rho^{-1}(P_5))$. Actually, the family $(\bar{a}', \bar{b}', \bar{c}, \bar{d}')$ gives rise to two admissible decompositions, each of which is determined by the orientation of $a' = \rho^{-1}(\bar{a}')$. The symplectic homology bases associated those decompositions are related by $-\mathrm{Id}$. Let (a', b', c', e') be the symplectic homology basis associated to $\mathcal{D}'$, we have

$$\begin{pmatrix} a' \\ b' \\ c' \\ e' \end{pmatrix} = \pm \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \\ e \end{pmatrix} = U^{\pm 1} \cdot \begin{pmatrix} a \\ b \\ c \\ e \end{pmatrix}.$$

We will call the transformation from $\mathcal{D}$ into $\mathcal{D}'$ the U move.

From the definitions, the following lemma is clear.

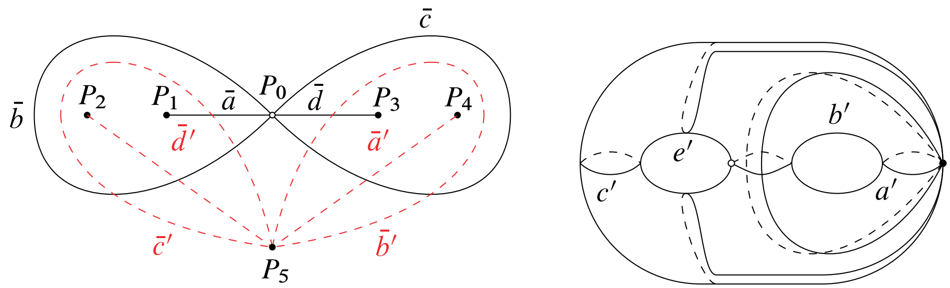

Figure 12. U move for admissible decompositions.

Lemma 4.2. *Let $\mathfrak{D}$ be an admissible decomposition for the pair (M, W) and γ an element of the group Γ. Then there exists an admissible decomposition $\mathfrak{D}'$ for the same pair such that the symplectic homology bases associated to $\mathfrak{D}$ and $\mathfrak{D}'$ are related by γ.*

From Lemma 3.4, we know that the family $\{T, S, R, U\}$ generates the group $\mathrm{Sp}(4, \mathbb{Z})$, therefore we have

Lemma 4.3. *Let $\mathfrak{D}$ be an admissible decomposition for the pair (M, W) and A an element of* $\mathrm{Sp}(4, \mathbb{Z})$. *Then there exist a Weierstrass point W' of M and an admissible decomposition $\mathfrak{D}'$ for the pair (M, W') such that the symplectic homology bases associated to $\mathfrak{D}$ and $\mathfrak{D}'$ are related by A.*

5. Symplectic homology bases associated to admissible decompositions and the group Γ

Let (M, W) be an element of the space $\mathcal{H}(2)/\mathbb{C}^*$. Our aim in this section is to prove the following theorem, which is the key ingredient of the proof of Theorem 1.1.

Theorem 5.1. *Let $\mathfrak{D}$ and $\mathfrak{D}'$ be two admissible decompositions for the pair (M, W) with the associated symplectic homology bases (a, b, c, e) and (a', b', c', e') respectively. Then there exists an element γ in Γ such that*

$$\begin{pmatrix} a' \\ b' \\ c' \\ e' \end{pmatrix} = \gamma \cdot \begin{pmatrix} a \\ b \\ c \\ e \end{pmatrix}.$$

In what follows, we denote by τ the hyperelliptic involution of M, by ρ the two-sheeted branched covering from M onto $\mathbb{CP}^1$, and by $P_0, \dots, P_5$ the images of the Weierstrass points of M by ρ, where $P_0 = \rho(W)$. We also equip M with the hyperbolic metric in the conformal class of the Riemann surface structure. Note that τ is now an isometry of M. Recall that, for any closed curve α in M which contains W, we denote by $[\alpha]$ the homotopy class of α in $\pi_1(M, W)$.

5.1. Admissible decompositions with common subfamily. Let us first prove the following

Proposition 5.2. *Let $\mathcal{D} = \{a, b_1, b_2, c_1, c_2, d\}$ and $\mathcal{D}' = \{a', b_1', b_2', c_1', c_2', d'\}$ be two admissible decompositions for the pair (M, W). Assume that $\bar{b} = \rho(b_1)$ and $\bar{b}' = \rho(b_1')$ are homotopic in $\pi_1(\mathbb{CP}^1 \setminus \{P_1, \dots, P_5\}, P_0)$. Then there exists an element $\gamma \in \Gamma$ such that the symplectic bases of $H_1(M, \mathbb{Z})$ associated to $\mathcal{D}$ and $\mathcal{D}'$ are related by γ.*

Proof. Since $\bar{b}$ and $\bar{b}'$ are homotopic in $\pi_1(\mathbb{CP}^1 \setminus \{P_1, \dots, P_5\}, P_0)$, there exists a homeomorphism Φ of $\mathbb{CP}^1$ isotopic to the identity relative to $\{P_0, \dots, P_5\}$ such that $\Phi(\bar{b}) = \bar{b}'$ (see [1], Theorem A.4, or [3], Lemma 2.9). The homotopy from the identity of $\mathbb{CP}^1$ to Φ can be lifted to a homotopy of M which is identity on the set of Weierstrass points of M. Therefore, we can assume that $b_1 \cup b_1' = b_1' \cup b_2'$ as subsets of M. *A priori*, the orientations of (b_1, b_2) and (b_1', b_2') may not be the same, but since $-\mathrm{Id} \in \Gamma$, we can assume that they have the same orientation which means that $b_1 = b_1'$ in $H_1(M, \mathbb{Z})$. Note that the orientation of (b_1, b_2) determines the orientation of a by the condition $\langle a, b_1 \rangle = 1$, and consequently, we get a unique numbering of the pair (b_1, b_2) by the condition that a goes from b_1 to b_2.

By definition, $b_1 \cup b_2$ is the boundary of an open annulus A_b which contains both $a \setminus \{W\}$ and $a' \setminus \{W\}$, therefore there exists an integer n such that a' is homotopic to the image of a by n Dehn twists in A_b. Thus, by applying the T move n times, we can assume that $a' = a$ as subsets of M. Let (a, b, c, e) and (a', b', c', e') be the symplectic bases of $H_1(M, \mathbb{Z})$ associated to $\mathcal{D}$ and $\mathcal{D}'$ respectively. It follows that we have the following equality in $H_1(M, \mathbb{Z})$

$$\begin{pmatrix} a' \\ b' \\ c' \\ e' \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_2 & 0 \\ X & Y \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \\ e \end{pmatrix},$$

with $X, Y \in \mathbf{M}_{2\times 2}(\mathbb{Z})$. Since $\left(\begin{smallmatrix} \mathrm{Id}_2 & 0 \\ X & Y \end{smallmatrix}\right)$ belongs to $\mathrm{Sp}(4, \mathbb{Z})$, simple computations show that we must have $X = 0$ and $Y \in \mathrm{SL}(2, \mathbb{Z})$. Since the group $\left(\begin{smallmatrix} \mathrm{Id}_2 & 0 \\ 0 & \mathrm{SL}(2,\mathbb{Z}) \end{smallmatrix}\right)$ is contained in Γ (cf. Lemma 3.3), the proposition follows. □

5.2. Standard decomposition. Let us now prove the following

Lemma 5.3. *Let $\mathcal{D} = \{a, b_1, b_2, c_1, c_2, d\}$ be an admissible decomposition of the pair (M, W). Let b_0 and c_0 be the simple closed geodesic in the free homotopy class of b_1 and c_1 respectively, then we have*

a) $b_0 \cap c_0 = \varnothing$,

b) $\tau(b_0) = -b_0, \tau(c_0) = -c_0$,

c) $W \notin b_0 \cup c_0$.

Proof. a) Let b (resp. c) be a simple closed curve freely homotopic to b_1 (resp. to c_1) which is contained in the annulus bounded by b_1 and b_2 (resp. c_1 and c_2). By construction b and c are freely homotopic to b_0 and c_0, respectively, and $b \cap c = \varnothing$. It is well known that $\mathrm{Card}\{b_0 \cap c_0\}$ is the intersection number $\iota(b, c)$ of the free homotopy classes of b and c, thus we have that $\mathrm{Card}\{b_0 \cap c_0\} = 0$.

b) By definition, we see that b_1 is freely homotopic to b_2 and $\tau(b_1) = -b_2$, therefore $\tau(b_1)$ is freely homotopic to $-b_1$. Now, since b_0 is the unique simple closed geodesic in the free homotopy class of b_1, $-b_0$ is then the unique simple closed geodesic in the free homotopy class of $-b_1$. Since τ is an isometry, $\tau(b_0)$ must be simple closed geodesic in the free homotopy class of $\tau(b_1)$, hence $\tau(b_0)$ is freely homotopic to $-b_1$, therefore we must have $\tau(b_0) = -b_0$.

c) Suppose that $W \in b_0$. Since b_0 is freely homotopic to b_1, there exists $[h] \in \pi_1(M, W)$ such that $[b_0] = [h][b_1][h]^{-1}$. Note that we have $[\tau(b_0)] = [b_0]^{-1}$, therefore

$$[h][b_1]^{-1}[h]^{-1} = [\tau(h)][\tau(b_1)][\tau(h)]^{-1} = [\tau(h)][a]^{-1}[b_1]^{-1}[a][\tau(h)]^{-1}.$$

It follows that

$$[a][\tau(h)]^{-1}[h][b_1]^{-1} = [b_1]^{-1}[a][\tau(h)]^{-1}[h]. \tag{3}$$

We deduce that $[b_1]^{-1}$ and $[a][\tau(h)]^{-1}[h]$ commute. But $[b_1]$ is a simple closed non-separating curve, therefore we have

$$[a][\tau(h)]^{-1}[h] = [b_1]^n \quad \text{with } n \in \mathbb{Z}. \tag{4}$$

Recall that τ acts by $-\mathrm{Id}$ on $H_1(M, \mathbb{Z})$, thus (4) implies the following equality in $H_1(M, \mathbb{Z})$

$$nb_1 - a = 2h. \tag{5}$$

It follows that $\langle a, b_1 \rangle = 0 \mod 2$, but by construction we know that $\langle a, b_1 \rangle = 1$ and we get a contradiction. We can then conclude that b_0 does not contain W. The same arguments apply to c_0 and the lemma follows. □

Remark. If a simple closed curve g satisfies $\tau(g) = -g$, then g contains exactly two fixed points of τ, which are Weierstrass points of M.

Let g be a simple closed geodesic in M. We will say that g has Property $(\mathscr{P})$, if it satisfies

$$\tau(g) = -g \quad \text{for } W \notin g.$$

By Lemma 5.3, we know that each admissible decomposition provides us with a pair of disjoint simple closed geodesics in M satisfying Property $(\mathscr{P})$. Conversely, let (g_1, g_2) be such a pair of simple closed geodesics, we will construct an admissible decomposition of (M, W) associated to (g_1, g_2) as follows:

- Cut open M along g_1 and g_2, we obtain a 4-holed sphere N which is equipped with a hyperbolic metric with geodesic boundary. Let g_i^+, g_i^- denote the boundary components of N corresponding to g_i, $i = 1, 2$. The curves $g_i^\pm$ inherit the orientation of g_i. We choose the notations so that the orientation of g_i^+ agrees with the one induced by the orientation of N. Note that the hyperelliptic involution τ of M induces an isometric involution of N which interchanges g_i^+ and g_i^-. By a slight abuse of notation, we will also denote by τ the involution of N.

  Let s_i^+ be a path of minimal length in N from W to g_i^+, $i = 1, 2$. Note that by definition, we have $s_1^+ \cap s_2^+ = \{W\}$. Let s_i^- denote the image of s_i^+ under τ. Since τ is an isometry, s_i^- is a path realizing the distance from W to g_i^- in N. In particular, it follows that $s_i^+ \cap s_j^- = \{W\}$. Note also that, since the action τ on the tangent space at W is $-\mathrm{Id}$, the union s_i of s_i^+ and s_i^- is a simple geodesic segment.

  Let us denote by $\tilde{s}_i^+, \tilde{s}_i^-, \tilde{s}_i$ the geodesic segments in M corresponding to s_i^+, s_i^- and s_i, respectively. Let P_i^+, P_i^- denote the endpoints of $\tilde{s}_i^+$ and $\tilde{s}_i^-$ in g_i, respectively (it may happen that $P_i^+ = P_i^-$), and let r_i be a simple arc in g_i which joins P_i^+ to P_i^-. It follows that $h_i = r_i \cup \tilde{s}_i$ is a simple closed curve in M containing W.

- We choose the orientations of h_1 and h_2 such that $\langle h_1, g_1\rangle = 1$ and $\langle g_2, h_2\rangle = 1$. Observe that the induced orientation of $\tilde{s}_1$ is from P_1^- to P_1^+, and the orientation of $\tilde{s}_2$ is from P_2^+ to P_2^-. There exist simple closed curves $\tilde{b}$ and $\tilde{c}$ containing W which satisfy the following:

  - $\tilde{b}$ and $\tilde{c}$ are disjoint from $g_1 \sqcup g_2$,
  - $\tilde{b}$ and g_1 bound an annulus C_1^+ which contains $\tilde{s}_1^+$,
  - $\tilde{c}$ and g_2 bound an annulus C_2^+ which contains $\tilde{s}_2^+$,
  - $\tilde{b} \cap \tilde{c} = \{W\}$,
  - $\tilde{b} \cap \tau(\tilde{b}) = \tilde{c} \cap \tau(\tilde{c}) = \{W\}$,
  - $\tilde{b} \cap \tau(\tilde{c}) = \tilde{c} \cap \tau(\tilde{b}) = \{W\}$.

  For $\tilde{b}$ we can take a curve in a neighborhood of $s_1^+ \cup g_1^+$ in N, and for $\tilde{c}$ we can take a curve in the neighborhood of $s_2^+ \cup g_2^+$ in N. By definition, we have

$$[\tilde{b}] = [\tilde{s}_1^+ * g_1 * (-\tilde{s}_1^+)] \quad \text{and} \quad [\tilde{c}] = [(-\tilde{s}_2^+) * g_2 * \tilde{s}_2^+] \quad \text{in } \pi_1(M, W).$$

  Since $\tau(g_i) = -g_i$, the curves $\tau(\tilde{b})$ and g_1 (resp. $\tau(\tilde{c})$ and g_2) also bound an annulus denoted by C_1^- (resp. C_2^-) which contains $\tilde{s}_1^-$ (resp. $\tilde{s}_2^-$). The union

C_i of C_i^+ and C_i^- is an annulus which contains g_i as a core curve. Following the orientation of g_2, we have two cases:

1. $[\tilde{b}][\tilde{c}][\tau(\tilde{b})][\tau(\tilde{c})] = 1$ in $\pi_1(M, W)$ (see Figure 13). In this case, we take $a = h_1, b_1 = \tilde{b}, b_2 = -\tau(\tilde{b}), c_1 = \tilde{c}, c_2 = -\tau(\tilde{c}), d = h_2$, then $\mathcal{D} = \{a, b_1, b_2, c_1, c_2, d\}$ is an admissible decomposition for (M, W).

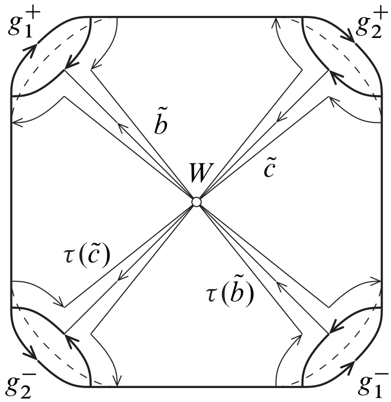

Figure 13. Admissible decomposition associated to a pair of geodesics satisfying $(\mathscr{P})$: Case 1.

2. $[\tilde{b}][\tau(\tilde{c})][\tau(\tilde{b})][c] = 1$ in $\pi_1(M, W)$. Consider the two-sheeted ramified covering $\rho\colon M \to \mathbb{CP}^1$. The images of g_1 and g_2 are two simple arcs in $\mathbb{CP}^1$ whose endpoints are images of the branched points of ρ. Recall that ρ has six branched points, one of which is W, four are contained in g_1 and g_2. We can assume that the last one is mapped to ∞. Let $\bar{u}_2$ be a simple arc in $\mathbb{CP}^1$ as shown on the right of Figure 14. Then $\tilde{u}_2 = \rho^{-1}(\bar{u}_2)$ is a simple arc which joins P_2^+ to P_2^- and passes through W. We denote by $\tilde{u}_2^+$ the subarc of $\tilde{u}_2$ from P_2^+ to W. Observe $\tilde{u}_2^- = \tau(\tilde{u}_2^+)$ is the subarc from P_2^- to W. One can easily check that

- $(-\tilde{s}_2^+) * \tilde{u}_2^+$ is homotopic to $\tilde{s}_1^+ * \tilde{g}_1^+ * (-\tilde{s}_1^+)$ in $\pi_1(M, W)$,
- $\tilde{u}_2^+ \cap \tau(\tilde{u}_2^-) = \{W\}$,
- $\tilde{u}_2 \cap C_1 = \{W\}$.

Consequently, there exists a simple closed curve $\tilde{c}'$ in M containing W such that

- $\tilde{c}'$ and g_2 bound an annulus C'^+_2 which contains $\tilde{u}_2^+$,
- $\tau(\tilde{c}')$ and g_2 bound an annulus C'^-_2 which contains $\tilde{u}_2^-$,
- $\tilde{c}' \cap \tilde{b} = \tilde{c}' \cap \tau(\tilde{b}) = \{W\}$,
- $\tilde{c}' \cap \tau(\tilde{c}') = \{W\}$.

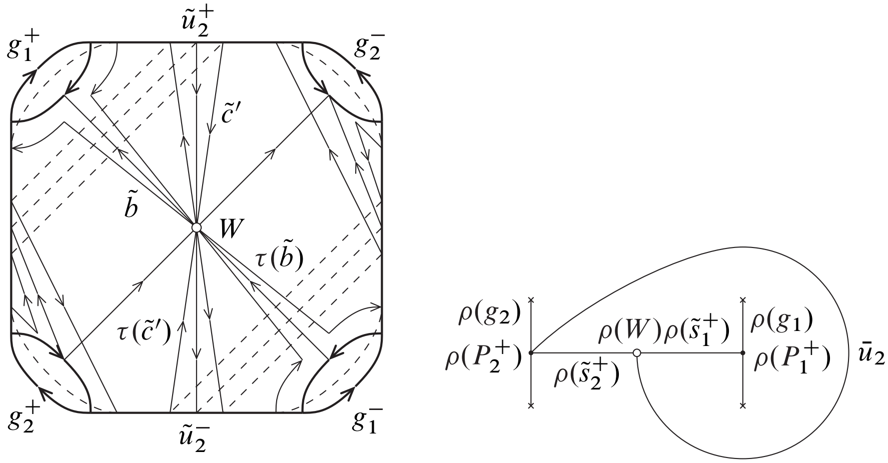

Figure 14. Admissible decomposition associated to a pair of geodesics satisfying ($\mathscr{P}$): Case 2.

We have

$$\begin{aligned}[\tilde{c}'] &= [(-\tilde{u}_2^+) * g_2 * \tilde{u}_2^+],\\ &= [(-\tilde{u}_2^+) * \tilde{s}_2^+][(-\tilde{s}_2^+) * g_2 * \tilde{s}_2^+][(-\tilde{s}_2^+) * u_2^+],\\ &= [\tilde{b}]^{-1}[\tilde{c}][\tilde{b}].\end{aligned}$$

Therefore, $[\tilde{b}][\tilde{c}'][\tau(\tilde{b})][\tau(\tilde{c}')] = 1$ in $\pi_1(M, W)$. Let h'_2 denote the simple closed curve $r_2 \cup \tilde{u}_2^+ \cup \tilde{u}_2^-$, we choose the orientation of h'_2 in such a way that $\langle g_2, h'_2 \rangle = 1$. It follows that, if we take $a = h_1$, $b_1 = \tilde{b}$, $b_2 = -\tau(\tilde{b})$, $c_1 = \tilde{c}'$, $c_2 = -\tau(\tilde{c}')$ and $d = h'_2$, then $\mathscr{D} = \{a, b_1, b_2, c_1, c_2, d\}$ is an admissible decomposition for (M, W) (see Figure 14).

In both cases, we will call $\mathscr{D}$ a standard decomposition associated to the pair of oriented geodesics (g_1, g_2).

From Lemma 5.3, we know that if $\mathscr{D} = \{a, b_1, b_2, c_1, c_2, d\}$ is an admissible decomposition for the pair (M, W), and b_0 (resp. c_0) is the simple closed geodesic in the free homotopy class of b_1 (resp. c_1), then b_0 and c_0 satisfy Property ($\mathscr{P}$). Hence, we can consider the standard decompositions associated to (b_0, c_0). The following proposition tells us that the symplectic bases of $H_1(M, \mathbb{Z})$ associated to the two decompositions are related by an element of the group Γ.

Proposition 5.4. *Let $\mathscr{D} = \{a, b_1, b_2, c_1, c_2, d\}$ be an admissible decomposition for the pair (M, W). Let b_0, c_0 be the simple closed geodesics in the free homotopy classes of b_1 and c_1 respectively. Let $\hat{\mathscr{D}} = \{\hat{a}, \hat{b}_1, \hat{b}_2, \hat{c}_1, \hat{c}_2, \hat{d}\}$ be the standard* *decomposition associated to the pair (b_0, c_0), then the symplectic bases of $H_1(M,\mathbb{Z})$ associated to $\mathcal{D}$ and $\hat{\mathcal{D}}$ are related by an element in Γ.*

Proof. Since b_1 and $\hat{b}_1$ are freely homotopic, there exists a closed curve h with basepoint W such that $[\hat{b}_1] = [h]^{-1}[b_1][h]$ in $\pi_1(M,W)$. We will show that $h \in \mathbb{Z}a \oplus \mathbb{Z}b_1 \oplus \mathbb{Z}c_1$ in $H_1(M,\mathbb{Z})$. By definition, we have

$$\begin{aligned}
&[\tau(\hat{b}_1)] = [\hat{a}]^{-1}[\hat{b}_1]^{-1}[\hat{a}]\\
&\implies [\tau(h)]^{-1}[\tau(b_1)][\tau(h)] = [\hat{a}]^{-1}[h]^{-1}[b_1]^{-1}[h][\hat{a}]\\
&\implies [\tau(h)]^{-1}[a]^{-1}[b_1]^{-1}[a][\tau(h)] = [\hat{a}]^{-1}[h]^{-1}[b_1]^{-1}[h][\hat{a}]\\
&\implies [b_1]^{-1}([a][\tau(h)][\hat{a}]^{-1}[h]^{-1}) = ([a][\tau(h)][\hat{a}]^{-1}[h]^{-1})[b_1]^{-1}.
\end{aligned}$$

It follows that $[b_1]$ and $[a][\tau(h)][\hat{a}]^{-1}[h]^{-1}$ commute. Since b_1 is a simple closed curve, there exists $k \in \mathbb{Z}$ such that

$$[a][\tau(h)][\hat{a}]^{-1}[h]^{-1} = [b_1]^k.$$

Therefore, in $H_1(M,\mathbb{Z})$, we have $\hat{a} = a - kb_1 - 2h$. We know that $\langle \hat{a}, \hat{c}_1\rangle = \langle \hat{a}, c_1\rangle = 0$ and $\langle a, c_1\rangle = \langle b_1, c_1\rangle = 0$, hence, $\langle h, c_1\rangle = 0$, which implies

$$h \in c_1^{\perp} = \mathbb{Z}a \oplus \mathbb{Z}b_1 \oplus \mathbb{Z}c_1.$$

Let (a,b,c,e) and $(\hat{a},\hat{b},\hat{c},\hat{e})$ be the symplectic bases of $H_1(M,\mathbb{Z})$ associated to $\mathcal{D}$ and $\hat{\mathcal{D}}$ respectively. We know that there exists $\gamma \in \mathrm{Sp}(4,\mathbb{Z})$ such that

$$\begin{pmatrix} \hat{a}\\ \hat{b}\\ \hat{c}\\ \hat{e}\end{pmatrix} = \gamma \cdot \begin{pmatrix} a\\ b\\ c\\ e\end{pmatrix}.$$

Since in $H_1(M,\mathbb{Z})$ we have $\hat{b} = b$, $\hat{c} = c$ and $\hat{a} = a + kb + 2h \in \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c$, it follows that γ is of the form

$$\gamma = \begin{pmatrix} x & y & z & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ x' & y' & z' & t'\end{pmatrix}.$$

Now, $\langle \hat{a}, \hat{b}\rangle = 1$ implies $x = 1$, and since z is the c-coordinate of $2h$ in $\mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c$, we have

$$\hat{a} = a + mb + 2\ell c \quad \text{with } m, \ell \in \mathbb{Z}.$$

It follows that

$$\begin{aligned}
\langle \hat{b}, \hat{e}\rangle = 0 &\implies x' = 0,\\
\langle \hat{c}, \hat{e}\rangle = 1 &\implies t' = 1,\\
\langle \hat{a}, \hat{e}\rangle = 0 &\implies y' = -z = -2\ell.
\end{aligned}$$

We deduce that

$$\gamma = \begin{pmatrix} 1 & m & 2\ell & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2\ell & n & 1 \end{pmatrix},$$

with ℓ, m, n in $\mathbb{Z}$. The proposition follows from Lemma 5.5 here below. □

Lemma 5.5. *For any integers ℓ, m, n, the matrix*

$$\gamma = \begin{pmatrix} 1 & m & 2\ell & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2\ell & n & 1 \end{pmatrix}$$

belongs to the group Γ.

Proof. We have shown that the group $\begin{pmatrix} \mathrm{Id}_2 & 0 \\ 0 & \mathrm{SL}(2,\mathbb{Z}) \end{pmatrix}$ is included in Γ. Thus

$$X = S \cdot \begin{pmatrix} \mathrm{Id}_2 & 0 \\ 0 & -\mathrm{Id}_2 \end{pmatrix} \cdot S \cdot \begin{pmatrix} \mathrm{Id}_2 & 0 \\ 0 & -\mathrm{Id}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix} \in \Gamma.$$

Set

$$Y = S^{-1}T^{-1}S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Now, straight computations show that T, X and Y commute, and $\gamma = T^m X^\ell Y^n$. □

Corollary 5.6. *Let $\mathfrak{D} = \{a, b_1, b_2, c_1, c_2, d\}$ and $\mathfrak{D}' = \{a', b'_1, b'_2, c'_1, c'_2, d'\}$ be two admissible decompositions of the pair (M, W) such that b_1 is freely homotopic to b'_1 and c_1 is freely homotopic to $\pm c'_1$. Then the symplectic bases of $H_1(M, \mathbb{Z})$ associated to $\mathfrak{D}$ and $\mathfrak{D}'$ are related by an element in Γ.*

Proof. Let b_0, c_0, c'_0 be the simple closed geodesics in the free homotopy classes of b_1, c_1 and c'_1 respectively. Let

$$\hat{\mathfrak{D}} = \{\hat{a}, \hat{b}_1, \hat{b}_2, \hat{c}_1, \hat{c}_2, \hat{d}\}$$

and

$$\hat{\mathfrak{D}}' = \{\hat{a}', \hat{b}'_1, \hat{b}'_2, \hat{c}'_1, \hat{c}'_2, \hat{d}'\}$$

be the standard decompositions associated to the pairs (b_0, c_0) and (b_0, c'_0) respectively. From Proposition 5.4, the symplectic bases of $H_1(M, \mathbb{Z})$ associated to $\mathcal{D}$ and $\hat{\mathcal{D}}$ (resp. $\mathcal{D}'$ and $\hat{\mathcal{D}}'$) are related by an element of Γ.

By assumption, we have $c_0 = \pm c'_0$. From the construction of standard decompositions, we can assume that $\hat{b}_1 = \hat{b}'_1$, $\hat{b}_2 = \hat{b}'_2$, and the corollary follows from Proposition 5.2. □

5.3. Reducing the number of intersections. By Proposition 5.4, we can now restrict ourselves into the case of standard decompositions. Our strategy for the proof of Theorem 5.1 is to find a sequence $\{(g_1^j, g_2^j)\}_{0 \leqslant j \leqslant k}$ of pairs of disjoint simple closed geodesics in M satisfying $(\mathscr{P})$ such that (g_1^0, g_2^0) is the pair associated to $\mathcal{D}$, (g_1^k, g_2^k) is the pair associated to $\mathcal{D}'$, and for any $0 < j \leqslant k$, up to a renumbering, we have $g_1^{j-1} = g_1^j$ and $\mathrm{Card}\{g_2^{j-1} \cap g_2^j\} = 1$. We start by proving the following key lemma

Lemma 5.7. *Let g, g_1, g_2 be three simple closed geodesics of M verifying Property $(\mathscr{P})$. Assume that*

$$g_1 \cap g_2 = \varnothing \quad \textit{and} \quad \mathrm{Card}\{g \cap (g_1 \cup g_2)\} = n > 1.$$

Then there exists a simple closed geodesic g_3 verifying Property $(\mathscr{P})$ such that $\mathrm{Card}\{g_3 \cap (g_1 \cup g_2)\} = 1$. *Also, if $g_3 \cap g_1 = \varnothing$, then* $\mathrm{Card}\{g \cap (g_1 \cup g_3)\} < \mathrm{Card}\{g \cap (g_1 \cup g_2)\}$. *Moreover, for $i = 1, 2$, if $g \cap g_i = \varnothing$, then we can find g_3 such that $g_3 \cap g_i = \varnothing$.*

Proof. We know that each of the curves g_1, g_2 contains two Weierstrass points. Let $W' \neq W$ be the other Weierstrass point of M which is not contained in $g_1 \cup g_2$. We have two possibilities:

Case 1: $W' \in g$. Let s be the segment of g which contains W' with endpoints in $g \cap (g_1 \cup g_2)$. We denote by Q_1, Q_2 the two endpoints of s and choose the orientation of s to be from Q_1 to Q_2. Since $\tau(g) = -g$ and $\tau(W') = W'$, we deduce that $\tau(s) = -s$ and Q_1, Q_2 are interchanged by τ. It follows that Q_1 and Q_2 are both contained in either g_1 or g_2. Without loss of generality, we can assume that Q_1, Q_2 are contained in g_2.

We know that τ preserves the orientation of M, since τ reverses the orientation of s and g_2, we deduce that s meets both sides of g_2. Let r be one of the two subsegments of g_2 with endpoints Q_1, Q_2, and let W'' be the Weierstrass point which is contained in r. Note that Q_1 and Q_2 must be distinct, otherwise $s = g$ and $\mathrm{Card}\{g \cap (g_1 \cup g_2)\} = 1$, which is discarded by the hypothesis.

Consider the simple closed curve g'_3 which is composed by s and r. Note that we have $\langle g'_3, g_2 \rangle = \pm 1$, therefore $g'_3 \neq 0$ in $H_1(M, \mathbb{Z})$ and, in particular, g'_3 is an essential, non-separating curve. By construction, we see that $\tau(g'_3) = -g'_3$. We can move g_3' slightly in its free homotopy class so that the following conditions are satisfied:

- $g_3' \cap s = \{W'\}$,
- $g_3' \cap g_2 = \{W''\}$,
- $\tau(g_3') = -g_3'$.

By construction, we have

- $g_3' \cap g_1 = \varnothing$.
- $\mathrm{Card}\{g_3' \cap g\} = \mathrm{Card}\{\mathrm{int}(r) \cap g\} + 1 \leqslant \mathrm{Card}\{g_2 \cap g\} - 2 + 1 = \mathrm{Card}\{g_2 \cap g\} - 1$.

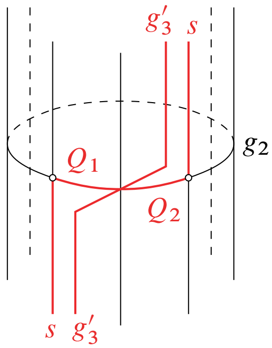

Figure 15. Case $W' \in g_3$.

Let g_3 be the simple closed geodesic in the free homotopy class of g_3'. Since $\tau(g_3') = -g_3'$, we have $\tau(g_3) = -g_3$, as $\tau(g_3)$ is the simple closed geodesic in the free homotopy class of $\tau(g_3')$. It follows from Lemma 5.8 below that g_3 does not contain W, therefore we can conclude that g_3 verifies Property $(\mathscr{P})$.

Let us now show that g_3 satisfies the conditions in the conclusion of the lemma. Let ι denote the geometric intersection number between free homotopy classes of simple closed curves in M. Recall that $\iota(\alpha, \beta) = \mathrm{Card}\{\alpha_0, \beta_0\}$, where α_0 and β_0 are the simple closed geodesics in the free homotopy classes of α and β respectively (see [3], [5]). We have

- $\mathrm{Card}\{g_3 \cap g_1\} = \iota(g_3', g_1) = 0$,
- $\mathrm{Card}\{g_3 \cap g_2\} = \iota(g_3', g_2) \leqslant \mathrm{Card}\{g_3' \cap g_2\} = 1$,
- $\mathrm{Card}\{g_3 \cap g\} = \iota(g_3', g) \leqslant \mathrm{Card}\{g_3' \cap g\} < \mathrm{Card}\{g_2 \cap g\}$.

By construction, we have $\langle g_3, g_2 \rangle = \langle g_3', g_2 \rangle = \pm 1$, therefore $g_3 \cap g_2 \neq \varnothing$. We deduce that $\mathrm{Card}\{g_3 \cap g_2\} = 1$, and the lemma is proven for this case.

Case 2: $W' \notin g$. Cutting M along the curves g_1, g_2, we then get a 4-holed sphere N which is equipped with a hyperbolic metric with geodesic boundary. Let $\hat{g}$ denote the union of geodesic arcs with endpoints in ∂N corresponding to sub-segments of g with endpoints in $g_1 \cup g_2$. Let $\hat{s}_1$ be a geodesic arc realizing the distance $\mathbf{d}_N(W', \hat{g})$. Note that $\hat{s}_1$ does not meet ∂N. The involution τ of M induces an involution on N, which will be denoted by τ_N. Let $\hat{s}_2$ denote the image of $\hat{s}_1$ by τ_N. Note that $\hat{s}_2$ is also a geodesic arc realizing the distance $\mathbf{d}_N(W', \hat{g})$.

From the fact that both $\hat{s}_1, \hat{s}_2$ realize the distance in N from W' to $\hat{g}$, we derive that $\hat{s}_1 \cap \hat{s}_2 = \{W'\}$, since if any other common point exists, it must be the other common endpoint in $\hat{g}$ of both segments, hence it is fixed by τ_N. But τ_N has only two fixed points in N corresponding to W and W', and by assumption $W, W' \notin \hat{g}$, therefore we get a contradiction. Since τ acts like $-\mathrm{Id}$ on the tangent plane at W', $\hat{s} = \hat{s}_1 \cup \hat{s}_2$ is in fact a geodesic arc.

Let s be the geodesic arcs in M corresponding to $\hat{s}$, and let Q_i, $i = 1, 2$, denote the endpoint of s. By construction, we have $\mathrm{int}(s) \cap (g_1 \cup g_2) = \varnothing$, $W \notin s$ and $\tau(s) = -s$. Let R_1 be an endpoint of the subarc of g (with endpoints in $g_1 \cap g_2$) that contains Q_1, and let r_1 denote the oriented subarc from Q_1 to R_1. We denote by R_2 and r_2 the images of R_1 and r_1 by τ respectively. The curve $c = (-r_1) * (-s_1) * s_2 * r_2$ is then a simple arc joining R_1 to R_2 verifying $\tau(c) = -c$.

Since $\tau(R_1) = R_2$, it follows that either $\{R_1, R_2\} \subset g_1$, or $\{R_1, R_2\} \subset g_2$. Without loss of generality, we can assume that R_1, R_2 are contained in g_2, then the same argument as in Case 1 shows that c meets both sides of g_2. Here we have two issues

- $R_1 = R_2$: in this case c is actually a simple closed curve which satisfies

  (a) $\tau(c) = -c$,

  (b) $c \cap g_1 = \varnothing$,

  (c) $\mathrm{Card}\{c \cap g_2\} = 1$.

  We can then find in the neighborhood of c a simple closed curve c', freely homotopic to c, satisfying (a), (b), (c) such that $c' \cap g = \{R_1\}$. Let g'_3 be the image of c' by the Dehn twist about g_2, observe that g'_3 satisfies (a), (b), (c), and $\mathrm{Card}\{g \cap g'_3\} = \mathrm{Card}\{g \cap g_2\} - 1$ (see Figure 16). We denote by g_3 the simple closed geodesic in the free homotopy class of g'_3.

- $R_1 \neq R_2$: let d be one of the two simples arcs in g_2 with endpoints R_1, R_2, then $c \cup d$ is a simple closed curve invariant under τ. We can choose d in such a way that there exits in the neighborhood of $c \cup d$ a simple closed curve g'_3 which satisfies (a), (b), (c), and $\mathrm{Card}\{g \cap g'_3\} \leqslant \mathrm{Card}\{g \cap g_2\} - 2$. We then take g_3 to be the closed geodesic in the free homotopy class of g'_3.

In both cases, using the same arguments as in Case 1, we see that g_3 verifies the required properties. It is also clear from the arguments above that, if in addition, we have $g \cap g_1 = \varnothing$, then $g_3 \cap g_1 = \varnothing$. The proof of the lemma is now complete. □

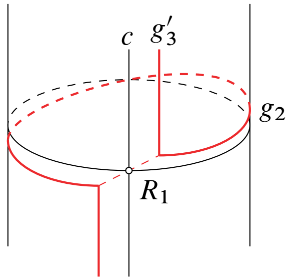

Figure 16. Case $W' \notin g$ and $R_1 = R_2$.

Lemma 5.8. *Let c be a simple closed curve in M such that $\tau(c) = -c$ and $W \notin c$. Let c_0 be the simple closed geodesic in the free homotopy class of c. Then we also have $\tau(c_0) = -c_0$ and $W \notin c_0$.*

Proof. Since $\tau(c) = -c$, the image of c by the ramified covering $\rho\colon M \to \mathbb{CP}^1$ is a simple arc $\bar{c}$ with endpoints in $\{P_1, \dots, P_5\}$ such that $\mathrm{int}(\bar{c}) \cap \{P_0, \dots, P_5\} = \varnothing$. Assume that P_1, P_2 are the endpoints of $\bar{c}$. Let $\tilde{c}$ be a simple closed curve in $\mathbb{CP}^1$ passing through $P_0 = \rho(W)$, which bounds a disc D containing $\bar{c}$ such that $\mathrm{int}(\mathrm{D}) \cap \{P_0, \dots, P_5\} = \{P_1, P_2\}$. Let $\tilde{d}$ be a simple arc in D which joins P_0 to an endpoint of $\bar{c}$. We have

- $\rho^{-1}(\tilde{c})$ is the union of two simple closed curve c_1, c_2, freely homotopic to c, such that $c_1 \cap c_2 = \{W\}$,
- $\rho^{-1}(\tilde{d})$ is a simple closed curve d passing through W such that $\tau(d) = -d$,
- $\rho^{-1}(\mathrm{int}(\mathrm{D}))$ is an open annulus bounded by c_1, c_2 which contains $d \setminus \{W\}$.

Therefore, in the group $\pi_1(M, W)$, we have $[\tau(c_1)] = [c_2]^{-1} = [d][c_1]^{-1}[d]^{-1}$, $[\tau(d)] = [d]^{-1}$. Clearly, in the homology level, we have $\langle c, d\rangle = \pm 1$. The lemma then follows from the same arguments as in Lemma 5.3 b) and c). □

Using Lemma 5.7, we can now prove

Proposition 5.9. *Let $\mathcal{D} = \{a, b_1, b_2, c_1, c_2, d\}$ and $\mathcal{D}' = \{a', b_1', b_2', c_1', c_2', d'\}$ be two admissible decompositions for (M, W). As usual, let (a, b, c, e) and (a', b', c', e') be the symplectic bases of $H_1(M, \mathbb{Z})$ associated to $\mathcal{D}$ and $\mathcal{D}'$ respectively. Assume that b_1 is freely homotopic to b_1', then there exists $\gamma \in \Gamma$ such that*

$$\begin{pmatrix} a' \\ b' \\ c' \\ e' \end{pmatrix} = \gamma \cdot \begin{pmatrix} a \\ b \\ c \\ e \end{pmatrix}.$$

Proof. Let b_0, c_0, c_0' be the simple closed geodesics in the free homotopy classes of b_1, c_1, c_1' respectively. According to Lemma 5.5, we know that b_0, c_0, c_0' verify Property $(\mathscr{P})$ and $b_0 \cap c_0 = b_0 \cap c_0' = \varnothing$.

If $c_0' = \pm c_0$, the proposition follows from Corollary 5.6. Hence, we only need to consider the case where $c_0' \neq c_0$ as subsets of M. In this case, since each of the curves b_0, c_0, c_0' contains exactly two Weierstrass points of M and $W \notin b_0 \cup c_0 \cup c_0'$, we deduce that $c_0' \cap c_0 \neq \varnothing$. Let n be the number of intersections of c_0 and c_0'. The proposition will be proved by induction.

Case $n = 1$. Note that in this case, the intersection of c_0 and c_0' must be a Weierstrass point. We will show that there exist two admissible decompositions $\{\hat{a}, \hat{b}_1, \hat{b}_2, \hat{c}_1, \hat{c}_2, \hat{d}\}$ and $\{\hat{a}', \hat{b}_1', \hat{b}_2', \hat{c}_1', \hat{c}_1', \hat{c}_2', \hat{d}'\}$ such that $\hat{b}_i = \hat{b}_i'$ and $\hat{b}_i, \hat{c}, \hat{c}'$ are freely homotopic to b_0, c_0 and c_0', respectively. We can then use Proposition 5.2 to conclude.

Observe that $\bar{b}_0 = \rho(b_0), \bar{c}_0 = \rho(c_0), \bar{c}_0' = \rho(c_0')$ are three simple arcs on $\mathbb{CP}^1$, which satisfy

- $P_0 \notin (\bar{b}_0 \cup \bar{c}_0 \cup \bar{c}_0')$.
- $\bar{b}_0 \cap (\bar{c}_0 \cup \bar{c}_0') = \varnothing$.
- $\bar{c}_0$ and $\bar{c}_0'$ have a common endpoint and $\mathrm{Card}\{\bar{c}_0 \cap \bar{c}_0'\} = 1$.

We choose the numbering of $\{P_1, \dots, P_5\}$ such that P_1 and P_2 are the endpoints of $\bar{b}_0$, P_3 and P_4 are the endpoints of $\bar{c}_0$ and P_4 and P_5 are the endpoints of $\bar{c}_0'$. Let $\tilde{b}$ be a simple closed curve in $\mathbb{CP}^1$, passing through P_0, which separates $\bar{b}_0$ from $\bar{c}_0 \cup \bar{c}_0'$ (see Figure 17). The pre-image of $\tilde{b}$ in M is the union of two simple closed curves passing through W, denoted by $\hat{b}_1$ and $\hat{b}_2$, which bound an open annulus containing b_0. Let $\tilde{c}$ (resp. $\tilde{c}'$) be a simple closed curve in $\mathbb{CP}^1$ passing through P_0 surrounding $\bar{c}_0$ (resp. $\bar{c}_0'$) as shown in Figure 17. Like $\rho^{-1}(\tilde{b})$, $\rho^{-1}(\tilde{c})$ (resp. $\rho^{-1}(\tilde{c}')$) is the union of two simple closed curves $\hat{c}_1$, $\hat{c}_2$ (resp. $\hat{c}_1'$, $\hat{c}_2'$) which bound an embedded open annulus in M containing c_0 (resp. c_0'). We denote by D_1, D_2 and D_2' the open disks in $\mathbb{CP}^1$ bounded by $\tilde{b}$, $\tilde{c}$ and $\tilde{c}'$ which contain $\bar{b}_0$, $\bar{c}_0$ and $\bar{c}_0'$, respectively.

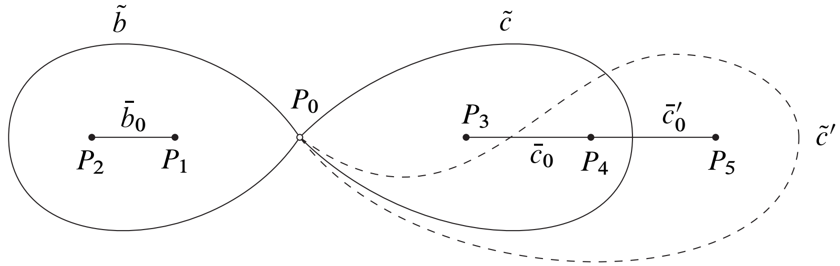

Figure 17. Case $b_0 = b_0'$, $\mathrm{Card}\{c_0 \cap c_0'\} = 1$.

We equip $\hat{b}_1$ and $\hat{b}_2$ with the orientation induced by b_0 (via free homotopy). The numbering is chosen such that the orientation of $\hat{b}_2$ is also the one induced by the orientation of the annulus $\rho^{-1}(\mathrm{D}_1)$. Observe that the family of curves $\{\hat{b}_1, \hat{b}_2, \hat{c}_1, \hat{c}_2\}$ (resp. $\{\hat{b}_1, \hat{b}_2, \hat{c}'_1, \hat{c}'_2\}$) cuts M into two open annulus and one open quadrilateral, which will be denoted by Q (resp. Q$'$). We will choose the numbering and the orientation of $\hat{c}_1, \hat{c}_2$ (resp. $\hat{c}'_1, \hat{c}'$) such that $\hat{b}_1 * \hat{c}_1 * (-\hat{b}_2) * (-\hat{c}_2)$ (resp. $\hat{b}_1 * \hat{c}'_1 * (-\hat{b}_2) * (-\hat{c}'_2)$) is the boundary of Q (resp. Q$'$) with the induced orientation. Remark that it can happen that $\hat{c}_i$ belongs to the free homotopy class of $-c_0$ and $\hat{c}'_i$ belongs to the free homotopy class of $-c'_0$.

We can then add to these families some simple closed curves to obtain two admissible decompositions for (M, W) as follows: let $\tilde{a}$ be a simple arc in D_1 joining P_0 to an endpoint of $\bar{b}_0$, and let $\tilde{d}$ (resp. $\tilde{d}'$) denote a simple arc in D_2 (resp. in D'_2) joining P_0 to an endpoint of $\bar{c}_0$ (resp. of $\bar{c}'_0$). Set $\hat{a} = \rho^{-1}(\tilde{a})$, $\hat{d} = \rho^{-1}(\tilde{d})$ and $\hat{d}' = \rho^{-1}(\tilde{d}')$. By choosing appropriate orientations for $\hat{a}, \hat{d}$, and $\hat{d}'$, we see that $\hat{\mathscr{D}} = \{\hat{a}, \hat{b}_1, \hat{b}_2, \hat{c}_1, \hat{c}_2, \hat{d}\}$ and $\hat{\mathscr{D}}' = \{\hat{a}, \hat{b}_1, \hat{b}_2, \hat{c}'_1, \hat{c}'_2, \hat{d}'\}$ are two admissible decompositions for (M, W).

Clearly, by construction, we have $\hat{c}_i$ (resp. $\hat{c}'_i$) is freely homotopic to $\pm c_0$ (resp. to $\pm c'_0$). Therefore, by Corollary 5.6, the symplectic bases of $H_1(M, \mathbb{Z})$ associated to $\mathscr{D}$ and $\hat{\mathscr{D}}$ are related by an element of Γ. Similarly, the symplectic bases of $H_1(M, \mathbb{Z})$ associated to $\mathscr{D}'$ and $\hat{\mathscr{D}}'$ are also related by an element of Γ. Now, by Proposition 5.2, we know that the symplectic bases associated to $\hat{\mathscr{D}}$ and $\hat{\mathscr{D}}'$ are related by an element of Γ. Hence, the proposition is proven for this case.

Case $n > 1$: By Lemma 5.7, there exists a simple closed geodesic c''_0 verifying Property $(\mathscr{P})$ such that

- $\mathrm{Card}\{c''_0 \cap b_0\} = 0$,
- $\mathrm{Card}\{c''_0 \cap c_0\} = 1$,
- $\mathrm{Card}\{c'_0 \cap c''_0\} < \mathrm{Card}\{c'_0 \cap c_0\} = n$.

Let $\mathscr{D}'' = \{a'', b''_1, b''_2, c''_1, c''_2, d''\}$ be the standard decomposition associated to the pair (b_0, c''_0). The arguments in Case $n = 1$ show that the symplectic bases associated to $\mathscr{D}''$ and $\mathscr{D}$ are related by an element of Γ. Now, since $\mathrm{Card}\{c'_0 \cap c''_0\} < n$, the induction hypothesis implies that the symplectic bases associated to $\mathscr{D}''$ and $\mathscr{D}'$ are also related by an element of Γ, and the proposition follows. □

5.4. Proof of Theorem 5.1. We can now give the proof of Theorem 5.1. Let b_0, c_0 be the simple closed geodesics in the free homotopy classes of b_1 and c_1 respectively. Note that the roles of b_1 and c_1 are interchanged by the S move, and the orientation of b_1 is irrelevant because $-\mathrm{Id} = S^2 \in \Gamma$. Let b'_0 denote the simple closed geodesic in the free homotopy class of b'_1. If $b'_0 = \pm b_0$, or $b'_0 = \pm c_0$, then Proposition 5.9 allows us to conclude immediately. Assume that $b'_0 \neq \pm b_0$ and $b'_0 \neq \pm c_0$. Let n be the number of intersections of b'_0 and $b_0 \cup c_0$. We first remark that $n > 0$, since each of the curves b_0, c_0, b'_0 contains two Weierstrass points, and by definition $W \notin (b_0 \cup c_0 \cup b'_0)$, but there are only six Weierstrass points on M. We proceed by induction.

Case $n = 1$: in this case, we can suppose that $b'_0 \cap b_0 = \varnothing$ and $\mathrm{Card}\{b'_0 \cap c_0\} = 1$. Let $\mathcal{D}''$ be the standard decomposition associated to the pair (b_0, b'_0). By Proposition 5.9, we know that the symplectic bases of $H_1(M, \mathbb{Z})$ associated to $\mathcal{D}$ and to $\mathcal{D}''$ are related by an element of Γ, and the symplectic bases associated to $\mathcal{D}'$ and $\mathcal{D}''$ are also related by an element of Γ, Theorem 5.1 is then proven for this case.

Case $n > 1$: by Lemma 5.7, there exists a simple closed geodesic g verifying Property $(\mathcal{P})$ such that

- $\mathrm{Card}\{g \cap (b_0 \cup c_0)\} = 1$,
- $\mathrm{Card}\{b'_0 \cap g\} < \mathrm{Card}\{b'_0 \cap c_0\}$ if $g \cap b_0 = \varnothing$, and $\mathrm{Card}\{b'_0 \cap g\} < \mathrm{Card}\{b'_0 \cap b_0\}$ if $g \cap c_0 = \varnothing$.

Without loss of generality, we can assume that $g \cap b_0 = \varnothing$. Let $\mathcal{D}''$ be the standard decomposition associated to the pair of geodesics (b_0, g). From Proposition 5.9, we know that the symplectic bases of $H_1(M, \mathbb{Z})$ associated to $\mathcal{D}$ and $\mathcal{D}''$ are related by an element of Γ. Since $\mathrm{Card}\{b'_0 \cap (b_0 \cup g)\} < \mathrm{Card}\{b'_0 \cap (b_0 \cup c_0)\}$, by the induction hypothesis, the symplectic bases of $H_1(M, \mathbb{Z})$ associated to $\mathcal{D}''$ and $\mathcal{D}'$ are related by an element Γ. Theorem 5.1 is then proven. $\square$

6. Proof of Theorem 1.1

6.1. The map Ξ. Let $\mathcal{M}$ denote the quotient $\mathcal{H}(2)/\mathbb{C}^*$. We define the map Ξ from $\mathcal{M}$ to $\Gamma\backslash\mathfrak{S}_2$ as follows: given a pair (M, W) in $\mathcal{M}$, we associate to (M, W) the Γ-orbit of the period matrix of the symplectic homology basis associated to an admissible decomposition for (M, W). It follows from Theorem 5.1 that the map Ξ is well defined. We will show that Ξ is a homeomorphism from $\mathcal{H}(2)/\mathbb{C}^*$ and $\Gamma\backslash\mathcal{J}_2$, which implies Theorem 1.1.

6.2. Injectivity of Ξ. Let (M, W) and (M', W') be two pairs in $\mathcal{M}$. Assume that M and M' are defined by the equations $w^2 = \prod_{i=0}^{5}(z - \lambda_i)$ and $w^2 = \prod_{i=0}^{5}(z - \lambda'_0)$ so that W and W' correspond to λ_0 and λ'_0 respectively. Let Π (resp. Π') be the period matrix of the symplectic basis associated to an admissible decomposition $\mathcal{D}$ (resp. $\mathcal{D}'$) for the pair (M, W) (reps. (M', W')). Assume that there exists an element γ of Γ such that $\Pi' = \gamma \cdot \Pi$. By Lemma 4.2, we know that there exists an admissible decomposition $\widetilde{\mathcal{D}}$ for the pair (M, W) such that the symplectic homology bases of M associated to $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ are related by γ. It follows that the period matrix of the basis associated to $\widetilde{\mathcal{D}}$ is equal to Π'.

Using an element of $\mathrm{PSL}(2,\mathbb{C})$ we can assume that $\lambda_1 = \lambda'_1 = 0, \lambda_2 = \lambda'_2 = 1$, and $\lambda_0 = \lambda'_0 = \infty$. Then from Theorem 2.2, we see that the values of λ_i and λ'_i ($i = 3,4,5$) can be computed by the same theta functions, with the same period matrix. Therefore we have $\lambda_i = \lambda'_i$, $i = 3,4,5$, and it follows that there exists a conformal homeomorphism $\phi\colon M \to M'$ such that $\phi(W) = W'$.

6.3. Surjectivity of Ξ. Let $\tilde{\Pi}$ be a matrix in $\mathfrak{J}_2$, we will show that there exists a pair (M,W) in $\mathscr{M}$ such that $\Xi((M,W)) = \Gamma\cdot\tilde{\Pi}$. Since $\tilde{\Pi}\in\mathfrak{J}_2$, there exists a Riemann surface of genus two M and a symplectic homology basis whose period matrix is $\tilde{\Pi}$. Let W_0 be a Weierstrass point of M, and let $\mathscr{D}$ be an admissible decomposition for the pair (M,W_0). Let Π be the period matrix of the symplectic homology basis of M associated to $\mathscr{D}$. By definition, there exists an element $A\in \mathrm{Sp}(4,\mathbb{Z})$ such that $\tilde{\Pi} = A\cdot\Pi$. According to Lemma 4.3, there exists an admissible decomposition $\tilde{\mathscr{D}}$ for a pair (M,W), where W is also a Weierstrass point of M, such that the symplectic homology bases associated to $\mathscr{D}$ and $\tilde{\mathscr{D}}$ is related by A. Consequently, the period matrix of the basis associated to $\tilde{\mathscr{D}}$ is $\tilde{\Pi}$ and by definition $\Gamma\cdot\tilde{\Pi} = \Xi((M,W))$.

6.4. Continuity of Ξ. To prove the continuity of Ξ we will consider the inverse map $\Xi^{-1}\colon \Gamma\backslash\mathfrak{J}_2 \to \mathscr{M}$. Let Π be a matrix in $\mathfrak{J}_2$, then Π is the period matrix of the symplectic homology basis associated to an admissible decomposition for a pair (M,W) in $\mathscr{M}$. There exist complex numbers $\{\lambda_0,\lambda_1,\dots,\lambda_5\}$ such that M is the surface defined by the equation $w^2 = \prod_{i=0}^5(z-\lambda_i)$. We can assume that W is the Weierstrass point corresponding to λ_0. A neighborhood of Π in $\mathfrak{H}_2$ consists of period matrices of the same symplectic homology basis on Riemann surfaces close to M.

Using $\mathrm{PSL}(2,\mathbb{C})$, we can assume that $\lambda_0 = 0$, it follows that $\omega = \frac{zdz}{w}$ is a holomorphic 1-form with double zero at W. Let $\mathscr{D} = \{a,b_1,b_2,c_1,c_2,d\}$ be an admissible decomposition for the pair (M,W). Let (a,b,c,e) be the symplectic homology basis associated to $\mathscr{D}$, then the map

$$\Phi\colon \mathcal{U}\to\mathbb{C}^4, \quad (M,\omega)\mapsto\Big(\int_a\omega, \int_b\omega, \int_c\omega, \int_e\omega\Big),$$

is a local chart for $\mathscr{H}(2)$ in the neighborhood $\mathcal{U}$ of (M,ω). Let $\rho\colon M\to\mathbb{CP}^1$ be the two-sheeted branched cover from M onto $\mathbb{CP}^1$. Recall that by construction, $\rho(a)=\bar{a}, \rho(b)=\bar{b}_*, \rho(c)=\bar{c}_*, \rho(e)=\bar{e}$ are simple arcs in $\mathbb{CP}^1$ with endpoints in $\{\lambda_0,\dots,\lambda_5\}$. We have

$$\int_a\omega = 2\int_{\bar{a}}\frac{zdz}{w}, \quad \int_b\omega = 2\int_{\bar{b}_*}\frac{zdz}{w}, \quad \int_c\omega = 2\int_{\bar{c}_*}\frac{zdz}{w}, \quad \int_e\omega = 2\int_{\bar{e}}\frac{zdz}{w}.$$

Clearly, the integrals of $\dfrac{zdz}{w}$ along $\bar{a},\bar{b}_*,\bar{c}_*,\bar{e}$ depend continuously on $(\lambda_1,\dots,\lambda_5)$. Since λ_i can be computed from Π by some theta functions, we get a continuous map Ψ from a neighborhood of Π in $\mathfrak{J}_2$ into $\mathbb{C}^4$. Now, in a neighborhood of Π, the map

Ξ^{-1} is the composition of Ψ and the natural projection $\mathbb{C}^4 \setminus \{0\} \to \mathbb{CP}^3$. It follows immediately that Ξ^{-1} is continuous. Since $\dim_{\mathbb{C}} \mathcal{J}_2 = \dim_{\mathbb{C}} \mathscr{M} = 3$, we conclude that Ξ is a homeomorphism.

6.5. $[\mathrm{Sp}(4,\mathbb{Z}) : \Gamma] = 6$. We have a natural projection from $\mathscr{M}$ onto the moduli space of closed Riemann surface of genus two $\mathfrak{M}_2$, which is homeomorphic to $\mathrm{Sp}(4,\mathbb{Z})\backslash\mathcal{J}_2$, by associating to any element (M, W) of $\mathscr{M}$ the point M in $\mathfrak{M}_2$. Every Riemann surface of genus two has six Weierstrass points, and the group of automorphisms of a generic one contains exactly two elements, the identity and the hyperelliptic involution, both fix all the Weierstrass points. Therefore, the pre-image of a generic point in $\mathfrak{M}_2$ contains exactly six points. Note that $-\mathrm{Id}_4 \in \mathrm{Sp}(4,\mathbb{Z})$ acts trivially on $\mathfrak{H}_2$ and the action of $\mathrm{Sp}(4,\mathbb{Z})/\{\pm\mathrm{Id}_4\}$ on $\mathfrak{H}_2$ is effective. Since $\{\pm\mathrm{Id}_4\} \subset \Gamma$, we derive $[\mathrm{Sp}(4,\mathbb{Z}) : \Gamma] = 6$. The proof Theorem 1.1 is now complete. □

Appendices

A. Existence of parallelogram decompositions

Recall that on a translation surface (M, ω), a *saddle connection* is a geodesic segment whose endpoints are singularities of the flat metric (the endpoints need not to be distinct). A *cylinder* is a subset of M which is isometric to $\mathbb{R}\times]0, h[/\mathbb{Z}$ and maximal with respect to this property, where the action of $\mathbb{Z}$ is generated by $(x, y) \mapsto (x+\ell, y)$. The boundary of a cylinder C is the set $\bar{C} \setminus C$, this boundary is a union of saddle connections. We say that C is a *simple cylinder* if the boundary of C is the union of only two saddle connections.

Proposition A.1 (Existence of parallelogram decompositions). *For any translation surface (M, ω) in $\mathcal{H}(2)$, there always exists a parallelogram decomposition on M.*

Proof. We first prove that there always exists a simple cylinder in M. By a well known theorem of Masur (see [8], or [9]), there always exists a closed geodesic γ in M. Let C_γ be the (open) cylinder consisting of closed geodesics freely homotopic to γ. Since M is of genus two, and has only one singular points, we have three cases:

a) C_γ is a simple cylinder.

b) $M \setminus \bar{C}_\gamma$ is a simple cylinder.

c) $\bar{C}_\gamma = M$.

If a) or b) occurs then we are done. If we are in case c), then the pair (M, ω) is obtained from a single parallelogram as shown in Figure 18. Therefore, in this situation, one can easily find a simple cylinder in another direction.

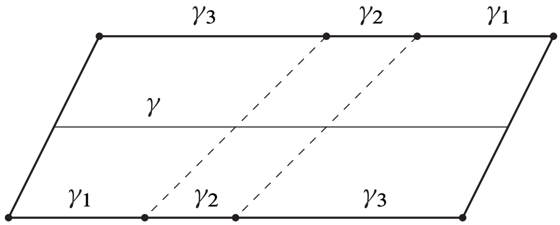

Figure 18. Surface admitting a decomposition in one cylinder.

Remark. This argument has been used in the proof of Theorem 7.1 in [10].

Let C_1 be a simple cylinder in M. Let b_1, b_2 denote the two saddle connections which bound C_1. Cutting off C_1 from M along b_1 and b_2, we get a torus minus two open disks whose boundaries meet at one point, which is the unique singular point p of M. Splitting the point p into two points, we obtain then a flat torus with an open disk removed whose boundary is the union of two parallel segments of same length. Gluing these two segments together, we obtain a flat torus M_1 with a marked simple geodesic arc b. Denote by p_1, p_2 the endpoints of b in M_1. Let us show that there always exist two parallel closed geodesic c_1, c_2 in M_1 such that $c_1 \cap b = \{p_1\}$, and $c_2 \cap b = \{p_2\}$.

Choose a direction $\theta \in [0; 2\pi]$ which is not parallel to b and consider the geodesic flow ψ_θ of M_1 in this direction. Let t_0 be the smallest value of t such that $t_0 > 0$ and $\psi_\theta^{t_0}(b) \cap b \neq \varnothing$. The value t_0 must be finite because otherwise the area of stripe $S_t = \bigcup_{0 \leqslant s \leqslant t} \psi_\theta^s(b)$ tends to infinity as $t \to \infty$, which is impossible.

Since $\psi_\theta^{t_0}(b)$ and b are parallel and have the same length, $\psi_\theta^{t_0}(b)$ contains at least one endpoint of b. Without loss of generality, we can assume that $\psi_\theta^{t_0}(b)$ contains p_1. Remark that the stripe $S_{t_0} = \bigcup_{0 \leqslant s \leqslant t_0} \psi_\theta^s(a)$ is the image of a parallelogram P under an isometric immersion $\varphi\colon \mathrm{P} \to M$ such that the restriction of φ to int(P) is injective, and b and $\psi_\theta^{t_0}(b)$ are the images of two opposite sides of P. By assumption, $\varphi^{-1}(p_1)$ consists of two points, which belong to two opposite sides of P. It follows that the image of the segment in P joining these two points by φ is a closed geodesic c_1 on M_1 which satisfies $c_1 \cap b = \{p_1\}$. Consequently, the closed geodesic c_2 passing through p_2 and parallel to c_1 also verifies $c_2 \cap b = \{p_2\}$.

By construction, we can identify $M_1 \setminus b$ with the complement of $\overline{C}_1$ in M. By this identification, the closed geodesics c_1 and c_2 correspond to two saddle connections which bound an open cylinder C_2 disjoint from C_1, and the complement of $\overline{C}_1 \cup \overline{C}_2$ is an open disk isometric to an open parallelogram in $\mathbb{R}^2$.

Let a (resp. d) be a saddle connection contained in $\overline{C}_1$ (resp. $\overline{C}_2$) which intersects all the closed geodesics parallel to b_1 and b_2 (resp. parallel to c_1 and c_2). One can easily check that the family $\{a, b_1, b_2, c_1, c_2, d\}$, with appropriate orientations, is a parallelogram decomposition of M. □

It follows from this proposition that, on any surface (M, ω) in $\mathcal{H}(2)$, there exist infinitely many parallelogram decompositions, since if we have one, we can get infinitely many others by using elementary moves T, S, R. It is also possible to show that any two parallelogram decompositions are related by elementary moves.

B. Proof of Lemma 3.4

For any $g \geqslant 1$, let σ be the permutation of $\{1, 2, \dots, 2g\}$ that transpose $2i$ and $2i-1$, for $i = 1, \dots, g$. The *elementary symplectic matrices* are the matrices

$$E_{ij} = \begin{cases} \mathrm{Id}_{2g} + e_{ij} & \text{if } i = \sigma(j); \\ \mathrm{Id}_{2g} + e_{ij} - (-1)^{i+j} e_{\sigma(j)\sigma(i)} & \text{otherwise,} \end{cases}$$

where $i \neq j$, and e_{ij} is the matrix whose the (i, j)-th entry is 1 and all other entries are 0. It is a classical fact that $\mathrm{Sp}(2g, \mathbb{Z})$ is generated by elementary symplectic matrices ([3], Chapter 7). For the case $g = 2$, we have

- $E_{12} = E_{21}^t = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E_{34} = E_{43}^t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$
- $E_{13} = E_{42}^{-1} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad E_{31} = E_{24}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$
- $E_{14} = E_{32} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E_{41} = E_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$

All we need is to verify that E_{ij} $(i \neq j)$ is contained in the group Γ' generated by $\{T, R, S, U\}$. It is clear that E_{12}, E_{34}, E_{43} belong to $\Gamma \subset \Gamma'$. We have

- $E_{21} = U^{-1}T^{-1}U \in \Gamma'$.
- $E_{13} = SU \in \Gamma'$.

Since Γ contains

$$\begin{pmatrix} \mathrm{Id}_2 & 0 \\ 0 & \mathrm{SL}(2, \mathbb{Z}) \end{pmatrix},$$

we have

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \Gamma,$$

therefore

$$S_2 = U^{-1} S_1 U = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma'.$$

It follows that

- $E_{31} = (S_1 S_2) E_{13}^{-1} (S_1 S_2)^{-1} \in \Gamma'$,
- $E_{14} = S_1 E_{13} S_1^{-1} \in \Gamma'$,
- $E_{41} = S_2 E_{13} S_2^{-1} \in \Gamma'$.

The lemma is then proven. □

C. A family of Γ right cosets in $\mathrm{Sp}(4, \mathbb{Z})$

In this section, we give a partition of the group $\mathrm{Sp}(4, \mathbb{Z})$ into Γ right cosets. Recall that Γ is not a normal subgroup of $\mathrm{Sp}(4, \mathbb{Z})$ (see Lemma 3.3), and the group $\mathrm{Sp}(4, \mathbb{Z})$ is generated by the matrices T, S, R and U (see Lemma 3.4). Set

$$\mathcal{F} = \{\Gamma, U \cdot \Gamma,\ RU \cdot \Gamma,\ SRU \cdot \Gamma,\ URU \cdot \Gamma,\ USRU \cdot \Gamma\}.$$

By Lemma 3.3, we know that the action of Γ on $(\mathbb{Z}/2\mathbb{Z})^4 \setminus \{0\}$ has two orbits $\mathcal{O}_1$ and $\mathcal{O}_2$, therefore we have a simple criterion to show that an element of $\mathrm{Sp}(4, \mathbb{Z})$ does *not* belong to Γ. Consequently, it is easy to verify that the elements in the family $\mathcal{F}$ are all distinct.

We will also determine explicitly the action of $T^{\pm 1}$, $R^{\pm 1}$, $S^{\pm 1}$, $U^{\pm 1}$ on $\mathcal{F}$ by multiplication from the left. Note that, since $S^{-1} = -S$ (resp. $U^{-1} = -U$) and $-\mathrm{Id}_4 \in \Gamma$, the actions of S and S^{-1} (resp. U and U^{-1}) are identical. Details of the calculations are lengthly and uninteresting, hence will be omitted. The final result is resumed in the following table.

	Γ	$U \cdot \Gamma$	$RU \cdot \Gamma$	$SRU \cdot \Gamma$	$URU \cdot \Gamma$	$USRU \cdot \Gamma$
T	Γ	$U \cdot \Gamma$	$RU \cdot \Gamma$	$URU \cdot \Gamma$	$SRU \cdot \Gamma$	$USRU \cdot \Gamma$
R	Γ	$RU \cdot \Gamma$	$U \cdot \Gamma$	$SRU \cdot \Gamma$	$URU \cdot \Gamma$	$USRU \cdot \Gamma$
S	Γ	$U \cdot \Gamma$	$SRU \cdot \Gamma$	$RU \cdot \Gamma$	$USRU \cdot \Gamma$	$URU \cdot \Gamma$
U	$U \cdot \Gamma$	Γ	$URU \cdot \Gamma$	$USRU \cdot \Gamma$	$RU \cdot \Gamma$	$SRU \cdot \Gamma$
T^{-1}	Γ	$U \cdot \Gamma$	$RU \cdot \Gamma$	$URU \cdot \Gamma$	$SRU \cdot \Gamma$	$USRU \cdot \Gamma$
R^{-1}	Γ	$RU \cdot \Gamma$	$U \cdot \Gamma$	$SRU \cdot \Gamma$	$URU \cdot \Gamma$	$USRU \cdot \Gamma$

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