

Residual properties of groups defined by basic commutators

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*For Pierre de la Harpe
on the occasion of his 70th birthday*

Abstract. In this paper we study the residual nilpotence of groups defined by basic commutators. We prove that the so-called Hydra groups as well as certain of their generalizations and quotients are, in the main, residually torsion-free nilpotent. By way of contrast we give an example of a group defined by two basic commutators which is not residually torsion-free nilpotent.

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1. Introduction

Let \mathcal{P} be a property or class of groups. Then a group G is termed residually \mathcal{P} if for each $g \in G$, $g \neq 1$, there exists a normal subgroup N of G such that $g \notin N$ and $G/N \in \mathcal{P}$. In 1935 Wilhelm Magnus [28] proved that free groups are residually torsion-free-nilpotent and as a corollary that an $n < \infty$ generator group with the same lower central quotients as a free group of rank n is free. The genesis of this paper is an earlier proof of ours, which we include here, that the so-called Hydra groups, which are one-relator groups defined by basic commutators and recently introduced by Dison and Riley [13], are residually torsion-free nilpotent. Whether all one-relator groups defined by basic commutators are residually torsion-free nilpotent remains to be determined. Here we prove that a number of one-relator groups defined by basic commutators, and some of their generalizations, are residually torsion-free nilpotent. Whether residual torsion-free nilpotence, can be used to further our understanding of the isomorphism problem for one-relator groups seems worth exploring further.

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A little history. There is now a large body of work devoted to residual properties of groups and in particular, to residual nilpotence. Perhaps the first residual property of free groups was obtained by F. W. Levi [25] in 1930 who proved in particular that free groups are residually finite 2-groups. In 1935 W. Magnus [28] proved an even stronger theorem, namely that free groups are residually torsion-free nilpotent. Since K. W. Gruenberg [15] later proved that finitely generated torsion-free nilpotent groups are residually finite p -groups for every prime p , Magnus' theorem is indeed a generalization of Levi's theorem. A. I. Malcev [32] subsequently extended Magnus' theorem to free products of torsion-free nilpotent groups by proving that the free product of residually torsion-free nilpotent groups is again residually torsion-free nilpotent. Proceeding in a different direction, A. I. Lichtman [26] has studied the residual nilpotence of the multiplicative group of a skew field generated by universal enveloping algebras. Much work has also focussed on the residual nilpotence of groups which are free in certain solvable varieties of groups, including the variety of all solvable groups of a given derived length [15]. The proof by B. Hartley [19] that the wreath products of torsion-free abelian groups by torsion-free nilpotent groups are residually torsion-free nilpotent touches on this work of Gruenberg [15]. The techniques used by Gruenberg as well as that of Hartley [19] make use of basic commutators, to be defined below, which go back to P. Hall's fundamental work on finite groups of prime power order [16] and his so-called collection process and the work of M. Hall [18]. A key ingredient of this work is the introduction and use of Lie and associative rings in furthering our understanding of the lower central sequence of a group that goes back to P. Hall [16], and was developed among others by M. Lazard [24] and W. Magnus (cf. [31]) and E. Witt [36] who proved that the sub-Lie rings of a free Lie ring are again free. Here we will make much use of the fundamental work of J. P. Labute [23] and his study of the Lie ring of a one-relator group, which given the right hypothesis, turns out to be a one-relator Lie ring. A brief survey of the residual nilpotence of a number of groups and some additional results and references can be found in the work of the first author [6].

Much of the foregoing discussion has focussed on the purely combinatorial group theoretic aspects of residually nilpotent groups. They arise naturally, not only in combinatorial group theory but also in many geometric problems which involve knots and links, arrangements of hyperplanes, homotopy theory, four manifolds and many other parts of mathematics. In particular, J. R. Stallings [35] proved that the lower central quotients of a fundamental group are invariant under homology cobordism of manifolds. His result underpins Milnor and Massey product invariants of links, which have analogues for knots in arbitrary 3-manifolds using suitable variations of the lower central series. The homological properties of finitely generated parafree groups, i.e., those finitely generated residually nilpotent groups with the same lower central quotients as free groups, play an important role in low dimensional topology, see, e.g., Cochran and Orr [12]. Finitely generated non-free parafree groups exist in profusion [2] and the work of Bousfield [11] contains a large number of infinitely generated examples.

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2. Our main results

2.1. The Hydra groups. In [13], Dison and Riley introduced a family of one-relator groups

$$G(k, a, t) = \langle a, t \mid [a, \underbrace{t, \dots, t}_k] = 1 \rangle \quad (k \geq 1)$$

which they termed Hydra groups. These groups are infinite cyclic extensions of finitely generated free groups. Indeed if we put $a_0 = a$, and $a_i = [a_{i-1}, t]$ for $i = 1, \dots, k-1$, then the subgroup H of $G(k, a, t)$ generated by a_0, \dots, a_{k-1} is a free normal subgroup of $G(k, a, t)$ and $G(k, a, t)$ is an infinite cyclic extension of H . Dison and Riley [13] proved that the subgroup of $G(k, a, t)$ generated by $a_0t, \dots, a_{k-1}t$ when $k > 1$, has extremely large distortion, by contrast with the finitely generated subgroups of free groups which have linear distortion. Here we will prove the following theorem.

Theorem 1. *The Hydra groups $G(k, a, t)$ are residually torsion-free nilpotent.*

So the Hydra groups like free groups are residually torsion-free nilpotent.

2.2. Generalizations of the Hydra groups. We shall use Theorem 1 in the proof of the following two theorems, Theorem 2 and Theorem 3, which taken together and given the right conditions, amount to a considerable generalization of Theorem 1.

Theorem 2. *Let X and Y be disjoint sets of generators and let F be the free group on $Z = X \cup Y$. Let u be an element in the subgroup of F generated by X and v an element in the subgroup of F generated by Y . If u and v are not proper powers, then for every $k > 1$*

$$G(k, u, v) = \langle Z \mid r(u, v) = 1 \rangle,$$

where $r(u, v) = [u, \underbrace{v, \dots, v}_k]$, is residually a finite p -group for every prime p .

If we ally Theorem 2 with Theorem L (see below), a theorem of J. P. Labute [23], which we will describe in due course, it is easy to deduce the following theorem, where we adopt the notation used in the formulation of Theorem 2.

Theorem 3. *Suppose that u and v are basic commutators and that $k > 1$. Then $G(k, u, v)$ is residually torsion-free nilpotent.*

Notice that we assume that $k > 1$. We will consider the case where $k = 1$ separately.

2.3. Big power groups and residually torsion-free nilpotent groups defined by simple commutators. Theorem 1 is at the heart of the proof of Theorem 2. Our first proof of Theorem 3 made heavy use of a rather different result which depends on a property of free groups now termed the *big powers property*. This property was used initially to prove that certain HNN extensions of a very simple type called extensions of centralizers, are residually free in Baumslag [1] and, as it turns out, a little earlier by R. C. Lyndon [27].

Theorem 4. *Suppose that*

$$G = \langle x_1, \dots, x_n, t \mid [u, t] = 1 \rangle,$$

where u is a word in x_1, \dots, x_n which is not a proper power. Then G is residually free.

We will still avail ourselves use of this property, which was the genesis of our next theorem, as well as some further results which are needed to prove Theorem 5.

Theorem 5. *Let u be an element in the free group F on X and v be an element in the free group E on Y . Suppose that u and v are not proper powers and that*

$$G = \langle X \cup Y \mid [u, v] = 1 \rangle.$$

Then

- (1) G is residually a finite p -group for every prime p .
- (2) If $u \in \gamma_j(F)$, $u \notin \gamma_{j+1}(F)$ and if $v \in \gamma_k(E)$, $v \notin \gamma_{k+1}(E)$ and if u is not a proper power modulo $\gamma_{j+1}(F)$ and v is not a proper power modulo $\gamma_{k+1}(E)$ then G is residually torsion-free nilpotent. In particular, if u and v are basic commutators, then G is residually torsion-free nilpotent.

So for instance the one-relator groups

$$G(x_1, \dots, x_n) = \langle x_1, \dots, x_n \mid [x_1, \dots, x_n] = 1 \rangle$$

and

$$G = \langle a, b, c, d \mid [[a, b], [c, d, d]] = 1 \rangle$$

are residually torsion-free nilpotent. Actually more is true in some special instances since the groups $G(x_1, \dots, x_n)$ are even residually free.

Labute's Theorem L, already cited above, adds to what is already known about some residual properties of certain groups, termed cyclically pinched one-relator groups, as detailed in Section 2.4 below.

2.4. Cyclically pinched one-relator groups. A one-relator group G is termed cyclically pinched if it is an amalgamated product of two free groups with a cyclic subgroup amalgamated:

$$G = \langle x_1, \dots, x_m, y_1, \dots, y_n \mid u(x_1, \dots, x_m) = v(y_1, \dots, y_n) \rangle,$$

where here $u = u(x_1, \dots, x_m)$ and $v(y_1, \dots, y_n)$ are non-trivial elements respectively in the free groups on $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$. There has been considerable attention paid to these cyclically pinched one-relator groups since they are generalizations of the fundamental groups of surfaces. Here we will add a little more to what is already known by invoking one of Labute's fundamental theorems in order to prove

Theorem 6. *Let G be the amalgamated product*

$$G = \langle x_1, \dots, x_m, y_1, \dots, y_n \mid u(x_1, \dots, x_m) = v(y_1, \dots, y_n) \rangle.$$

If u and v are basic commutators then G is residually torsion-free nilpotent.

So for example it follows that

$$G = \langle x_1, x_2, y_1, y_2 \mid [x_1, x_2] = [y_1, y_2, y_2] \rangle$$

is residually torsion-free nilpotent. In addition, it also follows that if u_1, \dots, u_q are basic commutators in disjoint sets of generators coming from the set $\{x_1, \dots, x_n\}$ then the following variation of Theorem 4 holds: the group

$$\langle x_1, \dots, x_n \mid [u_1, \dots, u_q] = 1 \rangle$$

is residually torsion-free nilpotent. These examples should be compared with a number of other related examples (cf., e.g., [3] and [8]).

2.5. Examples. In the closing sections of this paper, Sections 11 and 12, we will discuss some examples defined by two basic commutators. They are all quotients of the Hydra groups. Some of them are residually torsion-free nilpotent and some are not. The most important and most difficult to prove example is the following, which we record as our final theorem, in Section 11.

Theorem 7. *The group*

$$\langle a, t \mid [a, t, t] = [a, t, a, a, a] = 1 \rangle$$

is not residually torsion-free nilpotent.

We will discuss some additional examples in Section 12.

3. The arrangement of this paper

We will introduce some of the notation to be used here in Section 4.1. In Section 4.2 we record a little of what we will need in order to use the Lie ring of a group and we remind the reader of the definition of basic sequences and basic commutators in Section 4.3. Section 4.4 records some of the work of Kim and McCarron that we will need here, conveniently re-stated as Theorem KM, and Labute's important theorem is described in Section 4.5 as Theorem L. Finally we will need a theorem of P. Hall which we describe in Section 4.6. The proofs of our theorems are given in the subsequent sections labelled by the theorems being proved ending with Section 11 where Theorem 7 is proved, and Section 12, where some additional examples are discussed.

4. Notation, definitions and some basic theorems of Kim and McCarron, Labute and Philip Hall

4.1. Notation. As usual, if $x, y, a_1, \dots, a_{k+1}$ are elements of a group G we set $[x, y] = x^{-1}y^{-1}xy$, $x^y = y^{-1}xy$ and define

$$[a_1, a_2, \dots, a_{k+1}] = [[a_1, \dots, a_k], a_{k+1}] \quad (k > 1).$$

The lower central series of G is defined inductively by setting $\gamma_1(G) = G$, $\gamma_{n+1}(G) = [G, \gamma_n(G)]$.

4.2. The Lie ring of a group. Each of the factor groups $L_n = \gamma_n(G)/\gamma_{n+1}(G)$ is an abelian group and will often be written additively. We now put

$$L(G) = \bigoplus_{n=1}^{\infty} L_n.$$

$L(G)$ can be turned into a Lie ring (over \mathbb{Z}) by defining a binary operation, denoted $[x, y]$, first on the L_n and extended by linearity to all of $L(G)$, as follows:

$$[a\gamma_{i+1}(G), b\gamma_{j+1}(G)] = (a^{-1}b^{-1}ab)\gamma_{i+j+1}(G) \quad (a \in \gamma_i(G), b \in \gamma_j(G)).$$

We term $L(G)$ the Lie ring of G . The following references can be consulted by a reader interested in the construction of such Lie rings, see for example [16] and [31].

We will need also here the definition of a basic sequence and a basic commutator.

4.3. Basic commutators. Let

$$X = \{x_1, \dots, x_q\}$$

be a non-empty finite set and let G be the free groupoid generated by X . So the elements of G are simply the bracketed products of the elements of X and two such products are equal only if they are identical. The number of factors $|g|$ in such a product g is termed the length or weight of g . A sequence b_1, b_2, \dots of elements of G is termed a basic sequence in X if

- (1) every element of X occurs in the sequence;
- (2) if $|b_i| < |b_j|$, then $i < j$;
- (3) if $u = vw$ ($v, w \in G$) is an element of G of length at least 2, u occurs in the above sequence if and only if $v = b_i$, $w = b_j$ and $j < i$ and either $|b_i| = 1$ or $b_i = b_k b_\ell$ and $\ell \leq j$.

The terms in a basic sequence are called basic commutators. The proof of the existence of such basic sequences can be found for example in [18]. Now if F is a free group, freely generated by the set X and if $\gamma_n(F)$ denotes the n^{th} term of the lower central series of F then Wilhelm Magnus [28] proved that the basic commutators of weight n freely generate modulo $\gamma_{n+1}(F)$ the free abelian group $\gamma_n(F)/\gamma_{n+1}(F)$.

4.4. The theorems of Kim and McCarron. We will need special cases of Theorems 3.4 and 4.2 in Kim and McCarron [22] which, for convenience, we record here as Theorem KM. These theorems make use of what they call a p -preimage closed subgroup of a group. Here we will use a more customary notation, namely that of a p -group separated subgroup. This more directly reflects what is needed in the proof that certain amalgamated products of residually finite p -groups are again residually finite p -groups.

Definition. A subgroup H of a residually finite p -group G is termed p -group separated in G if for each element $g \in G$, $g \notin H$, there exists a homomorphism ϕ from G into a finite p -group such that $\phi(g) \notin \phi(H)$.

We are now in position to formulate the following theorem.

Theorem KM. (1) *If G is residually a finite p -group, if G' is a copy of G and if H' denotes the copy of H in G' , then the amalgamated product $G *_H H' G'$ is residually a finite p -group if and only if H is p -group separated in G .*

- (2) *Let p be a prime, A and B residually finite p -groups and let $c \in A$, $d \in B$ be elements of infinite order. If the subgroup of A generated by c is p -group separated in A and if the subgroup of B generated by d is p -group separated in B then the amalgamated product $A *_c B$ is residually a finite p -group.*

4.5. Labute's theorem. The theorem of Labute mentioned above is then the following one.

Theorem L. *Let F be a free group freely generated by X , let r be an element of F and let $G = \langle X \mid r = 1 \rangle$. Suppose that $r \in \gamma_n(F)$, $r \notin \gamma_{n+1}(F)$ and that $r\gamma_{n+1}(F)$ is not a proper power (i.e., multiple) in L_n . Then the Lie ring of G is additively free abelian. If G is residually nilpotent, then G is residually torsion-free nilpotent.*

4.6. Hall's theorem. There is yet another theorem that we will need in this paper, due to Philip Hall [17]:

Theorem H. *Let G be a group with a normal nilpotent subgroup H . Suppose that $G/[H, H]$ is nilpotent and that $H/[H, H]$ is nilpotent. Then G is nilpotent.*

5. The Hydra groups are residually torsion-free nilpotent

We begin with the proof of Proposition 1.

5.1. Proposition 1. We will show that the proof of Theorem 1 is a consequence of the following proposition which is itself an easy consequence of Philip Hall's Theorem H.

Proposition 1. *Let G be a group with a normal residually torsion-free nilpotent subgroup H . If G/H is torsion-free nilpotent and if $G/[H, H]$ is nilpotent, then G is residually torsion-free nilpotent.*

It suffices to prove that if $g \in H$, $g \neq 1$, then there is normal subgroup K_g of G such that $g \notin K_g$ with G/K_g torsion-free nilpotent. Since H is residually torsion-free nilpotent, there is a characteristic subgroup K_g of H which does not contain g with H/K_g torsion-free nilpotent. Moreover, since K_g is characteristic in H , K_g is normal in G . So G/K_g is an extension of a torsion-free nilpotent group H/K_g by a torsion-free nilpotent group G/H . Hence G/K_g is torsion-free. Now put $\bar{G} = G/K_g$, $\bar{H} = H/K_g$. Then $\bar{G}/[\bar{H}, \bar{H}] \cong G/[H, H]$ is nilpotent by assumption which by Hall's Theorem H, implies that $\bar{G} = G/K_g$ is nilpotent. Since $g \notin K_g$ and G/K_g is torsion-free nilpotent, this proves Proposition 1.

5.2. The proof of Theorem 1. We are now in a position to prove that the Hydra groups

$$G(k, a, t) = \langle a, t \mid \underbrace{[a, t, \dots, t]}_k = 1 \rangle \quad k \geq 1$$

are residually torsion-free nilpotent. Since $G(1, a, t)$ is free abelian of rank 2, it suffices to assume that $k > 1$. As we noted above, $G = G(k, a, t)$ is an infinite cyclic

extension of the free subgroup H generated by k elements $a_0 = a, a_i = [a_{i-1}, t]$ for $i = 1, \dots, k - 1$. Now free groups are residually torsion-free nilpotent. Moreover $G/\gamma_2(H)$ is clearly nilpotent of class k . So it follows by Proposition 1 that G is residually torsion-free nilpotent, as claimed.

6. The proof of Theorem 2

We turn now to the generalizations $G(k, u, v)$ of the Hydra groups discussed in Section 2.2, where

$$G(k, u, v) = \langle X, Y \mid r(u, v) = 1 \rangle,$$

$r(u, v) = [u, \underbrace{v, \dots, v}_k]$ and neither u nor v is a proper power. Our objective is to prove, given the appropriate hypothesis, that the $G(k, u, v)$ are residually torsion-free nilpotent. The first step in the proof, which depends heavily on Theorem KM, which is due to Kim and McCarron [22] and detailed in Section 4.4, is to prove that they are residually finite p -groups for every prime p . The proof is divided up into a number of steps which involve centralizers of elements and separation properties of various subgroups.

6.1. Centralizers of elements in the Hydra groups. We will need some more information about the Hydra groups:

Lemma 1. *The centralizer of a in*

$$G(k, a, t) = \langle a, t \mid \underbrace{[a, t, \dots, t]}_k = 1 \rangle \quad (k > 1)$$

is generated by a .

Proof. Let H be the normal closure in $G = G(k, a, t)$ of a . Then, adopting the notation introduced in Section 5.2 in the course of the proof of Theorem 1, we have already noted that H is free on the a_j and that G is the semi-direct product of H and the infinite cyclic group on t . In addition, t acts on H as follows:

$$t^{-1}a_0t = a_0a_1, \dots, t^{-1}a_{k-2}t = a_{k-2}a_{k-1}, t^{-1}a_{k-1}t = a_{k-1}.$$

Note that $a = a_0$.

Suppose that $g \in G(k, t, a)$ and that $[g, a] = 1$. Since $G(k, t, a) = H \rtimes \langle t \rangle$, $g = ht^n$ ($h \in H$). We can assume that $n \geq 0$. Then

$$a_0 = g^{-1}a_0g = t^{-n}h^{-1}a_0ht^n.$$

Now working modulo $\gamma_2(H)$, we find $a_0 = t^{-n}a_0t^n$. It follows, again working modulo $\gamma_2(H)$, that

$$t^{-n}a_0t^n = a_0a_1^n \dots,$$

where the terms following a_1^n are words in a_2, \dots, a_{k-1} . Hence these words are trivial in H and $n = 0$. It follows that we have proved that

$$a_0 = h^{-1}a_0h.$$

But H is free and therefore the centralizer of a_0 in H is a power of a_0 . This completes the proof. \square

Now denote the free group on Y by $F(Y)$. Then one of the consequences of Lemma 1 is

Lemma 2. *Let $k \geq 2$, let $v \in F(Y)$ be an element which is not a proper power in $F(Y)$ and let*

$$J = \langle Y, a, t \mid \underbrace{[a, t, \dots, t]}_k = 1, t = v \rangle.$$

Then the centralizer of a in J is the cyclic subgroup $\langle a \rangle$.

Proof. Observe that J is an amalgamated product of $G(k, t, a)$ and $F(Y)$:

$$J = G(k, t, a) *_{t=v} F(Y).$$

Suppose that g lies in the centralizer of a in J . Then g can be written in the form

$$g = e_1 \dots e_n,$$

where each e_i is from one of the factors $F(Y)$ or $G(k, a, t)$, and successive e_i, e_{i+1} come from different factors. Since $ag = ga$ and a does not belong to the subgroup generated by t it is not hard to see that $n = 1$. In this case, we have $ae_1 = e_1a$. It follows that $e_1 \in G(k, t, a)$ in which case Lemma 1 applies which means that e_1 is a power of a as claimed. \square

6.2. p -group separated subgroups. We will need now to make use of the work of Kim and McCarron on residually finite p -groups. Our objective is to use the fact that $G(k, u, v)$ is an amalgamated product in which the amalgamated subgroups satisfy a separation property. This is accomplished by the following lemma.

Lemma 3. *The subgroup $\langle t \rangle$ is p -group separated in $G(k, a, t)$ for all primes p .*

Proof. Finitely generated, residually torsion-free nilpotent groups are residually finite p -groups for every choice of the prime p ; see [15]. So $G(k, a, t)$ is residually a finite p -group by Theorem 1. Consequently, it follows from Kim and McCarron's

Theorem **KM** as described in Section 4.4, that $\langle a \rangle$ is p -group separated in $G(k, a, t)$ if and only if the group

$$\tilde{G}(k, a, t) := \langle a, t, c \mid \underbrace{[a, t, \dots, t]}_k = \underbrace{[c, t, \dots, t]}_k = 1 \rangle$$

is residually a finite p -group. Now $\tilde{G}(k, a, t)$ is the middle of a short exact sequence

$$1 \longrightarrow H_{2k} \longrightarrow \tilde{G}(k, a, t) \longrightarrow \langle t \rangle \longrightarrow 1$$

where H_{2k} is the free subgroup generated by $\{a, a^t, \dots, a^{t^{k-1}}, c, c^t, \dots, c^{t^{k-1}}\}$, and the quotient of $\tilde{G}(k, a, t)$ by H_{2k} is cyclic and generated by the image of t . It follows along the same lines as in the proof of Theorem 1 that the group $\tilde{G}(k, a, t)$, with a slight abuse of notation, is residually the semi-direct product of the groups $(H_{2k}/\gamma_i(H_{2k}) \rtimes \langle t \rangle)$ ($i \geq 1$). Moreover, the groups $H_{2k}/\gamma_i(H_{2k}) \rtimes \langle t \rangle$ are finitely generated, torsion-free nilpotent for all $i \geq 1$. Hence $\tilde{G}(k, a, t)$ is residually a finite p -group and therefore $\langle t \rangle$ is p -subgroup separated in $G(k, a, t)$, as claimed. \square

Next we have the following lemma.

Lemma 4. *Let*

$$J = \langle Y, a \mid \underbrace{[a, v, \dots, v]}_k = 1 \rangle,$$

where v is not a proper power in $F(Y)$. Then J is residually a finite p -group for every prime p .

Proof. Observe that J is an amalgamated product:

$$J = F(Y) *_{v=t} G(k, a, t).$$

In order to prove that J is residually a finite p -group we again have to appeal to Theorem **KM**. We have already proved that $\langle t \rangle$ is p -subgroup separated in $G(k, a, t)$. Now by Theorem 4, since v is not a proper power, $L = \langle Y, t \mid [t, v] = 1 \rangle$ is residually free. So the subgroup M of L generated by Y and $t^{-1}Yt$ is residually a finite p -group. But M is an amalgamated product of two copies of $gp(Y)$ amalgamating $gp(v)$. Hence $gp(v)$ is p -group separated in $gp(Y)$. It follows that J is residually a finite p -group by Theorem **KM**. \square

The last step before we come to the proof of Theorem 2 is the following lemma.

Lemma 5. *The subgroup $\langle a \rangle$ is p -subgroup separated in J .*

Proof. Suppose that $c \in J$ is such that for every homomorphism ϕ from J into a finite p -group, $\phi(c) \in \phi(\langle a \rangle)$. It follows that $[c, a]$ lies in every normal subgroup of J of index a power of p . Since J is residually a finite p -group, it follows that $[c, a] = 1$. So if $k > 1$, c is a power of a since the centralizer of a in J is generated by a . \square

We are now in position to complete the proof of Theorem 2. Present $G_k(u, v)$ as an amalgamated product

$$G(k, u, v) = J *_{a=u} F(X).$$

Since J is residually a finite p -group, $\langle a \rangle$ is p -subgroup separated in J , and $\langle u \rangle$ is p -subgroup separated in $F(X)$, the group $G(k, u, v)$ is residually a finite p -group by Theorem KM.

7. The proof of Theorem 3

The proof of Theorem 3 follows almost immediately from Labute's Theorem L. In order to explain why, suppose now that F is the free group on $Z = X \cup Y$. Because u and v are basic commutators, say of weights m and n , respectively, in disjoint sets of generators $r(u, v)$ is a basic commutator of weight $m + kn$. Consequently $r(u, v) \in \gamma_{m+kn}(F)$ and $r(u, v) \notin \gamma_{m+kn+1}(F)$. Moreover $r(u, v)\gamma_{m+kn+1}(F)$ is an element in a basis for $\gamma_{m+kn}(F)/\gamma_{m+kn+1}(F)$ and hence is not a proper power. Since $G(k, u, v)$ is residually a finite p -group, by Theorem 2, it follows by Theorem L, that $G(k, u, v)$ is residually torsion-free nilpotent.

8. The proof of Theorem 4

Suppose that u is an element of a free group F that is not a proper power and that G is the one-relator group on $X \cup \{t\}$ defined by the single relation $[u, t] = 1$. Then G is residually free by [2]. Now $G(x_1, \dots, x_n)$ is simply obtained from the free group on x_1, \dots, x_{n-1} by adding an additional generator x_n and the relation $[[x_1, \dots, x_{n-1}], x_n] = 1$. By a theorem of Magnus, Karrass, and Solitar [31] a non-trivial commutator in a free group is not a proper power. Hence Theorem 4 is an immediate consequence of [1].

9. The proof of Theorem 5

It turns out that the main step in the proof of Theorem 5 is the following lemma.

Lemma 6. *Let G be the one-relator group*

$$G = \langle X \cup \{t\} \mid [u, t] = 1 \rangle,$$

where u is an element in the free group E on X which is not a proper power. Then the following hold:

- (1) G is residually a finite p -group for each prime p ;
- (2) the subgroup of G generated by t is finitely p -group separable in G .

That G is residually a finite p -group follows, as previously noted, because G is even residually free. However in order to prove (2), we need some additional information. To this end, notice that if A is the free abelian group on s and t , if H is the subgroup of E generated by u , K is the subgroup of A generated by s , then G can be viewed as an amalgamated product of E and A with H amalgamated with K according to the isomorphism ϕ mapping u to s :

$$G = \langle E * A \mid H =_{\phi} K \rangle.$$

Now free groups and free abelian groups are residually finite p -groups. Moreover, H is finitely p -group separable in E since u generates its centralizer in E and K is clearly finitely p -group separable in A . So it follows from Theorem KM that G is residually a finite p -group. This again proves (1).

We are left with the proof of (2), that is if $g \in G$ and $g \notin T = \langle t \rangle$, then there is a homomorphism ϕ_g of G into a finite p -group such that $\phi_g(g) \notin \phi_g(T)$. The proof will be divided up into a number of cases.

- (1) $g \in H$, $g \neq 1$. Of course, $g \notin T$. Since E is residually a finite p -group, there exists a normal subgroup I of E of index a power of p such that $g \notin I$. Define $\phi_g : G \rightarrow E/I$ which maps E onto E/I and A onto E/I by sending s onto uI and t to the identity. Then $\phi_g(g) \notin \phi_g(T)$, as required.
- (2) $g \in E$, $g \notin H$. Then $[g, u] \neq 1$. Choose a normal subgroup I_1 of E of index a power of p such that $[g, u] \notin I_1$. Then $gI_1 \notin TI_1$ for otherwise $[g, u] \in I_1$. Then uI_1 is a non-trivial element of E/I_1 of order a power of p . $\phi_g : G \rightarrow E/I_1$ is defined first on E and then on A . We take it to be the canonical homomorphism of E onto E/I_1 . Next we define ϕ_g to be the homomorphism of A to G/I_1 which maps s to uI_1 and t to the identity. Let I be the kernel of ϕ_g . Then $\phi_g(g) \neq 1$ and $\phi_g(T) = \{1\}$. So $\phi_g(g) \notin \phi_g(T)$ as needed.
- (3) $g \in A$, $g \notin T$. Then $g = s^k t^\ell$ where $k \neq 0$. Since E is residually a finite p -group we can choose a homomorphism of E into a finite p -group so that the image of u^k has arbitrarily large order a power of p . In addition, there exists a homomorphism of A into a finite group so that the image of s has arbitrarily large finite order and the image of t is 1. It follows that there exists a homomorphism ϕ_g of G into a finite p -group P which maps T to $\{1\}$ and g to an element outside the image of T .

(4) $g = f_1 a_1 \dots f_n a_n$ where the $f_j \in E$, $f_j \notin H$, $a_j \in A$, $a_j \notin K$. Notice that if $g \notin A$, then

$$tg = t f_1 a_1 \dots f_n a_n \neq f_1 a_1 \dots f_n a_n t,$$

because $f_1 \notin A$. Now G is residually a finite p -group and $[g, t] \neq 1$. Hence there is a homomorphism ϕ_g of G into a finite p -group which maps $[g, t]$ to a non-trivial element. So $\phi_g(g) \notin \phi_g(T)$ since this implies that $\phi_g([g, t]) = 1$.

Since $G(u, v)$ is an amalgamated product of two residually finite p -groups where the amalgamated subgroups are cyclic and finitely p -group separable, $G(u, v)$ is residually a finite p -group. Now suppose that $u \in \gamma_j(F)$, $u \notin \gamma_{j+1}(F)$, $u\gamma_{j+1}(F)$ is not a proper power and that $v \in \gamma_k(F)$, $u \notin \gamma_{k+1}(F)$, $v\gamma_{k+1}(F)$ is not a proper power, then it is not hard to prove that $[u, v] \in \gamma_{j+k}(F)$, $[u, v] \notin \gamma_{j+k+1}(F)$ and that $[u, v]\gamma_{j+k+1}(F)$ is not a proper power. So Theorem L applies and therefore $G(u, v)$ is residually torsion-free nilpotent.

10. The proof of Theorem 6

Our objective now is prove Theorem 6, namely that if

$$G = \langle x_1, \dots, x_m, y_1, \dots, y_n \mid u(x_1, \dots, x_m) = v(y_1, \dots, y_n) \rangle$$

and if u and v are basic commutators, then G is residually torsion-free nilpotent. Now Magnus, Karrass and Solitar [30] have proved that in a free group, a non-trivial commutator is not a proper power. Consequently G is residually a finite p -group [3]. In addition, uv^{-1} is then a product of a basic commutator of weight k , say, and the inverse of a second basic commutator of weight ℓ , say, in a disjoint set of generators. So if q is the minimum of k and ℓ , then $uv^{-1} \in \gamma_q(F)$, $uv^{-1} \notin \gamma_{q+1}(F)$, where F is the free group on $x_1, \dots, x_m, y_1, \dots, y_n$ and $uv^{-1}\gamma_{q+1}(F)$ is not a proper power in $\gamma_q(F)/\gamma_{q+1}(F)$. So by Labute's Theorem L, G is residually torsion-free nilpotent.

11. The proof of Theorem 7

In this and the subsequent section we shall use Proposition 1 to define several examples of quotients of the Hydra groups which are residually torsion-free nilpotent. We begin first with the proof of Theorem 7, namely that the group

$$D = \langle a, t \mid [a, t, t] = [a, t, a, a, a] = 1 \rangle$$

is not residually a finite p -group if $p \neq 2$. As noted previously, Gruenberg [15] proved that finitely generated torsion-free nilpotent groups are residually finite p -groups for all primes p . So D is not residually torsion-free nilpotent. The proof is complicated and will be carried out by means of a series of lemmas.

Our proof uses the well-known Hall–Witt identity several times. Recall it for the readers convenience. For elements A, B, C of a group, the following identity is satisfied:

$$[A, B^{-1}, C]^B [B, C^{-1}, A]^C [C, A^{-1}, B]^A = 1. \quad (1)$$

In the rest of this section, we will use the notation $w = [a, t, a, a, t, a]$. We will show that $w \neq 1$ is a generalized 2-torsion element of G . That is, for every $n \geq 1$, the order of w is a power of 2 modulo $\gamma_n(G)$.

Lemma 7. *For every $c \in [G, G]$ and $n \geq 2$,*

$$[a, t, a, a, t, c] \in [\langle w \rangle^G, G] \gamma_n(G),$$

where $\langle w \rangle^G$ is the normal closure of w in G .

Proof. We will prove the statement for $c = [a, t]$. The general case clearly will follow, since the general element of $[G, G]$ can be presented as a product

$$\prod_j [a, t]^{\pm a^{l_j} q_j}, \quad l_j \in \mathbb{Z}, \quad q_j \in [G, G].$$

Denote $v := [a, t, a, a, t, [a, t]]$. The Hall–Witt identity (see (1) with $A = [a, t, a, a]$, $B = t^{-1}$, and $C = [a, t]$) implies that

$$[a, t, a, a, t, [a, t]]^{t^{-1}} [t^{-1}, [t, a], [a, t, a, a]]^{[a, t]} [a, t, [a, t, a, a]^{-1}, t^{-1}]^{[a, t, a, a]} = 1.$$

The relation $[a, t, t] = 1$ implies that $[t^{-1}, [t, a]] = 1$ and therefore,

$$[a, t, a, a, t, [a, t]] = [a, t, [a, t, a, a]^{-1}, t^{-1}]^{-[a, t, a, a]t}. \quad (2)$$

Applying the Hall–Witt identity one more time (see (1) with $A = a$, $B = t^{-1}$, and $C = [a, t, a, a]^{-1}$), we get

$$[a, t, [a, t, a, a]^{-1}]^{t^{-1}} [t^{-1}, [a, t, a, a], a]^{[a, t, a, a]^{-1}} [[a, t, a, a]^{-1}, a^{-1}, t^{-1}]^a = 1. \quad (3)$$

The relation $[a, t, a, a, a] = 1$ together with (2) implies that

$$\begin{aligned} [a, t, [a, t, a, a]^{-1}] &= [t^{-1}, [a, t, a, a], a]^{-[a, t, a, a]^{-1}t} \\ &= [a, t, a, a, t, a^t]^{-t^{-1}[a, t, a, a]^{-1}t} \\ &= [a, t, a, a, t, a[a, t]]^{-t^{-1}[a, t, a, a]^{-1}t} \in \langle w \rangle^G \langle v \rangle^G. \end{aligned}$$

Now, identity (2) implies that

$$v \in [\langle w \rangle^G, G][\langle v \rangle^G, G]$$

and the needed statement follows by induction on n . □

Lemma 8. For every $n \geq 2$, $[a, t, a^{-1}, [a, t, a]] \in [\langle w \rangle^G, G] \gamma_n(G)$.

Proof. First observe that

$$\begin{aligned} [a, t, a^{-1}, [a, t, a]] &= [[a, [a, t]]^{a^{-1}}, [a, t, a]] \\ &= [a, [a, t], a^{-1}, [a, t, a]] \\ &= [[a, t, a, a]^{a^{-1}}, [a, t, a]]^{[a, [a, t]]} \\ &= [[a, t, a, a], [a, t, a]]^{[a, [a, t]]}. \end{aligned} \tag{4}$$

Denote $E = [a, t, a, a]$. The relation $[E, a] = 1$ implies that $[E^{\pm 1}, a^{\pm 1}] = 1$. The Hall–Witt identity (see (1) with $A = [a, t]$, $B = a^{-1}$, $C = E$) implies that

$$[a, t, a, E]^{a^{-1}} [a^{-1}, E^{-1}, [a, t]]^E [E, [t, a], a^{-1}]^{[a, t]} = 1.$$

Hence

$$[E, [a, t, a]] = [E, [t, a], a^{-1}]^{[a, t]a}.$$

Applying the Hall–Witt identity one more time, we get

$$[E, [a, t, a]] = [[E, t^{-1}, a^{-1}]^{ta}, a^{-1}]^{[a, t]a} = [[t, E]^{t^{-1}}, a^{-1}]^{ta}, a^{-1}].$$

The needed statement follows from Lemma 7 and identity (4). \square

Lemma 9. For every n ,

$$[t, a, [a, t, a]^{-1}, a] \equiv w^{-1} \pmod{[\langle w \rangle^G, G] \gamma_n(G)}.$$

Proof. The Hall–Witt identity (see 1 with $A = t$, $B = a^{-1}$, $C = [a, [a, t]]$) implies that

$$[t, a, [a, [a, t]]]^{a^{-1}} [a^{-1}, [a, t, a], t]^{[a, t, a]^{-1}} [a, [a, t], t^{-1}, a^{-1}]^t = 1. \tag{5}$$

Since $[a, t, a, a, a] = 1$, one has

$$[a^{-1}, [a, t, a], t] = [a, t, a, a, t].$$

By Lemma 7, for every $n \geq 2$,

$$[[a, t, a, a, t]^{[a, t, a]^{-1}}, a] \equiv w \pmod{[\langle w \rangle^G, G] \gamma_n(G)}.$$

Now, identity (5) implies that the needed statement of the lemma is equivalent to the statement that, for every $n \geq 2$,

$$[[a, [a, t], t^{-1}, a^{-1}]^t, a] \in [\langle w \rangle^G, G] \gamma_n(G). \tag{6}$$

The Hall–Witt identity (see (1) with $A = a$, $B = [t, a]$, $C = t^{-1}$) and the relation $[a, t, t] = 1$ imply that

$$[a, [a, t], t^{-1}]^{[t, a]} [t, a, a^{-1}, [t, a]]^{t^{-1}a} = 1.$$

We can rewrite the last identity as

$$[a, t, a, t]^d [[a, t], [a, t, a]]^e = 1,$$

where $d = t^{-1}[a, [a, t]][t, a]$, $e = [a, t, a]a[t, a]^2t^{-1}a$. The Hall–Witt identity (see (1), with $A = a$, $B = t^{-1}$, and $C = [a, t, a]$) implies that

$$[[a, t], [a, t, a]]^{t^{-1}} [t^{-1}, [a, t, a]^{-1}, a]^{[a, t, a]} [a, t, a, a^{-1}, t^{-1}]^a = 1.$$

Therefore, there exist elements $c_1, c_2, c_3 \in G$, such that

$$[a, t, a, t] = [a, t, a, t, a^{c_1}]^{-c_2} [a, t, a, a, t]^{c_3}. \quad (7)$$

By Lemma 7, for every $n \geq 2$ and $g_1, g_2 \in \langle a \rangle^G$,

$$[[a, t, a, a, t], g_1, g_2] \in [\langle w \rangle^G, G] \gamma_n(G).$$

Now, identity (7) implies that, for every $n \geq 2$ and $g_1, g_2 \in \langle a \rangle^G$,

$$[a, t, a, t, g_1, g_2] \in [\langle w \rangle^G, G] \gamma_n(G)$$

and (6) follows. □

We are now in a position to complete the proof of Theorem 7.

Proof of Theorem 7. We claim that if $w = [a, t, a, a, t, a]$, then

$$w \notin \gamma_7(G), \quad w^2 \in \gamma_7(G).$$

This can be proved directly or by appealing to GAP, as the following GAP fragment shows:

```
gap > F:=FreeGroup(2);;
a:=F.1;; t:=F.2;;
gap>G:=F/[LeftNormedComm([a, t, t]),
LeftNormedComm([a, t, a, a, a])];;
gap>phi:=NqEpimorphismNilpotentQuotient(G, 6);;
gap>aa:=Image(phi, G.1);;
tt:=Image(phi, G.2);;
gap>xx:=LeftNormedComm([aa, tt, aa, aa, tt, aa]);;
gap>Order(xx);
2
```

Now we will show that w is a generalized 2-torsion element. That is, for every $n \geq 1$, the order of w is a power of 2 modulo $\gamma_n(G)$.

The Hall–Witt identity (see (1) with $A = [a, t, a, a]$, $B = t^{-1}$, and $C = a$) together with the relation $[a, t, a, a, a] = 1$ implies that

$$[a, t, a, a, t, a]^{t^{-1}} [t^{-1}, a^{-1}, [a, t, a, a]]^a = 1.$$

Hence,

$$[a, t, a, a, t, a]^{t^{-1}} [[a, t^{-1}], [a, t, a, a]] = 1. \tag{8}$$

The relation $[a, t, t] = 1$ implies that

$$[a, t^{-1}] = [t, a]^{t^{-1}} = [t, a].$$

Identity (8) can be rewritten as

$$[a, t, a, a, t, a]^{t^{-1}} [[t, a], [a, t, a, a]] = 1. \tag{9}$$

It follows from the Hall–Witt identity (see (1) with $A = [a, t, a]$, $B = a^{-1}$, and $C = [t, a]$) that

$$[a, t, a, a, [t, a]]^{a^{-1}} [a^{-1}, [a, t], [a, t, a]]^{[t, a]} [t, a, [a, t, a]^{-1}, a^{-1}]^{[a, t, a]} = 1. \tag{10}$$

The second term of (10) lies in $[\langle w \rangle^G, G]\gamma_n(G)$ for every $n \geq 2$, by Lemma 8. The third term of relation (10) is equivalent to w modulo $[\langle w \rangle^G, G]\gamma_n(G)$ for every $n \geq 2$, by Lemma 9. Now relations (9) and (10) imply that, for every n ,

$$w^2 \equiv [\langle w \rangle^G, G]\gamma_n(G). \quad \square$$

12. Two more examples

Example 1. Let

$$G_k = \langle a, t \mid \underbrace{[a, t, \dots, t]}_k = 1, [a, t, \dots, t, \underbrace{e, \dots, e}]_{k-2, \ell} = 1 \rangle,$$

where $e = \underbrace{[a, t, \dots, t]}_{k-1}$. Then G_k is residually torsion-free nilpotent for every $\ell \geq 1$.

It is worth noting that the simplest non-nilpotent group of such type is the following:

$$\langle a, t \mid [[a, t], [a, t, t]] = 1, [a, t, t, t] = 1 \rangle.$$

To show that the above groups satisfy the hypothesis of Proposition 1, consider the following generators of the normal closure of a :

$$c_1 = a, c_2 = [a, t], \dots, c_j = [a, t, \dots, t]_{j-1}, \quad j = 1, \dots, k.$$

The action of $\langle t \rangle$ on these generators is given as already discussed in the proof of Theorem 1. Writing the relator $[[a, \underbrace{t, \dots, t}_{k-2}, \underbrace{e, \dots, e}_\ell]$ in terms of the generators c_1, \dots, c_k , we find that it is $[c_{k-1}, \underbrace{c_k, \dots, c_k}_\ell]$. Recall that a free product of residually torsion-free nilpotent groups is residually torsion-free nilpotent [32]. The group $H = \langle a \rangle^{G_k}$ (using the notation in Proposition 1) is the free product

$$F(c_1, \dots, c_{k-2}) * \langle c_{k-1}, c_k \mid [c_{k-1}, \underbrace{c_k, \dots, c_k}_\ell] = 1 \rangle$$

which is residually torsion-free nilpotent by Theorem 1. The group G_k is residually torsion-free nilpotent by Proposition 1.

Next we have

Example 2. For $k, s \geq 1$, the group

$$\langle a, t \mid [[a, \underbrace{t, \dots, t}_s], [a, \underbrace{t, \dots, t}_{k-1}]] = 1, [a, \underbrace{t, \dots, t}_k] = 1 \rangle. \tag{11}$$

is residually torsion-free nilpotent.

Again, denoting the images of the c_i -s as before simply as c_i , we see that the subgroup H which is the normal closure of the element a can be presented in the form

$$\langle c_1, \dots, c_k \mid [c_i, c_k] = 1, i = s + 1, \dots, k - 1 \rangle$$

which is isomorphic to the group $F(c_1, \dots, c_s) * (F(c_{s+1}, \dots, c_{k-1}) \times \langle c_k \rangle)$, which is clearly residually torsion-free nilpotent. Conditions of the Proposition 1 are satisfied, hence the group defined by (11) is residually torsion-free nilpotent. A simple example of a group of this kind is

$$\langle a, t \mid [[a, t], [a, t, t, t]] = 1, [a, t, t, t, t] = 1 \rangle.$$

Remark. Observe that, for $k \geq 1$, the groups

$$\langle a, t \mid [a, \underbrace{t, \dots, t}_{k-1}, a] = 1, [a, \underbrace{t, \dots, t}_k] = 1 \rangle$$

are residually nilpotent by the following result from [33]: any central extension of a one-relator residually nilpotent group is residually nilpotent.

As we noted at the outset, we have been unable to determine whether a one-relator group defined by a basic commutator is residually torsion-free nilpotent, or residually a finite p-group or even residually finite. The best that we have managed to find is an example of a group defined by two relations, which are basic commutators, which is not even residually a finite p-group.

Remark. For a free group on two generators, the seventh term of the lower central series is the normal closure of all basic commutators of weight 7, 8, 9, and 10 (see [14]). It follows from the proof of Theorem 7 that the group

$$\langle a, t \mid [a, t, t] = [a, t, a, a, a] = 1, \text{ all basic commutators of weight } 7, 8, 9, 10 \rangle$$

has 2-torsion, namely

$$[a, t, a, a, t, a]^2 = 1, \quad [a, t, a, a, t, a] \neq 1.$$

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